Contact join-semilattices

Tatyana Ivanova

Institute of Mathematics and Informatics Bulgarian Academy of Sciences e-mail: tatyana.ivanova@math.bas.bg

Abstract

Contact algebra is one of the main tools in region-based theory of space. In [11, 12, 24, 23] it is generalized by dropping the operation Boolean complement. Furthermore we can generalize contact algebra by dropping also the operation meet. Thus we obtain structures, called contact join-semilattices (CJS) and structures, called distributive contact join-semilattices (DCJS). We obtain a set-theoretical representation theorem for CJS and a relational representation theorem for DCJS. As corollaries we get also topological representation theorems. We prove that the universal theory of CJS and of DCJS is the same and is decidable.

1 Introduction

In classical Euclidean geometry the notion of point is taken as one of the basic primitive notions. In contrast, region-based theory of space (RBTS) has as primitives the more realistic notion of *region* (abstraction of physical body) together with some basic relations and operations on regions. Some of these relations are mereological - part-of, overlap and its dual underlap. Other relations are topological - contact, nontangential part-of, dual contact and some others definable by means of the contact and part-of relations. This is one of the reasons that the extension of mereology with these new relations is commonly called mereotopology. There is no clear difference in literature between RBTS and mereotopology. The origin of RBTS goes back to Whitehead and de Laguna ([36, 25]). According to Whitehead points, as well as the other primitive notions in Euclidean geometry such as lines and planes, do not have separate existence in reality and because of this are not appropriate for primitive notions. Some papers on RBTS are [31, 6, 20, 26, 15, 32, 19, 17, 30, 18] (also the handbook [1] and [4], containing some logics of space).

RBTS has applications in computer science because of its simpler way of representing of qualitative spatial information. Mereotopology is used in the field of Artificial Intelligence, called Knowledge Representation (KR). RBTS initiated a special field in KR, called Qualitative Spatial Representation and Reasoning (QSRR) which is appropriate for automatization [5, 28]. RBTS is applied in geographic information systems, robot navigation. Surveys concerning various applications are for example [7, 8] and the book [21] (also special issues of Fundamenta Informaticae [10] and the Journal of Applied Nonclassical Logics [2]). One of the most popular systems in Qualitative Spatial Representation and Reasoning is the Region Connection Calculus (RCC) [27].

The notion of *contact algebra* is one of the main tools in RBTS. This notion appears in the literature under different names and formulations as an extension of Boolean algebra with some mereotopological relations [35, 29, 34, 33, 6, 14, 9, 13]. The simplest system, called just a contact algebra was introduced in [9] as an extension of Boolean algebra $B = (B, 0, 1, \cdot, +, *)$ with a binary relation C called *contact* and satisfying five simple axioms:

(C1) If aCb, then $a \neq 0$,

(C2) If aCb and $a \leq c$ and $b \leq d$, then cCd,

(C3) If aC(b+c), then aCb or aCc,

(C4) If aCb, then bCa,

(C5) If $a \neq 0$, then aCa.

The elements of the Boolean algebra are called regions and are considered as analogs of physical bodies. Boolean operations are considered as operations for constructing new regions from given ones. The unit element 1 symbolizes the region containing as its parts all regions, and the zero element 0 symbolizes the empty region.

The so called *extended contact algebras* ([22, 3]) extend the language of contact algebras by the predicate *covering* which gives the possibility to be defined the predicate *internal connectedness*.

Sometimes there is a problem in the motivation of the operation Boolean complement (*) of contact algebra. A question arises - if a represents some region, what region does a^* represent - it depends on the universe in which we consider a. Moreover if a represents a physical body, then a^* is unnatural - such a physical body does not exist. Because of this we can drop the operation of complement and replace the Boolean part of a contact algebra with distributive lattice. First steps in this direction were made in [11, 12], introducing the notion of distributive contact lattice. In a distributive contact lattice the only mereotopological relation is the contact relation. Non-tangential inclusion and dual contact (otherwise definable by contact and *) are not included in the language. In [24, 23] the language of distributive contact lattices is extended by considering these two relations as nondefinable primitives. The well known RCC-8 system of mereotopological relations is definable in this more expressive language and is not definable in the language of distributive contact lattices.

Furthermore we can generalize contact algebra by dropping also the operation meet. When the elements of a lattice represent physical bodies, the Boolean operation meet (\cdot) gives the closure of the interior of the intersection of two bodies (which in this case coincides with the intersection of the bodies). In some sense this is an unnatural body and it is reasonable not to consider it. In this paper we eliminate the operation meet from the language of distributive contact lattices. First we consider *contact join-semilattices (CJS)* and obtain a set-theoretical representation theorem and as a corollary - a topological representation theorem. We define also *distributive contact join-semilattices (DCJS)* and prove that every DCJS is also a CJS. The converse is not true. We obtain also a relational representation theorem for DCJS and as a corollary - a topological one. Finally we define a quantifier-free logic which is decidable.

2 Preliminaries

Further we will consider relational and topological contact algebras.

Let (W, R) be a relational system, where W is a nonempty set and R is a reflexive and symmetric binary relation in W and let B be a family of subsets of W closed under union, intersection and complement, containing \emptyset and W. We consider the structure $\underline{B} = (B, \leq, 0, 1, \cdot, +, *, C_R)$, where the interpretations of the constants, functional and predicate symbols are the following: $0 = \emptyset$; 1 = W; $a \leq b$ iff $a \subseteq b$; $a \cdot b = a \cap b$; $a + b = a \cup b$; $a^* = W \setminus a$; aC_Rb iff $\exists x \in a$ and $\exists y \in b$ such that xRy. The obtained structure \underline{B} is called **relational contact algebra over** (\mathbf{W}, \mathbf{R}) [31].

Topological spaces are among the first mathematical models of space, applied in practice. Standard models of contact algebras are topological. Let X be a topological space and a be its subset. We say that a is *regular closed* if a is the closure of its interior. It is a well known fact that the set RC(X) of all regular closed subsets of X is a Boolean algebra with respect to the following definitions: $a \leq b$ iff $a \subseteq b, 0$ is the empty set, 1 is the set X, $a+b = a \cup b, a \cdot b = Cl Int (a \cap b),$ $a^* = Cl(X \setminus a)$. If we define a contact by aCb iff $a \cap b$ is nonempty, then we obtain a contact algebra related to X, namely $RC(X) = (RC(X), \leq, 0, 1, \cdot, +, *, C)$ ([9], Example 2.1). It is called **the topological contact algebra over X**.

In the paper we consider also structures which extend by contact relation the language of the **join-semilattices**, which are defined in the following way:

Definition 2.1 [16] **Join-semilattice** with 0 is a structure $\underline{L} = (L, \leq, 0, +)$ such that are true the axioms

(1)
$$x \le x;$$

(2) $x \le y \land y \le x \to x = y;$
(3) $x \le y \le z \to x \le z;$
(4) $x + y = y + x;$
(5) $x \le x + y;$
(6) $x, y \le z \to x + y \le z;$
(7) $0 \le x.$

Definition 2.2 [16] *Distributive join-semilattice* with 0 is a join-semilattice with 0 $\underline{L} = (L, \leq, 0, +)$ such that is true the axiom

$$(ad) \ x \le a + b \to (\exists a' \le a) (\exists b' \le b) (x = a' + b').$$

Definition 2.3 [16] (page 80) A nonvoid subset I of a join-semilattice \underline{L} is an *ideal* iff for $a, b \in L$, we have $a + b \in I$ iff a and $b \in I$.

Definition 2.4 [16] (page 100) A subset F of a join-semilattice \underline{L} is called a **dual ideal** iff $a \in F$ and $a \leq x$ imply that $x \in F$, and $a, b \in F$ implies that there exists a lower bound d of $\{a, b\}$ such that $d \in F$.

Definition 2.5 [16] (page 100) An ideal I of a join-semilattice \underline{L} is prime iff $I \neq L$ and $L \setminus I$ is a dual ideal.

