# Undecidability

of first-order modal and intuitionistic logics with two variables and one monadic predicate letter\*

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#### Abstract

We prove that the positive fragment of first-order intuitionistic logic in the language with two individual variables and a single monadic predicate letter, without functional symbols, constants, and equality, is undecidable. This holds true regardless of whether we consider semantics with expanding or constant domains. We then generalise this result to intervals [QBL,QKC] and [QBL,QFL], where QKC is the logic of the weak law of the excluded middle and QBL and QFL are first-order counterparts of Visser's basic and formal logics, respectively. We also show that, for most "natural" first-order modal logics, the two-variable fragment with a single monadic predicate letter, without functional symbols, constants, and equality, is undecidable, regardless of whether we consider semantics with expanding or constant domains. These include all sublogics of QKTB, QGL, and QGrz—among them, QK, QT, QKB, QD, QK4, and QS4.

## 1 Introduction

While the (first-order) quantified classical logic **QC1** is undecidable [6], it contains a number of rather expressive decidable fragments [3]. This has long stimulated interest in drawing the borderline between decidable and undecidable fragments of **QC1** using a variety of criteria, in isolation or in combination, imposed on the language. One such criterion is the number and arity of predicate letters allowed in the language: while the monadic fragment is decidable [1], the fragment containing a single binary letter is not, as follows from [9]. Another criterion is the number of individual variables allowed in

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the language: while the two-variable fragment is decidable [17, 10], the three-variable fragment is not [24].

Similar questions have long been of interest in (first-order) quantified intuitionistic and modal logics. For languages without restrictions on the number of individual variables, Kripke [14] has shown that all "natural" quantified modal logics with two monadic predicate letters are undecidable, while Maslov, Mints, and Orevkov [15] and, independently, Gabbay [8] have shown that quantified intuitionistic logic with a single monadic predicate letter is undecidable.

The question of where the borderline lies in the intuitionistic and modal case when it comes to the number of individual variables allowed in the language has recently been investigated by Kontchakov, Kurucz, and Zakharyschev in [12]. It is shown in [12] that two-variable fragments of quantified intuitionistic and all "natural" modal logics are undecidable. Moreover, it is established in [12] that, to obtain undecidability of two-variable fragments, in the intuitionistic case, it suffices to use two binary and infinitely many monadic predicate letters, while in the modal case, it suffices to use only (infinitely many) monadic predicate letters.

Two questions were raised in [12] concerning the languages combining restrictions on the number of individual variables and predicate letters: first, how many monadic predicate letters are needed to obtain undecidability of the two-variable fragments in the modal case, and second, whether it suffices to use monadic predicate letters to obtain undecidability of the two-variable fragment in the intuitionistic case.

In the present paper, we address both of the aforementioned questions. First, we show that for two-variable fragments of most modal logics considered in [12], it suffices to use a single monadic predicate letter to obtain undecidability. Second, we show that the positive fragment of quantified intuitionistic logic **QInt** is undecidable in the language with two variables and a single monadic predicate letter. We also show that the latter result holds true for all logics in intervals [**QBL**, **QKC**] and [**QBL**, **QFL**], where **QKC** is the logic of the weak law of the excluded middle and **QBL** and **QFL** are first-order counterparts of Visser's basic and formal logics, respectively.

The paper is structured as follows. In section 2, we prove undecidability results about modal logics. In section 3, we do likewise for the intuitionistic and related logics. We conclude, in section 4, by discussing how our results can be applied in settings not considered in this paper and pointing out some open questions following from our work.

## 2 Modal logics

In this section, we prove undecidability results about two-variable fragments of quantified modal logics with a single monadic predicate letter. This is essentially achieved by adapting to the first-order language of Halpern's technique [11] for establishing complexity results for single-variable fragments of propositional modal logics.

#### 2.1 Syntax and semantics

A (first-order) quantified modal language contains countably many individual variables; countably many predicate letters of every arity; Boolean connectives  $\wedge$  and  $\neg$ ; modal connective  $\square$ ; and a quantifier  $\forall$ . Formulas as well as the symbols  $\vee$ ,  $\rightarrow$ ,  $\exists$ , and  $\diamondsuit$  are defined in the usual way. We also use the following abbreviations:  $\square^+\varphi = \varphi \wedge \square\varphi$  and  $\diamondsuit^+\varphi = \varphi \vee \diamondsuit\varphi$ .

A Kripke frame is a tuple  $\mathfrak{F} = \langle W, R \rangle$ , where W is a non-empty set (of worlds) and R is a binary (accessibility) relation on W. A predicate Kripke frame is a tuple  $\mathfrak{F}_D = \langle W, R, D \rangle$ , where  $\langle W, R \rangle$  is a Kripke frame and D is a function from W into a set of non-empty subsets of some set (the domain of  $\mathfrak{F}_D$ ), satisfying the condition that wRw' implies  $D(w) \subseteq D(w')$ . We call the set D(w) the domain of w. We will also be interested in predicate frames satisfying the condition that wRw' implies D(w) = D(w'); we refer to such frames as frames with constant domains.

A Kripke model is a tuple  $\mathfrak{M} = \langle W, R, D, I \rangle$ , where  $\langle W, R, D \rangle$  is a predicate Kripke frame and I is a function assigning to a world  $w \in W$  and an n-ary predicate letter P an n-ary relation I(w, P) on D(w). We refer to I as the interpretation of predicate letters with respect to worlds in W.

An assignment in a model is a function g associating with every individual variable x an element of the domain of the underlying frame.

The truth of a formula  $\varphi$  in a world w of a model  $\mathfrak M$  under an assignment g is inductively defined as follows:

- $\mathfrak{M}, w \models^g P(x_1, \dots, x_n) \text{ if } \langle g(x_1), \dots, g(x_n) \rangle \in I(w, P);$
- $\mathfrak{M}, w \models^g \varphi_1 \land \varphi_2$  if  $\mathfrak{M}, w \models^g \varphi_1$  and  $\mathfrak{M}, w \models^g \varphi_2$ ;
- $\mathfrak{M}, w \models^g \neg \varphi_1 \text{ if } \mathfrak{M}, w \not\models^g \varphi_1;$
- $\mathfrak{M}, w \models^g \Box \varphi_1$  if wRw' implies  $\mathfrak{M}, w' \models^g \varphi_1$ , for every  $w' \in W$ ;
- $\mathfrak{M}, w \models^g \forall x \varphi_1$  if  $\mathfrak{M}, w \models^{g'} \varphi_1$ , for every assignment g' such that g' differs from g in at most the value of x and such that  $g'(x) \in D(w)$ .

Note that, given a Kripke model  $\mathfrak{M} = \langle W, R, D, I \rangle$  and  $w \in W$ , the tuple  $\mathfrak{M}_w = \langle D_w, I_w \rangle$ , where  $D_w = D(w)$  and  $I_w(P) = I(w, P)$ , is a classical predicate model.

We say that  $\varphi$  is true at world w of model  $\mathfrak{M}$  and write  $\mathfrak{M}, w \models \varphi$  if  $\mathfrak{M}, w \models^g \varphi$  holds for every g assigning to free variables of  $\varphi$  elements of D(w). We say that  $\varphi$  is true in  $\mathfrak{M}$  and write  $\mathfrak{M} \models \varphi$  if  $\mathfrak{M}, w \models \varphi$  holds for every world w of  $\mathfrak{M}$ . We say that  $\varphi$  is true in predicate frame  $\mathfrak{F}_D$  and write  $\mathfrak{F}_D \models \varphi$  if  $\varphi$  is true in every model based on  $\mathfrak{F}_D$ . We say that  $\varphi$  is true in frame  $\mathfrak{F}$  and write  $\mathfrak{F} \models \varphi$  if  $\varphi$  is true in every predicate frame of the form  $\mathfrak{F}_D$ . Finally, we say that a formula is true in a class of frames if it is true in every frame from the class.

Let  $\mathfrak{M} = \langle W, R, D, I \rangle$  be a model,  $w \in W$ , and  $a_1, \ldots, a_n \in D(w)$ . Let  $\varphi(x_1, \ldots, x_n)$  be a formula whose free variables are among  $x_1, \ldots, x_n$ . We write  $\mathfrak{M}, w \models \varphi[a_1, \ldots, a_n]$  to mean  $\mathfrak{M}, w \models^g \varphi(x_1, \ldots, x_n)$ , where  $g(x_1) = a_1, \ldots, g(x_n) = a_n$ .

Given a propositional normal modal logic L, let  $\mathbf{Q}L$  be  $\mathbf{QCl} \oplus L$  where  $\oplus$  is the operation of closure under (predicate) substitution, modus ponens, generalization, and necessitation. Of particular interest to us are the quantified counterparts QGL, QGrz, and QKTB of propositinal logics GL, QGrz, and KTB. We recall that GL is the logic of Kripke frames whose accessibility relation is irreflexive, transitive, and contains no infinite ascending chains, while **Grz** is the logic of frames whose accessibility relation is reflexive, transitive, antisymmetric, and does not contain infinite ascending chains of pairwise distinct worlds (in other words, the accessibility relation on the frames for Grz is the reflexive closure of the one on the frames for GL). We also recall that QGL and **QGrz** are Kripke-incomplete [16, 20], but are valid on all the frames for **GL** and **Grz**, respectively. Thus, for technical reasons—namely, to avoid being distracted with Kripkecompleteness—we define logics  $\mathbf{QGL}^{sem}$  and  $\mathbf{QGrz}^{sem}$  as the sets of quantified formulas true in all the frames of GL and Grz, respectively. What is important for us is that  $\mathbf{QGL} \subseteq \mathbf{QGL}^{sem}$  and  $\mathbf{QGrz} \subseteq \mathbf{QGrz}^{sem}$ . Lastly, we recall that KTB is the logic of Kripke frames whose accessibility relation is reflexive and symmetric and that **QKTB** is complete with respect to this class of frames.

