A review of David Marker

Lectures on Infinitary Model Theory

for *Studia Logica*, by Dag Normann [1][2]

General

Infinitary model theory is the model theory of *infinitary languages*. The base *vocabulary* of an infinitary language is as for first order languages, consisting of symbols for objects, functions, predicates and finite dimensional relations in general. The infinitary aspect is that a language may allow conjunctions and disjunctions of infinite sets of formulas, typically

$$\bigvee\{A_n \mid n \in \mathbb{N}\} \text{ or } \bigvee_{n\in\mathbb{N}} A_n$$

and for some languages, infinitely long sequences of quantifiers. In this volume, mainly languages of the form $L_{\infty,\omega}(\tau)$, or sublanguages thereof, are studied, and the ω indicates that quantifier sequences are finite. In Sect. **2.4** we find a brief discussion of more general languages $L_{\kappa,\lambda}$ where κ and λ are cardinal numbers.

For the most general languages, there is no limitation of the class of variables available, while for some of the languages described below we may also restrict the set of variables. This means that there will be formulas with infinitely many free variables. Such formulas, as is shown in the form of an exercise, can however not be subformulas of sentences, and as a consequence, not subformulas of formulas in some n-type.

The ∞ in $L_{\infty,\omega}$ indicates that the disjunctions and conjunctions of *any* set of formulas are allowed for. One natural restriction will be that only countable conjunctions and disjunctions are permitted, this will be in the languages $L_{\omega_1,\omega}(\tau)$. More generally, but still without full freedom, we may consider languages $L_{\kappa,\omega}(\tau)$, where κ is a cardinal number that serves as a strict upper bound on the size of the infinite conjunctions and disjunctions permitted by the language, or we may restrict the language to formulas that live within some fixed set M, typically a model of Kripke-Platek set theory. The model theory of all such languages are considered in this volume.

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One obvious advantage of using infinitary languages is that the formulas have improved expressive power. This can be used to obtain distinctions between mathematical structures that cannot be expressed with first order formulas. For instance, formulas may contain some information about the cardinality of a structure satisfying it. Another example is the Scott formulas, single formulas that characterise structures up to elementary equivalence, and in the countable case, even up to isomorphism. In Chapter 1 there is a list of examples of classes of structures, many of them of interest in algebra, that can be characterised using infinitary formulas, but that are not first order axiomatisable. Actually, we find examples of applications to algebra throughout the book.

It is also the case that the model theory of infinitary languages can be used to investigate classes of models of regular first order theories. One classical example is the proof due to Michael D. Morley of the approximation to Vaught's conjecture: The number of isomorphism classes of countable models of some countable, complete first order theory is either $\leq \aleph_1$ or $2^{\aleph_0}$.

The increased expressive capacity comes with a cost, important tools in standard model theory like the Löwenheim-Skolem theorems and the compactness theorem do not hold. This is reflected in the book, where alternative methods for model construction are developed. We will be more specific below. Needless to say, the methodology of the model theory of first order theories will still be important, and it will be very hard to follow the exposition in the book without a solid knowledge of the classical model theory. Henkin-constructions, realising and omitting types, closure under elementary chains and so forth are ingredients in many proofs. It is also the case that many arguments, in particular those based on a knowledge of the base theory, are left for the reader as exercises. So, as warned by the author, this book has to be read on top of a standard introduction to model theory.

This said, the book is suitable as course-literature or for self study for students and scientist with a desire to master both the basics and the recent developments in the model theory of infinitary languages. The book is divided into three parts, with all together 12 chapters and two appendices. We will discuss the content of each chapter in more detail below. The main organisation of the book is as follows: In Part 1 the author develops the theory with an introduction to the basic results and the basic methodology. Part 2 brings the material further, with a focus on uncountable structures. Part 3 deals with effective aspects of the theory, linking it to effective descriptive set theory. For readers familiar with the fundaments of computability theory, this part actually gives a self-contained introduction to the hyperarithmetical sets, the basic results on Π^1_1- and Σ^1_1-sets and other results from *higher computability theory* that are useful for the, after all, main theme of model theory.

This book is a good and useful introduction to the model theory of infinitary languages, in particular for readers who want to learn the technicalities of the subject. Since many of the simpler proofs are left as exercises, the book may be demanding on the reader. On the positive side, solving the exercises will help the reader mastering the subject in question. So, the book can be recommended both for self study and as course literature in a graduate course.

On the negative side it can be said that there are too many typos and minor linguistic errors in the book, and that the proof reading of the printed version could have been better. There is no use in bringing up many specific examples, let us just consider one oddity: The distinction between "lightface" and "boldface" versions of Σ and Π in expressions like Σ^1_1 is discussed, but they appear equal in the printed text. The lack of this typographical distinction, and the other examples of poor proof reading and lack of a thorough language check, are however not so severe that the dedicated reader should have problems with the exposition.

The Chapters

In this section, we give a short summary of the content of each chapter. For some chapters, in particular in Part 2 and Part 3, one needs to know the content of earlier chapters in order to grasp the true content, and we will be purposely vague in some cases.

Part 1. Classical results in infinitary model theory

Chapter 1 introduces the basic concepts of infinitary languages and the related model theory, discusses to what extent results from classical model theory like the downward and upward Löwenheim-Skolem theorems and the compactness theorem will hold, and how the class of models of certain theories axiomatised using an infinitary language can be characterised as the class of models of a first order theory over a richer vocabulary omitting certain types. Important concepts introduced are the **PC**-classes (with indices). They are classes of τ-structures that can be characterised using finitary or ω_1-infinitary sentences in a vocabulary τ^* extending τ.

