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Executability and Connexivity in an Interpretation of Griss

Abstract. Although the work of G.F.C. Griss is commonly understood as a program of *negationless* mathematics, close examination of Griss's work suggests a more fundamental feature is its *executability*, a requirement that mental constructions are possible only if corresponding mental activity can be actively carried out. Emphasizing executability reveals that Griss's arguments against negation leave open several types of negation—including D. Nelson's strong negation—as compatible with Griss's intuitionism. Reinterpreting Griss's program as one of *executable mathematics*, we iteratively develop a pair of bilateral constructive logics and argue for their adequacy as accounts of the propositional basis of Griss's work. We conclude by observing connexive features exhibited by the two bilateral logics and by investigating the difficulties connexive principles reveal for the development of executable mathematics.

Keywords: Connexive logic, Constructive logic, G.F.C. Griss, Executable logic.

1. Introduction: Griss's Executable Mathematics

The mathematician G.F.C. Griss is best known for promoting a philosophy of mathematics that can be described as *superconstructive* in the sense of imposing stricter requirements on constructivity than his contemporary, L.E.J. Brouwer. The realization of Griss's project—a reconstruction of fundamental areas of mathematics under these stricter requirements—is offered across several papers ([17, 19–22]). The most well-recognized feature of Griss's work is its explicit rejection of the use of *negation* in constructive mathematics, e.g.,

> I contend that negation must be banished, I mean reasoning through negation. [18, p. 71][1]

Due to the apparent centrality of this rejection, Griss's program is known as *negationless intuitionistic mathematics*. A number of attempts at formalzing

[1] *Je prétends qu'il faut en bannir la négation, je veux dire le raisonnement par négation.*

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the underlying logic of Griss's program exist, including López-Escobar in [29], Minichiello in [32], and Krivtsov in [27]. Such attempts, taking Griss's choice of terminology literally, emphasize the rejection of negation to be the central feature of Griss's work. However, in spite of the choice of terminology, arguments are given in [13] that this label is infelicitous for several reasons, which can be summarized in two points.

First, although the rejection of negation is a hallmark feature of Griss's theory, it is not a *first principle*. Rather, it follows from a thesis about the intuitionistic conditional that the mental activity of constructive reasoning *presupposes the existence of an operand*. As Vredenduin remarks, a "fundamental feature of Griss's method is that he accepts that in constructing one is always constructing something and so never will construct nothing." [46, p. 206] In other words, Griss requires that the construction witnessing a conditional must be *executable*. Should one understand the negation of φ intuitionistically—as the construction of an absurdity $\perp$ from the supposition of φ—then negation can only be deployed in the case of *absurd propositions*, e.g. contradictions, for which no mental construction is possible. Thus, per Griss's strict requirements, the mental act of negating cannot be executed. As the rejection of intuitionistic negation is a consequence of Griss's requirement of executability, [13] concludes that executability is more fundamental to Griss's program than the rejection of negation.

Second, emphasizing the role of executability reveals that Griss's case against negation is not a critique of negation *in general*, but rather of intuitionistic negation insofar as the implicit conditional in Brouwerian account is a critical ingredient of Griss's argument. Constructive mathematics coheres with notions of negation beyond Brouwer's, most notably Nelson's strong negation of *constructible falsity* described in [33]. Given these alternatives, the interpreting Griss's argument as requiring the rejection of *all negation* is too hasty. Indeed, Nelson's setting is particularly well-suited to Griss's project. Griss's frequent use of pairs of contradictory (but "positive") relations, e.g., *identity* ($\equiv$) and *distinguishability* ($\not\equiv$), reflects some *implicit* notion of negation at the literal level. By identifying such pairs as the *extension* and *anti-extension* of a *common* relation, the techniques of e.g. [23] and [25] demonstrate how to lift negations from literals to complex formulae, revealing the existence of a sort of "holographic" theory of negation flattened in Griss's nominally negationless mathematics. This will be made explicit in Section 4.4.

If one recognizes that Nelson-style negation is compatible with Griss's aims, the emphasis of Griss's mathematics is more properly the insistence that constructive reasoning requires that a reasoner is able to bring to mind a construction of an antecedent. So reframed, Griss's project becomes one of *executable mathematics*.

1.1. Desiderata

Given this recentered interpretation of Griss's project as a mathematics of executability, the investigation into its logical basis becomes pressing. The goal in this paper is simply to introduce possible propositional bases for such a logic that is constructive and has an executable conditional.

As Francesco Berto remarks in [2], the fundamental *mental* activity of a construction is *targeted*; like acts of imagination, the constructive reasoner's "conscious acts... have a deliberate, explicit starting point: we set out to target a chosen content." [2, p. 1875]. The picture I envision is as follows: A reasoner, setting out to evaluate a conditional $\varphi \to \psi$, begins by attempting to *initialize a mental simulation* on the initial input φ. Griss's principle of executability requires that this attempt at initialization of the antecedent is *successful*. In its capacity as a *generator*, φ serves to *constrain* the ensuing mental simulation in several ways. Let us examine some natural requirements on the definition of a Grissian conditional.

We take *Criterion I* to be that an acceptable conditional will have both intuitionistic features and features of containment logics, i.e., logics imposing a topic inclusion requirement on conditionals. The requirement that the conditional is intuitionistic is obvious; Griss's commitment to the constructive nature of implication clearly follows from his embrace of Brouwerian principles, i.e., given the verification of the antecedent, Griss is largely faithful to Brouwer's understanding that verifications of $\varphi \to \psi$ are mental constructions that yield a construction of ψ when applied to a construction of φ.

Although topic-theoretic intuitions do not appear to have guided the development of Griss's program, the principle of executability leads to a number of topic-theoretic consequences. Clearly, one cannot execute a mental construction of a statement when one is unaware of its subject-matter. As initial input, an antecedent seeds a mental simulation and as the simulation evolves, the input's subject-matter serves as the core stock of concepts available to the simulation. A mental construction must make do with the tools at its disposal. Of course, because the ultimate product of an act of intuitionistic reasoning is constructed within a mental simulation, the constituent subject-matter of the end result must be limited to those concepts available to the reasoner; the only concepts universally guaranteed are those supplied by the initial input.[2] Criterion I thus demands a *topic-theoretic* constraint in the style of William Parry's *analytic implication* of [38] or Berto's *topic-sensitive intentional modals* of [4].

Parry's preferred example is particularly salient:

> If a system contains the assertion that two points determine a straight line, does the theorem necessarily follow that either two points determine a straight line or the moon is made of green cheese? No, for the system may contain no terms from which 'moon,' etc., can be defined. [40, p. 151]

In this case, *the concepts of geometry* serve as the raw materials through which constructions may be composed within the simulation. A construction establishing that "either two points determine a straight line or the moon is made of green cheese" could only be composed granted the availability of e.g. the concept of *green cheese*. But familiarity with the subject of geometry fails to ensure that a reasoner has access to the latter concept. The only conceptual materials that are *universally* available in a mental simulation are those materials provided by the input on which it was initiated.

As *Criterion II*, Griss's executability entails that the constructive verification of a conditional requires that one engage in mental activity in which the antecedent is clearly brought to mind. Over the course of an investigation, consequently, to verify $\varphi \rightarrow \psi$ requires the assurance of the possibility of future states of the investigation at which the antecedent is posited. Executability thus requires that an adequate conditional should exhibit a feature of *anti-vacuity*, which we roughly describe:

Definition 1. A conditional $\rightarrow$ enjoys *anti-vacuity* in a logical system if $\varphi \rightarrow \psi$ is verified (i.e., true) at a state only in case φ is possible at that state.

Such conditionals have appeared in the literature, e.g., David Lewis' "might" conditional $\varphi \diamond\!\!\rightarrow \psi$ of [28] is true at a world only in case an appropriate world at which the antecedent φ is true exists.

Finally, *Criterion III* follows from a surprising compatibility between Griss's philosophy and the account of negation described in David Nelson's [33]. [13] argues for a refinement of Griss's rejection of negation in which

[2] Thematically, Kant's rejection of the analyticity of arithmetic is closely related; Kant's remark that the concept of the number 12 is not discoverable within the union of the concepts of 5 and 7 could be given a topic-theoretic gloss.

two *types* of negation must be rejected: First, due to Griss's explicit commitment to Brouwerian principles, Griss follows Brouwer's rejection of *non-constructive negation* as unfit for constructive reasoning about infinitary contexts. Second, the BHK-style interpretation of negation makes clear that executability rules out *intuitionistic negation*. A proof of $\neg\varphi$—i.e. $\varphi \to \bot$—is only available in case φ is absurd. But to Griss, an absurdity cannot be brought to mind, whereby the necessary mental activity of verifiying an intuitionistic negation is impossible. Thus, Griss's commitments do not apply to a negation that is *constructive but not intuitionistic*, that is, are not of the form $\varphi \to \bot$ yet do not run afoul of constructive requirements like the disjunction property.

In light of the many competing notions of negation (see e.g. [47] or [24]), it is not surprising to find that there are negations satisfying this description. Most notably, Nelson's remarks on *constructible falsity* make clear that his account of negation—and others in its family, like those catalogued in [36]—are largely aligned with Griss's aims. Such negations are constructive (in satisfying the disjunction and existence properties) but not intuitionistic (in rejecting the BHK-style account of negation).

Griss clearly admits some activity resembling negation at the level of literals and Nelson-style bilateral techniques can lift Griss's implicit negation to the full language. The third criterion is thus to provide the system with the type of negation that helps to expose Griss's implicit theory of negation.

1.2. Prehensivist vs Nonprehensivist Interpretations

One further matter constrains the scope of this investigation. On the present interpretation, the primary concern of Griss's intuitionism as respecting the *mental realizability* of intuitionistic mental construction. Consequently, the truth of $\varphi \to \psi$ is in some ways a *two-step procedure*: *Initially*, by executability, its truth demands that the mathematician is capable of *bringing to mind a representation* of a situation satisfying φ. *Subsequently*, it requires that in any such representation, a means exists to mentally construct a representation satisfying ψ.

Let us linger on the former axis, taking Berto's *conceiving-as-imagining* in [3]—according to which an agent's ability to relate to a situation described by φ is a sufficient condition for its conceivability—as a starting point. Consider the range of propositional *simulanda* suitable for cognitive agents, that is, the extent of meaningful statements that agents can coherently imagine. Given the centrality of the activity of bringing to mind or mental simulation, this dimension—the extent of the class of propositions suitable as possible simulanda—plays a substantial role in generating two radically divergent theories.

To coin two phrases, let us define two (non-exhaustive) positions one could take:

- *Prehensivism* Reasoners can mentally simulate *any proposition.*
- *Nonprehensivism* Reasoners can not simulate *inconsistent* propositions.

To illustrate this dimension's significance to the underlying logic, informally consider how an axiom $\varphi \to \varphi$ might fare. On the *prehensivist* interpretation, any instance would likely be valid; irrespective of choice of φ, a reasoner can form a mental construction of φ and the identity function would trivially act to convert any such construction to a construction of of φ. Of course, executability means that certain instances—where φ is a contradiction—require a reasoner to clearly simulate a contradictory state. The impossibility of such activity by the *nonprehensivist* interpretation precludes the validity of $\varphi \to \varphi$ in general.

While prehensivism paints an extremely *permissive* picture of the bounds of imaginative activity, there is phenomenological evidence in its favor. Consider the activity in which one engages when one explains a proof by *reductio ad absurdum.* As a principle about the *act of reasoning*, its utility flows from an assumption that *one can consider falsehoods as though true.*

Proofs by *reductio ad absurdum* in mathematics are often phrased as invitations to intentional mental activity, e.g., '*assume* that such-and-such holds', etc.. Importantly, the respective conclusions are drawn *not* for having shown that the anticipated mental simulation *did not occur* but for being *shown to follow* in the situation considered by the participants.

Prehensivism, in short, assumes a model of intentionality that makes sense of reasoning about stipulations of falsehoods about mathematical objects. As Graham Priest remarks, such models of intentionality are particularly useful about the domain that most interests Griss:

> And when I imagine that 361 is a prime number (it isn't) I am imagining something about that very number. [43, p. 195]

From a phenomenological perspective, this example indeed provides an intuitive portrait of our mental activity when conducting proofs.

Despite its appeal, Griss's words suggest that he would have been unmoved by this type of example. His remark that "one cannot have a clear conception of a supposition that eventually proves to be a mistake" [17, p. 1127] suggests that the bounds of the conceivable are understood as roughly approximate to the bounds of mathematical possibility. Given the aim of developing an interpretation of Griss, taking a nonprehensivist posture is critical to the formulation of candidate propositional logics. We therefore are guided by a nonprehensivist view in the following, leaving the development of a prehensivist interpretation for future work.

Together, the three foregoing criteria and the commitment to nonprehensivism determine the structure of the paper and our investigation into propositional logics appropriate to Griss's project. Section 2 begins the satisfaction of Criterion I in introducing a JPAI, an intuitionistic version of Parry's logic of analytic implication. In the subsequent Section 3, we describe ExPAI, an *executable* version of this intuitionistic containment logic whose conditional satisfies the anti-vacuity constraint, satisfying Criterion II. Section 4 introduces two candidates $\mathsf{N}_1\mathsf{PAI}$ and $\mathsf{N}_1\mathsf{PAI}$ for the propositional logic of executable mathematics, defined by applying Nelsonian techniques to ExPAI differing on the falsification conditions for the conditional. Finally, Section 5 investigates several *connexive* features exhibited by these systems and considers their consequences for executable mathematics.

2. Criterion I: Intuitionistic Analytic Implication

In the first phase, we produce a propositional logic that meets Criterion I's demand for an intuitionistic conditional including a topic-inclusion filter. To meet the requirements of this condition, we initially survey the logic of William Parry's analytic implication PAI.

Although only recently has it become more-or-less unobjectionable to acknowledge the importance of topic or subject-matter to myriad formal investigations, considerations of topic have been taken up routinely by philosophical logicians for generations. Most famous and fully developed of these studies is Parry's work, whose logic PAI was first introduced in [39]. Parry's logic is designed to respect intuitions the *analyticity* of a conditional in part should require topic-theoretic inclusion.

Although Parry's motivations were independent of intuitionistic considerations, the example can be adapted to our particular case.

2.1. Fine's Analytic Implication

Parry introduced his PAI axiomatically in [39]. The initial semantic analyses of PAI began decades later by examinations of related systems by Dunn in

[6] and Urquhart in [45]. The first full model theory for Parry's logic itself was provided by Fine in [15].[3]

The starting point for the structures described in this paper is Fine's model theory of [15] for Parry's PAI. Interestingly, [15] provides two equivalent presentations of PAI based on different languages. The first—the type favored by Parry in [39]—enriches the classical propositional language with an intensional analytic implication connective $\rightarrow$. The second expands classical propositional logic with axioms describing both a unary $\mathsf{S4}$ necessity operator $\Box$ and a binary content inclusion connective $\preccurlyeq$, so that "$\varphi \preccurlyeq \psi$" is read as "the topic of φ is included in the topic of ψ."

To get more precise, first define the language $\mathcal{L}_F$:

$$\varphi ::= p \mid \neg\varphi \mid \Box\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \supset \varphi \mid \varphi \preccurlyeq \varphi$$

Fine's axiomatization—which we call $\mathsf{PAI/F}$—is as follows, where $var(\varphi)$ is the set of atoms appearing in φ:

I	1	The set of theorems of classical propositional logic
	2	Modus ponens for $\supset$
II	3	$\Box(\varphi \supset \psi) \supset (\Box\varphi \supset \Box\psi)$
	4	$\Box\varphi \supset \varphi$
	5	$\Box\varphi \supset \Box\Box\varphi$
	6	If $\vdash \varphi$ then $\vdash \Box\varphi$
III	7	$(\varphi \preccurlyeq \psi \wedge \psi \preccurlyeq \xi) \supset \varphi \preccurlyeq \xi$
	8	$(\varphi \preccurlyeq \xi \wedge \psi \preccurlyeq \xi) \supset (\varphi \wedge \psi \preccurlyeq \xi)$
	9	$\varphi \preccurlyeq \psi$ when $var(\varphi) \subseteq var(\psi)$
IV	10	$(\varphi \preccurlyeq \psi) \supset \Box(\varphi \preccurlyeq \psi)$

One can then define Parry's analytic implication connective with the scheme $(\varphi \rightarrow \psi) \equiv (\Box(\varphi \supset \psi) \wedge (\psi \preccurlyeq \varphi))$. Thus, Fine's presentation essentially gives an explicit conceptual analysis of Parry's conditional into a more expressive expansion of classical logic; in this sense, it is not unlike Gödel's work in [16], in which an explicit provability modality exposes the implicit proof-theoretic structure of intuitionistic logic.

Fine's semantics for Parry's system equips each world w of an $\mathsf{S4}$ Kripke model with join semilattices of *topics* $\langle \mathcal{T}_w, \oplus_w \rangle$.

Definition 2. A PAI model is a tuple $\langle W, R, \mathcal{T}, \oplus, v, t \rangle$ where:

- $\langle W, R \rangle$ is an $\mathsf{S4}$ Kripke frame

[3] Technically, [15], too, provides model theory for an *extension* of Parry's logic but the additional axiom was ultimately endorsed by Parry as true to the spirit of analytic implication in [37].