Given a logic L and a closed formula  $\varphi$  in the language of L, we say that  $\varphi$  is L-satisfiable if  $\neg \varphi \notin L$ . If L is complete with respect to a class  $\mathfrak{C}$  of frames, L-satisfiability of  $\varphi$  amounts to  $\varphi$  being true at a world of a model based on a frame in  $\mathfrak{C}$ .

We now turn to addressing the question, raised in [12], of how many monadic predicate letters are needed in the language of quantified modal logics to obtain undecidability of their two-variable fragments. Using suitable adaptations of a technique originally proposed in [11], and further refined in [4], [22], and [23], for propositinal languages, we show that all sublogics of **QGL**, **QGrz**, and **QKTB** are undecidable in the language with a single monadic predicate letter.

## 2.2 Sublogics of QGL and QGrz

In this section, we prove that all sublogics of **QGL** and **QGrz** are undecidable in languages with two individual variables and a single monadic predicate letter.

In the proof, we rely on the undecidability result by Kontchakov, Kurucz, and Za-kharyaschev [12] concerning first-order modal logics with two variables: we use their formulas encoding an undecidable tiling problem as a basis for our reduction. Even thought their formulas are not suitable as are for our purposes, they can be readily modified, as explained in the proof of Theorem 2.6 below, to be used in our argument.

We begin with a description of a satisfiability-preserving transformation of formulas that we subsequently, in the proof of Theorem 2.6, apply to a slightly modified version of the formulas obtained by Kontchakov, Kurucz, and Zakharyaschev [12].

Let  $\varphi$  be a (closed) formula containing monadic predicate letters  $P_1, \ldots, P_n$ . Let  $P_{n+1}$  be a monadic predicate letter distinct from  $P_1, \ldots, P_n$  and let  $B = \forall x P_{n+1}(x)$ . Define an embedding  $\cdot'$  as follows:

$$P_i(x)' = P_i(x), \text{ where } i \in \{1, \dots, n\};$$
  
 $(\neg \phi)' = \neg \phi';$   
 $(\phi \land \psi)' = \phi' \land \psi';$   
 $(\forall x \phi)' = \forall x \phi';$   
 $(\Box \phi)' = \Box (B \to \phi').$ 

**Lemma 2.1.** Let  $L \in \{QK, QGL^{sem}, QGrz^{sem}\}$ . Then,  $\varphi$  is L-satisfiable if, and only if,  $B \wedge \varphi'$  is L-satisfiable.

**Proof.** Assume that  $\mathfrak{M}, w_0 \models \varphi$ , for some  $\mathfrak{M}$  based on a frame for L and some  $w_0$ . Let  $\mathfrak{M}'$  be a model that extends  $\mathfrak{M}$  by setting  $I(w, P_{n+1}) = D(w)$ , for every  $w \in W$ . Then,  $\mathfrak{M}', w_0 \models B \land \varphi'$ . Conversely, assume that  $\mathfrak{M}, w_0 \models B \land \varphi'$ , for some  $\mathfrak{M}$  based on a frame for L. Let  $\mathfrak{M}'$  be a submodel of  $\mathfrak{M}$  with  $W' = \{w : \mathfrak{M}, w \models B\}$ . Then,  $\mathfrak{M}', w_0 \models \varphi$ . Note that, for every logic L in the statement of the Lemma,  $\mathfrak{M}'$  is based on a frame for L.  $\square$ 

**Remark 2.2.** In view of the proof of Lemma 2.1, if  $B \wedge \varphi'$  is satisfied in a model  $\mathfrak{M}$ , we can assume, without a loss of generality, that B is true in  $\mathfrak{M}$ .

Now, given a monadic predicate letter P, we inductively define the following sequence of formulas:

$$\begin{array}{lcl} \delta_1(x) & = & P(x) \land \Diamond (\neg P(x) \land \Diamond \Box^+ P(x)); \\ \delta_{m+1}(x) & = & P(x) \land \Diamond (\neg P(x) \land \Diamond \delta_m(x)). \end{array}$$

Using formulas from this sequence, define, for every  $k \in \{1, ..., n+1\}$ , the formula

$$\alpha_k(x) = \delta_k(x) \wedge \neg \delta_{k+1}(x) \wedge \Diamond \Box^+ \neg P(x).$$

We now define models associated with formulas  $\alpha_k(x)$ . For every  $k \in \{1, \ldots, n+1\}$ , let  $\mathfrak{F}_k = \langle W_k, R_k \rangle$  be a Kripke frame where  $W_k = \{w_k^0, \ldots, w_k^{2k}\} \cup \{w_k^*\}$  and  $R_k$  is the transitive closure of the relation  $\{\langle w_k^i, w_k^{i+1} \rangle : 0 \leq i < 2k\} \cup \{\langle w_k^0, w_k^* \rangle\}$ . For every such k, let  $\mathfrak{M}_k = \langle W_k, R_k, D, I \rangle$  be a model with constant domains and let a be an individual in the domain of every  $\mathfrak{M}_k$  (other than that, the relationship between the domains of  $\mathfrak{M}_k$ s is immaterial at this point). We say that  $\mathfrak{M}_k$  is a-suitable if

$$\mathfrak{M}_k, w \models P[a] \iff w = w_k^{2i}, \text{ for } i \in \{0, \dots, k\}.$$

**Lemma 2.3.** Let a be an individual in the domain of the models  $\mathfrak{M}_1, \ldots, \mathfrak{M}_{n+1}$  and let  $\mathfrak{M}_1, \ldots, \mathfrak{M}_{n+1}$  be a-suitable. Then,

$$\mathfrak{M}_k, w \models \alpha_m[a] \iff k = m \text{ and } w = w_k^0.$$

**Proof.** Straightforward.

**Remark 2.4.** Notice that the statement of Lemma 2.3 holds true if we replace the accessibility relations in  $\mathfrak{M}_1, \ldots, \mathfrak{M}_{n+1}$  with their reflexive closures.

Now, for every  $\alpha_k(x)$ , where  $k \in \{1, \ldots, n+1\}$ , define

$$\beta_k(x) = \neg P(x) \land \Diamond \alpha_k(x).$$

Let  $\varphi^*$  be the result of replacing in  $\varphi'$  of  $P_k(x)$  with  $\beta_k(x)$ , for every  $k \in \{1, \ldots, n+1\}$ .

Call a formula  $\psi$  containing only monadic predicate letters L-suitable if either  $\psi$  is not L-satisfiable or  $\psi$  is satisfiable in a model  $\mathfrak{M}$ , based on a frame with constant domains validating L, satisfying the downward inheritance property for monadic letters:  $\mathfrak{M} \models \Diamond P(x) \to P(x)$ , for every monadic predicate letter P (we call such a model L-suitable).

**Lemma 2.5.** Let  $L \in \{QK, QGL^{sem}, QGrz^{sem}\}$  and let  $\varphi$  be an L-suitable formula. Then,  $B \wedge \varphi'$  is L-satisfiable if, and only if,  $\forall x \beta_{n+1}(x) \wedge \varphi^*$  is L-satisfiable.

**Proof.** The right-to-left direction follows from the closure of L under predicate substitution. For the other direction, suppose that  $B \wedge \varphi'$  is **QK**-satisfiable. Let  $\mathfrak{M} = \langle W, R, D, I \rangle$  be a model such that  $\mathfrak{M}, w_0 \models B \wedge \varphi'$ , for some  $w_0 \in W$ . In view of Remark 2.2, we may assume, without a loss of generality, that  $\mathfrak{M} \models B$ .

For every  $w \in W$  and every frame  $\mathfrak{F}_k$   $(1 \leq k \leq n+1)$ , let  $\mathfrak{F}_k^w = \langle \{w\} \times W_k, R_k^w \rangle$  be an isomorphic copy of  $\mathfrak{F}_k$ . For every  $w \in W$  and  $k \in \{1, \ldots, n+1\}$ , add  $\{w\} \times W_k$  to W to obtain the set  $W^*$ . Define the relation  $R^*$  on  $W^*$  as follows:

$$R^* = R \cup \bigcup \{R_k^w \cup \{\langle w, (w, w_k^0) \rangle\} : w \in W, 1 \le k \le n+1\}.$$

Thus, for every  $w \in W$ , we make the roots of frames  $\mathfrak{F}_1^w, \ldots, \mathfrak{F}_{n+1}^w$  accessible from w. Next, for every  $u \in W^*$  let

$$D^*(u) = \begin{cases} D(u), & \text{if } u \in W, \\ D(w), & \text{if } u \in \{w\} \times W_k. \end{cases}$$

Finally, for every  $u \in W^*$  and every  $a \in D^*(u)$ , let

$$\langle a \rangle \in I^*(u, P) \iff u = (w, w_k^{2i}), \text{ for some } w \in W, k \in \{1, \dots, n+1\},$$
  
and  $i \in \{0, \dots, k\}; \text{ and } \mathfrak{M}, w \models P_k[a].$ 

Let  $\mathfrak{M}^* = \langle W^*, R^*, D^*, I^* \rangle$ . It immediately follows from Lemma 2.3 that, for every  $w \in W$ , every  $a \in D(w)$ , and every  $k \in \{1, \ldots, n+1\}$ ,

$$\mathfrak{M}, w \models P_k[a] \iff \mathfrak{M}^*, w \models \beta_k[a].$$

We can then show that, for every  $w \in W$ , every subformula  $\psi(x_1, \ldots, x_m)$  of  $\varphi$ , and every  $a_1, \ldots, a_m \in D(w)$ ,

$$\mathfrak{M}, w \models \psi'[a_1, \dots, a_m] \iff \mathfrak{M}^*, w \models \psi^*[a_1, \dots, a_m],$$

where  $\psi^*(x_1,\ldots,x_m)$  is obtained by substituting  $\beta_1(x),\ldots,\beta_{n+1}(x)$  for  $P_1(x),\ldots,P_{n+1}(x)$  in  $\psi'(x_1,\ldots,x_m)$ .