Lemma 2.6 [16] (page 100) Let I be an ideal and let F be a nonvoid dual ideal of a distributive join-semilattice \underline{L} . If $I \cap F = \emptyset$, then exists a prime ideal P of \underline{L} with $I \subseteq P$ and $P \cap F = \emptyset$.

3 Adding contact relation

We consider additionally the following axioms

$$\begin{array}{l} (8) \ x \leq 1; \\ (9) \ xCy \to x \neq 0; \\ (10) \ xCy \to yCx; \\ (11) \ xC(y+z) \to xCy \ {\rm or} \ xCz; \\ (12) \ xCy, \ y \leq y' \to xCy'; \\ (13) \ x \neq 0 \to xCx; \\ (14) \ {\rm for \ any } \ m, i \geq 1, \\ A^1_{m,i}: \ xCy, \ x \leq s_1, \ldots, s_m, \ y \leq t_1, \ldots, t_m, \ s_1 = s^1_1 + \ldots + s^i_1, \ldots, s_m = s^1_m + \ldots + s^i_m, \\ t_1 = t^1_1 + \ldots + t^i_1, \ldots, t_m = t^1_m + \ldots + t^i_m \to \\ \bigvee_{\substack{l_1 = 1, \ldots, i \\ k_1 = 1, \ldots, i}} \left(\left(\bigwedge_{1 \leq j \leq u \leq m} s^{l_j}_j Cs^{l_u}_u \wedge \bigwedge_{1 \leq j \leq u \leq m} t^{k_j}_j Ct^{k_u}_u \wedge \bigwedge_{\substack{j = 1, \ldots, m \\ u = 1, \ldots, m}} s^{l_j}_j Ct^{k_u}_u \right); \\ (15) \ {\rm for \ any } n, i \geq 1, \\ A_{n,i}: \ t \not\leq u, \ t \leq x_1, \ldots, x_n, \ x_1 = x^1_1 + \ldots + x^i_1, \ldots, x_n = x^1_n + \ldots + x^i_n \to \\ \bigvee_{\substack{j_1 = 1, \ldots, i \\ j_n = 1, \ldots, i}} \left(x^{j_1}_1, \ldots, x^{j_n}_n \not\leq u \wedge \bigwedge_{\substack{k = 1, \ldots, n \\ l = 1, \ldots, n}} x^{j_k}_k Cx^{j_l}_l \right); \\ \end{array} \right);$$

Definition 3.1 Contact join-semilattice (CJS for short) is a structure $\underline{B} = (B, \leq, 0, 1, +, C)$ such that are true the axioms $(1), \ldots, (10); (14)$ and (15).

Remark 3.2 • The axiom $A_{m,i}^1$ says that if a is in contact with b, $a \leq s_1, \ldots, s_m, b \leq t_1, \ldots, t_m$ and $s_1, \ldots, s_m, t_1, \ldots, t_m$ are presented as finite

joins, then one element can be chosen of every join in such a way that every two chosen elements are in contact;

- The axiom $A_{n,i}$ says that if $t \leq u, t \leq a_1, \ldots, a_n$ and a_1, \ldots, a_n are presented as finite joins, then one element can be chosen of every join in such a way that every chosen element is not $\leq u$ and every two chosen elements are in contact.
- The axiom $A_{1,1}^1$ is xCy, $x \leq s_1$, $y \leq t_1$, $s_1 = s_1^1$, $t_1 = t_1^1 \rightarrow s_1^1Ct_1^1$ and obviously $A_{1,1}^1$ is equivalent to the axiom (C2) of contact algebra.
- The axiom $A_{1,2}^1$ is xCy, $x \le s_1$, $y \le t_1$, $s_1 = s_1^1 + s_1^2$, $t_1 = t_1^1 + t_1^2 \rightarrow s_1^1Ct_1^1 \lor s_1^1Ct_1^2 \lor s_1^2Ct_1^1 \lor s_1^2Ct_1^2$. By it we easily obtain that in every CJS is true axiom (11).
- The axiom $A_{1,1}$ is $t \not\leq u, t \leq x_1, x_1 = x_1^1 \rightarrow x_1^1 \not\leq u$ and $x_1^1 C x_1^1$. By it, taking u = 0, we easily obtain that in every CJS is true axiom (13).

Definition 3.3 Distributive contact join-semilattice (DCJS for short) is a structure $\underline{B} = (B, \leq, 0, 1, +, C)$ such that are true the axioms $(1), \ldots, (13)$ and the axiom (ad).

We will prove that every DCJS is also a CJS. Let \underline{B} be a DCJS. We will prove that in \underline{B} are true axioms (14) and (15). For the purpose first we will prove two lemmas.

Lemma 3.4 In \underline{B} is true the formula

 $(d_n) \ x \le a_1 + \ldots + a_n \to (\exists a'_1 \le a_1) \ldots (\exists a'_n \le a_n)(x = a'_1 + \ldots + a'_n),$

where $n \geq 2$.

Proof. We will prove the lemma by induction on n. The base of induction is obvious. Let n > 2 and $\underline{B} \models d_{n-1}$. We will prove that $\underline{B} \models d_n$. Let $x \le a_1 + \ldots + a_n = a_1 + \ldots + (a_{n-1} + a_n)$. By the induction hypothesis, there are $a'_1 \le a_1, \ldots, a'_{n-2} \le a_{n-2}, y \le a_{n-1} + a_n$ such that $x = a'_1 + \ldots + a'_{n-2} + y$. Since $y \le a_{n-1} + a_n$, by axiom (ad), there are $a'_{n-1} \le a_{n-1}, a'_n \le a_n$ such that $y = a'_{n-1} + a'_n$. \Box

Lemma 3.5 Let $x = s_1^1 + \ldots + s_1^i = \ldots = s_m^1 + \ldots + s_m^i$. Then there are t_1, \ldots, t_n such that $x = t_1 + \ldots + t_n$ and for every $j \in \{1, \ldots, n\}$, there are $l_1, \ldots, l_m \in \{1, \ldots, i\}$ such that $t_j \leq s_1^{l_1}, \ldots, s_m^{l_m}$.

Proof. Induction on m. The base of induction is trivial. Let m > 1 and the lemma is true for m - 1. We will prove that it is true for m. Let $x = s_1^1 + \ldots + s_1^i = \ldots = s_m^1 + \ldots + s_m^i$. By the induction hypothesis, there are t_1, \ldots, t_n such that $x = t_1 + \ldots + t_n$ and for every $j \in \{1, \ldots, n\}$, there are $l_1, \ldots, l_{m-1} \in \{1, \ldots, i\}$ such that $t_j \leq s_1^{l_1}, \ldots, s_{m-1}^{l_{m-1}}$. Now we consider the finite joins $x = s_m^1 + \ldots + s_m^i = t_1 + \ldots + t_n$. We have that for every $j \in \{1, \ldots, n\}$,

 $\begin{array}{l} t_j \leq t_1 + \ldots + t_n = s_m^1 + \ldots + s_m^i. \text{ Using this fact and Lemma 3.4, we get that for every } j \in \{1, \ldots, n\}, \text{ there are } v_j^1 \leq s_m^1, \ldots, v_j^i \leq s_m^i \text{ such that } t_j = v_j^1 + \ldots + v_j^i. \\ \text{Thus } x = v_1^1 + \ldots + v_1^i + \ldots + v_n^i \text{ and for any } j \in \{1, \ldots, n\}, k \in \{1, \ldots, i\}, \\ v_j^k \leq t_j \leq s_1^{l_1}, \ldots, s_{m-1}^{l_{m-1}} \text{ and } v_j^k \leq s_m^k. \end{array}$

Lemma 3.6 Let $m, i \ge 1$. Then $\underline{B} \models A^1_{m,i}$.