- For each $w \in W$, $\langle \mathcal{T}_w, \oplus_w \rangle$ is a join semilattice
- v is a valuation from atomic formulae to W
- For each $w \in W$, t_w is a function mapping atomic formulae to $\mathcal{T}_w$

with the assumption that for atoms p and q, whenever wRw', $t_w(p) \leq_w t_w(q)$ implies $t_{w'}(p) \leq_{w'} t_{w'}(q)$ (where $a \leq_w b$ if $a \oplus_w b = b$).

Notably, Definition 2's requirement of *topic-persistence*—that wRw' and $t_w(p) \leq_w t_w(q)$ implies $t_{w'}(p) \leq_{w'} t_{w'}(q)$ for atoms p, q—guarantees that topic inclusion between formulae will persist across accessible worlds.

These join semilattices reflect an intuition that topics can be *fused* together into more complex topics; e.g., the topics of **cats** and **dogs** together form the complex topic **cats-and-dogs**. The functions t_w are responsible for assigning topics to formulae; for PAI, the assignment is determined as follows:

Definition 3. The topic assignment function t_w is extended through the language:

- $t_w(\neg\varphi) = t_w(\Box\varphi) = t_w(\varphi)$
- $t_w(\varphi \circ \psi) = t_w(\varphi) \oplus_w t_w(\psi)$ for binary connectives $\circ$

For example, a conjunction's topic is simply the fusion of its conjuncts' topics.

Truth at a world is defined as follows:

Definition 4. Truth conditions are defined recursively:

- $w \Vdash p$ if $w \in v(p)$
- $w \Vdash \neg\varphi$ if $w \nVdash \varphi$
- $w \Vdash \varphi \wedge \psi$ if $w \Vdash \varphi$ and $w \Vdash \psi$
- $w \Vdash \varphi \vee \psi$ if $w \Vdash \varphi$ or $w \Vdash \psi$
- $w \Vdash \varphi \supset \psi$ if $w \nVdash \varphi$ or $w \Vdash \psi$
- $w \Vdash \Box\varphi$ if for all $w' \in w{\uparrow}$, $w' \Vdash \varphi$
- $w \Vdash \varphi \preccurlyeq \psi$ if $t_w(\varphi) \leq_w t_w(\psi)$

where $w{\uparrow}$ is the set $\{w' \in W \mid wRw'\}$.

This allows us to provide a definition of validity:

Definition 5. $\Gamma \vDash_{\mathsf{PAI}} \varphi$ iff for all PAI models and points w, if $\mathfrak{M}, w \Vdash \psi$ for each $\psi \in \Gamma$, then $\mathfrak{M}, w \Vdash \varphi$

To satisfy Criterion I, we introduce a constructive version of PAI in the language intended by Parry.

2.2. Intuitionistic Analytic Implication

Our emphasis on Fine's PAI/F rather than Parry's formulation is unusual but explained by its increased expressivity, allowing a more precise clarification of Parry's intuitions. This expressivity will soon bear fruit; for now, we revert to Parry's original language. This language $\mathcal{L}_J$—in which we will primarily work—is defined:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi$$

That an S4 Kripke frame is a feature common to model theories for both PAI/F and intuitionistic logic alike points to a natural semantics for a system JPAI of intuitionistic PAI. Imposing a constraint that truth persists along R—that once a statement has been verified in an investigation, it remains true—determines a subclass of PAI models suitable for intuitionistic purposes:

Definition 6. A JPAI model is a tuple $\langle W, R, \mathcal{T}, \oplus, v, t\rangle$ augmenting Definition 2 with the condition of *truth-persistence* that for all atoms p, if wRw' and $w \in v(p)$, also $w' \in v(p)$.

On such models, we introduce the following truth conditions:

Definition 7. JPAI truth conditions are defined recursively:

- $w \Vdash p$ if $w \in v(p)$
- $w \Vdash \neg\varphi$ if for all $w' \in w{\uparrow}$, $w' \nVdash \varphi$
- $w \Vdash \varphi \wedge \psi$ if $w \Vdash \varphi$ and $w \Vdash \psi$
- $w \Vdash \varphi \vee \psi$ if $w \Vdash \varphi$ or $w \Vdash \psi$
- $w \Vdash \varphi \rightarrow \psi$ if $\begin{cases} \text{for all } w' \text{ such that } wRw', \text{ if } w' \Vdash \varphi \text{ then } w' \Vdash \psi \\ t_w(\psi) \leq_w t_w(\varphi) \end{cases}$

Finally, a definition of validity in JPAI can be offered:

Definition 8. $\Gamma \vDash_{\mathsf{JPAI}} \varphi$ iff for all JPAI models $\mathfrak{M}$ and points w, if $\mathfrak{M}, w \Vdash \psi$ for each $\psi \in \Gamma$, then $\mathfrak{M}, w \Vdash \varphi$

Before proceeding to provide a proof theory for JPAI, we provide lemmas characterizing two features of its model theory, i.e., the *persistence of topic inclusion* and the *persistence of truth*.

Lemma 1. *In any* JPAI *model, if* $t_w(\varphi) \leq_w t_w(\psi)$ *and* wRw' *then* $t_{w'}(\varphi) \leq_{w'} t_{w'}(\psi)$*.*

Proof. Topic persistence follows identically from the proof for PAI in [15]. ∎

Lemma 2. *In any* JPAI *model, if* $w \Vdash \varphi$ *and* wRw' *then* $w' \Vdash \varphi$*.*

Proof. The basis step is guaranteed by the R-closure of each set $v(p)$. Induction steps for stock intuitionistic connectives are standard, while induction for formulae $\psi \rightarrow \xi$ follows from the standard case in conjunction with Lemma 1. ∎

We provide JPAI—and each related system in this paper—with two distinct styles of proof theory. A direct proof theory is provided by means of a tableau calculus in the style of [42] while an indirect Hilbert-style proof theory follows by means of a Gödel-McKinsey-Tarski-style translation into Fine's axiomatic calculus for PAI.[4]

First we review a tableau calculus for JPAI, following closely to Priest's presentation of intuitionistic logic in [42]. Nodes on a tableau are decorated with a label $\langle i, k\rangle$, admitting informal interpretatons of i as analogous to a possible world and k as an indication of truth or falsity. The symbol ρ will be used as a syntactic device intended to mirror accessibility between possible worlds. Note also that the presentation includes rules concerning *pseudoformulae*—formulae not in $\mathcal{L}_J$ that are included *virtually* on a branch in a tableau. Such pseudoformulae are distinguished by their appearance inside a dashed box in the below.

Definition 9. A tableau calculus for JPAI—intuitionistic PAI—is given by the following rules:

$$\begin{array}{c} \varphi\ \langle i\,1\rangle \\ i\rho j \\ \downarrow \\ \varphi\ \langle j\,1\rangle \end{array} \quad \begin{array}{c} \cdot \\ \downarrow \\ i\rho i \end{array} \quad \begin{array}{c} i\rho j \\ j\rho k \\ \downarrow \\ i\rho k \end{array} \quad \begin{array}{c} \neg\varphi\ \langle i\,1\rangle \\ i\rho j \\ \downarrow \\ \varphi\ \langle j\,0\rangle \end{array} \quad \begin{array}{c} \neg\varphi\ \langle i\,0\rangle \\ \downarrow \\ i\rho j \\ \varphi\ \langle j\,1\rangle \end{array}$$

[4]There would unquestionably be utility in axiomatic presentations of JPAI and related systems. Such a presentation is made difficult given the constraints of Griss's executability and nonprehensivism. As we noted earlier, these constraints conflict with axioms of the form $\varphi \rightarrow \psi$. The metavariable φ can, after all, receive a contradictory substitution instance.

$$
\begin{array}{c}
\varphi \wedge \psi \ \langle i\, 1\rangle \\
\downarrow \\
\varphi \ \langle i\, 1\rangle \\
\psi \ \langle i\, 1\rangle
\end{array}
\qquad
\begin{array}{c}
\varphi \wedge \psi \ \langle i\, 0\rangle \\
\swarrow \quad \searrow \\
\varphi \ \langle i\, 0\rangle \quad \psi \ \langle i\, 0\rangle
\end{array}
\qquad
\begin{array}{c}
\varphi \vee \psi \ \langle i\, 1\rangle \\
\swarrow \quad \searrow \\
\varphi \ \langle i\, 1\rangle \quad \psi \ \langle i\, 1\rangle
\end{array}
\qquad
\begin{array}{c}
\varphi \vee \psi \ \langle i\, 0\rangle \\
\downarrow \\
\varphi \ \langle i\, 0\rangle \\
\psi \ \langle i\, 0\rangle
\end{array}
$$

$$
\begin{array}{c}
\varphi \rightarrow \psi \ \langle i\, 1\rangle \\
i\rho j \\
\downarrow \\
\boxed{\psi \preccurlyeq \varphi} \ \langle j\, 1\rangle \\
\swarrow \quad \searrow \\
\varphi \ \langle j\, 0\rangle \quad \psi \ \langle j\, 1\rangle
\end{array}
\qquad
\begin{array}{c}
\varphi \rightarrow \psi \ \langle i\, 0\rangle \\
\swarrow \quad \searrow \\
\boxed{\psi \preccurlyeq \varphi} \ \langle i\, 0\rangle \quad
\begin{array}{c} i\rho j \\ \varphi \ \langle j\, 1\rangle \\ \psi \ \langle j\, 0\rangle \end{array}
\end{array}
$$

$$
\begin{array}{c}
\cdot \\
\downarrow \\
\boxed{\varphi \preccurlyeq \psi} \ \langle j\, 1\rangle \ \star
\end{array}
\qquad
\begin{array}{c}
\boxed{\varphi \preccurlyeq \xi} \ \langle i\, 1\rangle \\
\boxed{\psi \preccurlyeq \xi} \ \langle i\, 1\rangle \\
\downarrow \\
\boxed{\varphi \wedge \psi \preccurlyeq \xi} \ \langle i\, 1\rangle
\end{array}
\qquad
\begin{array}{c}
\boxed{\varphi \preccurlyeq \psi} \ \langle i\, 1\rangle \\
i\rho j \\
\downarrow \\
\boxed{\varphi \preccurlyeq \psi} \ \langle j\, 1\rangle
\end{array}
\qquad
\begin{array}{c}
\boxed{\varphi \preccurlyeq \psi} \ \langle i\, 1\rangle \\
\boxed{\psi \preccurlyeq \xi} \ \langle i\, 1\rangle \\
\downarrow \\
\boxed{\varphi \preccurlyeq \xi} \ \langle i\, 1\rangle
\end{array}
$$

with the proviso $\star$ that $var(\varphi) \subseteq var(\psi)$.

Following standard presentations, we say a branch $\mathcal{B}$ is *closed* if labels $\ulcorner \varphi \langle i, 1\rangle \urcorner$ and $\ulcorner \varphi \langle i, 0\rangle \urcorner$ appear on $\mathcal{B}$ for some formula or pseudoformula φ. We then say that a tableau is closed if every branch $\mathcal{B}$ is closed.

This allows the definition of provability for JPAI:

DEFINITION 10. $\Gamma \vdash_{\mathsf{JPAI}} \varphi$ if there is a closed JPAI tableau whose initial segment is the sequence $\ulcorner \psi \langle 0, 1\rangle \urcorner$ for every $\psi \in \Gamma$ followed by $\ulcorner \varphi \langle 0, 0\rangle \urcorner$.

2.3. Metatheoretical Remarks on JPAI

Now, consider some metatheoretic results concerning JPAI. First, we prove soundness and completeness between the foregoing model theory and proof theory. Second, we examine adequate translations between JPAI and PAI/F.

First, a definition:

DEFINITION 11. A model $\mathfrak{M}$ is faithful to a branch $\mathcal{B}$ of a tableau if there exists a function f from labels to the points in $\mathfrak{M}$ such that:

- If φ is a formula and $\ulcorner \varphi, \langle i, 1\rangle \urcorner$ is on $\mathcal{B}$ then $f(i) \Vdash \varphi$
- If φ is a formula and $\ulcorner \varphi, \langle i, 0\rangle \urcorner$ is on $\mathcal{B}$ then $f(i) \nVdash \varphi$
- If $i\rho j$ is on $\mathcal{B}$ then $f(i)Rf(j)$
- If $\ulcorner \boxed{\varphi \preccurlyeq \psi} \langle i, 1\rangle \urcorner$ is on $\mathcal{B}$ then $t_{f(i)}(\psi) \leq_{f(i)} t_{f(i)}(\varphi)$

- If $\ulcorner \boxed{\varphi \preccurlyeq \psi} \langle i, 0 \rangle \urcorner$ is on $\mathcal{B}$ then $t_{f(i)}(\psi) \not\leq_{f(i)} t_{f(i)}(\varphi)$

This definition allows us to formulate a soundness lemma for the system:

Lemma 3. (Soundness Lemma for JPAI) *Let $\mathcal{B}$ be a branch of a tableau and let $\mathfrak{M}$ be a* JPAI *model faithful to $\mathcal{B}$. Then if any tableau rule is applied to $\mathcal{B}$, $\mathfrak{M}$ remains faithful to at least one of the resulting extensions $\mathcal{B}'$.*

Proof. The rules for persistence follow from Lemmas 2 and 4 while steps for negation, conjunction, disjunction, and frame conditions are identical to those found in the treatment of intuitionistic logic discussed in Lemma 6.7.3 of [42]. This leaves us to consider applications of rules for intuitionistic analytic implication and pseudoformulae:

- If $\ulcorner \varphi \to \psi \langle i, 1 \rangle \urcorner$ and $i\rho j$ are on branch $\mathcal{B}$ then because $f(i) \Vdash \varphi \to \psi$, $t_{f(i)}(\psi) \leq_{f(i)} t_{f(i)}(\varphi)$. By hypothesis, $f(i)Rf(j)$ and by topic persistence, $t_{f(j)}(\psi) \leq_{f(j)} t_{f(j)}(\varphi)$, so the inclusion of $\boxed{\psi \preccurlyeq \varphi} \langle j, 1 \rangle$ is respected by the model. Additionally, at $f(j)$, either $f(j) \Vdash \varphi$ or $f(j) \nVdash \psi$, so the model will remain faithful to at least one of the two branches induced by the rule.
- If $\ulcorner \varphi \to \psi \langle i, 0 \rangle \urcorner$ is on branch $\mathcal{B}$ then by hypothesis, $f(i) \nVdash \varphi \to \psi$. Thus, one of two cases holds. In case $t_{f(i)}(\psi) \not\leq_{f(i)} t_{f(i)}(\varphi)$, the model is faithful to the branch including $\ulcorner \boxed{\psi \preccurlyeq \varphi} \langle i, 0 \rangle \urcorner$. In case there exists a $w \in f(i)\uparrow$ such that $w \Vdash \varphi$ and $w \nVdash \psi$, by introducing a new j and modifying the function f so that $f(j) = w$, the model will remain faithful to the branch in which $\ulcorner i\rho j \urcorner$, $\ulcorner \varphi \langle j, 1 \rangle \urcorner$, and $\ulcorner \psi \langle j, 0 \rangle \urcorner$ appear.
- Properties of the topic semilattices—the definition of assignment of concepts to complex formula, the persistence of topic, and the transitivity of $\leq_w$—establish the lemma for the four rules concerning pseudoformulae $\boxed{\varphi \preccurlyeq \psi}$.

■

This lemma quickly gives us soundness through a standard argument:

Theorem 1. (Soundness of JPAI) *If* $\Gamma \vdash_{\mathsf{JPAI}} \varphi$ *then* $\Gamma \vDash_{\mathsf{JPAI}} \varphi$

Proof. We prove the contrapositive. Suppose that $\Gamma \nvDash_{\mathsf{JPAI}} \varphi$. Then there is a JPAI model $\mathfrak{M}$ and point w such that $\mathfrak{M}, w \Vdash \psi$ for each $\psi \in \Gamma$ and $\mathfrak{M}, w \nVdash \varphi$. Thus, setting $f(0) = w$, $\mathfrak{M}$ is faithful to the branch consisting of the initial list of a tableau proof. By Lemma 3, any attempt to apply rules to this initial list will always have at least one branch to which $\mathfrak{M}$ remains faithful (and which will not close). Thus, $\Gamma \nvdash_{\mathsf{JPAI}} \varphi$. ■

With soundness of JPAI established, we turn to completeness. We offer a definition by which we can introduce a model that is induced by an open branch on a tableau, first by describing a method of extracting a topic semilattice for each i on an open branch $\mathcal{B}$.