The proof proceeds by induction. We only consider the modal case, leaving the rest to the reader. In this case,  $\psi'(x_1,\ldots,x_m) = \Box(\forall x\,P_{n+1}(x)\to\chi'(x_1,\ldots,x_m))$  and  $\psi^*(x_1,\ldots,x_m) = \Box(\forall x\,\beta_{n+1}(x)\to\chi^*(x_1,\ldots,x_m))$ . If  $\mathfrak{M}^*,w\not\models\psi^*[a_1,\ldots,a_m]$ , then there exists  $w'\in W^*$  with  $wR^*w'$  such that  $\mathfrak{M}^*,w'\models\forall x\,\beta_{n+1}(x)$  and  $\mathfrak{M}^*,w'\not\models\chi^*[a_1,\ldots,a_m]$ . The condition  $\mathfrak{M}^*,w'\models\forall x\,\beta_{n+1}(x)$  guarantees that  $w'\in W$ ; therefore, we may apply the inductive hypothesis to conclude that  $\mathfrak{M},w'\not\models\chi'[a_1,\ldots,a_m]$ . The other direction is straightforward.

Thus,  $\mathfrak{M}^*, w_0 \models \forall x \, \beta_{n+1}(x) \wedge \varphi^*$ , i. e.,  $\forall x \, \beta_{n+1}(x) \wedge \varphi^*$  is **QK**-satisfiable.

For  $\mathbf{QGL}^{sem}$  and  $\mathbf{QGrz}^{sem}$ , the proof is similar. The only difference is that, when defining the model  $\mathfrak{M}^*$ , instead of  $R^*$  mentioned above, we take as the accessibility relations its transitive, and its reflexive and transitive, closure, respectively. We only observe that, for atomic formulas, the proof relies on L-suitability, for  $L \in \{\mathbf{QGL}^{sem}, \mathbf{QGrz}^{sem}\}$ , of  $\varphi$  and, hence, of  $B \wedge \varphi'$ : in the construction described above, we begin with an L-suitable model for  $B \wedge \varphi'$ .

We can now prove our main result in this section.

**Theorem 2.6.** Let L be a logic such that  $\mathbf{QK} \subseteq L \subseteq \mathbf{QGL}$  or  $\mathbf{QK} \subseteq L \subseteq \mathbf{QGrz}$ . Then, L is undecidable in the language with two individual variables and a single monadic predicate letter.

**Proof.** We first establish the following:

**Sublemma 1.** Let  $L \in \{QK, QGL^{sem}, QGrz^{sem}\}$ . Then, the problem of L-satisfiability of L-suitable formulas containing only two individual variables and only monadic predicate letters is undecidable.

**Proof.** By reduction from an undecidable [2]  $\mathbb{N} \times \mathbb{N}$  tiling problem.

Kontchakov, Kurucz, and Zakharyaschev [12] define, for a finite set T of tile types (for a brief description of the tiling problem considered here, see Section 3.2), the formula  $\chi_T$ , a conjunction of the formulas (we write H(x,y) and V(x,y) for, respectively,  $succ_H(x,y)$  and  $succ_V(x,y)$  used by Kontchakov, Kurucz, and Zakharyaschev)

$$\forall x \bigvee_{t \in T} (P_t(x) \land \bigwedge_{t' \neq t} P_{t'}(x));$$

$$\forall x \forall y (H(x, y) \rightarrow \bigwedge_{right(t) \neq left(t')} \neg (P_t(x) \land P_{t'}(y)));$$

$$\forall x \forall y (V(x, y) \rightarrow \bigwedge_{up(t) \neq down(t')} \neg (P_t(x) \land P_{t'}(y)));$$

$$\forall x \exists y H(x, y) \land \forall x \exists y V(x, y);$$

$$\forall x \forall y (H(x, y) \rightarrow \Box H(x, y));$$

$$\forall x \forall y (V(x, y) \rightarrow \Box V(x, y));$$

$$\forall x \forall y ( \Diamond V(x, y) \rightarrow V(x, y));$$

$$\forall x \forall y ( \Diamond V(x, y) \land \exists x (D(x) \land H(x, y)) \rightarrow \forall y (H(x, y) \rightarrow \forall x (D(x) \rightarrow V(y, x)))],$$

$$\Box \forall x \forall y [V(x, y) \land \exists x (D(x) \land H(x, y)) \rightarrow \forall y (H(x, y) \rightarrow \forall x (D(x) \rightarrow V(y, x)))],$$

and show that  $\chi_T$  satisfies the condition

$$\chi_T$$
 is L-satisfiable  $\iff$  T tiles  $\mathbb{N} \times \mathbb{N}$ . (\*)

We effectively construct from  $\chi_T$  an L-suitable formula  $\chi_T^*$  and show that the condition (\*) remains satisfied if  $\chi_T$  is replaced with  $\chi_T^*$ . Since the tiling of  $\mathbb{N} \times \mathbb{N}$  by T is undecidable [2], the statement of the Sublemma follows.

We first consider the case  $L = \mathbf{QK}$ . Let  $\chi_T^{\circ}$  be the formula obtained from  $\chi_T$  by substituting  $\neg D(x)$  for D(x). It should be clear that  $\chi_T^{\circ}$  is  $\mathbf{QK}$ -satisfiable if, and only if,  $\chi_T$  is  $\mathbf{QK}$ -satisfiable.

Replace in  $\chi_T^{\circ}$  every occurrence of  $\Box \psi$  by  $\Box (\forall x \, Q(x) \to \psi)$  and substitute into so obtained formula the formulas  $\Diamond (\neg Q_1^H(x) \land \neg Q_2^H(y))$  and  $\Diamond (\neg Q_1^V(x) \land \neg Q_2^V(y))$  for, respectively, H(x,y) and V(x,y). Denote the resultant formula by  $\bar{\chi}_T^{\circ}$ . Lastly, put  $\chi_T^{\star} = \forall x \, Q(x) \land \bar{\chi}_T^{\circ}$ .

We first show that  $\chi_T^{\star}$  is satisfiable if, and only if,  $\chi_T^{\circ}$  is satisfiable (and so  $\chi_T^{\star}$  is satisfiable if, and only if, T tiles  $\mathbb{N} \times \mathbb{N}$ ).

Suppose  $\chi_T^{\circ}$  is satisfiable. Then, by (\*), there exists a tiling  $\tau \colon \mathbb{N} \times \mathbb{N} \to T$ . We use  $\tau$  to define the model  $\mathfrak{M}$ : let

$$\begin{aligned} W &= \{w^*\} \cup \{w_{ij}: i, j \in \mathbb{N}\} \cup \{w'_{ij}: i, j \in \mathbb{N}\}; \\ R &= (\{w^*\} \times \{w_{ij}: i, j \in \mathbb{N}\}) \cup ((\{w^*\} \cup \{w_{ij}: i, j \in \mathbb{N}\}) \times \{w'_{ij}: i, j \in \mathbb{N}\}); \\ D_w &= \mathbb{N} \times \mathbb{N}, & \text{for every } w \in W'; \\ I(w,Q) &= \begin{cases} \varnothing & \text{if } w = w'_{ij}, \\ \mathbb{N} \times \mathbb{N} & \text{if } w \neq w'_{ij}; \end{cases} \\ I(w,Q_1^H) &= \begin{cases} \mathbb{N} \times \mathbb{N} - \{\langle i,j \rangle\} & \text{if } w = w'_{ij}, \\ \mathbb{N} \times \mathbb{N} & \text{if } w \neq w'_{ij}; \end{cases} \\ I(w,Q_2^H) &= \begin{cases} \mathbb{N} \times \mathbb{N} - \{\langle i,j \rangle\} & \text{if } w = w'_{ij}, \\ \mathbb{N} \times \mathbb{N} & \text{if } w \neq w'_{ij}; \end{cases} \\ I(w,Q_1^V) &= \begin{cases} \mathbb{N} \times \mathbb{N} - \{\langle i,j \rangle\} & \text{if } w = w'_{ij}, \\ \mathbb{N} \times \mathbb{N} & \text{if } w \neq w'_{ij}; \end{cases} \\ I(w,Q_2^V) &= \begin{cases} \mathbb{N} \times \mathbb{N} - \{\langle i,j \rangle\} & \text{if } w = w'_{ij}, \\ \mathbb{N} \times \mathbb{N} & \text{if } w \neq w'_{ij}; \end{cases} \\ I(w^*,D) &= \mathbb{N} \times \mathbb{N}; \\ I(w_{ij},D) &= \mathbb{N} \times \mathbb{N} - \{\langle i,j \rangle\}; \\ I(w_{ij},D) &= \varnothing; \\ I(w,P_t) &= \{\langle i,j \rangle \in \mathbb{N} \times \mathbb{N} : \tau(i,j) = t\}, \end{cases} & \text{for every } w \in W'; \end{cases}$$

The existence of  $\tau$  implies that  $\mathfrak{M}, w^* \models \chi_T^*$ .