The concept of *elementary equivalence* extends in a canonical way to these infinitary languages. In **Chapter 2** it is first shown how elementary equivalence can be characterised via the existence of systems of partial morphisms. Then, the Scott sentence of a structure is introduced. The Scott sentence $\Phi^{\mathcal{M}}$ of a structure $\mathcal{M}$ is a single formula in $L_{\infty,\omega}$ satisfied by exactly the structures $\mathcal{N}$ elementary equivalent to $\mathcal{M}$. It even characterises structures up to isomorphism in the countable case.

The focus in **Chapter 3** is on Vaught's conjecture and Morley's theorem. Vaught's original conjecture is that the isomorphism class of countable models of a complete first order theory over a countable vocabulary will either be countable or has the cardinality of the continuum. The conjecture is naturally extended to single sentences in $L_{\omega_1,\omega}(\tau)$. Morley's theorem (see above) is proved in two ways, in a soft way and by the original argument. The soft argument shows how general facts from contemporary descriptive set theory may answer intricate questions in model theory in a simple way.
The concept of *Vaught counterexamples* is introduced. They represent possible counterexamples to the generalisation of Vaught's conjecture. The concept liberates the conjecture from the Continuum Hypothesis. Vaught counterexamples are revisited in later chapters.

The main tool introduced in **Chapter 4** is that of a *consistency property*, due to Makkai. A consistency property will be a collection of countable fragments of $L_{\omega_1,\omega}$ with certain closure properties so that they permit Henkin-style constructions of models. Important applications are adjustments of the omitting type theorem and Craig's interpolation theorem to $L_{\omega_1,\omega}$ and the observation that the concept of well orderings still is undefinable in $L_{\omega_1,\omega}$.

The theme of **Chapter 5** is to what extent $L_{\omega_1,\omega}$ can be used to distinguish between possible cardinalities and to allow combinations of cardinalities. Key results are the identification of the Hanf number for $L_{\omega_1,\omega}$, Morley's two cardinal theorem, and Knight's axiomatisation of $\aleph_1$. This chapter requires familiarity with indiscernibles, and makes use of a more sophisticated combinatorial result from set theory.

Part 2. Building uncountable models

The main theme in **Chapter 6** is the construction of models of cardinality $\aleph_1$, using criteria for when a countable structure may have an elementary end extension with respect to a countable fragment of $L_{\omega_1,\omega}$. In this chapter we find constructions of structures realising few types and of structures with a two cardinality property.

Vaught Counterexamples were introduced in Chapter 3. In **Chapter 7**, they are investigated further, focussing on the possible number of models of a Vaught counterexample of cardinality $\aleph_1$ and the possibility of having a model of cardinality $\aleph_2$ independently of the continuum hypothesis.

In **Chapter 8** a construction due to Zilber is given. It is a class $\mathcal{K}$ of structures that is categorical in all uncountable cardinalities. $\mathcal{K}$ cannot be characterised in L_∞, ω alone, also the quantifier $\exists$ *uncountably many* is needed. As the construction is inspired by field theory, and some knowledge of algebraic geometry is needed for the understanding, we will not go into further details.

Part 3. Effective considerations

Chapter 9 introduces the reader to basic computability theory and effective descriptive set theory.

Chapter 10 is a continuation of Chapter 9. Together, the two chapters provide the reader with the tools from standard computability theory, higher computability theory and effective descriptive set theory most frequently used in model theory. Key topics are *hyperarithmetical sets and functions*, Π^1_1- and Σ^1_1-sets, both lightface and boldface, and the general mathematics around these.

In **Chapter 11**, the tools from the two previous chapters are used. It is shown how formulas in $L_{\omega_1,\omega}$ may be coded as elements in a Polish space, typically $\mathbb{N}^{\mathbb{N}}$. One application is the Kreisel-Barwise compactness theorem for Π^1_1 sets of hyperarithmetical sentences. Another is an "evaluation" of the Scott formula for a nicely represented structures.

Chapter 12 is devoted to a theorem by Montalbán. The *degree spectrum* of a countable structure $\mathcal{M}$ is the set of "reals" (subsets of $\mathbb{N}$ or functions from $\mathbb{N}$ to $\mathbb{N}$) relative to which there is a computable structure $\mathcal{N}$ elementary equivalent to $\mathcal{M}$. Under the assumption of *projective determinacy*, Montalbán characterises the Vaught counterexamples Φ in terms of which spectra we may have when $\mathcal{M}$ varies over the models of Φ.

The appendices

An Abelian group is $\aleph_1$ *free* if all countable subgroups are free. In **Appendix A** it is shown that there are $2^{\aleph_1}$ non-isomorphic $\aleph_1$-free Abelian groups of cardinality $\aleph_1$ (and, of course, only one of them will be free).

In **Appendix B** the author introduces the Kripke-Platek set theory and admissible structures, and demonstrates how infinitary model theory can be used to prove theorems in descriptive set theory and higher computability theory.

Conclusion

For the intended reader, this book gives an accessible introduction to the model theory of infinite languages, and it can be recommended for students and established scientists with an interest in the subject. The author assumes that the reader is familiar with the basics of classical model theory, and without this background, the reader may find the book hard to read. The book could have benefitted from an extra round of proof-reading, but the exposition is clear, so this should not cause serious problems for the reader.