Proof. Let $xCy, x \leq s_1, \ldots, s_m, y \leq t_1, \ldots, t_m, s_1 = s_1^1 + \ldots + s_1^i, \ldots, s_m = s_m^1 + \ldots + s_m^i, t_1 = t_1^1 + \ldots + t_1^i, \ldots, t_m = t_m^1 + \ldots + t_m^i$. Using Lemma 3.4, we obtain that there are $s_{\alpha 2}^\beta \leq s_{\alpha}^\beta, t_{\alpha 2}^\beta \leq t_{\alpha}^\beta$ for $\alpha = 1, \ldots, m$ and $\beta = 1, \ldots, i$ such that $x = s_{12}^1 + \ldots + s_{12}^i = \ldots = s_{m2}^1 + \ldots + s_{m2}^i; y = t_{12}^1 + \ldots + t_{12}^i = \ldots = t_{m2}^1 + \ldots + t_{m2}^i$. Using Lemma 3.5, we obtain that there are $u_1, \ldots, u_n, v_1, \ldots, v_k$ such that $x = u_1 + \ldots + u_n, y = v_1 + \ldots + v_k$; for every $z \in \{1, \ldots, n\}$, there are $l_1, \ldots, l_m \in \{1, \ldots, i\}$ such that $u_z \leq s_{12}^{l_1}, \ldots, s_{m2}^{l_m}$; for every $z \in \{1, \ldots, k\}$, there are $j_1, \ldots, j_m \in \{1, \ldots, i\}$ such that $u_z \leq t_{12}^{l_1}, \ldots, t_{m2}^{l_m}$. By axiom (11) it can be easily verified that $(u_1 + \ldots + u_n)C(v_1 + \ldots + v_k)$ implies that there are $z_1 \in \{1, \ldots, n\}, z_2 \in \{1, \ldots, k\}$ such that $u_{z_1}Cv_{z_2}$. Clearly there are $l_1, \ldots, l_m \in \{1, \ldots, l_m \in \{1, \ldots, k\}$ such that $u_{z_1}Cv_{z_2}$ and axiom (9), $u_{z_1} \neq 0$ and hence by axiom (13), $u_{z_1}Cu_{z_1}$; so using axiom (12) and $u_{z_1} \leq s_1^{l_1}, \ldots, s_m^{l_m}$, we obtain that every two elements among $s_1^{l_1}, \ldots, s_m^{l_m}$ are in contact. Similarly every two elements among $t_1^{j_1}, \ldots, t_m^{j_m}$ are in contact.

Lemma 3.7 Let $n, i \ge 1$. Then $\underline{B} \models A_{n,i}$.

Proof. Let $t \not\leq u, t \leq x_1, \ldots, x_n, x_1 = x_1^1 + \ldots + x_1^i, \ldots, x_n = x_n^1 + \ldots + x_n^i$. By Lemma 3.4 and Lemma 3.5, there are t_1, \ldots, t_m such that $t = t_1 + \ldots + t_m$ and for every $j \in \{1, \ldots, m\}$, there are $l_1, \ldots, l_n \in \{1, \ldots, i\}$ such that $t_j \leq x_1^{l_1}, \ldots, x_n^{l_n}$. Suppose for the sake of contradiction that $t_1, \ldots, t_m \leq u$. By axiom (6) we get that $t_1 + \ldots + t_m \leq u$, i.e. $t \leq u$ - a contradiction. Consequently there is $j \in \{1, \ldots, m\}$ such that $t_j \not\leq u$ and hence $t_j \neq 0$; so $t_j C t_j$. There are $l_1, \ldots, l_n \in \{1, \ldots, i\}$ such that $t_j \leq x_1^{l_1}, \ldots, x_n^{l_n}$. Thus every two elements among $x_1^{l_1}, \ldots, x_n^{l_n}$ are in contact. Let $k \in \{1, \ldots, n\}$. Suppose for the sake of contradiction that $x_k^{l_k} \leq u$ but we have $t_j \leq x_k^{l_k}$, so $t \leq u$ - a contradiction. Consequently $x_k^{l_k} \not\leq u$. \Box

By Lemma 3.6 and Lemma 3.7 we obtain

Proposition 3.8 Every DCJS is also a CJS.

4 Examples of contact join-semilattices and distributive contact join-semilattices

In this section we will give concrete examples of CJS and DCJS. These examples are considered as "standard examples" because later on we will prove representation theorems of CJS and DCJS by algebras of such standard type.

We will need the following proposition

Proposition 4.1 Every contact algebra is a DCJS.

Proof. Let <u>B</u> be a contact algebra. Obviously axioms $(1), \ldots, (13)$ are true in <u>B</u>. We will prove that <u>B</u> \models (ad). Let $x \leq a + b$. We have $x \cdot a \leq a, x \cdot b \leq b$ and $x = x \cdot (a + b) = x \cdot a + x \cdot b$, because <u>B</u> is a distributive lattice. \Box

The following lemma shows a set-theoretical example of CJS

Lemma 4.2 Let W be a nonempty set and B be a family of subsets of W, containing \emptyset , W and closed under \cup . We define in B: $0 = \emptyset$, 1 = W, $a + b = a \cup b$, $a \leq b$ iff $a \subseteq b$, aCb iff $a \cap b \neq \emptyset$. Then the obtained structure $\underline{B} = (B, \leq, 0, 1, +, C)$ is a CJS.

Proof. We consider $\underline{B_1}$ - the relational contact algebra over (W, =), where $B_1 = 2^W$. From Proposition 4.1 we get that $\underline{B_1}$ is a DCJS and by Proposition 3.8, $\underline{B_1}$ is a CJS. Clearly \underline{B} is a substructure of $\underline{B_1}$. But we also have that the axioms of CJS can be considered as universal formulas and therefore \underline{B} is also a CJS. \Box

The following lemma shows a relational example of DCJS

Lemma 4.3 Let (W, R) be a relational system with a reflexive and symmetric relation R and let B be a family of subsets of W, containing \emptyset , W and closed under \cup . We define in $B: 0 = \emptyset$, 1 = W, $a + b = a \cup b$, $a \leq b$ iff $a \subseteq b$, aCb iff $(\exists U \in a)(\exists V \in b)(URV)$. If in $\underline{B} = (B, \leq, 0, 1, +, C)$ is fulfilled the axiom (ad), then \underline{B} is a DCJS.

Proof. We consider $\underline{B_1}$ - the relational contact algebra over (W, R), where $B_1 = 2^W$. By Proposition 4.1, $\underline{B_1}$ is a DCJS. Clearly \underline{B} is a substructure of $\underline{B_1}$. Axioms $(1), \ldots, (13)$ can be considered as universal formulas, they are true in $\underline{B_1}$ (since $\underline{B_1}$ is a DCJS); so they are true also in the substructure \underline{B} . We have that in \underline{B} is true (ad) and consequently \underline{B} is a DCJS. \Box

The following lemmas show topological examples of CJS and of DCJS

Lemma 4.4 Let X be a topological space and B be a subset of RC(X), containing \emptyset , X and closed under \cup . We define in B: $0 = \emptyset$, 1 = X, $a+b = a \cup b$, $a \leq b$ iff $a \subseteq b$, aCb iff $a \cap b \neq \emptyset$. Then the obtained structure $\underline{B} = (B, \leq, 0, 1, +, C)$ is a CJS.

Proof. Clearly <u>B</u> is a substructure of the topological contact algebra over X and similarly as in the proof of Lemma 4.2 we get that <u>B</u> is a CJS. \Box

Lemma 4.5 Let X be a topological space and B be a subset of RC(X), containing \emptyset , X and closed under \cup . We define in B: $0 = \emptyset$, 1 = X, $a + b = a \cup b$, $a \leq b$ iff $a \subseteq b$, aCb iff $a \cap b \neq \emptyset$. If $\underline{B} = (B, \leq, 0, 1, +, C)$ satisfies the axiom (ad), then \underline{B} is a DCJS.

Proof. The proof is similar to the proof of Lemma 4.4, using that <u>B</u> is a substructure of RC(X). \Box

Proposition 4.6 There is a standard set-theoretical example of CJS which is not a DCJS.