Conversely, suppose  $\chi_T^*$  is satisfiable, i.e.  $\mathfrak{M}_0, w_0 \models \chi_T^*$ , for some model  $\mathfrak{M}_0$  and world  $w_0$ . If we remove from  $\mathfrak{M}_0$  the worlds refuting  $\forall x \, Q(x)$  and define the interpretation of letters H and V to be the sets of pairs satisfying, respectively,  $\diamondsuit(\neg Q_1^H(x) \land \neg Q_2^H(y))$  and  $\diamondsuit(\neg Q_1^V(x) \land \neg Q_2^V(y))$ , then  $\chi_T^{\circ}$  is satisfied at  $w_0$  in the resultant model.

It remains to show that  $\chi_T^*$  is **QK**-suitable. To see that it is, observe that the model  $\mathfrak{M}$  defined above is **QK**-suitable.

Next, suppose  $L \in \{\mathbf{QGL}^{sem}, \mathbf{QGrz}^{sem}\}$ . The argument given above for  $\mathbf{QK}$  applies as is to  $\mathbf{QGL}^{sem}$  since the model  $\mathfrak{M}$  defined above is a  $\mathbf{QGL}^{sem}$ -model. For  $\mathbf{QGrz}^{sem}$ , the argument is similar—the only difference is that, in defining a  $\mathbf{QGrz}^{sem}$ -suitable model satisfying  $\chi_T^{\star}$ , we take as the accessibility relation the reflexive closure of the relation R defined above.

Since  $\chi_T^{\star}$  contains only two individual variables and only monadic predicate letters, the statement of the sublemma follows.

Now, let  $F = \{\neg \chi_T^{\star} : T \text{ tiles } \mathbb{N} \times \mathbb{N} \}$  (thus, F contains only L-suitable formulas with two individual variables and only monadic predicate letters). It follows from the proof of Sublemma 1 that  $\mathbf{QK} \cap F = \mathbf{QGL}^{sem} \cap F = \mathbf{QGrz}^{sem} \cap F$  and that  $\mathbf{QK} \cap F$  is undecidable.

By Lemmas 2.1 and 2.5, for  $L \in \{QK, QGL^{sem}, QGrz^{sem}\}$ ,

$$\neg \chi_T^{\star} \in L \cap F \iff \forall x \, \beta_{n+1}(x) \to \neg (\neg \chi_T^{\star})^* \in L,$$

which implies the statement of the theorem.

Corollary 2.7. QK, QT, QD, QK4, QS4, QGL, and QGrz are undecidable in the language with two individual variables and a single monadic predicate letter.

**Remark 2.8.** Theorem 2.6 and Corollary 2.7 hold true if we replace every logic L mentioned in their statements with  $L \oplus bf$ , where  $bf = \forall x \Box P(x) \rightarrow \Box \forall x P(x)$ ; adding bf to L forces us to consider only predicate frames for L with constant domains.

We conclude this section by noticing that the results obtained herein are quite tight. In has been shown in [26], Theorem 5.1, that for logics **QK**, **QT**, **QK4**, and **QS4**, adding—on top of the restriction to at most two individual variables and a single monadic predicate letter—the very slight restriction that modal operators apply only to formulas with at most one free individual variable results in decidable fragments. As noticed in [26], the same holds true for the other logics mentioned in Corollary 2.7.

## 2.3 Sublogics of QKTB

We now prove results similar to those established in the preceding section for logics in the interval [QK, QKTB], where QKTB is the predicate logic of reflexive and symmetric frames. In so doing, we use an adaptation of a technique used in [23] for proving results about computational complexity of finite-variable fragments of sublogics of the propositional logic KTB.

We proceed as in the previous section right up to the point where formulas  $\alpha_k$  and models  $\mathfrak{M}_k$  are defined. Then, we define the formulas  $\alpha_k$  as follows. First, let

Next, inductively define, for every  $k \in \{1, ..., n+1\}$ , the following sequence of formulas:

$$\begin{array}{lcl} \delta(x) & = & \Box^+ P(x); \\ \delta^k_k(x) & = & \Box^{\leqslant k} \neg P(x) \wedge \diamondsuit^{k+1} P(x) \wedge \diamondsuit^{k+2} \delta(x); \\ \delta^k_i(x) & = & \Box^{\leqslant i} \neg P(x) \wedge \diamondsuit^{i+1} P(x) \wedge \Box \diamondsuit^{i+1} P(x) \wedge \diamondsuit^{2i+3} \delta^k_{i+1}(x), \text{ where } 1 \leqslant i < k. \end{array}$$

For notational convenience, let  $\delta_2^1(x) = \delta(x)$ .

Lastly, let, for every  $k \in \{1, ..., n+1\}$ ,

$$\alpha_k(x) = P(x) \wedge \diamondsuit^2 \delta_1^k(x) \wedge \neg \diamondsuit^3 \delta_2^k(x).$$

Now we define models  $\mathfrak{M}_k$  associated with formulas  $\alpha_k$ . Given an individual a and  $k \in \{1, \ldots, n+1\}$ , a model  $\mathfrak{M}_k$ , whose domain contains a, looks as follows. For brevity, we call some worlds a-worlds; if a world is not an a-world, we call it an  $\bar{a}$ -world. The model is a chain of worlds whose root,  $r_k$ , is an a-world. The root is part of a pattern of worlds, described below, which is in turn succeeded by three final a-worlds. The pattern looks as follows: a single a-world is followed by 2i+1  $\bar{a}$ -worlds, for  $1 \leq i \leq k$ . Thus the chain looks as follows: the root (an a-world), then three  $\bar{a}$ -worlds, then an a-world, then five  $\bar{a}$ -worlds, then an a-world, ..., then an a-world, then 2k+1  $\bar{a}$ -worlds, then three a-worlds. The accessibility relation between the worlds of  $\mathfrak{M}_k$  is both reflexive and symmetric.

We say that  $\mathfrak{M}_k$  is a-suitable if

$$\mathfrak{M}_k, w \models P[a] \iff w \text{ is an } a\text{-world.}$$

We can, then, prove the following analogue of Lemma 2.3.

**Lemma 2.9.** Let a be an individual in the domain of the models  $\mathfrak{M}_1, \ldots, \mathfrak{M}_{n+1}$  and let  $\mathfrak{M}_1, \ldots, \mathfrak{M}_{n+1}$  be a-suitable. Then,

$$\mathfrak{M}_k, w \models \alpha_m[a] \iff k = m \text{ and } w = r_k.$$

**Proof.** Straightforward.

Let

$$\beta_k(x) = \neg P(x) \land \Box \Diamond P(x) \land \Diamond \alpha_k(x),$$

and let  $\varphi^*$  be the result of replacing in  $\varphi'$  of  $P_k(x)$  with  $\beta_k(x)$ , for every  $k \in \{1, \ldots, n+1\}$ . We can then prove the following analogue of Lemma 2.5:

**Lemma 2.10.** Let  $L \in \{QK, QKTB\}$ . Then,  $B \wedge \varphi'$  is L-satisfiable if, and only if,  $\forall x \beta_{n+1}(x) \wedge \varphi^*$  is L-satisfiable.

**Proof.** Analogous to the proof of Lemma 2.5, with the observation that the truth status of formulas  $\alpha_k$  is not changed at the worlds of the models  $\mathfrak{M}_k$  once they get attached to the model  $\mathfrak{M}$  satisfying the formula  $B \wedge \varphi'$  to obtain the model  $\mathfrak{M}^*$  satisfying the formula  $\forall x \beta_{n+1}(x) \wedge \varphi^*$ , even though their roots can now see the worlds of  $\mathfrak{M}$  due to the symmetry of the accessibility relation of  $\mathfrak{M}^*$ . For a detailed argument showing that the truth status of formulas  $\alpha_k$  in  $\mathfrak{M}^*$  at worlds from  $\mathfrak{M}_k$  is not affected, we refer the reader

to the proof of Lemma 3.9 in [23].

Then, using an argument analogous to the one used in the proof of Theorem 2.6, we obtain the following:

**Theorem 2.11.** Let L be a logic such that  $\mathbf{QK} \subseteq L \subseteq \mathbf{QKTB}$ . Then, L is undecidable in the language with two individual variables and a single monadic predicate letter.

Corollary 2.12. QKB and QKTB are undecidable in the language with two individual variables and a single monadic predicate letter.

## 3 Intuitionistic and related logics

We now consider logics closely related to the quantified intuitionistic logic QInt.

#### 3.1 Syntax and semantics

The (first-order) quantified intuitionistic language contains countably many individual variables; countably many predicate letters of every arity; propositional constants  $\bot$  ("falsehood") and  $\top$  ("truth"); propositional connectives  $\land$ ,  $\lor$ , and  $\rightarrow$ ; and quantifiers  $\exists$  and  $\forall$ . Formulas are defined in the usual way; when parentheses are left out,  $\land$  and  $\lor$  are understood to bind tighter than  $\rightarrow$ . We also use the following abbreviations:  $\Box \varphi = \top \rightarrow \varphi$ ,  $\Box^0 \varphi = \varphi$ , and  $\Box^{n+1} \varphi = \Box \Box^n \varphi$ .

A Kripke frame is a tuple  $\mathfrak{F} = \langle W, R \rangle$ , where W is a non-empty set (of worlds) and R is a binary (accessibility) relation on W that is reflexive, anti-symmetric, and transitive.

A Kripke model  $\mathfrak{M} = \langle W, R, D, I \rangle$  is defined as in the modal case, except that the interpretation function I satisfies the additional condition that wRw' implies  $I(w, P) \subseteq I(w', P)$ . An assignment is defined as in the modal case.