Definition 12. For $\mathcal{B}$ an open branch of a JPAI tableau in which an integer i appears, let $\leq_{\langle \mathcal{B},i\rangle}$ denote the transitive-reflexive closure of

$$\{\langle \varphi, \psi\rangle \mid \ulcorner \boxed{\varphi \preccurlyeq \psi}\, \langle i, 1\rangle\urcorner \text{ occurs in } \mathcal{B}\}.$$

Then $\langle \mathcal{T}^{\sim}_{\langle \mathcal{B},i\rangle}, \leq^{\sim}_{\langle \mathcal{B},i\rangle}\rangle$—the canonical poset for i on $\mathcal{B}$—is the posetal projection of $\langle \mathcal{L}_j, \leq_{\langle \mathcal{B},i\rangle}\rangle$, i.e., the quotient identifying $\varphi, \psi \in \mathcal{L}_j$ iff $\varphi \leq_{\langle \mathcal{B},i\rangle} \psi$ and $\psi \leq_{\langle \mathcal{B},i\rangle} \varphi$.

That $\langle \mathcal{T}^{\sim}_{\langle \mathcal{B},i\rangle}, \leq^{\sim}_{\langle \mathcal{B},i\rangle}\rangle$ is a partially ordered set follows by its construction (i.e., the posetal projection of a transitively and reflexively closed preorder is a partial ordering). This implies that we can without loss of generality refer to the structure as a join semilattice $\langle \mathcal{T}^{\sim}_{\langle \mathcal{B},i\rangle}, \oplus^{\sim}_{\langle \mathcal{B},i\rangle}\rangle$ via the usual definitions.

Letting $[\![\varphi]\!]^{\sim}$ denote the equivalence class of φ in $\mathcal{T}^{\sim}_{\langle \mathcal{B},i\rangle}$, we now describe a canonical model:

Definition 13. Let $\mathcal{B}$ be an open branch of a JPAI tableau. Then define $\mathfrak{M}_{\mathcal{B}}$ as the model induced by the branch so that:

- $W = \{w_k \mid k \text{ occurs in } \mathcal{B}\}$
- $v(p) = \{w_k \mid \ulcorner p\, \langle k, 1\rangle\urcorner \text{ is on } \mathcal{B}\}$
- $w_i R w_j$ iff $\ulcorner i\rho j\urcorner$ appears in $\mathcal{B}$
- Each w_i has the semilattice $\langle \mathcal{T}_{w_i}, \oplus_{w_i}\rangle = \langle \mathcal{T}^{\sim}_{\langle \mathcal{B},i\rangle}, \oplus^{\sim}_{\langle \mathcal{B},i\rangle}\rangle$
- Each t_{w_i} is defined so that $t_{w_i}(p) = [\![p]\!]^{\sim}$

Now, we prove three lemmas that establish the suitability of models $\mathfrak{M}_{\mathcal{B}}$ for the task of proving completeness. First, we introduce a stalwart lemma that appropriate topic semilattices for the points of $\mathfrak{M}_{\mathcal{B}}$ exist:

Lemma 4. *For every i on an open branch $\mathcal{B}$, the canonical poset $\langle \mathcal{T}^{\sim}_{\langle \mathcal{B},i\rangle}, \leq^{\sim}_{\langle \mathcal{B},i\rangle}\rangle$ ensures that for all $\varphi, \psi \in \mathcal{L}_J$:*

- *If $\ulcorner \boxed{\varphi \preccurlyeq \psi}\, \langle i, 1\rangle\urcorner$ occurs in $\mathcal{B}$ then $[\![\varphi]\!]^{\sim} \leq^{\sim}_{w_i} [\![\psi]\!]^{\sim}$ (i.e. $[\![\varphi]\!]^{\sim} \leq^{\sim}_{\langle \mathcal{B},i\rangle} [\![\psi]\!]^{\sim}$)*
- *If $\ulcorner \boxed{\varphi \preccurlyeq \psi}\, \langle i, 0\rangle\urcorner$ occurs in $\mathcal{B}$ then $[\![\varphi]\!]^{\sim} \nleq^{\sim}_{w_i} [\![\psi]\!]^{\sim}$ (i.e. $[\![\varphi]\!]^{\sim} \nleq^{\sim}_{\langle \mathcal{B},i\rangle} [\![\psi]\!]^{\sim}$)*

Proof. The first bullet point follows from the construction of $\langle \mathcal{T}^{\sim}_{\langle \mathcal{B},i\rangle}, \leq^{\sim}_{\langle \mathcal{B},i\rangle}\rangle$ so we cover the second bullet point by contraposition, supposing that

$[\![\varphi]\!]^{\sim} \leq^{\sim}_{w_i} [\![\psi]\!]^{\sim}$. Then there are $\varphi' \in [\![\varphi]\!]^{\sim}$ and $\psi' \in [\![\psi]\!]^{\sim}$ such that $\varphi' \leq_{\langle \mathcal{B}, i\rangle} \psi'$. Without loss of generality, just assume that there are φ and ψ themselves. Now, $\langle \varphi, \psi \rangle$ entered the relation $\leq_{\langle \mathcal{B}, i\rangle}$ through one of three modes:

- If $\langle \varphi, \psi \rangle$ was a member of the initial relation of which $\leq_{\langle \mathcal{B}, i\rangle}$ is the transitive-reflexive closure, then $\ulcorner \boxed{\varphi \preccurlyeq \psi} \langle i, 1\rangle \urcorner$ occurs in $\mathcal{B}$ by hypothesis.
- If $\langle \varphi, \psi \rangle$ entered through reflexive closure, then $\varphi = \psi$, whence $\ulcorner \boxed{\varphi \preccurlyeq \psi} \langle i, 1\rangle \urcorner$ occurs in $\mathcal{B}$ by exhaustivity, i.e., that every possible rule has been applied.
- If $\langle \varphi, \psi \rangle$ entered in virtue of transitive closure, then there is a sequence $\xi_0, ..., \xi_n$ such that $\ulcorner \boxed{\varphi \preccurlyeq \xi_0} \langle i, 1\rangle \urcorner$, $\ulcorner \boxed{\xi_j \preccurlyeq \xi_{j+1}} \langle i, 1\rangle \urcorner$ (for $0 \leq j \leq n-1$), and $\ulcorner \boxed{\xi_n \preccurlyeq \psi} \langle i, 1\rangle \urcorner$ occur in $\mathcal{B}$. By n many applications of the rules, $\ulcorner \boxed{\varphi \preccurlyeq \psi} \langle i, 1\rangle \urcorner$ would have to appear in $\mathcal{B}$.

So in each case, $\ulcorner \boxed{\varphi \preccurlyeq \psi} \langle i, 1\rangle \urcorner$ occurs in $\mathcal{B}$ and, by the assumption that $\mathcal{B}$ is open, $\ulcorner \boxed{\varphi \preccurlyeq \psi} \langle i, 0\rangle \urcorner$ does *not* occur in $\mathcal{B}$, as required. ■

We must prove that the construction satisfies Definition 6:

LEMMA 5. *Every model $\mathfrak{M}_{\mathcal{B}}$ induced by an open branch $\mathcal{B}$ of a* JPAI *tableau is a* JPAI *model.*

PROOF. The steps for frame conditions are as in Lemma 6.7.6 of [42]. That the topic semilattices have the correct properties follows from Lemma 4. That topic persistence holds follows from the third of the rules governing pseudoformulae $\boxed{\varphi \preccurlyeq \psi}$. ■

Finally, we prove a completeness lemma for the system:

LEMMA 6. (Completeness Lemma for JPAI) *For any open completed branch $\mathcal{B}$ of a* JPAI *tableau, for every formula φ and point w in $\mathfrak{M}_{\mathcal{B}}$,*

- *If $\ulcorner \varphi \langle i, 1\rangle \urcorner$ is on $\mathcal{B}$ then $\mathfrak{M}_{\mathcal{B}}, w_i \Vdash \varphi$*
- *If $\ulcorner \varphi \langle i, 0\rangle \urcorner$ is on $\mathcal{B}$ then $\mathfrak{M}_{\mathcal{B}}, w_i \nVdash \varphi$*

PROOF. The basis step is established by construction of $\mathfrak{M}_{\mathcal{B}}$ and most steps in the induction are identical to Lemma 6.7.7 of [42]. The only difference lies in the case in which $\varphi = \psi \rightarrow \xi$:

- Suppose that $\ulcorner \psi \rightarrow \xi \langle i, 1\rangle \urcorner$ is on $\mathcal{B}$. Then for all j such that $\ulcorner i\rho j \urcorner$ is on $\mathcal{B}$, two things can be inferred: First, $\mathcal{B}$ also includes $\ulcorner \boxed{\xi \preccurlyeq \psi} \langle j, 1\rangle \urcorner$;

by Lemma 4, $\mathfrak{M}_{\mathcal{B}}$'s topic semilattices have the right properties. Second, either $\ulcorner \psi \langle j, 0\rangle \urcorner$ or $\ulcorner \xi \langle j, 1\rangle \urcorner$ is on $\mathcal{B}$; the induction hypothesis establishes, then, that for all w' such that $w_i R w'$, if $\mathfrak{M}_{\mathcal{B}}, w' \Vdash \psi$ then $\mathfrak{M}_{\mathcal{B}}, w' \Vdash \xi$, whence $\mathfrak{M}_{\mathcal{B}}, w_i \Vdash \psi \to \xi$.

- If $\ulcorner \psi \to \xi \langle i, 0\rangle \urcorner$ is on $\mathcal{B}$, then either $\ulcorner \boxed{\xi \preccurlyeq \psi} \langle j, 0\rangle \urcorner$ is on $\mathcal{B}$ or $\ulcorner i\rho j \urcorner$, $\ulcorner \psi \langle j, 1\rangle \urcorner$, and $\ulcorner \xi \langle j, 0\rangle \urcorner$ are on $\mathcal{B}$. In the former case, Lemma 4 ensures that $\mathfrak{M}_{\mathcal{B}}, w_i \nVdash \psi \to \xi$ on topic-theoretic grounds. In the latter case, the induction hypothesis ensures the existence of a $w_j \in w_i{\uparrow}$ at which ψ is true and ξ is false, whence $\mathfrak{M}_{\mathcal{B}}, w_i \nVdash \psi \to \xi$ on truth-theoretic grounds. Either way, the formula fails at w_i. ■

Again, these lemmas give us completeness by standard techniques:

THEOREM 2. (Completeness of JPAI) *If* $\Gamma \vDash_{\mathsf{JPAI}} \varphi$ *then* $\Gamma \vdash_{\mathsf{JPAI}} \varphi$

PROOF. We prove the contrapositive. If $\Gamma \nvdash_{\mathsf{JPAI}} \varphi$, then there is an open branch $\mathcal{B}$ of a JPAI tableau and an i such that $\ulcorner \psi \langle i, 1\rangle \urcorner$ is on $\mathcal{B}$ for each $\psi \in \Gamma$ while $\ulcorner \varphi \langle i, 0\rangle \urcorner$ is on $\mathcal{B}$. By Lemma 6, the model $\mathfrak{M}_{\mathcal{B}}$ and point w_i make true all formulae in Γ although $\mathfrak{M}_{\mathcal{B}}, w_i \nVdash \varphi$, i.e., $\mathfrak{M}_{\mathcal{B}}$ is a countermodel showing that $\Gamma \nvDash_{\mathsf{JPAI}} \varphi$. ■

Before proceeding to apply Criterion II's requirement of executability to our conditional, we pause to consider a further matter: Gödel-Tarski-McKinsey-style translations from the language $\mathcal{L}_J$ to $\mathcal{L}_F$. Investigating such translations—the first of which is Gödel's [16] translation from intuitionistic logic to the modal logic $\mathsf{S4}$—have both theoretical and practical upshots.

From a theoretical perspective, such translations help make explicit the character of the intuitionistic connectives. The merits of such translations have been definitively demonstrated in Artemov's explicit *logic of proofs* in [1], which has been extended to analyze bilateral systems of the type we are ultimately interested in [10] and [12].

As a practical matter, the adequacy of a translation to $\mathsf{PAI/F}$ immediately allows us to complement the proof theory via semantic tableaux in Definition 9 with a second, Hilbert-style proof theory via the axiomatization of $\mathsf{PAI/F}$. Two proof theories, after all, are better than one.

DEFINITION 14. Define a translation $\tau : \mathcal{L}_J \to \mathcal{L}_F$

- $p^\tau = \Box p$
- $(\neg\varphi)^\tau = \Box\neg(\varphi^\tau)$
- $(\varphi \wedge \psi)^\tau = (\varphi^\tau) \wedge (\psi^\tau)$
- $(\varphi \vee \psi)^\tau = (\varphi^\tau) \vee (\psi^\tau)$

- $(\varphi \to \psi)^\tau = \Box((\varphi^\tau) \supset (\psi^\tau)) \wedge ((\psi^\tau) \preccurlyeq (\varphi^\tau))$

Before proving the adequacy of the translation τ, we must introduce two lemmas. First, we prove that JPAI models get the translation correct *locally*, that is, at any individual point w, w cannot distinguish φ from φ^τ.

LEMMA 7. *Let* $\mathfrak{M}$ *be a* JPAI *model and let* $\Vdash_J$ *and* $\Vdash_F$ *represent the truth conditions for the* JPAI *and* PAI/F *conditions, respectively. Then for a* φ *in the language* $\mathcal{L}_J$*,*

$$\mathfrak{M}, w \Vdash_J \varphi \textit{ iff } \mathfrak{M}, w \Vdash_F \varphi^\tau$$

PROOF. Fix a model $\mathfrak{M}$. As basis step, because $v(p)$ is R-closed in $\mathfrak{M}$, $w \Vdash_J p$ holds iff $w \Vdash_F \Box p$. As induction hypothesis, suppose that this holds for all ψ and ξ that are subformulae of φ.

- If $\varphi = \neg\psi$, then $w \Vdash_J \neg\psi$ iff for all $w' \in w\uparrow$, $w' \nVdash_J \psi$. By hypothesis, for all such w', also $w' \nVdash_F \psi^\tau$, whence $w \Vdash_F \Box\neg(\psi^\tau)$.
- If $\varphi = \psi \wedge \xi$, then $w \Vdash_J \psi \wedge \xi$ iff $w \Vdash_J \psi$ and $w \Vdash_J \xi$, which the induction hypothesis assures us holds iff $w \Vdash_F \psi^\tau$ and $w \Vdash_F \xi^\tau$, i.e., iff $w \Vdash_F (\psi^\tau) \wedge (\xi^\tau)$.
- The case in which $\varphi = \psi \vee \xi$ follows by similar lines to the case of conjunction.
- If $\varphi = \psi \to \xi$, then $w \Vdash_J \psi \to \xi$ breaks into two cases:
  - ◦ First, for all $w' \in w\uparrow$, either $w' \nVdash_J \psi$ or $w' \Vdash_J \xi$. By induction hypothesis, for all $w' \in w\uparrow$ either $w' \nVdash_F \psi^\tau$ or $w' \Vdash_F \xi^\tau$, whence $w \Vdash_F \Box((\psi^\tau) \supset (\xi^\tau))$.
  - ◦ Second, $t_w(\xi) \leq_w t_w(\psi)$. It is easy to confirm that $t_w(\xi) = t_w(\xi^\tau)$ (and *mutatis mutandis* for ψ), whence $t_w(\xi^\tau) \leq_w t_w(\psi^\tau)$, so $w \Vdash_F (\xi^\tau) \preccurlyeq (\psi^\tau)$.

The two cases are thus equivalent with $w \Vdash_F \Box((\psi^\tau) \supset (\xi^\tau)) \wedge (\xi^\tau \preccurlyeq \psi^\tau)$. ■

This lemma exposes a complication: All JPAI models are PAI models but the converse is not true, e.g., a PAI model in which a set $v(p)$ is not closed under R will not satisfy Definition 6. Thus, arbitrary PAI countermodels to a formula ψ^τ may not qualify as a JPAI countermodel of ψ. We are thus obliged to introduce a method of constructing a suitable countermodel if any should exist.