Proof. We consider the set $W = \{1, 2, 3, 4\}$. Let $B = \{\emptyset, W, \{1, 3\}, \{2, 4\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 4\}\}$. It can be easily verified that *B* is closed under \cup . We define in *B*: $0 = \emptyset, 1 = W$, $a + b = a \cup b, a \le b$ iff $a \subseteq b, aCb$ iff $a \cap b \ne \emptyset$. By Lemma 4.2, the structure $\underline{B} = (B, \le, 0, 1, +, C)$ is a CJS. But \underline{B} does not satisfy the axiom (ad), because $\{1, 2\} \le \{1, 3\} + \{2, 4\}$ but $(\forall a' \le \{1, 3\})(\forall b' \le \{2, 4\})(\{1, 2\} \ne a' + b')$. □

Proposition 4.7 There is a standard topological example of CJS which is not a DCJS.

Proof. We consider the same W and B as in the proof of Proposition 4.6. We define topology on W, taking for open all subsets of W. It can be easily verified that $RC(W) = 2^W$. We define in B: $0 = \emptyset$, 1 = W, $a + b = a \cup b$, $a \leq b$ iff $a \subseteq b$, aCb iff $a \cap b \neq \emptyset$. By Lemma 4.4, the obtained structure $\underline{B} = (B, \leq, 0, 1, +, C)$ is a CJS. The structure \underline{B} is the same as the structure \underline{B} in the proof of Proposition 4.6 and therefore \underline{B} is not a DCJS. \Box

5 Representation theorems for contact join-semilattices

First we will prove a set-theoretical representation theorem of CJS. For this purpose we will need the following definition, taken from the theory of contact algebras

Definition 5.1 [9] Let \underline{B} be a CJS. A subset of $B \Gamma$ is called a **clan** in \underline{B} if the following conditions are true:

1) $1 \in \Gamma$; 2) $0 \notin \Gamma$; 3) $x \in \Gamma, x \leq y \to y \in \Gamma$; 4) $x, y \in \Gamma \to xCy$; 5) $x + y \in \Gamma \to x \in \Gamma$ or $y \in \Gamma$. We denote by $Clans(\underline{B})$ the set of the clans in \underline{B} .

Example 5.2 Let W be a nonempty set and <u>B</u> be the standard set-theoretical example of CJS of all subsets of W. Let $x \in W$. Then it can be easily verified that $P_x = \{P \subseteq W : x \in P\}$ is a clan.

Let \underline{B} be an arbitrary CJS. We will prove several lemmas. The first lemma has two variants - the first variant contains the text in the brackets, the second one - no.

Lemma 5.3 (Let $u \neq 1$.) Let Γ be a subset of B and Γ satisfies condition 3) from Definition 5.1 and the condition:

(*) $x_1, \ldots, x_n \in \Gamma \to \text{for every presentation of } x_1, \ldots, x_n \text{ as finite joins, one}$ element can be chosen of every join $(\not\leq u)$ in such a way that every two chosen elements are in contact.

Let $x + y \in \Gamma$. Then there exists a set Γ_1 , satisfying the same conditions and such that $\Gamma_1 = \Gamma \cup \{z : x \leq z\}$ or $\Gamma_1 = \Gamma \cup \{z : y \leq z\}$.

Proof. We will prove only the first variant of the lemma. The second variant is proved similarly. Suppose for the sake of contradiction that the following two conditions are true:

(\clubsuit) there are $x_1, \ldots, x_m \in \Gamma, z_1, \ldots, z_k \ge x$ and presentations of $x_1, \ldots, x_m, z_1, \ldots, z_k$ as finite joins such that it is impossible to be chosen one element $\le u$ of every join in such a way that every two chosen are in contact;

(\bigstar) there are $y_1, \ldots, y_n \in \Gamma, t_1, \ldots, t_r \ge y$ and presentations of $y_1, \ldots, y_n, t_1, \ldots, t_r$ as finite joins such that it is impossible to be chosen one element $\le u$ of every join in such a way that every two chosen are in contact.

Let the presentations as finite joins be:

$x_1 = x_1^1 + \ldots + x_1^{i_1}$	$y_1 = y_1^1 + \ldots + y_1^{j_1}$
:	:
$x_m = x_m^1 + \ldots + x_m^{i_m}$	$y_n = y_n^1 + \ldots + y_n^{j_n}$
$z_1 = z_1^1 + \ldots + z_1^{i_{11}}$	$t_1 = t_1^1 + \ldots + t_1^{j_{11}}$
:	:
$z_k = z_k^1 + \ldots + z_k^{i_{1k}}$	$t_r = t_r^1 + \ldots + t_r^{j_{1r}}$

Let $i \in \{1, \ldots, k\}$, $j \in \{1, \ldots, r\}$. It can be easily verified that $x + y \leq z_i + t_j$. We have also that $x + y \in \Gamma$. Consequently $z_i + t_j \in \Gamma$. We have that $x_1, \ldots, x_m, y_1, \ldots, y_n, z_1 + t_1, \ldots, z_1 + t_r, \ldots, z_k + t_1, \ldots, z_k + t_r \in \Gamma$. Thus by property (*) of Γ we obtain that one element can be chosen of every of the

following joins $(\not\leq u)$:

$$\begin{array}{c} x_{1}^{1}+\ldots+x_{1}^{i_{1}} \\ \vdots \\ x_{m}^{1}+\ldots+x_{m}^{i_{m}} \\ y_{1}^{1}+\ldots+y_{1}^{j_{1}} \\ \vdots \\ y_{n}^{1}+\ldots+y_{n}^{j_{n}} \\ z_{1}^{1}+\ldots+z_{1}^{i_{11}}+t_{1}^{1}+\ldots+t_{1}^{j_{11}} \\ \vdots \\ z_{1}^{1}+\ldots+z_{1}^{i_{11}}+t_{r}^{1}+\ldots+t_{r}^{j_{r}} \\ \vdots \\ z_{k}^{1}+\ldots+z_{k}^{i_{1k}}+t_{1}^{1}+\ldots+t_{1}^{j_{11}} \\ \vdots \\ z_{k}^{1}+\ldots+z_{k}^{i_{1k}}+t_{1}^{1}+\ldots+t_{r}^{j_{r}} \end{array}$$

in such a way that every two chosen are in contact. Suppose for the sake of contradiction that for every $s \in \{1, \ldots, k\}$, among $z_s^1, \ldots, z_s^{i_{1s}}$ some element is chosen. But this is a contradiction with (\clubsuit) . Consequently there is $s \in \{1, \ldots, k\}$ such that among $z_s^1, \ldots, z_s^{i_{1s}}$ no one is chosen. Consequently one element $(\not\leq u)$ is chosen from every of the joins:

$$\begin{array}{c} t_{1}^{1}+\ldots+t_{1}^{j_{11}}\\ \vdots\\ t_{r}^{1}+\ldots+t_{r}^{j_{1r}}\\ y_{1}^{1}+\ldots+y_{1}^{j_{1}}\\ \vdots\\ y_{n}^{1}+\ldots+y_{n}^{j_{n}}\\ \end{array}$$

in such a way that every two chosen elements are in contact. But this is a contradiction with (\spadesuit) . Consequently condition (\clubsuit) is not true or condition (\clubsuit) is not true. Without loss of generality (\clubsuit) is not true i.e. the following condition is satisfied:

 (\heartsuit) for any $x_1, \ldots, x_m \in \Gamma$, $z_1, \ldots, z_k \ge x$ and presentations of x_1, \ldots, x_m , z_1, \ldots, z_k as finite joins, one element $\le u$ can be chosen of every join in such a way that every two chosen elements are in contact.

We consider the set $\Gamma_1 = \Gamma \cup \{z : x \leq z\}$. It can be easily proved that Γ_1 satisfies property 3) from Definition 5.1. We will prove that Γ_1 satisfies the property (*). Let $a_1, \ldots, a_p, b_1, \ldots, b_q \in \Gamma_1$, where $p + q > 0, a_1, \ldots, a_p \in \Gamma$,

 $b_1, \ldots, b_q \ge x$. We will prove that for every presentation of $a_1, \ldots, a_p, b_1, \ldots, b_q$ as finite joins, one element $(\not\le u)$ can be chosen of every join in such a way that every two chosen are in contact.