The truth of a formula  $\varphi$  in a world w of a model  $\mathfrak{M}$  under an assignment g is inductively defined as follows:

- $\mathfrak{M}, w \not\models^g \bot;$
- $\mathfrak{M}, w \models^g \top$ ;
- $\mathfrak{M}, w \models^g P(x_1, \dots, x_n) \text{ if } \langle g(x_1), \dots, g(x_n) \rangle \in I(w, P);$
- $\mathfrak{M}, w \models^g \varphi_1 \wedge \varphi_2$  if  $\mathfrak{M}, w \models^g \varphi_1$  and  $\mathfrak{M}, w \models^g \varphi_2$ ;
- $\mathfrak{M}, w \models^g \varphi_1 \vee \varphi_2$  if  $\mathfrak{M}, w \models^g \varphi_1$  or  $\mathfrak{M}, w \models^g \varphi_2$ ;
- $\mathfrak{M}, w \models^g \varphi_1 \to \varphi_2$  if wRw' and  $\mathfrak{M}, w' \models^g \varphi_1$  imply  $\mathfrak{M}, w' \models^g \varphi_2$ , for every  $w' \in W$ ;
- $\mathfrak{M}, w \models^g \exists x \varphi_1$  if  $\mathfrak{M}, w \models^{g'} \varphi_1$ , for some assignment g' that differs from g at most in the value of x and such that  $g'(x) \in D(w)$ ;

•  $\mathfrak{M}, w \models^g \forall x \varphi_1$  if  $\mathfrak{M}, w' \models^{g'} \varphi_1$ , for every  $w' \in W$  such that wRw' and every assignment g' such that g' differs from g in at most the value of x and such that  $g'(x) \in D(w')$ .

Truth in models, frames, and classes of frames is defined as in the modal case. **QInt** is the set of formulas true in all frames.

We also consider some logics closely related to **QInt**. First, **QKC** is the quantified counterpart of the propositional logic **KC** = **Int** +  $\neg p \lor \neg \neg p$ . Semantically, **QKC** is characterized by the frames that satisfy the (convergence) condition that  $wRv_1$  and  $wRv_2$  imply the existence of a world u such that  $v_1Ru$  and  $v_2Ru$ .

Second, we consider quantified counterparts of Visser's basic propositional logic **BPL** and formal propositional logic **FPL** [25]: **BPL** and **FPL** are logics in the intuitionistic language whose modal companions are **K4** and **GL**—that is, given the Gödel's translation t of the intuitionistic language into the modal one (see, for example, [5], § 3.9),  $\mathbf{BPL} = t^{-1}(\mathbf{K4})$  and  $\mathbf{FPL} = t^{-1}(\mathbf{GL})$ . Therefore, we define their quantified counterparts as logics  $\mathbf{QBL} = T^{-1}(\mathbf{QK4})$  and  $\mathbf{QFL} = T^{-1}(\mathbf{QGL})$ , where T is the extension of t with the following clauses:  $T(\exists x \varphi) = \exists x T(\varphi)$ ; and  $T(\forall x_1 \dots \forall x_n \varphi) = \Box \forall x_1 \dots \forall x_n T(\varphi)$ , where  $\varphi$  does not begin with a universal quantifier. To give the semantic account of  $\mathbf{QBL}$  and  $\mathbf{QFL}$ , we use Kripke frames and models as defined for  $\mathbf{QInt}$ , except that the accessibility relation is now only required to be anti-symmetric and transitive. The relation  $\mathfrak{M}, w \models^g \varphi$  is defined as in the intuitionistic case, with the following modification for the universal quantifiers:

•  $\mathfrak{M}, w \models^g \forall x_1 \dots \forall x_n \varphi_1$ , where  $\varphi_1$  does not begin with a universal quantifier, if  $\mathfrak{M}, w' \models^{g'} \varphi_1$ , for every  $w' \in W$  such that wRw' and every assignment g' such that g' differs from g in at most the values of  $x_1, \dots, x_n$  and such that  $g'(x_1), \dots, g'(x_n) \in D(w')$ .

This clause is required to make, in the absence of reflexivity of the accessibility relation, the formula  $\forall x \forall y \varphi$  equivalent to the formula  $\forall y \forall x \varphi$ . Then, **QBL** is sound (and complete) with respect to all such frames, while **QFL** is sound (but not complete) with respect to the subclass where the converse of the accessibility relation is well-founded (i. e., with respect to the frames of the logic **GL**). For technical reasons, namely to avoid being distracted with Kripke-completeness, we define the logic **QFL**<sup>sem</sup> as the set of formulas valid in all frames where the converse of the accessibility relation is well-founded; all that matters to us is that **QFL**  $\subseteq$  **QFL**<sup>sem</sup>.

## 3.2 Undecidability results

We now address the question, raised in [12], of whether it suffices to use only monadic predicate letters to obtain undecidability of the two-variable fragment **QInt**(2) of **QInt**. We show that, in fact, it suffices to use a *single* monadic predicate letter to obtain undecidability of **QInt**(2). We do so by suitably adapting the technique used in [21] to (polynomially) reduce satisfiability in propositional intuitionistic logic **Int** to satisfiability in the fragment of **Int** with only two propositional variables. As the technique from [21]

requires that we work with positive formulas, we first show that the *positive* monadic fragment of  $\mathbf{QInt}(2)$  is undecidable. We note here that transitioning from the propositional language to the first-order one, we "strengthen" the result from [21] in the following sense: while in the propositional case, (the positive fragment of)  $\mathbf{Int}$  is polynomially reducible to its two-variable subfragment, in the the first-order case, we (polynomially) reduce (the positive fragment of)  $\mathbf{QInt}(2)$  to its subfragment containing a single predicate letter. Working with the positive fragment of  $\mathbf{QInt}$  also allows us to extend our results to the interval  $[\mathbf{QInt}, \mathbf{QKC}]$ , as all logics in this interval share the positive fragment. Moreover, a modification of this construction allows us to obtain analogous results for logics in  $[\mathbf{QBL}, \mathbf{QFL}]$ .

It is proven in [12] that  $\mathbf{QInt}(2)$  is undecidable by reducing the following undecidable tiling problem [2] to the complement of  $\mathbf{QInt}(2)$ : given a finite set T of tile types that are tuples of colours  $t = \langle left(t), right(t), up(t), down(t) \rangle$ , decide whether T tiles the grid  $\mathbb{N} \times \mathbb{N}$  in the sense that there exists a function  $\tau : \mathbb{N} \times \mathbb{N} \to T$  such that, for every  $i, j \in \mathbb{N}$ , we have  $up(\tau(i,j)) = down(\tau(i,j+1))$  and  $right(\tau(i,j)) = left(\tau(i+1,j))$ . The results in this section build on this proof.

We start off by proving that the positive fragment of  $\mathbf{QInt}(2)$  containing two binary and an unlimited number of monadic predicate letters, as well as two propositional variables, is undecidable. This is achieved by eliminating the constant  $\perp$  from the formulas used in the proof of undecidability of  $\mathbf{QInt}(2)$  from [12]. For most formulas from [12], all we do is replace  $\perp$  with a propositional variable q. The resultant formulas are listed below for the reader's convenience; for ease of reference, we preserve the numbering from [12]:

$$\forall x \bigvee_{t \in T} (P_t(x) \land \bigwedge_{t' \neq t} (P_{t'}(x) \to q)), \tag{1}$$

$$\bigwedge_{right(t)\neq left(t')} \forall x \, \forall y \, (H(x,y) \wedge P_t(x) \wedge P_{t'}(y) \to q), \tag{2}$$

$$\bigwedge_{up(t)\neq down(t')} \forall x \, \forall y \, (V(x,y) \wedge P_t(x) \wedge P_{t'}(y) \to q), \tag{3}$$

$$\forall x \,\exists y \, H(x,y) \land \forall x \,\exists y \, V(x,y), \tag{4}$$

$$\forall x \, \forall y \, (V(x,y) \vee (V(x,y) \to q)), \tag{5}$$

$$\forall x \,\forall y \,[V(x,y) \land \exists x \,(D(x) \land H(y,x)) \to \forall y \,(H(x,y) \to \forall x \,(D(x) \to V(y,x)))]. \tag{6}$$

Let  $\psi_T^+$  be the conjunction of formulas (1) through (6). Then, define

$$\varphi_T^+ = \psi_T^+ \to ((\exists x (D(x) \to q) \to p) \to p),$$

where p is a propositional variable distinct from q.

<sup>&</sup>lt;sup>1</sup>In light of [19], the reduction of **Int** to its single-variable fragment would imply that the complexity classes **P** and **PSPACE** are equivalent.

**Lemma 3.1.**  $\varphi_T^+ \notin \mathbf{QInt}(2)$  if, and only if, T tiles  $\mathbb{N} \times \mathbb{N}$ .

**Proof.** The proof is a minor modification of the proof of Theorem 1 from [12], with q essentially playing the role that "falsehood" plays in [12].

For the left to right direction, we observe that, given a model  $\mathfrak{M}$  and a world w such that  $\mathfrak{M}, w \not\models \varphi_T^+$ , as well as an arbitrary  $d \in D(w)$ , there exists a world u in  $\mathfrak{M}$  with wRu such that  $\mathfrak{M}, u \models D[d]$  and  $\mathfrak{M}, u \not\models q$ . This is a straightforward consequence of the fact that  $\mathfrak{M}, w \not\models (\exists x (D(x) \to q) \to p) \to p$ . Given this, the argument from [12] applies.

For the other direction, the model falsifying  $\varphi_T^+$  is different from the one used in [12] only in the evaluation of p and q. Thus, we use the same frame and interpretation of predicate letters as in [12], and additionally make q false at every world of the model and make p false at  $w_0$  and true at every other world.

Since  $\varphi_T^+$  is a positive formula, this immediately gives us the following:

Corollary 3.2. The positive fragment of QInt with two individual variables, two binary predicate letters, an unlimited number of monadic predicate letters, and two propositional variables is undecidable.