Definition 15. For a PAI model $\mathfrak{M}$, let $\mathfrak{M}^{\dagger}$ be a model modifying $\mathfrak{M}$ so that for all atoms p,

$$v(p) = \{w \in W \mid \mathfrak{M}, w \Vdash \Box p\}$$

$\mathfrak{M}^{\dagger}$ considers an atom p to be true at a world only if p was *necessary* at that world in $\mathfrak{M}$, i.e., $\mathfrak{M}^{\dagger}$ is what remains when contingent truths are stripped from $\mathfrak{M}$. Such models will serve to bridge between PAI/F and JPAI countermodels for the fragment of $\mathcal{L}_F$ in which we are interested:

Lemma 8. *For* PAI *model* $\mathfrak{M}$ *and formula* φ *in the language of* JPAI*,*

$$\mathfrak{M}, w \Vdash \varphi^{\tau} \text{ iff } \mathfrak{M}^{\dagger}, w \Vdash \varphi^{\tau}$$

Proof. In the case of atoms, this follows from construction of $\mathfrak{M}^{\dagger}$; $\mathfrak{M}, w \Vdash \Box p$ precisely when $\mathfrak{M}^{\dagger}, w \Vdash \Box p$. As induction hypothesis, suppose that this has been established for all formulae ψ^{τ} and φ^{τ} of lesser complexity than ψ^{τ}

- For negation, $\mathfrak{M}, w \Vdash \Box\neg(\psi^{\tau})$ iff for all $w' \in w\uparrow$, $w' \nVdash \psi^{\tau}$. By induction hypothesis, this holds iff for all $w' \in w\uparrow$, $\mathfrak{M}^{\dagger}, w' \nVdash \psi^{\tau}$, whence $\mathfrak{M}^{\dagger}, w \Vdash \Box\neg(\psi^{\tau})$.
- For conjunction, $\mathfrak{M}, w \Vdash \psi^{\tau} \wedge \xi^{\tau}$ iff $\mathfrak{M}, w \Vdash \psi^{\tau}$ and $\mathfrak{M}, w \Vdash \xi^{\tau}$. By induction hypothesis, this is equivalent to $\mathfrak{M}^{\dagger}, w \Vdash \psi^{\tau}$ and $\mathfrak{M}^{\dagger}, w \Vdash \xi^{\tau}$, i.e., $\mathfrak{M}^{\dagger}, w \Vdash \psi^{\tau} \wedge \xi^{\tau}$.
- Disjunction follows through a dual argument to that of conjunction.
- For conditionals, $\mathfrak{M}, w \Vdash \Box((\psi^{\tau}) \supset (\xi^{\tau})) \wedge (\xi^{\tau} \preccurlyeq \psi^{\tau})$ iff two clauses hold:
  - $\mathfrak{M}, w \Vdash \Box((\psi^{\tau}) \supset (\xi^{\tau}))$ holds iff for all $w' \in w\uparrow$, either $\mathfrak{M}, w' \nVdash \psi^{\tau}$ or $\mathfrak{M}, w' \Vdash \xi^{\tau}$. By induction hypothesis, this holds iff $\mathfrak{M}^{\dagger}, w' \nVdash \psi^{\tau}$ or $\mathfrak{M}^{\dagger}, w' \Vdash \xi^{\tau}$ for all $w' \in w\uparrow$, so $\mathfrak{M}^{\dagger}, w \Vdash \Box((\psi^{\tau}) \supset (\xi^{\tau}))$.
  - As t_w is shared, $\mathfrak{M}, w \Vdash \psi^{\tau} \preccurlyeq \xi^{\tau}$ iff $\mathfrak{M}, w \Vdash \psi^{\tau} \preccurlyeq \xi^{\tau}$.

So $\mathfrak{M}, w \Vdash (\psi \rightarrow \xi)^{\tau}$ iff $\mathfrak{M}^{\dagger}, w \Vdash (\psi \rightarrow \xi)^{\tau}$. ■

Agreement with respect to the fragment $\mathcal{L}_J^{\tau}$—the image of $\mathcal{L}_J$ under τ—is all that is necessary to establish the adequacy of τ. Thus, although a PAI model $\mathfrak{M}$ may disagree with $\mathfrak{M}^{\dagger}$ about some formulae, Lemma 8 ensures that their theories agree *when it counts*. Let $\vdash_{\mathsf{PAI/F}}$ denote derivability in Fine's axiom system of [15]. Then:

Theorem 3. $\Gamma \vdash_{\mathsf{JPAI}} \varphi$ *iff* $\Gamma^{\tau} \vdash_{\mathsf{PAI/F}} \varphi^{\tau}$

Proof. Retain the notation of $\Vdash_J$ and $\Vdash_F$. Consider two directions:

- For right-to-left, suppose that $\Gamma \nvdash_{\mathsf{JPAI}} \varphi$. By completeness, there exists a JPAI model $\mathfrak{M}$ and a point such that $\mathfrak{M}, w \Vdash_J \psi$ for all $\psi \in \Gamma$ while $\mathfrak{M}, w \nVdash_J \varphi$. By Lemma 7, $\mathfrak{M}, w \Vdash_F \psi^\tau$ for each $\psi^\tau \in \Gamma^\tau$ while $\mathfrak{M}, w \nVdash_F \varphi^\tau$. Thus, $\mathfrak{M}$ serves as a countermodel showing that $\Gamma^\tau \nvDash_{\mathsf{PAI/F}} \varphi^\tau$. By soundness, $\Gamma^\tau \nvdash_{\mathsf{PAI/F}} \varphi^\tau$.
- For left-to-right, suppose that $\Gamma^\tau \nvdash_{\mathsf{PAI/F}} \varphi^\tau$. By completeness, find a PAI model $\mathfrak{M}$ including a point w such that $\mathfrak{M}, w \Vdash_F \psi^\tau$ for each $\psi^\tau \in \Gamma^\tau$ and $\mathfrak{M}, w \nVdash_F \varphi^\tau$. By Lemma 8, there exists a JPAI model $\mathfrak{M}^\dagger$ agreeing with $\mathfrak{M}$ on all formulae in $\mathcal{L}_J^\tau$, so $\mathfrak{M}^\dagger, w \Vdash_F \psi^\tau$ for each $\psi^\tau \in \Gamma^\tau$ and $\mathfrak{M}^\dagger, w \nVdash_F \varphi^\tau$. But Lemma 7 again allows us to infer that $\mathfrak{M}^\dagger, w \Vdash_J \psi$ for each $\psi \in \Gamma$ and $\mathfrak{M}^\dagger, w \nVdash_J \varphi$. Thus $\mathfrak{M}^\dagger$ witnesses that $\Gamma \nvDash_{\mathsf{JPAI}} \varphi$ and, by soundness, that $\Gamma \nvdash_{\mathsf{JPAI}} \varphi$. ■

Criterion I—the first of our desiderata—is met by the conditional of JPAI: It is *intuitionistic* and it imposes a *topic inclusion* constraint.[5] However, the conditional of JPAI can not yet be said to be *executable* in the sense required by Griss. We now turn to imposing constraints to satisfy the next criterion.

3. Criterion II: Executable Analytic Implication

To satisfy Criterion II, we enhance the picture given by JPAI by imposing a constraint that a conditional $\varphi \to \psi$ can be considered to be verified only in case constructions of the antecedent φ can be *initialized*. A natural interpretation of this notion of executability is that the antecedent is necessarily possible, that is, at every stage in an investigation, there is a potential future stage at which φ can be modeled coherently.

Thus, as JPAI models are built on $\mathsf{S4}$ Kripke frames, a plausible way to characterize executability might be to identify executability of φ at w with the truth of the $\mathsf{S4}$ formula $\Box\Diamond\varphi$ at w. Although $\mathcal{L}_J$ lacks such modal operators—and thus we have introduced no corresponding truth conditions—that the intuitive truth conditions of $\Box\Diamond\varphi$ coincide with those of the intuitionistic formula $\neg\neg\varphi$ ensures that we have adequate tools to express this reading of executability. We call the system introduced in this section ExPAI—executable intuitionistic PAI.

[5] I should acknowledge, however, that the notion of topic inclusion for the intuitionistic case might be more complicated than the straightforward Parry treatment acknowledges. In the BHK setting, the topic of an intuitionistic conditional, after all, arguably adds something to the topics of its subformulae. For this reason, the framework of conditional-agnostic analytic implication in [14] might provide ways to improve what we have described so far.

3.1. Executable Analytic Implication

The constraint of executability requires revisions to only the satisfaction relation rather than a wholesale revision of the models, whence a great deal of model theory can be ported over from Section 2.2.

In particular, the models for ExPAI are just models for JPAI from Definition 6; the difference lies in how the truth conditions are evaluated. We will as a matter of convention refer to a single structure as a JPAI model to emphasize the adoption of the truth conditions of Definition 7 and as an ExPAI model when truth conditions are intended as follows:

Definition 16. ExPAI truth conditions are defined by modifying Definition 7 as follows:

- $w \Vdash \varphi \to \psi$ if $\begin{cases} \text{for all } w' \in w{\uparrow} \text{ there is a } w'' \in w'{\uparrow} \text{ s.t. } w'' \Vdash \varphi \\ \text{for all } w' \in w{\uparrow}, \text{ if } w' \Vdash \varphi \text{ then } w' \Vdash \psi \\ t_w(\psi) \leq_w t_w(\varphi) \end{cases}$

This allows the definition of ExPAI validity:

Definition 17. $\Gamma \vDash_{\mathsf{ExPAI}} \varphi$ iff for all ExPAI models $\mathfrak{M}$ and points w, if $\mathfrak{M}, w \Vdash \psi$ for each $\psi \in \Gamma$, then $\mathfrak{M}, w \Vdash \varphi$

Just as a great deal of model theory carries over from JPAI with only modest revisions, a proof theory for ExPAI requires only small updates to Definition 9:

Definition 18. A tableau calculus for ExPAI is given revising the tableau calculus in Definition 9 with the following rules:

$$\begin{array}{c} \varphi \to \psi \ \langle i\,1 \rangle \\ i\rho j \\ \downarrow \\ \neg\neg\varphi \ \langle i\,1 \rangle \\ \boxed{\psi \preccurlyeq \varphi} \ \langle j\,1 \rangle \\ \begin{array}{cc} \swarrow & \searrow \\ \varphi \ \langle j\,0 \rangle & \psi \ \langle j\,1 \rangle \end{array} \end{array} \qquad \begin{array}{c} \varphi \to \psi \ \langle i\,0 \rangle \\ \begin{array}{ccc} \swarrow & \downarrow & \searrow \\ \neg\neg\varphi \ \langle i\,0 \rangle & \boxed{\psi \preccurlyeq \varphi} \ \langle i\,0 \rangle & \begin{array}{c} i\rho j \\ \varphi \ \langle j\,1 \rangle \\ \psi \ \langle j\,0 \rangle \end{array} \end{array} \end{array}$$

Provability is naturally defined:

Definition 19. $\Gamma \vdash_{\mathsf{ExPAI}} \varphi$ if there is a closed ExPAI tableau whose initial segment is the sequence $\ulcorner \psi \, \langle 0, 1 \rangle \urcorner$ for every $\psi \in \Gamma$ followed by $\ulcorner \varphi \, \langle 0, 0 \rangle \urcorner$.

The executability of the conditional in ExPAI can be read from the foregoing truth definitions and tableau rules.

On our reading of executability, for the truth of $\varphi \to \psi$ at a stage in an investigation, one must be able to posit a possible state in which φ is satisfied. Moreover, because verification of $\varphi \to \psi$ must persist throughout the investigation, the possibility of such initializations of φ must persist into all stages of the investigation. Our model theory and proof theory appear to adequately reflect this reading in, e.g., the three distinct clauses for the truth of a conditional.

3.2. Metatheoretical Remarks on ExPAI

As before, we pause to examine some metatheory for ExPAI. Most importantly, we prove soundness and completeness between the proof theory and model theory offered above. The definition of faithfulness can be retained without modification from Definition 11 while soundness requires only a very modest modification to Lemma 3.

Lemma 9. (Soundness Lemma for ExPAI) *Let $\mathcal{B}$ be a branch of a tableau and let $\mathfrak{M}$ be a* ExPAI *model faithful to $\mathcal{B}$. Then if any tableau rule is applied to $\mathcal{B}$, $\mathfrak{M}$ remains faithful to at least one of the resulting extensions $\mathcal{B}'$.*

Proof. As in Lemma 3, negation, conjunction, disjunction, and frame conditions follow from Lemma 6.7.3 of [42]. The rules for the ExPAI conditional, however, essentially conjoin or disjoin a double negation with the JPAI rules for the conditional. Thus, synthesizing the case of negation with that of the JPAI conditional establishes the lemma for the ExPAI conditional rules as well. ■

From Lemma 9, we establish:

Theorem 4. (Soundness of ExPAI) *If* $\Gamma \vdash_{\mathsf{ExPAI}} \varphi$ *then* $\Gamma \vDash_{\mathsf{ExPAI}} \varphi$

As for a completeness proof, recall that completeness for JPAI followed from three lemmas. Luckily, proofs of Lemmas 4 and 5 make no appeal to particular properties of the conditional; because ExPAI models and JPAI models coincide, these two lemmas continue to hold without loss of generality for ExPAI. The only lemma requiring modification is Lemma 6. Continue to define $\mathfrak{M}_{\mathcal{B}}$ as in Definition 13. Then:

Lemma 10. (Completeness Lemma for ExPAI) *For any open completed branch $\mathcal{B}$ of a* ExPAI *tableau, for every formula φ and point w in $\mathfrak{M}_{\mathcal{B}}$,*

- *If* $\ulcorner \varphi \langle i, 1 \rangle \urcorner$ *is on* $\mathcal{B}$ *then* $\mathfrak{M}_{\mathcal{B}}, w_i \Vdash \varphi$
- *If* $\ulcorner \varphi \langle i, 0 \rangle \urcorner$ *is on* $\mathcal{B}$ *then* $\mathfrak{M}_{\mathcal{B}}, w_i \nVdash \varphi$

Proof. The basis step and most inductive steps are established along similar lines to the proof of Lemma 6, leaving us to treat the cases in which $\ulcorner\psi \rightarrow \xi\, \langle i, 1\rangle\urcorner$ or $\ulcorner\psi \rightarrow \xi\, \langle i, 1\rangle\urcorner$ appear on $\mathcal{B}$. In these cases, the elements of Lemma 6 concerning JPAI negations and conditionals can be synthesized to treat the ExPAI conditional. We consider the former case as an illustration:

- Let $\ulcorner\psi \rightarrow \xi\, \langle i, 1\rangle\urcorner$ be found on $\mathcal{B}$; we establish each of the three clauses of the truth conditions for the conditional.

  ∘ First, $\ulcorner\neg\neg\psi\, \langle i, 1\rangle\urcorner$ must be on $\mathcal{B}$ and standard arguments from Lemma 6.7.7 of [42] establish that $f(i) \Vdash \neg\neg\psi$ assuming the induction hypothesis holds for ψ. Consequently, for all $w \in f(i)\!\uparrow$ there exists a $w' \in w\!\uparrow$ such that $w' \Vdash \psi$.
  ∘ Pick an arbitrary $f(j) \in f(i)\!\uparrow$; either $\ulcorner\psi\, \langle j, 0\rangle\urcorner$ or $\ulcorner\xi\, \langle j, 1\rangle\urcorner$ is on $\mathcal{B}$. By induction hypothesis, either $f(j) \nVdash \psi$ or $f(j) \Vdash \xi$. As $f(j)$ was chosen arbitrarily from $f(i)\!\uparrow$, for all $w \in f(i)\!\uparrow$, if $w \Vdash \psi$ then $w \Vdash \xi$.
  ∘ For arbitrary $f(j) \in f(i)\!\uparrow$, $\ulcorner\boxed{\xi \preccurlyeq \psi}\langle j, 1\rangle\urcorner$ is on $\mathcal{B}$ and, by Lemma 4, $t_{f(j)}(\xi) \leq_{f(j)} t_{f(j)}(\psi)$.

Together, these imply that $f(i) \Vdash \psi \rightarrow \xi$, as needed. ∎

Completeness follows from this Lemma 10:

Theorem 5. (Completeness of ExPAI) *If* $\Gamma \vDash_{\mathsf{ExPAI}} \varphi$ *then* $\Gamma \vdash_{\mathsf{ExPAI}} \varphi$

One may note that the modifications needed to establish Lemmas 3 and 6 work because the executability of an ExPAI conditional $\varphi \rightarrow \psi$ has been identified with the necessity of the antecedent's possibility, a condition which coincides with truth of the intuitionistic formula $\neg\neg\varphi$ in JPAI. This identification tacitly relies on the the adequacy of drawing a parallel between $\varphi \rightarrow \psi$ in ExPAI and $\neg\neg\varphi \wedge (\varphi \rightarrow \psi)$ in JPAI.

Explicitly capturing these intuitions with a precise definition spurs a continued investigation of the Gödel-McKinsey-Tarski-style relationships, focusing on that between ExPAI and Fine's $\mathsf{PAI/F}$. As an intermediate step, we consider the relationship between ExPAI and JPAI:

Definition 20. Define a translation $\sigma : \mathcal{L}_J \rightarrow \mathcal{L}_J$ as:

- $p^\sigma = p$
- $(\neg\varphi)^\sigma = \neg(\varphi^\sigma)$
- $(\varphi \wedge \psi)^\sigma = (\varphi^\sigma) \wedge (\psi^\sigma)$
- $(\varphi \vee \psi)^\sigma = (\varphi^\sigma) \vee (\psi^\sigma)$
- $(\varphi \rightarrow \psi)^\sigma = \neg\neg(\varphi^\sigma) \wedge ((\varphi^\sigma) \rightarrow (\psi^\sigma))$

We have seen that—unlike in the case of JPAI and PAI models—the classes of JPAI and ExPAI models coincide. This simplifies the matter of demonstrating the adequacy of σ by sparing us the step of defining intermediate models as in Definition 15.