Case 1: q = 0The proof is obvious. Case 2: p = 0

Let us have the following presentation of b_1, \ldots, b_q as finite joins:

$$b_1 = b_1^1 + \ldots + b_1^{l_1}$$

$$\vdots$$

$$b_q = b_q^1 + \ldots + b_q^{l_q}$$

We have also $x + y \in \Gamma$ and we finish the proof, using condition (\heartsuit). Case 3: p, q > 0

Again we use condition (\heartsuit). \Box

Lemma 5.4 Let tCt_1 . Then there is a clan Γ such that $t, t_1 \in \Gamma$.

Proof. We consider $M = \{P \subseteq B :$

t, $t_1 \in P;$ $0 \notin P;$ $x \in P, x \leq y \rightarrow y \in P;$ $x_1, \dots, x_k \in P \rightarrow \text{ for every presentatio}$

 $x_1, \ldots, x_k \in P \to$ for every presentation of x_1, \ldots, x_k as finite joins, one element can be chosen of every join in such a way that every two chosen elements are in contact}.

We will prove that (M, \subseteq) has a maximal element. Let L be a chain in (M, \subseteq) . We will prove that L has an upper bound in M.

Case 1: $L = \emptyset$

We consider the set $P = \{x \in B : t \leq x \text{ or } t_1 \leq x\}$. We will prove that $P \in M$. For the purpose we will prove only the last condition for the elements of M. The other conditions are obviously true. Let $x_1, \ldots, x_k \in P$. Let $\{x \in \{x_1, \ldots, x_k\} : t \leq x\} = \{a_1, \ldots, a_m\}$, where $m \geq 0$. Let b_1, \ldots, b_n $(n \geq 0)$ be the rest elements of $\{x_1, \ldots, x_k\}$, i.e. $t_1 \leq b_1, \ldots, b_n$.

Let us have the following presentations of $a_1, \ldots, a_m, b_1, \ldots, b_n$ as finite joins:

$$a_{1} = a_{1}^{1} + \ldots + a_{1}^{i_{1}}$$

$$\vdots$$

$$a_{m} = a_{m}^{1} + \ldots + a_{m}^{i_{m}}$$

$$b_{1} = b_{1}^{1} + \ldots + b_{1}^{j_{1}}$$

$$\vdots$$

$$b_{n} = b_{n}^{1} + \ldots + b_{n}^{j_{n}}$$

We will consider only the case when $m \ge n$. The other case $(n \ge m)$ is

symmetric. We have

$$tCt_1, t \le a_1, \dots, a_m, t_1 \le \underbrace{b_1, \dots, b_n, 1, \dots, 1}_{m \text{ times}}, m > 0$$

Let $i = max(i_1, \ldots, i_m, j_1, \ldots, j_n)$. We supplement every join with its first element in such a way that to have *i* elements. We use also that $1 = \underbrace{1 + \ldots + 1}_{i \text{ times}}$.

By axiom $A^1_{m,i}$ we get that one element can be chosen from the new joins in such a way that every two chosen elements are in contact. Consequently one element can be chosen from every of the initial joins in such a way that every two chosen elements are in contact. Consequently $P \in M$. P is an upper bound of L.

Case 2: $L \neq \emptyset$

It can be easily verified that $\bigcup L \in M$. Obviously $\bigcup L$ is an upper bound of L.

By Zorn Lemma, (M, \subseteq) has a maximal element Γ . We will prove that Γ is a clan. It is easily seen that Γ satisfies conditions 1),...,4) of Definition 5.1. Now we will prove that Γ satisfies condition 5) of Definition 5.1. Let $x + y \in \Gamma$. By the second variant of Lemma 5.3, without loss of generality there exists a set Γ_1 such that satisfies properties 3) and 4) of Definition 5.1, the last condition of the definition of M and $\Gamma_1 = \Gamma \cup \{z : x \leq z\}$. We will prove that $\Gamma_1 \in M$. Since $\Gamma \in M, t, t_1 \in \Gamma$ and hence $t, t_1 \in \Gamma_1$. Suppose for the sake of contradiction that $0 \in \Gamma_1$. Since Γ_1 satisfies condition 4) of Definition 5.1, 0C0 and hence $0 \neq 0$ a contradiction. Consequently $0 \notin \Gamma_1$. Clearly Γ_1 satisfies the rest conditions of the definition of M. Consequently $\Gamma_1 \in M$. We have also that Γ is a maximal element of M and $\Gamma \subseteq \Gamma_1$. Thus $\Gamma = \Gamma_1$. Clearly $x \in \Gamma_1$. Consequently $x \in \Gamma$. Thus Γ satisfies condition 5) of Definition 5.1; so Γ is a clan. We have that $\Gamma \in M$ and therefore $t, t_1 \in \Gamma$. \Box

Lemma 5.5 Let $t \leq u$. Then there is a clan Γ such that $t \in \Gamma$, $u \notin \Gamma$.

Proof. We consider the set $M = \{P \subseteq B :$

 $t \in P, \ u \notin P;$

 $x \in P, x \le y \to y \in P;$

 $x_1, \ldots, x_k \in P \to$ for every presentation of x_1, \ldots, x_k as finite joins, one element can be chosen of every join, $\leq u$, in such a way that every two chosen elements are in contact}.

We will prove that (M, \subseteq) has a maximal element. Let L be a chain in (M, \subseteq) . We will prove that L has an upper bound in M.

Case 1: $L = \emptyset$

We consider the set $P = \{x \in B : t \leq x\}$. We will prove that $P \in M$. The conditions for the elements of M, without the last one, are obviously true. We will prove the last condition. Let $x_1, \ldots, x_k \in P$. Let us have the following

presentations of x_1, \ldots, x_k as finite joins:

$$x_1 = x_1^1 + \ldots + x_1^{i_1}$$

$$\vdots$$

$$x_k = x_k^1 + \ldots + x_k^{i_k}$$

Let $i = max(i_1, \ldots, i_k)$. We supplement every join with its first element in such a way that all joins to have *i* elements. By axiom $A_{k,i}$, one element can be chosen of every join, $\leq u$, in such a way that every two chosen elements are in contact. Consequently the last condition of the definition of M is fulfilled. Thus $P \in M$. The set P is an upper bound of L. **Case 2:** $L \neq \emptyset$

It can be easily verified that $\bigcup L \in M$. Clearly $\bigcup L$ is an upper bound of L.

From Zorn Lemma we obtain that (M, \subseteq) has a maximal element Γ . We will prove that Γ is a clan. It can be easily verified that Γ fulfills conditions $1, \ldots, 4$) of Definition 5.1. We will prove that Γ satisfies condition 5) of Definition 5.1. Let $x + y \in \Gamma$. We must prove that $x \in \Gamma$ or $y \in \Gamma$. Since $t \not\leq u, u \neq 1$. By the first variant of Lemma 5.3, without loss of generality there exists a set Γ_1 such that fulfills property 3) of Definition 5.1, the property (*) from the first variant of Lemma 5.3 and $\Gamma_1 = \Gamma \cup \{z : x \leq z\}$. We will prove that $\Gamma_1 \in M$. Suppose for the sake of contradiction that $u \in \Gamma_1$. Since Γ_1 fulfills property (*), $u \in \Gamma_1$, u = u (a presentation of u as a finite join), we have $u \not\leq u$ - a contradiction. Consequently $u \notin \Gamma_1$. The rest conditions of the definition of M can be verified easily. Consequently $\Gamma_1 \in M$. Thus $\Gamma_1 = \Gamma$ and $x \in \Gamma$. Consequently Γ satisfies condition 5) of Definition 5.1. Thus Γ is a clan. We have $\Gamma \in M$ and hence $t \in \Gamma$ and $u \notin \Gamma$. \Box

Now we can prove

Theorem 5.6 (Set-theoretical representation theorem of CJS) Let \underline{B} be a CJS. Then there is a nonempty set W and an isomorphic embedding of \underline{B} in the standard set-theoretical example of CJS of all subsets of W.

Proof. Let $W = Clans(\underline{B})$. We define a function h from B to 2^W in the following way: $h(a) = \{\Gamma \in Clans(\underline{B}) : a \in \Gamma\}$. We will prove that h is an isomorphic embedding.