We now show how, drawing on an idea of Kripke's for modal logics [14], one can, in the positive fragment of **QInt**, simulate binary predicate letters using monadic predicate letters and propositional variables. As this does not increase the number of individual variables in a formula, it will allow us to eliminate binary predicate letters from the formula  $\varphi_T^+$ .

**Lemma 3.3.** Let  $\chi$  be a positive formula in **QInt** containing an occurrence of a binary predicate letter Q, and let  $Q_1$  and  $Q_2$  be monadic predicate letters, and r and s be propositional variables, not occurring in  $\chi$ . Let  $\chi'$  be the result of uniformly replacing every subformula of  $\chi$  of the form Q(x,y) with  $(Q_1(x) \wedge Q_2(y) \rightarrow r) \vee s$ . Then,  $\chi \in \mathbf{QInt}$  if, and only if,  $\chi' \in \mathbf{QInt}$ .

**Proof.** The left-to-right direction follows from the closure of **QInt** under substitution. For the other direction, assume that there exist  $\mathfrak{M} = \langle W, R, D, I \rangle$  and  $w_0 \in W$  such that  $\mathfrak{M}, w_0 \not\models \chi$ . We modify  $\mathfrak{M}$  to obtain a model  $\mathfrak{M}'$  falsifying  $\chi'$  as follows. For every  $w \in W$  and every  $a, b \in D(w)$  such that  $\mathfrak{M}, w \not\models Q[a, b]$ , add to W a world  $w_{a,b}$  with  $wR'w_{a,b}$  and let

$$\mathfrak{M}', w_{a,b} \not\models r;$$
  
 $\mathfrak{M}', w_{a,b} \models s;$   
 $\mathfrak{M}', w_{a,b} \models Q_1[d] \iff d = a;$   
 $\mathfrak{M}', w_{a,b} \models Q_2[d] \iff d = b;$ 

and let all the predicate letters different from  $Q_1$  and  $Q_2$  and occurring in  $\chi'$  be universally true at every such world; likewise for propositional variables different from r and s. Also, let  $\mathfrak{M}', w \not\models s$ .

Then we can show that  $\mathfrak{M}, w \models \theta[a_1, \ldots, a_m]$  if, and only if,  $\mathfrak{M}', w \models \theta'[a_1, \ldots, a_m]$ , for every subformula  $\theta$  of  $\chi$ , every  $w \in W$ , and every  $a_1, \ldots, a_m \in D(w)$ , where  $\theta'$  is the

result of substituting in  $\theta$  every occurrence of Q(x,y) with  $(Q_1(x) \wedge Q_2(y) \to r) \vee s$ . The proof is by induction on  $\theta$ .

For the base case, first note that if  $\mathfrak{M}, w \not\models Q[a, b]$ , then the presence in  $\mathfrak{M}'$  of the world  $w_{a,b}$  guarantees that  $\mathfrak{M}', w \not\models (Q_1[a] \land Q_2[b] \to r) \lor s$ ; on the other hand, if  $\mathfrak{M}, w \models Q[a, b]$ , then  $\mathfrak{M}', w \models (Q_1[a] \land Q_2[b] \to r) \lor s$ , as  $\mathfrak{M}, u \not\models Q_1[a]$  or  $\mathfrak{M}, u \not\models Q_2[b]$ , for every u with wR'u.

The cases for  $\theta = \theta_1 \vee \theta_2$ ,  $\theta = \theta_1 \wedge \theta_2$ , and  $\theta = \exists x \, \theta_1$  are straightforward.

Let  $\theta = \theta_1 \to \theta_2$ . Assume that  $\mathfrak{M}', w \not\models \theta'[a_1, \ldots, a_m]$ . Then,  $\mathfrak{M}', u \models \theta'_1[a_1, \ldots, a_m]$  and  $\mathfrak{M}', u \not\models \theta'_2[a_1, \ldots, a_m]$ , for some  $u \in W'$  with wR'u. If we could apply the inductive hypothesis to u, we would be done. To see that we can, notice that  $\theta'_2$  is built out of atomic formulas and the formula  $(Q_1(x) \land Q_2(y) \to r) \lor s$ , all of which are true under every assignment in every  $w' \in W' - W$ , using only  $\land$ ,  $\lor$ ,  $\to$ ,  $\exists$ , and  $\forall$ . Therefore,  $\theta'_2$  is true in every  $w' \in W' - W$  under every assignment; hence,  $u \in W$  and the inductive hypothesis is, therefore, applicable. Thus,  $\mathfrak{M}, w \not\models \theta[a_1, \ldots, a_m]$ . The other direction is straightforward.

The case  $\theta = \forall x \, \theta_1$  is similarly argued.

Now, let  $\xi_T^+$  be the result of replacing in  $\varphi_T^+$  of

$$H(x,y)$$
 with  $(H_1(x) \wedge H_2(y) \rightarrow r_1) \vee s_1;$   
 $V(x,y)$  with  $(V_1(x) \wedge V_2(y) \rightarrow r_2) \vee s_2.$ 

In view of Lemma 3.3,  $\xi_T^+ \notin \mathbf{QInt}(2)$  if, and only if, T tiles  $\mathbb{N} \times \mathbb{N}$ . As we can replace in  $\xi_T^+$  a propositional variable such as q with, say,  $\exists x \, Q(x)$ , we immediately obtain the following:

**Theorem 3.4.** The positive monadic fragment of QInt with two individual variables is undecidable.

We now embed the positive monadic fragment of **QInt**(2) into its subfragment containing formulas with only one monadic predicate letter, suitably adapting the technique from [21]. As this embedding does not introduce any fresh variables, our main result in this section immediately follows.

We begin by defining the frame  $\mathfrak{F} = \langle W, R \rangle$  to be used in the construction of a refuting countermodel. The frame  $\mathfrak{F}$ , depicted in Figure 1, is made up of levels of worlds. The top-most, unnumbered, level comprises  $d_1$ ,  $d_2$ ,  $d'_2$  and  $d_3$ ; level 0 comprises  $a_1^0$ ,  $a_2^0$ ,  $b_1^0$  and  $b_2^0$ ; level 1 comprises  $a_1^1$ ,  $a_2^1$ ,  $a_3^1$ ,  $b_1^1$ ,  $b_2^1$  and  $b_3^1$ ; the accessibility relation between these worlds is depicted by arrows (the arrows that can be inferred by reflexivity and transitivity are omitted). The other levels are defined recursively.

For each  $k \ge 2$ , level k contains worlds  $a_l^k$  and  $b_l^k$ , for every  $l \in \{1, \ldots, s_k\}$ , where  $s_k$  is defined by recursion:  $s_1 = 3$ ;  $s_{k+1} = (s_k - 1)^2$ . To define instances of the accessibility relation between worlds of level k + 1 and worlds of level of k, for each  $k \ge 1$ , take the lexicographic ordering of pairs  $\langle i, j \rangle$ , where  $i, j \in \{2, \ldots, s_k\}$ , and, provided  $\langle i, j \rangle$  is the mth pair in this ordering, put

$$\begin{aligned} &a_m^{k+1}R\,b_1^k, & a_m^{k+1}R\,a_i^k, & a_m^{k+1}R\,b_j^k, \\ &b_m^{k+1}R\,a_1^k, & b_m^{k+1}R\,a_i^k, & b_m^{k+1}R\,b_j^k. \end{aligned}$$

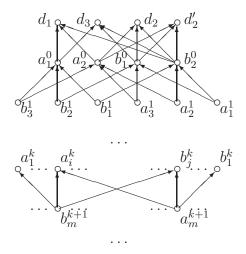


Figure 1: Frame 3

Let  $\mathfrak{N}_a = \langle W, R, D, I \rangle$  be an intuitionistic Kripke model with a constant domain  $\mathcal{A}$  containing element a; we assume that  $\mathcal{A}$  contains at least three elements—as we shall see, this assumption does not lead to a loss of generality. We say that  $\mathfrak{N}_a$  is a-suitable if, for some  $a' \in \mathcal{A} - \{a\}$ ,

- $I(d_2, P) = A \{a\};$
- $I(d'_2, P) = \{a'\};$
- $I(d_3, P) = \{a, a'\};$
- $I(b_1^0, P) = \{a'\};$
- $I(w, P) = \emptyset$ , for every  $w \in W \{d_2, d'_2, d_3, b_1^0\}$ .

We now define formulas of one free variable, x, so that each so defined formula  $\psi(x)$  is associated with a world of an a-suitable model based on  $\mathfrak{F}$  (or a frame isomorphic to  $\mathfrak{F}$ ), in the following sense: for every  $w \in W$ , the relation  $w \not\models \psi(a)$  holds if, and only if, w sees the world associated with  $\psi$ . For these formulas, we use notation making it clear which worlds they correspond to: the formula  $D_i$ , for  $i \in \{1,3\}$ , corresponds to  $d_i$ , the formula  $D_i$  corresponds to  $d_i$  and  $d_i$  to  $d_i$  to  $d_i$  and the formula  $d_i$  to  $d_i$  to  $d_i$ .