LEMMA 11. *Let $\mathfrak{M}$ be an* ExPAI *model and let $\Vdash_E$ and $\Vdash_J$ represent the satisfaction relations for the* ExPAI *and* JPAI *conditions, respectively. Then for a φ in the language $\mathcal{L}_J$,*

$$\mathfrak{M}, w \Vdash_E \varphi \textit{ iff } \mathfrak{M}, w \Vdash_J \varphi^\sigma$$

PROOF. The basis step in which φ is an atom is immediate. The common truth conditions for negation, conjunction, and disjunction make the induction step for corresponding formulae trivial, leaving us to consider the conditional:

- If $\varphi = \psi \rightarrow \xi$ then $w \Vdash_E \psi \rightarrow \xi$ iff two clauses hold:
  - First, the induction hypothesis guarantees that the condition that for all $w' \in w\uparrow$ there is a w'' such that $w'Rw''$ and $w'' \Vdash_E \varphi$ is equivalent to the analogous one in which $w'' \Vdash_J \varphi^\sigma$. But this condition is precisely what is meant by $w \Vdash_J \neg\neg(\varphi^\sigma)$.
  - Second, consider the second and third clauses in parallel; each is equivalent to the case for JPAI. That for all $w' \in w\uparrow$ if $w' \Vdash_E \psi$ then $w' \Vdash_E \xi$ is equivalent by induction hypothesis to the condition that for all such w', if $w' \Vdash_J \psi^\sigma$ then $w' \Vdash_J \xi^\sigma$. Likewise, the shared interpretation of the topic semilattices of $\mathfrak{M}$ ensures that $t_w(\xi) \leq_w t_w(\psi)$ holds iff $t_w(\xi^\sigma) \leq_w t_w(\psi^\sigma)$. Together, then, the second and third clauses are jointly equivalent to $w \Vdash_J \psi^\sigma \rightarrow \xi^\sigma$.

Thus, $w \Vdash_E \psi \rightarrow \xi$ iff $w \Vdash_J \neg\neg(\psi^\sigma) \wedge (\psi^\sigma \rightarrow \xi^\sigma)$. ■

This implies the adequacy of our translation σ:

THEOREM 6. $\Gamma \vdash_{\mathsf{ExPAI}} \varphi$ *iff* $\Gamma^\sigma \vdash_{\mathsf{JPAI}} \varphi^\sigma$

PROOF. By Lemma 11, a countermodel to one of the above inferences serves as a countermodel to the other, whence we infer equivalence of semantic validity. By soundness and completeness, this equivalence holds for provability as well. ■

Additionally, a composite translation $\tau \circ \sigma$ allows us to translate from the language of ExPAI to Fine's PAI/F, yielding the following corollary:

COROLLARY 1. $\Gamma \vdash_{\mathsf{ExPAI}} \varphi$ *iff* $\Gamma^{\tau\circ\sigma} \vdash_{\mathsf{PAI/F}} \varphi^{\tau\circ\sigma}$

The conditional of ExPAI seems to capture many of the desiderata of a model of Griss's view of the intuitionistic conditional; it is executable as it demands the satisfiability of its antecedent, it is constructive in respecting the spirit of the BHK interpretation, and it imposes the topic-theoretic filter necessary to model the substrate needed for reasoners to arrive at conclusions.

As a model of Griss's propositional logic, however, ExPAI is infelicitous for very clear reasons: It includes the same intuitionistic negation that is incompatible with Griss's philosophy of mathematics. In order to provide a model, we would need to kick the ladder away in a sense, preserving only ExPAI^+—the negation-free fragment of ExPAI—as the acceptable kernel of Griss's account of mathematical reasoning.

But as argued in [13], Griss's arguments against negation do not extend to Nelson-style strong negation. We can lift the negations implicit in Griss's use of contrary predicates like identity ($=$) and apartness ($\#$) to complex formulae in order to extract a full theory of negation from Griss's use of contrary atoms, e.g., by identifying $m\#n$ with $\sim(m = n)$. We tackle the problem of determining an appropriate propositional logic meeting Criterion III now.

4. Criterion III: Bilateral Executable Systems

Like Griss, David Nelson's philosophy of mathematics is *superconstructive* in offering a critique that Brouwer's intuitionism is insufficiently constructive. On Nelson's critique, a satisfactory treatment of constructivity should treat *truth* and *falsity* on a par, i.e., judgments that a statement is false should face the same demands as judgments of the statement's truth.

This asymmetry can be easily recognized by reviewing the *disjunction property* of intuitionistic logic. This principle is reflected in the BHK conditions by demanding that a verification of a disjunction $\varphi \vee \psi$ requires one to identify either φ or ψ and furnish it with a verification. More informally, a disjunction cannot be true unless one can identify which disjunct is responsible for its truth.

Nelson observes that no such expectation is made of *falsity*. An intuitionist can consider a *conjunction* to be refuted without the constructive demand that the responsible conjunct be identified. As a refutation with a witness is conceptually more informative than a refutation without, one can clearly distinguish constructive refutation from non-constructive refutation. So Nelson advocates for a version of intuitionism in which refutations, like verifications, must be constructive.

As mentioned in the introduction, the strong negation Nelson introduced in [33]—receiving variations in [34] and [35]—evades a refined reading of Griss's rejection of negation for its exhibiting two features: Nelson's goals of providing an account of constructible falsity ensures that his negation is both *constructive* (evading Brouwer's arguments of [5]) but distinct from intuitionistic negation (evading Griss's arguments against the BHK-style interpretation).

Following [23] or [44], Nelson's intuitions are illuminated by the model theory. The single truth-at-a-world relation $\Vdash$ is retired in favor of distinct verification ($\Vdash^+$) and falsification ($\Vdash^-$) relations. Each connective, then, receives separate conditions for its truth and falsity, while strong negation serves to exchange proofs for refutations and *vice versa*.

Determining the falsity conditions is a separate task. The strong duality between conjunction and disjunction virtually demands a particular set of falsification conditions for each connective. In contrast, when examining the conditional, a surprising variety of possible falsity conditions is available; as Wansing shows in [48] and [49], there are a number of semantic interpretations of $\neg(\varphi \to \psi)$ that are consistent with the Nelsonian intuition. Consequently, a Nelson-style interpretation of Griss does not come with a commitment to embrace those adopted by Nelson in [33]. In this section, we will identify two potential falsification conditions for the executable conditional as particularly compelling, before introducing corresponding bilateral systems $\mathsf{N_1PAI}$ and $\mathsf{N_2PAI}$.

4.1. Common Features

Our two systems differ only on their assumptions concerning the falsification of a conditional, leaving a great deal of common ground. In this section, we introduce the machinery common to both systems before moving to offer complete definitions individually.

The language $\mathcal{L}_N$ will be defined by replacing the intuitionistic negation $\neg$ with a strong negation $\sim$ in the style of Nelson that intuitively toggles between verification and falsification.

$$\varphi ::= p \mid {\sim}\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi$$

The two systems will share a common definition of model, again, largely inspired by [44]:

Definition 21. An NPAI model is a tuple $\langle W, R, \mathcal{T}, \oplus, v^+, v^-, t\rangle$ augmenting Definition 6 by replacing the single valuation function v with a positive v^+ and negative v^- such that for all atoms p:

- $v^+(p) \cap v^-(p) = \varnothing$
- $v^+(p)$ and $v^-(p)$ are R-closed

We read the functions v^+ and v^- as mapping atomic variables to stages of an investigation at which there is direct proof and direct refutation, respectively.

Definition 22. The truth conditions common to both systems are:

- $w \Vdash^+ p$ if $w \in v^+(p)$
- $w \Vdash^+ {\sim}\varphi$ if $w \Vdash^- \varphi$
- $w \Vdash^+ \varphi \wedge \psi$ if $w \Vdash^+ \varphi$ and $w \Vdash^+ \psi$
- $w \Vdash^+ \varphi \vee \psi$ if $w \Vdash^+ \varphi$ or $w \Vdash^+ \psi$
- $w \Vdash^- p$ if $w \in v^-(p)$
- $w \Vdash^- {\sim}\varphi$ if $w \Vdash^+ \varphi$
- $w \Vdash^- \varphi \wedge \psi$ if $w \Vdash^- \varphi$ or $w \Vdash^- \psi$
- $w \Vdash^- \varphi \vee \psi$ if $w \Vdash^- \varphi$ & $w \Vdash^- \psi$

$$w \Vdash^+ \varphi \to \psi \text{ if } \begin{cases} \text{for all } w' \in w{\uparrow} \text{ there is a } w'' \in w'{\uparrow} \text{ s.t. } w'' \Vdash^+ \varphi \\ \text{for all } w' \in w{\uparrow}, \text{ if } w' \Vdash^+ \varphi \text{ then } w' \Vdash^+ \psi \\ t_w(\psi) \leq_w t_w(\varphi) \end{cases}$$

Importantly, the verification conditions continue to reflect those of ExPAI, substituting the undecorated $\Vdash$ of Definition 16 for $\Vdash^+$.

In the proof theoretic presentation, we again use pseudoformulae i.e. formulae not in the language $\mathcal{L}_N$. Bearing in mind the use of pseudoformulae as auxiliary devices, we add the following truth condition for pseudoformulae of the form $\boxed{\neg\varphi}$ for its utility in proving soundness and completeness:

Definition 23. The truth conditions for pseudoformulae $\boxed{\neg\varphi}$ are:

- $w \Vdash^+ \boxed{\neg\varphi}$ if for all $w' \in w{\uparrow}$, $w' \nVdash^+ \varphi$

taking note that φ may be a pseudoformula itself.

As we will see, we do not require a falsification condition for $\boxed{\neg\varphi}$ as the proof theory compels only the above case.

We delay definitions of validity for the bilateral systems until their introduction, introducing now the common rules for tableau calculi for our systems.

Definition 24. The common rules for bilateral tableau calculi for the systems retain structural rules (i.e., those for frame conditions and content inclusion) from Definition 9 and add the following rules for $n \in \{0, 1\}$:

$$\begin{array}{c} \varphi\ \langle i\,1\rangle^+ \\ i\rho j \\ \downarrow \\ \varphi\ \langle j\,1\rangle^+ \end{array} \quad \begin{array}{c} \varphi\ \langle i\,1\rangle^- \\ i\rho j \\ \downarrow \\ \varphi\ \langle j\,1\rangle^- \end{array} \quad \begin{array}{c} {\sim}\varphi\ \langle i\,n\rangle^+ \\ \downarrow \\ \varphi\ \langle i\,n\rangle^- \end{array} \quad \begin{array}{c} {\sim}\varphi\ \langle i\,n\rangle^- \\ \downarrow \\ \varphi\ \langle i\,n\rangle^+ \end{array} \quad \begin{array}{c} \boxed{\neg\chi}\ \langle i\,1\rangle^+ \\ i\rho j \\ \downarrow \\ \chi\ \langle j\,0\rangle^+ \end{array} \quad \begin{array}{c} \boxed{\neg\chi}\ \langle i\,0\rangle^+ \\ \downarrow \\ i\rho j \\ \chi\ \langle j\,1\rangle^+ \end{array}$$

$$\begin{array}{c} \varphi\wedge\psi\ \langle i\,1\rangle^{+} \\ \downarrow \\ \varphi\ \langle i\,1\rangle^{+} \\ \psi\ \langle i\,1\rangle^{+} \end{array} \qquad \begin{array}{cc} \multicolumn{2}{c}{\varphi\wedge\psi\ \langle i\,0\rangle^{+}} \\ \swarrow & \searrow \\ \varphi\ \langle i\,0\rangle^{+} & \psi\ \langle i\,0\rangle^{+} \end{array} \qquad \begin{array}{cc} \multicolumn{2}{c}{\varphi\vee\psi\ \langle i\,1\rangle^{+}} \\ \swarrow & \searrow \\ \varphi\ \langle i\,1\rangle^{+} & \psi\ \langle i\,1\rangle^{+} \end{array} \qquad \begin{array}{c} \varphi\vee\psi\ \langle i\,0\rangle^{+} \\ \downarrow \\ \varphi\ \langle i\,0\rangle^{+} \\ \psi\ \langle i\,0\rangle^{+} \end{array}$$

$$\begin{array}{cc} \multicolumn{2}{c}{\varphi\wedge\psi\ \langle i\,1\rangle^{-}} \\ \swarrow & \searrow \\ \varphi\ \langle i\,1\rangle^{-} & \psi\ \langle i\,1\rangle^{-} \end{array} \qquad \begin{array}{c} \varphi\wedge\psi\ \langle i\,0\rangle^{-} \\ \downarrow \\ \varphi\ \langle i\,0\rangle^{-} \\ \psi\ \langle i\,0\rangle^{-} \end{array} \qquad \begin{array}{c} \varphi\vee\psi\ \langle i\,1\rangle^{-} \\ \downarrow \\ \varphi\ \langle i\,1\rangle^{-} \\ \psi\ \langle i\,1\rangle^{-} \end{array} \qquad \begin{array}{cc} \multicolumn{2}{c}{\varphi\vee\psi\ \langle i\,0\rangle^{-}} \\ \swarrow & \searrow \\ \varphi\ \langle i\,0\rangle^{-} & \psi\ \langle i\,0\rangle^{-} \end{array}$$

$$\begin{array}{cc} \multicolumn{2}{c}{\varphi\to\psi\ \langle i\,1\rangle^{+}} \\ \multicolumn{2}{c}{i\rho j} \\ \multicolumn{2}{c}{\downarrow} \\ \multicolumn{2}{c}{\boxed{\neg\neg\varphi}\ \langle j\,1\rangle^{+}} \\ \multicolumn{2}{c}{\boxed{\psi\preccurlyeq\varphi}\ \langle j\,1\rangle} \\ \swarrow & \searrow \\ \varphi\ \langle j\,0\rangle^{+} & \psi\ \langle j\,1\rangle^{+} \end{array} \qquad \begin{array}{ccc} \multicolumn{3}{c}{\varphi\to\psi\ \langle i\,0\rangle^{+}} \\ \swarrow & \downarrow & \searrow \\ \boxed{\neg\neg\varphi}\ \langle i\,0\rangle^{+} & \boxed{\psi\preccurlyeq\varphi}\ \langle i\,0\rangle & i\rho j \\ & & \varphi\ \langle j\,1\rangle^{+} \\ & & \psi\ \langle j\,0\rangle^{+} \end{array}$$

taking note that χ may be a pseudoformula (and would thus properly appear as $\boxed{\chi}$).

We update the notion of closure as well. A branch $\mathcal{B}$ on a tableau is considered closed if one of the three conditions is met:

- $\ulcorner\varphi\,\langle i,1\rangle^{+}\urcorner$ and $\ulcorner\varphi\,\langle i,0\rangle^{+}\urcorner$ both appear
- $\ulcorner\varphi\,\langle i,1\rangle^{-}\urcorner$ and $\ulcorner\varphi\,\langle i,0\rangle^{-}\urcorner$ both appear
- $\ulcorner\varphi\,\langle i,1\rangle^{+}\urcorner$ and $\ulcorner\varphi\,\langle i,1\rangle^{-}\urcorner$ both appear

In other words, at a single point a formula can not be both verified and not verified, nor both falsified and not falsified, nor both verified and falsified.

Having reviewed the elements common to our two Griss-inspired bilateral systems, let us examine them individually. Asking after the appropriate understanding of the falsification of Griss's executable conditional lead us to two accounts and two propositional logics completing the above. We examine these systems—$\mathsf{N_1PAI}$ and $\mathsf{N_2PAI}$—sequentially.

4.2. $\mathsf{N_1PAI}$: Impossibility of Verification

First, one might convincingly argue that to falsify an executable intuitionistic conditional is simply to establish the impossibility that one of the three clauses necessary for the conditional's verification could be verified. In other words, this is to say that the refutation of $\varphi \to \psi$ is an object serving to authoritatively rule out any possible verification.

We introduce $\mathsf{N_1PAI}$ as the completion of the foregoing, incomplete definitions encapsulating this intuition formally. Thus, we define the full truth conditions by completing Definition 22:

DEFINITION 25. The $\mathsf{N_1PAI}$ truth conditions add to Definition 22 the following:

- $w \Vdash^- \varphi \to \psi$ if for all $w' \in w{\uparrow}$, $w' \nVdash^+ \varphi \to \psi$

Validity is then understood as the preservation of verification:

DEFINITION 26. $\Gamma \vDash_{\mathsf{N_1PAI}} \varphi$ iff for all NPAI models evaluated under Definition 25 and points w, if $\mathfrak{M}, w \Vdash^+ \psi$ for each $\psi \in \Gamma$, then $\mathfrak{M}, w \Vdash^+ \varphi$

A corresponding proof theory emerges by similarly augmenting Definition 24:

DEFINITION 27. The bilateral tableau calculus for $\mathsf{N_1PAI}$ adds to Definition 24 the following rules:

$$\begin{array}{cc} \varphi \to \psi \ \langle i\, 1\rangle^- & \varphi \to \psi \ \langle i\, 0\rangle^- \\ i\rho j & \\ \downarrow & \downarrow \\ & i\rho j \\ \varphi \to \psi \ \langle j\, 0\rangle^+ & \varphi \to \psi \ \langle j\, 1\rangle^+ \end{array}$$

DEFINITION 28. $\Gamma \vdash_{\mathsf{N_1PAI}} \varphi$ if there is a $\mathsf{N_1PAI}$ tableau proof every branch of which closes whose initial segment is the sequence $\ulcorner \psi \langle i, 1\rangle^+ \urcorner$ for every $\psi \in \Gamma$ followed by $\ulcorner \varphi \langle i, 0\rangle^+ \urcorner$.

The adequacy of these definitions is established by proving soundness and completeness.