We will show that h is an injection. Let $a \neq b$. Suppose for the sake of contradiction that $a \leq b$ and $b \leq a$. By axiom (2), a = b - a contradiction. Consequently $a \not\leq b$ or $b \not\leq a$. Without loss of generality $a \not\leq b$. By Lemma 5.5, there is a clan Γ such that $a \in \Gamma$, $b \notin \Gamma$. Consequently $\Gamma \in h(a)$ and $\Gamma \notin h(b)$, i.e. $h(a) \neq h(b)$.

Clearly $h(0) = \emptyset$ and h(1) = W.

We will prove that h preserves the operation +. By condition 5) from Definition 5.1, $h(a+b) \subseteq h(a) \cup h(b)$. Let $\Gamma \in h(a) \cup h(b)$. We will prove $\Gamma \in h(a+b)$. Without loss of generality $\Gamma \in h(a)$ and hence $\Gamma \in Clans(\underline{B}), a \in \Gamma$; so using condition 3) from Definition 5.1 and $a \le a + b$, we obtain that $a + b \in \Gamma$ and therefore $\Gamma \in h(a + b)$. Consequently $h(a + b) = h(a) \cup h(b)$.

We will prove that h preserves the relation \leq . We have $a \leq b$ iff a + b = b, $h(a) \subseteq h(b)$ iff $h(a) \cup h(b) = h(b)$, h preserves the operation + and h is an injection, so h preserves the relation \leq .

We will prove that h preserves the relation C. We have $h(a)Ch(b) \iff h(a) \cap h(b) \neq \emptyset$. We must prove aCb iff $h(a) \cap h(b) \neq \emptyset$. By Lemma 5.4, aCb implies $h(a) \cap h(b) \neq \emptyset$. Let $h(a) \cap h(b) \neq \emptyset$. Consequently there is $\Gamma \in h(a)$, h(b) and therefore Γ is a clan, $a \in \Gamma$ and $b \in \Gamma$. By condition 4) from Definition 5.1, aCb.

Thus h is an isomorphic embedding. \Box

Theorem 5.7 (Topological representation theorem of CJS) Let \underline{B} be a CJS. Then there is a compact, semiregular, T_0 topological space X and an isomorphic embedding h of \underline{B} in the topological contact algebra over X (considered as a standard topological example of CJS).

Proof. As in the proof of Theorem 5.6, we see that there is a relational system (W, =) and an isomorphic embedding h_1 of \underline{B} in the relational contact algebra $\underline{B_1}$ of all subsets of W. It is shown in [9] (Theorem 5.1) that every contact algebra is isomorphically embedded in the topological contact algebra over some compact, semiregular, T_0 topological space. Therefore there is an embedding h_2 of $\underline{B_1}$ in the topological contact algebra over some compact, semiregular, T_0 topological space X. The desired embedding h is $h_2 \circ h_1$. \Box

6 Representation theorems of distributive contact join-semilattices

For proving a relational representation theorem of DCJS we will need the following definition

Definition 6.1 Let <u>B</u> be a DCJS. We define abstract point of <u>B</u> as a subset of $B \Gamma$ such that:

 1 ∈ Γ;
 0 ∉ Γ;
 x ∈ Γ, x ≤ y → y ∈ Γ;
 x, y ∈ Γ → there is a lower bound of {x, y} z such that z ∈ Γ;
 x + y ∈ Γ → x ∈ Γ or y ∈ Γ. We denote by AP(B) the set of all abstract points of B.

We consider an arbitrary DCJS \underline{B} . We will prove several lemmas.

Lemma 6.2 Let P be a prime ideal, $0 \in P$, $1 \notin P$. Then $U = B \setminus P$ is an abstract point.

Proof. By Definition 2.5, U is a dual ideal. Consequently U satisfies conditions 3) and 4) of Definition 6.1. Obviously U fulfills conditions 1) and 2) of Definition 6.1. Let $x + y \in U$. Suppose for the sake of contradiction that x, $y \notin U$. Consequently $x, y \in P$ but P is a prime ideal, so P is an ideal, so $x + y \in P$ - a contradiction. Consequently $x \in U$ or $y \in U$. Consequently U satisfies condition 5) of Definition 6.1. Thus U is an abstract point. \Box

Lemma 6.3 Let Γ be a clan and $a \in \Gamma$. Then there is an abstract point U such that $a \in U, U \subseteq \Gamma$.

Proof. We consider the set $[a) \stackrel{def}{=} \{x : a \leq x\}$. It can be easily verified that [a) is a dual ideal and $[a) \subseteq \Gamma$. We denote $I = B \setminus \Gamma$. Since $[a) \subseteq \Gamma$, $[a) \cap I = \emptyset$. It can be easily verified that I is an ideal. By Lemma 2.6, there exists a prime ideal P of \underline{B} with $I \subseteq P$ and $P \cap [a] = \emptyset$. We denote $U = B \setminus P$. We have $P \cap [a] = \emptyset$, so $[a] \subseteq U$, so $1 \in U$, so $1 \notin P$. Since Γ is a clan, $0 \in I$, so $0 \in P$. By Lemma 6.2, U is an abstract point. Clearly $a \in U$ and $U \subseteq \Gamma$. \Box

Lemma 6.4 Let Γ be a clan. Then there is a set of abstract points Σ such that $\Gamma = \bigcup \Sigma$ and for any $U, V \in \Sigma, x \in U$ and $y \in V$ imply xCy.

Proof. Let $a \in \Gamma$. By Lemma 6.3, there is an abstract point U_a such that $a \in U_a, U_a \subseteq \Gamma$. We denote $\Sigma = \{U_a : a \in \Gamma\}$. It can be easily verified that $\Gamma = \bigcup \Sigma$. Let $U, V \in \Sigma$. Let $x \in U, y \in V$. We must prove that xCy. Since $U, V \in \Sigma, U = U_b$ and $V = U_c$ for some $b, c \in \Gamma$ and moreover $U_b, U_c \subseteq \Gamma$. Consequently $x, y \in \Gamma$ but Γ is a clan, so xCy. \Box

Lemma 6.5 Every two elements of an abstract point are in contact.

Proof. The lemma can be easily proved using axioms (13), (12) and (10) from Definition 3.3. \Box

Corollary 6.6 Every abstract point is a clan.

Lemma 6.7 Let $t \leq u$. Then there is an abstract point U such that $t \in U$, $u \notin U$.

Proof. Since <u>B</u> is a DCJS, <u>B</u> is a CJS and we can apply Lemma 5.5. Thus there is a clan Γ such that $t \in \Gamma$, $u \notin \Gamma$. By Lemma 6.3, there is an abstract point U such that $t \in U$, $U \subseteq \Gamma$. Obviously $u \notin U$. \Box

Now we can prove

Theorem 6.8 (Relational representation theorem of DCJS) Let \underline{B} be a DCJS. Then there is a relational structure (W, R) with a reflexive and symmetric relation R and an isomorphic embedding of \underline{B} in the relational contact algebra of all subsets of W (considered as the standard relational example of DCJS of all subsets of W).

Proof. Let $W = AP(\underline{B})$. We define R in the following way: URV iff $(\forall a \in U)(\forall b \in V)(aCb)$.

By Corollary 6.6, R is reflexive. Obviously R is symmetric. We define a function h from B to 2^W in the following way: $h(a) = \{U \in AP(\underline{B}) : a \in U\}$. We will prove that h is an isomorphic embedding.

Using Lemma 6.7, we prove that h is an injection.

Clearly $h(0) = \emptyset$ and h(1) = W.

Similarly as in Theorem 5.6 we prove that h preserves the operation + and the relation \leq .

We will prove that h preserves the relation C. Let $a, b \in B$. We have h(a)Ch(b) iff there are $U \in h(a), V \in h(b)$ such that $(\forall x \in U)(\forall y \in V)(xCy)$. Clearly h(a)Ch(b) implies aCb. Now let aCb. Since <u>B</u> is also a CJS, using Lemma 5.4, we obtain that there is a clan Γ such that $a, b \in \Gamma$. By Lemma 6.4 we see that h(a)Ch(b). Consequently h preserves the relation C.