First, we define formulas associated with the worlds of the three top-most levels:

We proceed by recursion. Assume formulas associated with the worlds of level k, where  $k \ge 1$ , have been defined. Take the lexicographic ordering of pairs  $\langle i, j \rangle$ , where  $i, j \in \{2, \ldots, s_k\}$ , and, provided  $\langle i, j \rangle$  is the mth pair in this ordering, put

$$\begin{array}{rcl} A_m^{k+1}(x) & = & A_1^k(x) \to B_1^k(x) \lor A_i^k(x) \lor B_j^k(x); \\ B_m^{k+1}(x) & = & B_1^k(x) \to A_1^k(x) \lor A_i^k(x) \lor B_i^k(x). \end{array}$$

**Lemma 3.5.** Let  $\mathfrak{M} = \langle W, R, D, I \rangle$  be an a-suitable model and let  $w \in W$ . Then,

$$\mathfrak{M}, w \not\models A_m^k[a] \iff wRa_m^k \quad and \quad \mathfrak{M}, w \not\models B_m^k[a] \iff wRb_m^k.$$

**Proof.** Induction on k.

Now, let  $\varphi$  be a positive formula containing monadic predicate letters  $P_1, \ldots, P_n$  (we may assume  $n \ge 2$ ). For each  $i \in \{1, \ldots, n\}$ , define

$$\alpha_i(x) = A_i^{n+1}(x) \vee B_i^{n+1}(x).$$

Finally, let  $\varphi^*$  be the result of substituting, for every  $i \in \{1, ..., n\}$ , of  $\alpha_i(x)$  for  $P_i(x)$  into  $\varphi$ .

Lemma 3.6.  $\varphi \in QInt$  if, and only if,  $\varphi^* \in QInt$ .

**Proof.** The right-to-left direction follows from the closure of **QInt** under predicate substitution. For the other direction, assume that  $\mathfrak{M}_{\varphi}, w_0 \not\models \varphi$  for some  $\mathfrak{M}_{\varphi} = \langle W_{\varphi}, R_{\varphi}, D_{\varphi}, I_{\varphi} \rangle$  and  $w_0 \in W_{\varphi}$ . (We may assume without a loss of generality that the domain of  $\mathfrak{M}_{\varphi}$  contains at least three individuals; we use this fact in the construction of  $\mathfrak{M}^*$  below.) We need to construct a model  $\mathfrak{M}^*$  falsifying  $\varphi^*$  at some world.

For every  $w \in W_{\varphi}$  and  $a \in D_{\varphi}(w)$ , let  $\mathfrak{F}_a^w = \langle \{\langle w, a \rangle\} \times W, R_a^w \rangle$  be an isomorphic copy of the frame  $\mathfrak{F}$  under the isomorphism  $f \colon v \mapsto \langle w, a, v \rangle$ .

Let

$$W^* = W_{\varphi} \cup \big(\bigcup_{w \in W_{\varphi}} \big(\{w\} \times D_{\varphi}(w)\big) \times W\big).$$

Let S be the smallest relation on  $W^*$  such that

•  $R_{\varphi} \subseteq S$ ;

- $\bullet \bigcup_{w \in W_{\varphi}, \ a \in D_{\varphi}(w)} R_a^w \subseteq S;$
- for every  $w \in W_{\varphi}$ , every  $v \in W^* W_{\varphi}$ , every  $a \in D_{\varphi}(w)$  and every  $i \in \{1, \dots, n\}$ ,

$$wSv \iff \text{either } v \in \{\langle w, a, a_i^{n+1} \rangle, \langle w, a, b_i^{n+1} \rangle\} \text{ and } \mathfrak{M}_{\varphi}, w \not\models P_i[a]$$
  
or  $v \in \{\langle w, a, a_{n+1}^{n+1} \rangle, \langle w, a, b_{n+1}^{n+1} \rangle\},$ 

and let  $R^*$  be the reflexive transitive closure of S.

Let  $D^*(u) = D_{\varphi}(u)$  if  $u \in W_{\varphi}$  and  $D^*(u) = D_{\varphi}(w)$  if  $u = \langle w, a, v \rangle$ , for some  $w \in W_{\varphi}$ ,  $a \in D_{\varphi}(w)$  and  $v \in W$ .

Let  $I^*$  be an interpretation function on  $\langle W^*, R^*, D^* \rangle$  such that, for every  $w \in W_{\varphi}$  and every  $a \in D_{\varphi}(w)$ ,

- $I^*(\langle w, a, d_2 \rangle, P) = D_{\varphi}(w) \{a\};$
- $I^*(\langle w, a, d_2' \rangle, P) = \{a'\}$ , where a' is a fixed element of  $D_{\varphi}(w) \{a\}$ ;
- $I^*(\langle w, a, d_3 \rangle, P) = \{a, a'\}$ , where a' is a fixed element of  $D_{\varphi}(w) \{a\}$ ;
- $I^*(\langle w, a, b_1^0 \rangle, P) = \{a'\}$ , where a' is a fixed element of  $D_{\varphi}(w) \{a\}$ ;
- $I^*(u, P) = \emptyset$ , for every  $u \in W^* (\{d_2, d'_2, d_3, b_1^0\} \times \{\langle w, c \rangle : w \in W_{\varphi}, c \in D_{\varphi}(w)\})$ .

Finally, let  $\mathfrak{M}^* = \langle W^*, R^*, D^*, I^* \rangle$ . Evidently,  $I^*$  satisfies the heredity condition; hence,  $\mathfrak{M}^*$  is an intuitionistic Kripke model.

We can now show, by induction on  $\psi$ , that  $\mathfrak{M}_{\varphi}, w \models \psi[a_1, \ldots, a_m]$ , if and only if,  $\mathfrak{M}^*, w \models \psi^*[a_1, \ldots, a_m]$ , for every  $w \in W_{\varphi}$ , every  $a_1, \ldots, a_m \in D^*(w)$ , and every subformula  $\psi$  of  $\varphi$ . We only consider the cases where  $\psi$  is atomic and where  $\psi = \psi_1 \to \psi_2$ .

We shall rely on the following sublemmas, whose proof we leave to the reader:

**Sublemma 2.** For every  $w \in W_{\varphi}$  and every  $a \in D_{\varphi}(w)$ ,

$$\mathfrak{M}^*, w \not\models A_1^n[a]$$
 and  $\mathfrak{M}^*, w \not\models B_1^n[a]$ .

**Sublemma 3.** Let  $\mathfrak{N}_a = \langle W, R, D, I \rangle$  be an a-suitable model with a constant domain  $\mathcal{A}$  and let  $b \in \mathcal{A} - \{a\}$ . Then, for every  $w \in W$ , every  $k \geq 2$  and every  $m \in \{1, \ldots, s_k\}$ ,

$$\mathfrak{N}_a, w \models A_m^k[b]$$
 and  $\mathfrak{N}_a, w \models B_m^k[b]$ .

Observe that, for every  $w \in W_{\varphi}$  and every  $a \in D_{\varphi}(w)$ , the submodel of  $\mathfrak{M}^*$  whose set of worlds is  $\{\langle w, a, v \rangle : v \in \{\langle w, a \rangle\} \times W\}$  is a generated submodel of  $\mathfrak{M}^*$  and is a-suitable; hence, Lemma 3.5 and Sublemma 3 apply to such submodels of  $\mathfrak{M}^*$ .

We now proceed with induction.

Let  $\psi = P_i(x)$ , and so  $\psi^* = A_i^{n+1}(x) \vee B_i^{n+1}(x)$ , for some  $i \in \{1, ..., n\}$ .

Assume  $\mathfrak{M}_{\varphi}, w \not\models P_i[a]$ . By definition of  $\mathfrak{M}^*$ , both  $wR^*\langle w, a, a_i^{n+1} \rangle$  and  $wR^*\langle w, a, b_i^{n+1} \rangle$ . By Lemma 3.5, both  $\mathfrak{M}^*, \langle w, a, a_i^{n+1} \rangle \not\models A_i^{n+1}[a]$  and

 $\mathfrak{M}^*, \langle w, a, b_i^{n+1} \rangle \not\models B_i^{n+1}[a]$ . Hence, by heredity,  $\mathfrak{M}^*, w \not\models A_i^{n+1}[a]$  and  $\mathfrak{M}^*, w \not\models B_i^{n+1}[a]$ . Therefore,  $\mathfrak{M}^*, w \not\models A_i^{n+1}[a] \vee B_i^{n+1}[a]$ . Conversely, assume  $\mathfrak{M}^*, w \not\models A_i^{n+1}[a] \vee B_i^{n+1}[a]$ . Then,  $\mathfrak{M}^*, w \not\models A_i^{n+1}[a]$  and

 $\mathfrak{M}^*, w \not\models B_i^{n+1}[a]$ . Hence, there exist  $u', u'' \in W^*$  and  $i, j \in \{2, \ldots, s_n\}$  such that  $u', u'' \in R^*(w)$  and

$$u' \models A_1^n[a]; \quad u' \not\models B_1^n[a]; \quad u' \not\models A_i^n[a]; \quad u' \not\models B_j^n[a]; u'' \models B_1^n[a]; \quad u'' \not\models A_1^n[a]; \quad u'' \not\models A_i^n[a]; \quad u'' \not\models B_i^n[a].$$
(\*)

We show that  $u' = \langle w, a, a_i^{n+1} \rangle$  and  $u'' = \langle w, a, b_i^{n+1} \rangle$ . Since  $u' \models A_1^s[a]$  and  $u'' \models B_1^s[a]$ , by Sublemma 2,  $u', u'' \in W^* - W_{\varphi}$ . Therefore, from  $u' \not\models B_1^n(a)$  and  $u'' \not\models A_1^n[a]$  we obtain, by Sublemma 3, that  $u', u'' \in \{\langle w, a \rangle\} \times W$ . Hence, from (\*) we obtain by Lemma 3.5 that, for some  $i, j \in \{2, \ldots, s_n\}$ ,

$$\neg u'R^*\langle w, a, a_1^n \rangle; \quad u'R^*\langle w, a, b_1^n \rangle; \quad u'R^*\langle w, a, a_i^n \rangle; \quad u'R^*\langle w, a, b_j^n \rangle;$$
 
$$\neg u''R^*\langle w, a, b_1^n \rangle; \quad u''R^*\langle w, a, a_1^n \rangle; \quad u''R^*\langle w, a, a_i^n \rangle; \quad u''R^*\langle w, a, b_j^n \rangle.$$

Now, in  $\mathfrak{F}$ , and hence in  $\mathfrak{F}_{w}^{a}$ , only worlds of level greater than n see more than one world of level n. Hence, u' and u" belong to a level greater than n. Since every world of  $\mathfrak{F}_w^a$  of level greater that n+1 sees  $a_1^n$  and  $b_1^n$ , by Lemma 3.5, for every world w of level greater than n+1, both  $w \not\models A_1^n[a]$  and  $w \not\models B_1^n[a]$ . Therefore, u' and u'' are worlds of level n+1. Since  $u'R^*u'$  and  $u''R^*u''$ , in view of (\*),  $\mathfrak{M}^*, u' \not\models A_i^{n+1}[a]$  and  $\mathfrak{M}^*, u'' \not\models B_i^{n+1}[a]$ . Hence, by Lemma 3.5,  $u'R^*\langle w, a, a_i^{n+1} \rangle$  and  $u''R''\langle w, a, b_i^{n+1} \rangle$ . Therefore,  $u' = \langle w, a, a_i^{n+1} \rangle$ and  $u'' = \langle w, a, b_i^{n+1} \rangle$ .