In order to prove soundness, we must establish the correctness of several rules of persistence; where Lemma 1 continues to guarantee topic persistence, we need to reexamine the case of persistence of both verification and falsification. As these steps are not as familiar in the literature on intuitionistic logic, we err on the side of prolixity in the proof of the following:

LEMMA 12. *In* $\mathsf{N_1PAI}$*, if* $w \Vdash^+ \varphi$ *(*$w \Vdash^- \varphi$*, respectively) and* wRw'*, then* $w' \Vdash^+ \varphi$ *(*$w' \Vdash^- \varphi$*, respectively).*

PROOF. As basis step, that $v^+(p)$ and $v^-(p)$ are closed under R establishes the case in which φ is an atom.

- If $\varphi = {\sim}\psi$, then without loss of generality suppose that $w \Vdash^+ {\sim}\psi$ and wRw'. That $w \Vdash^+ {\sim}\psi$ means that $w \Vdash^- \psi$ and by the induction hypothesis, also $w' \Vdash^- \psi$, whence $w' \Vdash^+ {\sim}\psi$.
- If $\varphi = \psi \wedge \xi$, then suppose that wRw' and consider the cases of verification and falsification individually:

  ◦ If $w \Vdash^+ \psi \wedge \xi$ then $w \Vdash^+ \psi$ and $w \Vdash^+ \xi$. By induction hypothesis, $w' \Vdash^+ \psi$ and $w' \Vdash^+ \xi$, whence $w' \Vdash^+ \psi \wedge \xi$.
  ◦ If $w \Vdash^- \psi \wedge \xi$ then either $w \Vdash^- \psi$ or $w \Vdash^- \xi$. By induction hypothesis, it follows that either $w' \Vdash^- \psi$ or $w' \Vdash^- \xi$, whence $w' \Vdash^- \psi \wedge \xi$.
- If $\varphi = \psi \vee \xi$, arguments dual to those used for conjunction establish this case.
- If $\varphi = \psi \rightarrow \xi$, suppose that wRw' and consider two cases individually:

  ◦ If $w \Vdash^+ \psi \rightarrow \xi$, then three clauses must be examined. First, because $w'{\uparrow} \subseteq w{\uparrow}$, the condition that for all $u \in w{\uparrow}$ there exists a $v \in u{\uparrow}$ such that $v \Vdash^+ \psi$ holds *a fortiori* of $w'{\uparrow}$. Second, that $w'{\uparrow} \subseteq w{\uparrow}$ again entails that for every $u \in w'{\uparrow}$ such that $u \Vdash^+ \psi$ also $u \Vdash^+ \xi$. Third, by persistence of topic inclusion, if $t_w(\xi) \leq_w t_w(\psi)$, also $t_{w'}(\xi) \leq_{w'} t_{w'}(\psi)$. Together, one may infer that $w' \Vdash^+ \psi \rightarrow \xi$.
  ◦ If $w \Vdash^- \psi \rightarrow \xi$, then for all $u \in w{\uparrow}$, $u \nVdash^+ \varphi \rightarrow \psi$. But because $w'{\uparrow} \subseteq w{\uparrow}$, this holds *a fortiori* for w', i.e., for all $u \in w'{\uparrow}$, $u \nVdash^+ \varphi \rightarrow \psi$. Thus $w' \Vdash^- \psi \rightarrow \xi$. ■

Just as the definition of closure required augmentation, we need to have a slightly more complex definition of faithfulness for the bilateral systems:

Definition 29. A model $\mathfrak{M}$ is faithful to a branch $\mathcal{B}$ of a tableau if there exists a function f from labels to the points in $\mathfrak{M}$ satisfying clauses modifying Definition 11 so that:

- If φ is a formula and $\ulcorner \varphi, \langle i, 1\rangle^* \urcorner$ is on $\mathcal{B}$ then $f(i) \Vdash^* \varphi$
- If φ is a formula and $\ulcorner \varphi, \langle i, 0\rangle^* \urcorner$ is on $\mathcal{B}$ then $f(i) \nVdash^* \varphi$
- If $\ulcorner \boxed{\neg\varphi} \langle i, 1\rangle \urcorner$ is on $\mathcal{B}$ then for all $w \in f(i){\uparrow}$, $w \nVdash^+ \varphi$
- If $\ulcorner \boxed{\neg\varphi} \langle i, 0\rangle \urcorner$ is on $\mathcal{B}$ then there is an $w \in f(i){\uparrow}$ such that $w \Vdash^+ \varphi$

where $*$ is either the sign "+" or sign "−".

Definition 29 allows the expression of a soundness lemma:

Lemma 13. (Soundness Lemma for $\mathsf{N_1PAI}$) *Let $\mathcal{B}$ be a branch of a tableau and let $\mathfrak{M}$ be a $\mathsf{N_1PAI}$ model faithful to $\mathcal{B}$. Then if any tableau rule is applied to $\mathcal{B}$, $\mathfrak{M}$ remains faithful to at least one of the resulting extensions $\mathcal{B}'$.*

Proof. This can be proven by induction over the union of formulae and pseudoformulae. Again, the rules for persistence follow from Lemma 12 while rules for frame conditions are identical to those found in the treatment of intuitionistic logic discussed in Lemma 6.7.3 of [42], whose treatment of intuitionistic negation immediately applies to the rules for the pseudoformulae $\ulcorner\neg\varphi\urcorner$. Assume that we are working with a model that is thus far faithful to $\mathcal{B}$.

- When $\varphi = {\sim}\psi$, the arguments are extremely straightforward. If $\ulcorner{\sim}\psi\,\langle i,1\rangle^{+}\urcorner$ is on $\mathcal{B}$, then by faithfulness to $\mathcal{B}$, $f(i) \Vdash^{+} {\sim}\psi$ and $f(i) \Vdash^{-} \psi$. Thus, when $\ulcorner\psi\,\langle i,1\rangle^{-}\urcorner$ is added to the model, the model remains faithful. The arguments for cases in which $\ulcorner{\sim}\psi\,\langle i,0\rangle^{+}\urcorner$, $\ulcorner{\sim}\psi\,\langle i,1\rangle^{-}\urcorner$, or $\ulcorner{\sim}\psi\,\langle i,0\rangle^{-}\urcorner$ are on $\mathcal{B}$ follow by analogous, simple arguments.
- When $\varphi = \psi\wedge\xi$, the cases in which $\ulcorner\psi\wedge\xi\,\langle i,1\rangle^{+}\urcorner$ or $\ulcorner\psi\wedge\xi\,\langle i,0\rangle^{+}\urcorner$ are on $\mathcal{B}$ are identical to the arguments for conjunction of Lemma 6.7.3 of [42], exchanging the undecorated $\Vdash$ relation with the decorated $\Vdash^{+}$ relation. Dually, in case $\ulcorner\psi\wedge\xi\,\langle i,1\rangle^{-}\urcorner$ or $\ulcorner\psi\wedge\xi\,\langle i,0\rangle^{-}\urcorner$, standard arguments from [42] for *disjunction* apply, in this case replacing $\Vdash$ with $\Vdash^{-}$.
- When $\varphi = \psi \vee \xi$, the steps are dual to the case of conjunction.
- We examine the distinct signs individually for cases in which $\varphi = \psi \to \xi$:

  ∘ The cases in which $\ulcorner\psi \to \xi\,\langle i,1\rangle^{+}\urcorner$ or $\ulcorner\psi \to \xi\,\langle i,0\rangle^{+}\urcorner$ appear on $\mathcal{B}$ are identical to the case in Lemma 9, exchanging the undecorated $\Vdash$ for the decorated $\Vdash^{+}$ at appropriate steps.
  ∘ If $\ulcorner\psi \to \xi\,\langle i,1\rangle^{-}\urcorner$ and $\ulcorner i\rho j\urcorner$ are on $\mathcal{B}$, then by faithfulness to $\mathcal{B}$, $f(i) \Vdash^{-} \psi \to \xi$ and $f(i)Rf(j)$ and $f(j) \nVdash^{+} \psi \to \xi$. Thus, when $\ulcorner\psi \to \xi\,\langle j,0\rangle^{+}\urcorner$ is added to the branch, the model remains faithful.
  ∘ If $\ulcorner\psi \to \xi\,\langle i,0\rangle^{-}\urcorner$ is on a branch, then by faithfulness of the model, $f(i) \nVdash^{-} \psi \to \xi$ and there exists a $w \in f(i)\!\uparrow$ such that $w \Vdash^{+} \varphi \to \psi$. If $\ulcorner i\rho j\urcorner$ is added to the branch, update f so that $f(j) = w$; then when $\ulcorner\psi \to \xi\,\langle j,1\rangle^{+}\urcorner$ is added, the model remains faithful. ■

Additionally, to ensure that the appearance of both $\ulcorner\varphi\langle i,1\rangle^{+}\urcorner$ and $\ulcorner\varphi\langle i,1\rangle^{-}\urcorner$ on the same branch is an appropriate closure condition, note that no formula can be both verified and falsified at the same point:

Lemma 14. *In* $\mathsf{N_1PAI}$*, for no point w in a model and formula φ does both $w \Vdash^{+} \varphi$ and $w \Vdash^{-} \varphi$.*

Proof. As basis step, disjointness of $v^{+}(p)$ and $v^{-}(p)$ establishes the property when φ is an atom. Suppose as induction hypothesis that this holds for all subformulae of φ.

- If $\varphi = {\sim}\psi$, then the induction hypothesis means that it is not the case that both $w \Vdash^+ \psi$ and $w \Vdash^- \psi$, which respectively entail that not both $w \Vdash^- {\sim}\psi$ and $w \Vdash^+ {\sim}\psi$. Thus, it cannot hold that both $w \Vdash^+ \varphi$ and $w \Vdash^- \varphi$.
- If $\varphi = \psi \wedge \xi$, then suppose for contradiction that both $w \Vdash^+ \psi \wedge \xi$ and $w \Vdash^- \psi \wedge \xi$. By the former clause, both $w \Vdash^+ \psi$ and $w \Vdash^+ \xi$. By induction hypothesis, then, both $w \nVdash^- \psi$ and $w \nVdash^- \xi$, ruling out the requirement for the latter clause to hold that either $w \Vdash^- \psi$ or $w \Vdash^- \xi$ would have to obtain.
- If $\varphi = \psi \vee \xi$, the argument is dual to the case of conjunction.
- If $\varphi = \psi \rightarrow \xi$, then because $w \in w\!\uparrow$, the assumption that $w \Vdash^- \psi \rightarrow \xi$ requires that $w \nVdash^+ \psi \rightarrow \xi$, whence it is impossible that both $w \Vdash^+ \varphi$ and $w \Vdash^- \varphi$. ■

By Lemmas 13 and 14, we infer soundness:

THEOREM 7. (Soundness of $\mathsf{N_1PAI}$) *If* $\Gamma \vdash_{\mathsf{N_1PAI}} \varphi$ *then* $\Gamma \vDash_{\mathsf{N_1PAI}} \varphi$

Completeness for $\mathsf{N_1PAI}$ requires the same sorts of modifications to arguments for ExPAI as encountered when proving soundness. We must first revise our definition yielding models induced by open branches of tableaux as follows.

DEFINITION 30. Let $\mathcal{B}$ be an open branch of a $\mathsf{N_1PAI}$ tableau. Then define $\mathfrak{M}_{\mathcal{B}}$ as the model induced by the branch modifying Definition 13 as follows:

- $v^*(p) = \{w_k \mid \ulcorner p\,\langle k, 1\rangle^* \urcorner \text{ is on } \mathcal{B}\}$

where $*$ is either the sign "+" or the sign "−".

Although our stalwart Lemma 4 continues to apply without modification, we need to produce minor revisions to its companion lemmas for completeness of $\mathsf{N_1PAI}$

LEMMA 15. *Every model* $\mathfrak{M}_{\mathcal{B}}$ *induced by open branch* $\mathcal{B}$ *of an* $\mathsf{N_1PAI}$ *tableau is an* NPAI *model.*

PROOF. By modifying the steps of the proof of Lemma 5 ensuring that the steps for v are duplicated for both v^+ and v^-. ■

LEMMA 16. (Completeness Lemma for $\mathsf{N_1PAI}$) *For any open completed branch* $\mathcal{B}$ *of a* $\mathsf{N_1PAI}$ *tableau, for every formula* φ *and point* w *in* $\mathfrak{M}_{\mathcal{B}}$*,*

- *If* $\ulcorner \varphi\,\langle i, 1\rangle^* \urcorner$ *is on* $\mathcal{B}$ *then* $\mathfrak{M}_{\mathcal{B}}, w_i \Vdash^* \varphi$
- *If* $\ulcorner \varphi\,\langle i, 0\rangle^* \urcorner$ *is on* $\mathcal{B}$ *then* $\mathfrak{M}_{\mathcal{B}}, w_i \nVdash^* \varphi$

where $*$ is either the sign "$+$" or the sign "$-$".

PROOF. The basis step is established by construction of the functions v^+ and v^- in $\mathfrak{M}_{\mathcal{B}}$. Assuming the induction hypothesis that the property holds for all subformulae of φ, the cases in which $\varphi = {\sim}\psi$ is nearly trivial, e.g., they are all variations on the following argument:

- If $\ulcorner {\sim}\psi \langle i, 1\rangle^{+} \urcorner$ is on $\mathcal{B}$, so is $\ulcorner \psi \langle i, 1\rangle^{-} \urcorner$. By induction hypothesis, $w_i \Vdash^- \psi$, whence $w_i \Vdash^+ {\sim}\psi$ as required.

All other steps in the induction in which $*$ is the sign "$+$" are identical to the case of Lemma 10, substituting $\Vdash^+$ in place of $\Vdash$. In particular, note that the case in which $\varphi = \psi \to \xi$ and $*$ is the sign "$+$" follow immediately by making this adaptation of Lemma 10.

Similarly, steps for conjunction and disjunction when $*$ is the sign "$-$" follow dually by adapting the arguments for disjunction and conjunction, respectively, from Lemma 10 by swapping out instances of $\wedge$ for $\vee$ (and *vice versa*) and substituting $\Vdash^-$ in place of the undecorated $\Vdash$. This leaves only the cases in which $\varphi = \psi \to \xi$ and $*$ is the sign "$-$".

- If $\ulcorner \psi \to \xi \langle i, 1\rangle^{-} \urcorner$ is on $\mathcal{B}$, then fix an arbitrary j such that $\ulcorner i\rho j \urcorner$ is on $\mathcal{B}$. By construction of $\mathfrak{M}_{\mathcal{B}}$, $w_i R w_j$. As $\mathcal{B}$ is exhausted, it includes $\ulcorner \psi \to \xi \langle j, 0\rangle^{+} \urcorner$. As mentioned earlier, the argument of Lemma 10 is immediately adapted by decorating each $\Vdash$ relation with a "$+$", whence $w_j \nVdash^+ \psi \to \xi$. But as j was arbitrary, this holds for all $w \in w_i{\uparrow}$, whence $w_i \Vdash^- \psi \to \xi$.
- If $\ulcorner \psi \to \xi \langle i, 0\rangle^{-} \urcorner$ is on $\mathcal{B}$. Then $\mathcal{B}$ includes $\ulcorner i\rho j \urcorner$ (whence $w_i R w_j$ by construction) as well as $\ulcorner \psi \to \xi \langle j, 1\rangle^{+} \urcorner$. Again, an immediate adaptation of Lemma 10 establishes that $w_j \Vdash^+ \psi \to \xi$; as $w_i R w_j$, then, $w_i \nVdash^- \psi \to \xi$. ■

Lemma 16 yields completeness:

THEOREM 8. (Completeness of $\mathsf{N_1PAI}$) *If* $\Gamma \vDash_{\mathsf{N_1PAI}} \varphi$ *then* $\Gamma \vdash_{\mathsf{N_1PAI}} \varphi$

One further observation that is worth making is that the system $\mathsf{N_1PAI}$ is authentically nonprehensivist. Reviewing the consequences of Lemma 14 establishes that $\mathsf{N_1PAI}$ is nonprehensivist via the corollary:

COROLLARY 2. *In* $\mathsf{N_1PAI}$, *no conditional of the form* $(\varphi \wedge {\sim}\varphi) \to \psi$ *can be true at any world.*

Let us take stock of $\mathsf{N_1PAI}$ as a propositional logic forming the kernel of the present interpretation of Griss's executable mathematics. Its conditional meets the desiderata and its falsification condition follows from an intelligible—and intuitionistically plausible—principle.

However, the falsification condition of $\mathsf{N_1PAI}$ may face the following criticism: As argued in [13], the Nelson-style understanding of refutation is an acceptable device in the formalisation of Griss's project; nevertheless, *intuitionistic* negation remains proscribed in a model of Griss's reasoning. Nevertheless, there is a sense in which $\mathsf{N_1PAI}$ has employed intuitionistic negation *virtually* in several ways. This feature exposes a possible conflict between the $\mathsf{N_1PAI}$ interpretation and its aspirations as an interpretation of Griss.