Thus h is an isomorphic embedding. \Box

Theorem 6.9 (Topological representation theorem of DCJS) Let \underline{B} be a DCJS. Then there is a compact, semiregular, T_0 topological space X and an isomorphic embedding h of \underline{B} in the topological contact algebra over X (considered as a standard topological example of DCJS).

Proof. The proof is similar of the proof of Theorem 5.7. \Box

Remark 6.10 It is possible to prove Theorem 6.8 (also Lemma 6.7) without using of clans.

Second proof of Lemma 6.7. We consider $[t) = \{x \in B : t \leq x\}$ and $(u] = \{x \in B : x \leq u\}$. It can be easily verified that (u] is an ideal and that [t) is a dual ideal. Suppose for the sake of contradiction that $(u] \cap [t) \neq \emptyset$, i.e. there is $x \in (u] \cap [t)$. We have $t \leq x \leq u$ and hence $t \leq u$ - a contradiction. Consequently $(u] \cap [t] = \emptyset$. By Lemma 2.6, there exists a prime ideal P of \underline{B} with $(u] \subseteq P$ and $P \cap [t] = \emptyset$. Using Lemma 6.2, we obtain that $U = B \setminus P$ is an abstract point of \underline{B} . Clearly $t \in U$ and $u \notin U$. \Box

Second proof of Theorem 6.8. The proof is the same as before with two differences.

For proving the reflexivity of R we use Lemma 6.5.

We prove that aCb implies h(a)Ch(b) in a similar way as in [24] (Lemma 3.8 (i)). Let aCb. We consider $P = \{x : x\overline{C}b\}$. We will prove that P is an ideal. It suffices to show that $x + y \in P$ iff $x, y \in P$. Let $x + y \in P$. Consequently $(x + y)\overline{C}b$. Suppose for the sake of contradiction that $x \notin P$ or $y \notin P$. Without loss of generality $x \notin P$ and hence xCb; so (x + y)Cb - a contradiction. Consequently $x, y \in P$. Now let $x, y \in P$ and suppose for the sake of contradiction that $x + y \notin P$. Thus P is an ideal.

We have also that [a) is a dual ideal and $[a) \cap P = \emptyset$; so by Lemma 2.6, there exists a prime ideal P' of \underline{B} with $P \subseteq P'$ and $P' \cap [a] = \emptyset$. By Lemma 6.2, $F = B \setminus P'$ is an abstract point.

We consider $I = \{x : (\exists y \in F)(x\overline{C}y)\}$. We will prove that I is an ideal. Let $x, y \in B$. It can be easily seen that $x + y \in I$ implies $x, y \in I$. Now let $x, y \in I$. We will prove $x + y \in I$. We have that $(\exists z_1 \in F)(x\overline{C}z_1)$ and $(\exists z_2 \in F)(y\overline{C}z_2)$. Since $z_1, z_2 \in F$ and F is an abstract point, there is a lower bound of $\{z_1, z_2\}$ z such that $z \in F$. Suppose for the sake of contradiction that (x + y)Cz. Consequently zCx or zCy. Without loss of generality zCx but $z \leq z_1$; so xCz_1 - a contradiction. Consequently $(x + y)\overline{C}z$ and hence $x + y \in I$. Consequently I is an ideal. Suppose for the sake of contradiction that there is $x \in [b) \cap I$. We have $(\exists y \in F)(b \leq x\overline{C}y)$. Since $y \in F$, $y \notin P$; so yCb; so yCx- a contradiction. Consequently $[b) \cap I = \emptyset$. We have also that [b) is a dual ideal, I is an ideal; so by Lemma 2.6, there is a prime ideal I' with $I \subseteq I'$ and $I' \cap [b] = \emptyset$. By Lemma 6.2, $F_1 = B \setminus I'$ is an abstract point.

It remains to prove that there are $U \in h(a)$, $V \in h(b)$ such that $(\forall x \in U)(\forall y \in V)(xCy)$. Clearly $F \in h(a)$ and $F_1 \in h(b)$. Let $x \in F$, $y \in F_1$. Suppose for the sake of contradiction that yCx. Consequently $y \in I$ and hence $y \in I'$; so $y \notin F_1$ - a contradiction. Consequently yCx. Thus h(a)Ch(b). \Box

7 A quantifier-free logic

We consider a quantifier-free language \mathcal{L} which has

- constants: 0, 1;
- functional symbols: +;
- predicate symbols: \leq , C.

We consider a quantifier-free logic L which has axioms these of CJS and an only rule of inference - modus ponens.

Theorem 7.1 (Completeness theorem) Let φ be a formula in \mathcal{L} . Then the following conditions are equivalent:

1) φ is a theorem of L;

2) φ is true in all topological contact algebras;

- 3) φ is true in all DCJS;
- 4) φ is true in all CJS;

5) φ is true in all finite CJS with number of the elements $\leq 2^n + 1$, where n is the number of the variables of φ .

Proof. Let T be the set of the axioms of L. Condition 1) is equivalent to 1') $T \vdash \varphi$. By the well known Completeness theorem, Condition 1') is equivalent to 1") $T \models \varphi$.

1") \rightarrow 2) It can be easily verified.

 $2) \rightarrow 3$) Let \mathcal{A} be a DCJS and v be a valuation in \mathcal{A} . We will prove that $(\mathcal{A}, v) \models \varphi$. By Theorem 6.9, there is a topological space X and an isomorphic embedding h of \mathcal{A} in $\underline{RC(X)}$. Let the variables of φ be $p_1, \ldots, p_n, n \ge 0$. We define a valuation v_1 in $\overline{RC(X)}$ in the following way:

$$v_1(p) = \begin{cases} h(v(p)) & \text{if } p = p_1 \text{ or } p = p_2 \text{ or } \dots \text{ or } p = p_n \\ \emptyset & \text{otherwise} \end{cases}$$

Clearly $(\mathcal{A}, v) \models \varphi$ iff $(RC(X), v_1) \models \varphi$. Using 2), we get that $(\mathcal{A}, v) \models \varphi$.

 $3) \rightarrow 4$) Let \mathcal{A} be a $\overline{\text{CJS}}$ and v be a valuation in \mathcal{A} . We will prove that $(\mathcal{A}, v) \models \varphi$. By Theorem 5.7, there is a topological space X and an isomorphic embedding h of \mathcal{A} in RC(X). We define a valuation v_1 in RC(X) as above and we have $(\mathcal{A}, v) \models \varphi$ iff $(RC(X), v_1) \models \varphi$. By Proposition 4.1, RC(X) is a DCJS and by 3), $(RC(X), v_1) \models \varphi$.

 $4) \rightarrow 5)$ Obviously.

 $5) \rightarrow 1$ ") Let $\mathcal{A} \models T$, i.e. \mathcal{A} is a CJS. Let v be a valuation in \mathcal{A} . We will prove that $(\mathcal{A}, v) \models \varphi$. Let the variables of φ be p_1, \ldots, p_n . We consider the set $S = \{v(p_{i_1}) + \ldots + v(p_{i_m}) : i_1 < \ldots < i_m \leq n, m \geq 1\} \cup \{0, 1\}$. Clearly $|S| \leq 2^n + 1$. The structure \mathcal{S} with universe S is a substructure of \mathcal{A} and since \mathcal{A} is a CJS and the axioms of CJS can be considered as universal formulas, \mathcal{S} is a CJS. We define a valuation v_1 in \mathcal{S} in the following way:

$$v_1(p) = \begin{cases} v(p) & \text{if } p = p_1 \text{ or } p = p_2 \text{ or } \dots \text{ or } p = p_n \\ 0 & \text{otherwise} \end{cases}$$

By 5), $(\mathcal{S}, v_1) \models \varphi$ and hence $(\mathcal{A}, v) \models \varphi$. \Box

Corollary 7.2 L is decidable.