Thus,  $wR^*\langle a, a_i^{n+1}\rangle$  and  $wR^*\langle w, a, b_i^{n+1}\rangle$ . Hence,  $\mathfrak{M}^*, w \not\models P_i[a]$ .

Let  $\psi = \psi_1 \to \psi_2$ . Assume  $\mathfrak{M}^*, w \not\models \psi^*[a_1, \ldots, a_m]$ . Then,  $\mathfrak{M}^*, u \models \psi_1^*[a_1, \ldots, a_m]$ and  $\mathfrak{M}^*, u \not\models \psi_2^*[a_1, \ldots, a_m]$ , for some  $u \in W^*$  with  $wR^*u$ . If we could apply the inductive hypothesis to u, we would be done. To see that we can, notice that  $\psi_2^*$  is built out of formulas of the form  $A_i^{n+1}(x) \vee B_i^{n+1}(x)$  using only  $\wedge$ ,  $\vee$ ,  $\rightarrow$ ,  $\exists$ , and  $\forall$ . As, in view of Lemma 3.5 and Sublemma 3, formulas  $A_i^{n+1}(x) \vee B_i^{n+1}(x)$  are true at every world in  $W^*-W_{\varphi}$  accessible from  $W_{\varphi}$ , it follows that  $u\in W_{\varphi}$ ; the inductive hypothesis is, therefore, applicable. Thus,  $\mathfrak{M}_{\varphi}, w \not\models \psi[a_1, \ldots, a_m]$ . The other direction is straightforward.

We conclude that 
$$\mathfrak{M}^*, w_0 \not\models \varphi^*$$
 and, thus,  $\varphi^* \notin \mathbf{QInt}$ .

As the construction of  $\varphi^*$  from  $\varphi$  did not introduce any fresh individual variables, we have the following:

**Theorem 3.7.** The positive fragment of QInt with two individual variables and a single predicate letter is undecidable.

We now extend the argument presented above to the logics in the intervals [QBL, QKC] and [QBL, QFL].

First, to establish the undecidability of the two-variable fragments of logics whose semantics might contain irreflexive worlds, we need to slightly modify formulas (1) through

(6) listed above. Therefore, we define  $\psi_T^*$  to be the conjunction of  $\psi_T^+$  and following formula:

$$\forall x \, \forall y \, (H(x,y) \vee (H(x,y) \to q)), \tag{5a}$$

and define

$$\varphi_T^* = \psi_T^* \to [(\exists x (D(x) \to \Box^5 q) \to p) \to \Box p].$$

This enables us to prove, using the tiling problem described above, that T tiles  $\mathbb{N} \times \mathbb{N}$  if and only if  $\varphi_T^* \notin L(2)$ , where  $L \in \{\mathbf{QBL}, \mathbf{QFL}^{sem}\}$ . We leave the details of the proof to the reader. As the construction of  $\varphi_T^*$  is uniform for both logics, it follows that the claim holds for every  $L \in [\mathbf{QBL}, \mathbf{QFL}^{sem}]$ . Notice that the same proof also works for logics in  $[\mathbf{QBL}, \mathbf{QKC}]$ . We simulate binary predicate letters by monadic ones as for  $\mathbf{QInt}$ . We now show how to simulate all monadic predicate letters with a single one.

For the interval [QBL, QKC], notice that if we add to the model  $\mathfrak{M}^*$  built in the proof of Lemma 3.6 a world d accessible from every element of  $W^*$  and such that  $I^*(d, P) = D(d)$ , the resultant model is a model of every logic in the interval [QBL, QKC]. Thus, we have the following:

**Theorem 3.8.** Let L be a logic in the interval [QBL, QKC]. Then, the positive fragment of L with two individual variables and a single predicate letter is undecidable.

We next consider the interval [QBL, QFL<sup>sem</sup>]. In this case, we need to make a more substantial modification to the frame  $\mathfrak{F}$ , as the semantics of QFL<sup>sem</sup> prohibits the existence of reflexive worlds. We then proceed as follows. First, add to W worlds  $\bar{d}_2$ ,  $\bar{d}'_2$ , and  $\bar{d}_3$  with  $d_2R\bar{d}_2$ ,  $d'_2R\bar{d}'_2$ , and  $d_3R\bar{d}_3$ . Second, for every  $k \geq 0$ , do the following: for every world  $a_i^k$ , add to W the world  $\bar{a}_i^k$  and, for every world  $b_i^k$ , add to W the world  $\bar{b}_i^k$ ; also, let  $a_i^kR\bar{a}_i^k$  and  $b_i^kR\bar{b}_i^k$ , for every k and i. Lastly, whenever in  $\mathfrak{F}$  we had  $a_i^{k+1}Ra_j^k$  or  $a_i^{k+1}Rb_j^k$ , let  $\bar{a}_i^{k+1}Ra_j^k$  and  $\bar{a}_i^{k+1}Rb_j^k$ ; also, whenever we had  $b_i^{k+1}Ra_j^k$  or  $b_i^{k+1}Rb_j^k$ , let  $\bar{b}_i^{k+1}Ra_j^k$  and  $\bar{b}_i^{k+1}Rb_j^k$ . We then define a-suitable models so that  $I(\bar{d}_2,P) = I(d_2,P)$ ,  $I(\bar{d}_2,P) = I(d'_2,P)$ ,  $I(\bar{d}_3,P) = I(d_3,P)$ , and for every k and k, k, k, k, k, k, k, which serve to evaluate formulas whose main connective is k, or k, at the worlds whose doubles they are. Then, k-suitable models satisfy the condition in the statement of Lemma 3.5, and the model  $\mathfrak{M}^*$  built in the proof of Lemma 3.6 becomes a model of every logic in [QBL, QFL<sup>sem</sup>]. As QFL  $\subseteq$  QFL<sup>sem</sup>, we have the following:

**Theorem 3.9.** Let L be a logic in the interval [QBL, QFL]. Then, the positive fragment of L with two individual variables and a single predicate letter is undecidable.

**Remark 3.10.** Note that the results of this section hold true if we only consider frames with constant domains.

## 4 Discussion

As already noticed, the results presented in the present paper concerning sublogics of **QGL** and **QGrz** are quite tight: as shown in [26], for all "natural" sublogics of **QGL** 

and QGrz—including QK, QT, QD, QK4, QS4, QGL, and QGrz—adding to the restriction to two individual variables and a single monadic predicate letter considered in section 2 a minor restriction that the modal operators only apply to formulas with at most one free variable, results in decidable fragments of those logics. It is not difficult to notice that the results analogous to those obtained in section 2 can be obtained for quasi-normal logics such as QS (Solovay's logic) and Lewis's QS1, QS2, and QS3 [7].

A notable exception in our consideration of modal logics is **QS5**, whose two-variable monadic fragment was shown to be undecidable in [12]. While it is not difficult to extend our results to the multimodal version of **QS5**—we need to modify the construction used for sublogics **QKTB** by substituting a succession of two steps along distinct accessibility relations for a single step along a single accessibility relation in the frames of a-suitable models—nor is it difficult to show, by encoding the tiling problem used in [12], that the two-variable fragment of **QS5** with two monadic predicate letters and infinitely many propositional symbols is undecidable, the case of **QS5** remains elusive. We conjecture that the fragment of **QS5** with two variables and a single monadic predicate letter is decidable.

On the other hand, it is relatively straightforward to show that the two-variable fragment of **QS5** with a single monadic predicate letter and an infinite supply of individual variables is undecidable. Indeed, let **SIB** be the first-order theory of a symmetric irreflexive binary relation S; it is well-known that **SIB** is undecidable [18, 13]. We can then simulate S(x,y) as  $\Box(\neg P(x) \lor \neg P(y))$  and show that, if a quantified modal logic L is valid on a frame containing a world that can see infinitely many worlds, then L is undecidable in the language with a single monadic predicate letter (and infinitely many individual variables). This observation covers all modal logics considered in [12], but not covered by the results of section 2, including **QS5**, **QGL.3**, and **QGrz.3**.

By contrast, we can say nothing about superintuitionistic logics not included in the interval [QInt, QKC], as our proof relies on the fact that we are working with the positive fragment of those logics. It is not essential to our proof that formulas  $A_i^k(x)$  and  $B_i^k(x)$  be positive; however, by discarding their positivity we would weaken, rather than strengthen, our results.

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