While Griss's executability requirement—the necessary possibility of the antecedent of $\varphi \to \psi$ ($\Box\Diamond\varphi$)—has been encoded by a double intuitionistic negation ($\neg\neg\varphi$), this seems on its face to be a matter of convenience insofar as we could have provided a proof theory with pseudoformulae of the form $\Box\varphi$ and $\Diamond\varphi$ for the same purposes. The use of intuitionistic negation, in other words, is merely an *expediency*; as such, there is no philosophical conflict on its face.

In contrast, the falsification condition for $\varphi \to \psi$ in the first option is arguably *essentially* an intuitionistically negated formula $\neg(\varphi \to \psi)$. One might argue, then, that Griss's rejection of intuitionistic negation renders the falsification condition of $\mathsf{N_1PAI}$ objectionable. If one takes the position that all that has been accomplished is smuggling in the same negation Griss rejects by applying the veneer of Nelson's notation, a plausible alternative would be to leverage Nelson's own account of negation to provide the falsification conditions for subformulae by which a falsification of the conditional is yielded.

4.3. $\mathsf{N_2PAI}$: Constructive Refutation

Nelson's account of the falsification conditions for a conditional $\varphi \to \psi$ are a pair consisting of a verification of φ and a refutation of ψ. A reasonable, Nelson-style account of the falsification condition of Griss's executable conditional therefore might be two-fold: A falsification requires either one to furnish an explicit refutation of the antecedent (showing the failure of its executability) or to furnish a Nelson-style counterexample.

More explicitly, one might take the falsification of an executable intuitionistic conditional as either a falsification of the antecedent or the Nelsonian condition of a verification of the antecedent paired with a refutation of the consequent. We formalize this intuition by the bilateral system $\mathsf{N_2PAI}$.

Its falsification conditions complete Definition 22 by adding the following clause:

Definition 31. The truth conditions for $\mathsf{N_2PAI}$ add to Definition 22:

- $w \Vdash^{-} \varphi \rightarrow \psi$ if either $w \Vdash^{-} \varphi$ or both $w \Vdash^{+} \varphi$ and $w \Vdash^{-} \psi$

We then define $\mathsf{N_2PAI}$ validity:

Definition 32. $\Gamma \vDash_{\mathsf{N_2PAI}} \varphi$ iff for all NPAI models evaluated under Definition 31 and points w, if $\mathfrak{M}, w \Vdash^{+} \psi$ for each $\psi \in \Gamma$, then $\mathfrak{M}, w \Vdash^{+} \varphi$

Having defined an appropriate model theory, we can introduce a corresponding update to Definition 24 as follows:

Definition 33. The bilateral tableau calculus for $\mathsf{N_2PAI}$ adds to the rules of Definition 24 the following rules:

$$
\begin{array}{c}
\varphi \rightarrow \psi \; \langle i\, 1\rangle^{-} \\
\swarrow \qquad \searrow \\
\begin{array}{c|c}
\varphi \; \langle i\, 1\rangle^{-} & \begin{array}{c} \varphi \; \langle i\, 1\rangle^{+} \\ \psi \; \langle i\, 1\rangle^{-} \end{array}
\end{array}
\end{array}
\qquad
\begin{array}{c}
\varphi \rightarrow \psi \; \langle i\, 0\rangle^{-} \\
\downarrow \\
\varphi \; \langle i\, 0\rangle^{-} \\
\swarrow \qquad \searrow \\
\begin{array}{c|c}
\varphi \; \langle i\, 0\rangle^{+} & \psi \; \langle i\, 0\rangle^{-}
\end{array}
\end{array}
$$

Definition 34. $\Gamma \vdash_{\mathsf{N_2PAI}} \varphi$ if there is a $\mathsf{N_2PAI}$ tableau proof every branch of which closes whose initial segment is the sequence $\ulcorner \psi \, \langle i, 1\rangle^{+} \urcorner$ for every $\psi \in \Gamma$ followed by $\ulcorner \varphi \, \langle i, 0\rangle^{+} \urcorner$.

As in the case of $\mathsf{N_1PAI}$, several lemmas are necessary to support soundness:

Lemma 17. *In* $\mathsf{N_2PAI}$*, if* $w \Vdash^{+} \varphi$ *(*$w \Vdash^{-} \varphi$*, respectively) and* wRw'*, then* $w' \Vdash^{+} \varphi$ *(*$w' \Vdash^{-} \varphi$*, respectively).*

Proof. The closure of $v^{+}(p)$ and $v^{-}(p)$ under R establishes the basis step. As induction hypothesis, suppose that the property holds for all subformulae of φ. All cases except falsification of conditionals are identical to those from the previous proof for $\mathsf{N_1PAI}$, so consider this outstanding case:

- Suppose that $\varphi = \psi \rightarrow \xi$ and that $w \Vdash^{-} \psi \rightarrow \xi$. Then one of two cases holds: either $w \Vdash^{-} \psi$ or both $w \Vdash^{+} \psi$ and $w \Vdash^{-} \xi$. In the former case, the induction hypothesis establishes that $w' \Vdash^{-} \psi$; in the latter, the induction hypothesis means that both $w' \Vdash^{+} \psi$ and $w' \Vdash^{-} \xi$. In either case, it follows that $w' \Vdash^{-} \psi \rightarrow \xi$.

■

Lemma 18. (Soundness Lemma for $\mathsf{N_2PAI}$) *Let* $\mathcal{B}$ *be a branch of a tableau and let* $\mathfrak{M}$ *be a* $\mathsf{N_2PAI}$ *model faithful to* $\mathcal{B}$*. Then if any tableau rule is applied to* $\mathcal{B}$*,* $\mathfrak{M}$ *remains faithful to at least one of the resulting extensions* $\mathcal{B}'$*.*

Proof. Along lines identical to the proof of Lemma 13 save for an appeal to Lemma 17 (rather than Lemma 12) and a revised argument for cases in which $*$ is the sign "$-$" and $\varphi = \psi \to \xi$. Assume that the model is faithful to $\mathcal{B}$:

- If $\ulcorner \psi \to \xi \langle i, 1\rangle^{-} \urcorner$ is on $\mathcal{B}$, then there is a branch $\mathcal{B}'$ introducing $\ulcorner \psi \langle i, 1\rangle^{-} \urcorner$ and a branch $\mathcal{B}''$ in which $\ulcorner \psi \langle i, 1\rangle^{+} \urcorner$ and $\ulcorner \xi \langle i, 1\rangle^{-} \urcorner$ have been added. By faithfulness to $\mathcal{B}$, $f(i) \Vdash^{-} \psi \to \xi$. Thus, either $f(i) \Vdash^{-} \psi$ (in which case the model is faithful to $\mathcal{B}'$) or $f(i) \Vdash^{+} \psi$ and $f(i) \Vdash^{-} \xi$ (in which case the model is faithful to $\mathcal{B}''$). Either way, the model remains faithful to one of the branches.
- If $\ulcorner \psi \to \xi \langle i, 0\rangle^{-} \urcorner$ is on the $\mathcal{B}$ then there are branches $\mathcal{B}'$ and $\mathcal{B}''$—both including $\ulcorner \psi \langle i, 0\rangle^{-} \urcorner$—introducing $\ulcorner \psi \langle i, 0\rangle^{+} \urcorner$ and $\ulcorner \xi \langle i, 0\rangle^{-} \urcorner$, respectively. By faithfulness of the model, $f(i) \nVdash^{-} \psi \to \xi$; this entails that $f(i) \nVdash^{-} \psi$. Also, one may infer that either $f(i) \nVdash^{+} \psi$ or $f(i) \nVdash^{-} \xi$, which ensures that the model is faithful to either $\mathcal{B}'$ or $\mathcal{B}''$, respectively.

■

Again, we must show that the new closure conditions on branches are appropriate, requiring us to prove that in $\mathsf{N_2PAI}$, a formula's verification and falsification cannot hold simultaneously:

LEMMA 19. *In* $\mathsf{N_2PAI}$*, for no point* w *in a model and formula* φ *does both* $w \Vdash^{+} \varphi$ *and* $w \Vdash^{-} \varphi$*.*

PROOF. The basis step again follows from disjointness of $v^{+}(p)$ and $v^{-}(p)$. Suppose as induction hypothesis that this holds for all subformulae of φ. The cases of negation, conjunction, and disjunction are identical to the previous lemma, leaving us only to treat the case of executable intuitionistic conditionals.

- If $\varphi = \psi \to \xi$, then suppose that $w \Vdash^{+} \psi \to \xi$ and $w \Vdash^{-} \psi \to \xi$. By the latter, either $w \Vdash^{-} \psi$ or both $w \Vdash^{+} \psi$ and $w \Vdash^{-} \xi$. Consider each subcase:

  ◦ From the supposition that $w \Vdash^{-} \psi$, heredity ensures that for any $w' \in w{\uparrow}$, $w' \Vdash^{-} \psi$. But because $w \Vdash^{+} \psi \to \xi$, there must exist a $w' \in w{\uparrow}$ such that $w' \Vdash^{+} \psi$. By induction hypothesis, it cannot hold that $w' \Vdash^{+} \psi$ and $w' \Vdash^{-} \psi$, so we have a contradiction.
  ◦ In case both $w \Vdash^{+} \psi$ and $w \Vdash^{-} \xi$, because $w \Vdash^{+} \psi \to \xi$ and $w \in w{\uparrow}$, the fact that $w \Vdash^{+} \psi$ entails that $w \Vdash^{+} \xi$. But the induction hypothesis precludes a case in which both $w \Vdash^{+} \xi$ and $w \Vdash^{-} \xi$ hold. ■

Lemmas 18 and 19 establish soundness straightforwardly:

THEOREM 9. (Soundness of $\mathsf{N_2PAI}$) *If* $\Gamma \vdash_{\mathsf{N_2PAI}} \varphi$ *then* $\Gamma \vDash_{\mathsf{N_2PAI}} \varphi$

For completeness for $\mathsf{N_2PAI}$, we can continue to assume the well-worn Lemma 4 as well as the newer Lemma 15 without loss of generality. The corresponding completeness lemma follows from very modest updates to Lemma 16 as well.

Lemma 20. (Completeness Lemma for $\mathsf{N_2PAI}$) *For any open completed branch $\mathcal{B}$ of a $\mathsf{N_2PAI}$ tableau, for every formula φ and point w in $\mathfrak{M}_{\mathcal{B}}$,*

- *If $\ulcorner\varphi\,\langle i,1\rangle^{*}\urcorner$ is on $\mathcal{B}$ then $\mathfrak{M}_{\mathcal{B}}, w_i \Vdash^{*} \varphi$*
- *If $\ulcorner\varphi\,\langle i,0\rangle^{*}\urcorner$ is on $\mathcal{B}$ then $\mathfrak{M}_{\mathcal{B}}, w_i \nVdash^{*} \varphi$*

where $*$ is either the sign "+" or the sign "−".

Proof. The proof of Lemma 16 requires only updates to the cases in which $\varphi = \psi \rightarrow \xi$ and $*$ is the sign "−".

- If $\ulcorner\psi \rightarrow \xi\,\langle i,1\rangle^{-}\urcorner$ is on $\mathcal{B}$, then the branch includes either $\ulcorner\psi\,\langle i,1\rangle^{-}\urcorner$ or it includes $\ulcorner\psi\,\langle i,1\rangle^{+}\urcorner$ and $\ulcorner\xi\,\langle i,1\rangle^{-}\urcorner$. By induction hypothesis, the cases respectively ensure that either $w_i \Vdash^{-} \psi$ or both $w_i \Vdash^{+} \psi$ and $w_i \Vdash^{-} \xi$. In either case, $w_i \Vdash^{-} \psi \rightarrow \xi$, as required.
- If $\ulcorner\psi \rightarrow \xi\,\langle i,0\rangle^{-}\urcorner$ is on $\mathcal{B}$, then so is $\ulcorner\psi\,\langle i,0\rangle^{-}\urcorner$, whence by induction hypothesis $w_i \nVdash^{-} \psi$. Likewise, either the branch introduces $\ulcorner\psi\,\langle i,0\rangle^{+}\urcorner$ or it introduces $\ulcorner\xi\,\langle i,0\rangle^{-}\urcorner$, which respectively ensure by induction hypothesis that $w_i \nVdash^{+} \psi$ or $w_i \nVdash^{-} \xi$. In other words, $w_i \nVdash^{-} \psi$ and either $w_i \nVdash^{+} \psi$ or $w_i \nVdash^{-} \xi$, i.e., $w_i \nVdash^{-} \psi \rightarrow \xi$. ■

Completeness for $\mathsf{N_2PAI}$ follows from standard arguments:

Theorem 10. (Completeness of $\mathsf{N_2PAI}$) *If $\Gamma \vDash_{\mathsf{N_2PAI}} \varphi$ then $\Gamma \vdash_{\mathsf{N_2PAI}} \varphi$*

As before, a corollary of Lemma 19 ensures that the second bilateral system is authentically nonprehensivist as well:

Corollary 3. *In $\mathsf{N_2PAI}$, no conditional of the form $(\varphi \wedge {\sim}\varphi) \rightarrow \psi$ can be true at any world.*

Having introduced the systems $\mathsf{N_1PAI}$ and $\mathsf{N_2PAI}$ and having proven soundness and completeness, we can complete the investigation returning to a final examination of translations from these systems.

4.4. Further Metatheoretical Remarks

We now can return to the matter of Gödel-McKinsey-Tarski-style translations for our bilateral executable systems. For such purposes, the techniques of [23] and [25] are especially fruitful. Kamide, notably, captures proof and refutation at the atomic level, eliminating negation entirely by adding a new set of falsified atoms. Define $\mathcal{L}_J^\star$ as the language in the signature of $\mathcal{L}_J$ in which every atom p is complemented by a new atom $p^\star$. One can interpret $p^\star$ as the statement that p is falsified. Then define the following partial translation:

Definition 35. Define a partial translation $\upsilon : \mathcal{L}_N \to \mathcal{L}_J^\star$ as:

$$\bullet\ p^\upsilon = p \qquad \bullet\ (\sim p)^\upsilon = p^\star \quad \bullet\ (\sim\sim\varphi)^\upsilon = \varphi^\upsilon$$
$$\bullet\ (\varphi \wedge \psi)^\upsilon = (\varphi^\upsilon) \wedge (\psi^\upsilon) \qquad \bullet\ (\sim(\varphi \wedge \psi))^\upsilon = (\sim\varphi)^\upsilon \vee (\sim\psi)^\upsilon$$
$$\bullet\ (\varphi \vee \psi)^\upsilon = (\varphi^\upsilon) \vee (\psi^\upsilon) \qquad \bullet\ (\sim(\varphi \vee \psi))^\upsilon = (\sim\varphi)^\upsilon \wedge (\sim\psi)^\upsilon$$
$$\bullet\ (\varphi \to \psi)^\upsilon = (\varphi^\upsilon) \to (\psi^\upsilon)$$

In order to get the appropriate translations, we can not naively translate into the system ExPAI, which lacks the truth-theoretic and topic-theoretic guarantees for relationships between atoms p and $p^\star$. ExPAI, for example, views the atoms p and $p^\star$ as independent of one another. While in the bilateral systems $v^+(p)$ and $v^-(p)$ are disjoint, no constraint prevents a point's membership in both $v(p)$ and $v(p^\star)$. Likewise, while the bilateral systems ensure that $t_w(p) = t_w(\sim p)$, ExPAI—without further constraints—is free to assign p and $p^\star$ distinct topics.

We must therefore provide a modest extension to ExPAI that we will call ExPAI*. We can succinctly introduce this extension:

Definition 36. An ExPAI* model in language $\mathcal{L}_J^\star$ is a model in the sense of Definition 6 in which $v(p) \cap v(p^\star) = \varnothing$ and $t_w(p) = t_w(p^\star)$ for all atoms p and points w.

Clearly, the classes of NPAI models and ExPAI* models have a one-to-one correspondence in that the models can be paired together in an obvious way.

Definition 37. Let $\mathfrak{M}$ be an NPAI model. Then the unilateral ExPAI* model $\mathfrak{M}^\star$ in language $\mathcal{L}_J^\star$ is defined

- $v(p) = v^+(p)$
- $v(p^\star) = v^-(p)$
- $t_w(p^\star) = t_w(p)$

Definition 38. A tableau calculus for ExPAI* is given by adding the following rules to the tableau calculus in Definition 18:

$$\begin{array}{c} p\ \langle i\,1\rangle \\ \downarrow \\ p^\star\ \langle i\,0\rangle \end{array} \qquad \begin{array}{c} p^\star\ \langle i\,1\rangle \\ \downarrow \\ p\ \langle i\,0\rangle \end{array} \qquad \begin{array}{c} \cdot \\ \downarrow \\ \boxed{p^\star \preccurlyeq p}\ \langle i\,1\rangle \end{array} \qquad \begin{array}{c} \cdot \\ \downarrow \\ \boxed{p \preccurlyeq p^\star}\ \langle i\,1\rangle \end{array}$$

where p or $p^\star$ appear in a formula on the branch and i appears on the branch.