8 Conclusion

Some possible future research directions are for example:

- the complexity of the considered logic;
- is the theory of CJS finitely axiomatizable or not; is it possible axioms $A_{m,i}^1$ and $A_{n,i}$ to be simplified;
- to be obtained representations in T_1 and T_2 topological spaces by considering axiomatic extensions of CJS and DCJS;
- the language to be extended by considering as nondefinable primitives of the relations non-tangential inclusion and dual contact.

Acknowledgements. This paper is supported by National program "Young scientists and Postdoctoral candidates" 2020 of Ministry of Education and Science of Bulgaria.

References

- M. Aiello, I. Pratt-Hartmann and J. van Benthem (Eds.), Handbook of spatial logics. Springer, 2007.
- [2] P. Balbiani (Ed.), Special Issue on Spatial Reasoning, J. Appl. Non-Classical Logics, vol. 12, (3-4), 2002.
- [3] P. Balbiani and T. Ivanova, "Relational representation theorems for extended contact algebras," *Stud Logica*, 2020. https://doi.org/10.1007/s11225-020-09923-0
- [4] P. Balbiani, T. Tinchev and D. Vakarelov, "Modal logics for regionbased theory of space," Fundamenta Informaticae, Special Issue: Topics in Logic, Philosophy and Foundation of Mathematics and Computer Science in Recognition of Professor Andrzej Grzegorczyk, vol. 81, (1-3), 2007, pp. 29–82.
- [5] B. Bennett, "Determining consistency of topological relations," Constraints, vol. 3, 1998, pp. 213–225.
- [6] B. Bennett and I. Düntsch, "Axioms, algebras and topology," in Handbook of Spatial Logics, M. Aiello, I. Pratt, and J. van Benthem (Eds.), Springer, 2007, pp. 99–160.
- [7] A. Cohn and S. Hazarika, "Qualitative spatial representation and reasoning: An overview," *Fundamenta Informaticae*, vol. 46, 2001, pp. 1–20.
- [8] A. Cohn and J. Renz, "Qualitative spatial representation and reasoning," in F. van Hermelen, V. Lifschitz and B. Porter (Eds.) Handbook of Knowledge Representation, Elsevier, 2008, pp. 551–596.
- [9] G. Dimov and D. Vakarelov, "Contact algebras and region-based theory of space: A proximity approach I," *Fundamenta Informaticae*, vol. 74, (2-3), 2006, pp. 209–249.
- [10] I. Düntsch (Ed.), Special issue on Qualitative Spatial Reasoning, Fundam. Inform., vol. 46, 2001.
- [11] I. Düntsch, W. MacCaull, D. Vakarelov and M. Winter, "Topological representation of contact lattices," *Lecture Notes in Computer Science*, vol. 4136, 2006, pp. 135–147.
- [12] I. Düntsch, W. MacCaull, D. Vakarelov and M. Winter, "Distributive contact lattices: Topological representation," *Journal of logic and Algebraic Programming*, vol. 76, 2008, pp. 18–34.
- [13] I. Düntsch and D. Vakarelov, "Region-based theory of discrete spaces: A proximity approach," in M. Nadif, A. Napoli, E. SanJuan and A. Sigayret (Eds.) Proceedings of Fourth International Conference Journées

de l'informatique Messine, Metz, France, 2003, pp. 123–129, Journal version in Annals of Mathematics and Artificial Intelligence, vol. 49, (1-4), 2007, pp. 5–14.

- [14] I. Düntsch and M. Winter, "A representation theorem for Boolean contact algebras," *Theoretical Computer Science (B)*, vol. 347, 2005, pp. 498–512.
- [15] G. Gerla, "Pointless geometries", in Handbook of Incidence Geometry, F. Buekenhout (Ed.), Elsevier, 1995, pp. 1015–1031.
- [16] G. Grätzer, "General Lattice Theory", Birkhäuser, Basel, 1978.
- [17] R. Gruszczyński and A. Pietruszczak, "A study in Grzegorczyk point-free topology Part I: Separation and Grzegorczyk structures," *Stud Logica*, vol. 106, 2018, pp. 1197–1238, "Part II: Spaces of points," *Stud Logica*, vol. 107, 2019, pp. 809–843.
- [18] R. Gruszczyński and A. Pietruszczak, "Full development of Tarski's geometry of solids," *The Bulletin of Symbolic Logic*, vol. 14(4), 2008, pp. 481–540.
- [19] A. Grzegorczyk, "Axiomatizability of geometry without points," in: The Concept and the Role of the Model in Mathematics and Natural and Social Sciences, Synthese Library (A Series of Monographs on the Recent Development of Symbolic Logic, Significs, Sociology of Language, Sociology of Science and of Knowledge, Statistics of Language and Related Fields), vol. 3, Springer, Dordrecht, 1961.
- [20] T. Hahmann and M. Gruninger, "Region-based theories of space: Mereotopology and beyond," S. Hazarika (ed.): Qualitative Spatio-Temporal Representation and Reasoning: Trends and Future Directions, 2012, pp. 1–62, IGI Publishing.
- [21] Qualitative spatio-temporal representation and reasoning: Trends and future directions. S. M. Hazarika (Ed.), IGI Global, 1st ed., 2012.
- [22] T. Ivanova, "Extended contact algebras and internal connectedness," Stud Logica, vol. 108, 2020, pp. 239–254.
- [23] T. Ivanova, "Logics for extended distributive contact lattices," Journal of Applied Non-Classical Logics, vol. 28(1), 2018, pp. 140–162.
- [24] T. Ivanova and D. Vakarelov, "Distributive mereotopology: extended distributive contact lattices," Annals of Mathematics and Artificial Intelligence, vol. 77(1), 2016, pp. 3–41.
- [25] T. de Laguna, "Point, line and surface as sets of solids," J. Philos, vol. 19, 1922, pp. 449–461.
- [26] I. Pratt-Hartmann, "First-order region-based theories of space," in Logic of Space, M. Aiello, I. Pratt-Hartmann and J. van Benthem (Eds.), Springer, 2007.

- [27] D. A. Randell, Z. Cui, and A. G. Cohn., "A spatial logic based on regions and connection," in B. Nebel, W. Swartout, C. Rich (Eds.) Proceedings of the 3rd International Conference Knowledge Representation and Reasoning, Morgan Kaufmann, Los Allos, CA, 1992, pp. 165–176.
- [28] J. Renz and B. Nebel, "On the complexity of qualitative spatial reasoning: a maximal tractable fragment of the region connection calculus," *Artificial Intelligence*, vol. 108, 1999, pp. 69–123.
- [29] J. Stell, "Boolean connection algebras: A new approach to the Region Connection Calculus," Artif. Intell., vol. 122, 2000, pp. 111–136.
- [30] A. Tarski, "Foundations of the geometry of solids," *Logic, semantics, meta-mathematics*, papers from 1923 to 1938, Clarendon Press, Oxford, 1956, pp. 24–29.
- [31] D. Vakarelov, "Region-based theory of space: Algebras of regions, representation theory and logics," in D. Gabbay, S. Goncharov and M. Zakharyaschev (Eds.) Mathematical Problems from Applied Logic II. Logics for the XXIst Century, Springer, 2007, pp. 267–348.
- [32] D. Vakarelov, "Point-free theories of space and time," Journal of Applied Logics: IfCoLog Journal of Logics and their Applications, vol. 7(6), 2020, pp. 1243–1322.
- [33] D. Vakarelov, G. Dimov, I. Düntsch, and B. Bennett, "A proximity approach to some region-based theories of space," *Journal of applied non-classical logics*, vol. 12, (3-4), 2002, pp. 527–559.
- [34] D. Vakarelov, I. Düntsch and B. Bennett, "A note on proximity spaces and connection based mereology," in C. Welty and B. Smith (Eds.) Proceedings of the 2nd International Conference on Formal Ontology in Information Systems (FOIS'01), ACM, 2001, pp. 139–150.
- [35] H. de Vries, "Compact spaces and compactifications," Van Gorcum, 1962.
- [36] A. N. Whitehead, "Process and Reality," New York, MacMillan, 1929.