We can just as quickly prove some metatheoretic features of this system:

THEOREM 11. (Soundness/Completeness of $\mathsf{ExPAI}^\star$) $\Gamma \vdash_{\mathsf{ExPAI}^\star} \varphi$ *iff* $\Gamma \vDash_{\mathsf{ExPAI}^\star} \varphi$

PROOF. Note the following:

- Adapting Lemma 9 to the case of $\mathsf{ExPAI}^\star$ is nearly trivial. For example, by Definition 36, a model faithful to a branch in which $\ulcorner p\,\langle i, 1\rangle \urcorner$ appears must be faithful to the branch extended with $\ulcorner p^\star\,\langle i, 0\rangle \urcorner$. Likewise, *every* model is trivially faithful to branches including $\ulcorner \boxed{p^\star \preccurlyeq p}\,\langle i, 1\rangle \urcorner$.
- Adapting Lemma 10 to this case is likewise straightforward. The construction of the model establishes the basis step for the new atoms of language $\mathcal{L}^\star_J$. And a version of Lemma 4 is trivially established for the broader language, guaranteeing the necessary topic-preservation for the conditional.

As a soundness lemma and a completeness lemma for $\mathsf{ExPAI}^\star$ can be simply inferred, soundness and completeness of the extension follow immediately. ■

$\mathsf{ExPAI}^\star$'s additional features allow it to serve as an appropriate target into which validity for our bilateral systems can be translated.

First, we provide a completion of our partial translation υ:

DEFINITION 39. Define a translation $\upsilon_1 : \mathcal{L}_N \to \mathcal{L}^\star_J$ by completing υ with:

- $(\sim(\varphi \to \psi))^{\upsilon_1} = \neg(\varphi^{\upsilon_1}) \vee \neg((\varphi^{\upsilon_1}) \to (\psi^{\upsilon_1}))$

The following observation will be useful; although it is a well-known fact in the intuitionistic setting, we restate it for reference:

OBSERVATION 1. In an JPAI model, $w \Vdash \neg\varphi$ iff $w \Vdash \neg\neg\neg\varphi$.

Of course, insofar as $\mathsf{ExPAI}^\star$ models are a special class of JPAI models, Observation 1 holds *a fortiori* of $\mathsf{ExPAI}^\star$.

Local adequacy of υ_1 between $\mathsf{N}_1\mathsf{PAI}$ and $\mathsf{ExPAI}^\star$ models can be established:

LEMMA 21. *Let $\mathfrak{M}$ be an NPAI model with $\mathsf{N}_1\mathsf{PAI}$ verification and falsification relations $\Vdash^+_{N_1}$ and $\Vdash^-_{N_1}$ and let $\mathfrak{M}^\star$ have the satisfaction relation $\Vdash_E$. Then for a φ in the language $\mathcal{L}_N$,*

- $\mathfrak{M}, w \Vdash^+_{N_1} \varphi$ *iff* $\mathfrak{M}^\star, w \Vdash_E \varphi^{\upsilon_1}$
- $\mathfrak{M}, w \Vdash^-_{N_1} \varphi$ *iff* $\mathfrak{M}^\star, w \Vdash_E (\sim\varphi)^{\upsilon_1}$

PROOF. The basis step is achieved by definition of $\mathfrak{M}^\star$, e.g.,

$$\mathfrak{M}, w \Vdash^+_{N_1} p \text{ iff } w \in v^+(p) \text{ iff } w \in v(p) \text{ iff } \mathfrak{M}^\star \Vdash_E p^{\upsilon_1}$$

The induction steps for $\Vdash^+_{N_1}$ are extremely straightforward as the verification conditions of $\mathsf{N_1PAI}$ and ExPAI coincide. Those for conjunction and disjunction for $\Vdash^-_{N_1}$ are just as straightforward by dual arguments, e.g.,

$$\begin{array}{lll} \mathfrak{M}, w \Vdash^-_{N_1} \psi \wedge \xi & \text{iff} & \mathfrak{M}, w \Vdash^-_{N_1} \psi \text{ or } \mathfrak{M}, w \Vdash^-_{N_1} \xi \\ & \text{iff} & \mathfrak{M}^\star \Vdash_E ({\sim}\psi)^{\upsilon_1} \text{ or } \mathfrak{M}^\star \Vdash_E ({\sim}\xi)^{\upsilon_1} \\ & \text{iff} & \mathfrak{M}^\star \Vdash_E ({\sim}\psi)^{\upsilon_1} \vee ({\sim}\xi)^{\upsilon_1} \\ & \text{iff} & \mathfrak{M}^\star \Vdash_E ({\sim}(\psi \wedge \xi)))^{\upsilon_1} \end{array}$$

The only condition to examine in detail, then, is when $\mathfrak{M}, w \Vdash^-_{N_1} \psi \to \xi$.

- $\mathfrak{M}, w \Vdash^-_{N_1} \psi \to \xi$ holds iff for all $w' \in w{\uparrow}$, $w' \nVdash^+_{N_1} \psi \to \xi$. This holds iff the disjunction of the following two clauses holds:
  - ∘ For all $w' \in w{\uparrow}$ there exists a $w'' \in w'{\uparrow}$ such that for all $w''' \in w''{\uparrow}$, $w''' \nVdash^+_{N_1} \psi$. By induction hypothesis and definitions, this holds iff $w \Vdash_E \neg\neg\neg(\psi^{\upsilon_1})$. But by Observation 1, this holds iff $w \Vdash_E \neg(\psi^{\upsilon_1})$.
  - ∘ There exists a $w'' \in w'{\uparrow}$ such that either: 1) $w'' \Vdash^+_{N_1} \psi$ and $w'' \nVdash^+_{N_1} \xi$ or 2) $t_{w''}(\xi) \nleq_{w''} t_{w''}(\psi)$. By induction hypothesis, the former is equivalent to the conjunction that $w'' \Vdash_E \psi^{\upsilon_1}$ and $w'' \nVdash_E \xi^{\upsilon_1}$ and the latter transfers directly to $\mathfrak{M}^\star$. But this is equivalent to $w'' \nVdash_E (\psi)^{\upsilon_1} \to (\xi)^{\upsilon_1}$, or $w' \Vdash_E \neg((\psi)^{\upsilon_1} \to (\xi)^{\upsilon_1})$. This condition holds iff $w \Vdash_E \neg((\psi)^{\upsilon_1} \to (\xi)^{\upsilon_1})$.

In other words, either $w \Vdash_E \neg(\psi^{\upsilon_1})$ or $w \Vdash_E \neg((\psi)^{\upsilon_1} \to (\xi)^{\upsilon_1})$, *i.e*, $w \Vdash_E ({\sim}(\psi \to \xi))^{\upsilon_1}$. This completes the induction. ■

This local adequacy easily gives us the adequacy of υ_1 in general:

THEOREM 12. $\Gamma \vdash_{\mathsf{N_1PAI}} \varphi$ *iff* $\Gamma^{\upsilon_1} \vdash_{\mathsf{ExPAI}^\star} \varphi^{\upsilon_1}$

PROOF. By Lemma 21, a countermodel $\mathfrak{M}$ to one of the above inferences exists iff $\mathfrak{M}^\star$ is a countermodel to the other. By soundness and completeness, this equivalence holds for provability as well. ■

Now, let us investigate similar translations for $\mathsf{N_2PAI}$:

DEFINITION 40. Define a translation $\upsilon_2 : \mathcal{L}_N \to \mathcal{L}^\star_J$ by completing υ with:

- $({\sim}(\varphi \to \psi))^{\upsilon_2} = ({\sim}\varphi)^{\upsilon_2} \vee (\varphi^{\upsilon_2} \wedge ({\sim}\psi)^{\upsilon_2})$

As before, we have a sort of local adequacy:

Lemma 22. *Let* $\mathfrak{M}$ *be an* NPAI *model with* $\mathsf{N_2PAI}$ *verification and falsification relations* $\Vdash^+_{N_2}$ *and* $\Vdash^-_{N_2}$ *and let* $\mathfrak{M}^\star$ *have the satisfaction relation* $\Vdash_E$*. Then for a* φ *in the language* $\mathcal{L}_N$*,*

- $\mathfrak{M}, w \Vdash^+_{N_2} \varphi$ *iff* $\mathfrak{M}^\star, w \Vdash_E \varphi^{\upsilon_2}$
- $\mathfrak{M}, w \Vdash^-_{N_2} \varphi$ *iff* $\mathfrak{M}^\star, w \Vdash_E (\sim\varphi)^{\upsilon_2}$

Proof. As in the proof of Lemma 21, the basis step and all induction steps besides that in which $w \Vdash^-_{N_2} \psi \to \xi$ are trivial. So suppose that $w \Vdash^-_{N_2} \psi \to \xi$. This holds iff either $w \Vdash^-_{N_2} \psi$ or both $w \Vdash^+_{N_2} \psi$ and $w \Vdash^-_{N_2} \xi$. By induction hypothesis, this holds iff either $w \Vdash_E (\sim\psi)^{\upsilon_2}$ or both $w \Vdash_E \psi^{\upsilon_2}$ and $w \Vdash_E (\sim\xi)^{\upsilon_2}$. But this holds iff $w \Vdash_E (\sim\psi)^{\upsilon_2} \vee (\psi^{\upsilon_2} \wedge (\sim\xi)^{\upsilon_2})$, i.e., iff $w \Vdash_E (\sim(\psi \to \xi))^{\upsilon_2}$. ■

This local adequacy of υ_2 leads to a proof of its general adequacy:

Theorem 13. $\Gamma \vdash_{\mathsf{N_2PAI}} \varphi$ *iff* $\Gamma^{\upsilon_2} \vdash_{\mathsf{ExPAI}^\star} \varphi^{\upsilon_2}$

Note that we could use similar techniques to define appropriate extensions $\mathsf{JPAI}^\star$ and $\mathsf{PAI/F}^\star$ into which our bilateral systems can be translated. For considerations of space, we set this aside.

5. Connexivity and Executable Mathematics

Having defined two systems intended to formalize the propositional logic of Griss's executable mathematics, we conclude by investigating the presence of features closely related to the principles of *connexive* logics. Although a number of distinct definitions of a connexive logic are available, it is fairly natural to define a connexive logic as a deductive system that contains one or more of the following as theorems:

Aristotle's Thesis	$\sim(\varphi \to \sim\varphi)$
Boethius' Thesis	$(\varphi \to \psi) \to \sim(\varphi \to \sim\psi)$

where $\to$ represents a binary conditional connective, possibly interpreted as material implication or as an intensional entailment connective. The reader can find detailed resources discussing the history and philosophy of this field in [30] and [50].

5.1. Connexive Features of $\mathsf{N_1PAI}$ and $\mathsf{N_2PAI}$

Of course, there are weaker properties in the constellation of the above definition that remain interesting from a connexive perspective. For example, consider a metatheoretic property considered in [8] that for no formula φ is

$\varphi \models \sim\varphi$ valid. Although such a property is distinct from Aristotle's Thesis, the two arguably reflect a common—if coarse—thesis concerning reasoning.

Even if not connexive in a strict sense, our logics $\mathsf{N}_1\mathsf{PAI}$ and $\mathsf{N}_2\mathsf{PAI}$ exhibit several such *proto-connexive features* which ultimately bear on the interpretation of Griss. Recall that the features *executability* and *nonprehensivism* assumed among our criteria lead to a property of *anti-vacuity*. It is plausible that it is this feature of anti-vacuity to which the specter of connexivity can be traced. Indeed, the recent paper [7] investigates connections between intuitionism and connexivity through the introduction of an anti-vacuous conditional that can be embedded in intuitionistic logic via a translation resembling that described in Definition 20.

To clarify the relationship between anti-vacuity and connexivity, we recall the discussion of [51] in which the *elimination of counterexamples* is identified as a strategy to validate Aristotle's Thesis. In a number of propositional logics—including classical logic—only contradictions imply their own negations. In such cases, the class of contradictions coincides with the class of counterexamples to Aristotle's Thesis. Because the class of contradictions acts as a barrier to connexivity, one might expect that semantic constraints preventing this class from service as counterexamples will in many cases correspond to the validity of Aristotle's Thesis. The assumption of anti-vacuity has a history of serving in this capacity.

Indeed, the verification of Aristotle's Thesis of Priest's connexive logic P_N of [41] can be attributed precisely to the feature of anti-vacuity ruling out this class of counterexamples. The model theory for Priest's system—built on an $\mathsf{S5}$ Kripke frame—is designed so that $\varphi \rightarrow \psi$ is true at a world w only in case φ is possible, i.e., true at some accessible world w'. As contradictions cannot be true at *any* world in the semantics of P_N, this requirement eliminates any potential counterexamples to Aristotle's Thesis. Thus, the connexivity of P_N—i.e., the validity of Aristotle's Thesis—is the direct result of an assumption of anti-vacuity.

A similar phenomenon concerning connexivity and anti-vacuity is uncovered in Andreas Kapsner's analysis of "empty promise conversions" in [26]. The discussion of [51] notes that "the vacuity inherent in classical counterexamples [to Aristotle's Thesis] shares important traits with... Kapsner's critique of 'empty promise conversions' in constructive semantics" [51, p. 285] To illustrate the notion of an empty promise conversion, consider an intuitionistic conditional $\varphi \rightarrow \psi$ whose antecedent is a contradiction. Because no constructions of φ exist, *all* constructions of φ can be "converted" into constructions of the consequent ψ and the BHK clause for conditionals is satisfied vacuously. Further:

> Ideally, the BHK-type interpretations of intuitionistic connectives are robust and non-vacuous.... [I]n cases in which the antecedent of a conditional is logically impossible, the BHK clause can be satisified in a much less satisfying way... in this case one may fulfill the letter of the BHK requirement by an "empty promise." One does not *need* to design or construct an appropriate algorithm. [51, p. 285–286]

These offensive empty promise conversions occur only in case a conditional has an unsatisfiable or contradictory antecedent, i.e., the cases licensing empty promise conversions and the barriers to Aristotle's Thesis are one and the same. Consequently, to cleanse a constructive logic of empty promise conversions likely eliminates counterexamples to Aristotle's Thesis.

Clearly, Griss's requirement of executability directly eliminates the possibility of empty promise conversions. Supposing executability, constructions of $\varphi \rightarrow \psi$ are recognized as such only in case constructions of φ exists. A construction witnessing $\varphi \rightarrow \psi$ is guaranteed to encounter constructions of φ upon initialization and is thereby compelled to maintain an *executable* algorithm by which the constructions it advertises can be *carried out*.

Given these considerations, it is not surprising that the impossibility of counterexamples to Aristotle's Thesis should place both bilateral logics in the neighborhood of connexive logics:

Observation 2. In both $\mathsf{N_1PAI}$ and $\mathsf{N_2PAI}$, $\varphi \rightarrow {\sim}\varphi$ and ${\sim}\varphi \rightarrow \varphi$ are countertheorems.

Proof. Without loss of generality, consider an NPAI model evaluated under either semantics and suppose that $w \Vdash^{+} \varphi \rightarrow {\sim}\varphi$. Then there exists a $w' \in w{\uparrow}$ such that $w' \Vdash^{+} \varphi$; moreover, because wRw', also $w' \Vdash^{+} {\sim}\varphi$, whence $w' \Vdash^{-} \varphi$. But Lemmas 14 and 19 show consistency holds on either interpretation, so it cannot hold that φ is both verified and falsified at a single point. So in for no w does $w \Vdash^{+} \varphi \rightarrow {\sim}\varphi$. ■

The impossibility of counterexamples to formulae $\varphi \rightarrow {\sim}\varphi$ is a metatheoretical counterpart to Aristotle's Thesis. If insufficient to establish the authentic *connexivity* of the bilateral executable systems, Observation 2 reflects that $\mathsf{N_1PAI}$ and $\mathsf{N_2PAI}$ exhibit at least some *proto-Aristotelian* features.

Turning to Aristotle's Thesis proper, we find that the two bilateral systems part ways. In $\mathsf{N_1PAI}$, the axiom form of Aristotle's Thesis is indeed a theorem:

Observation 3. In $\mathsf{N_1PAI}$, ${\sim}(\varphi \rightarrow {\sim}\varphi)$ and ${\sim}({\sim}\varphi \rightarrow \varphi)$ are theorems.

Proof. Without loss of generality, suppose for contradiction that $w \nVdash^{+} {\sim}(\varphi \rightarrow {\sim}\varphi)$. Then $w \nVdash^{-} \varphi \rightarrow {\sim}\varphi$ and there exists a $w' \in w{\uparrow}$ such that

$w' \Vdash^+ \varphi \to {\sim}\varphi$. But we have just observed that no such w' is possible. Thus, $w \Vdash^+ {\sim}(\varphi \to {\sim}\varphi)$. ■

In contrast, the mere absence of counterexamples does not suffice to establish theoremhood of Aristotle's Thesis in $\mathsf{N_2PAI}$:

Observation 4. In $\mathsf{N_2PAI}$, neither ${\sim}(\varphi \to {\sim}\varphi)$ nor ${\sim}({\sim}\varphi \to \varphi)$ is a theorem.

Proof. Consider a model with a single point w such that $v^+(p) = v^-(p) = \varnothing$. For it to hold that $w \Vdash^+ {\sim}(p \to {\sim}p)$ would require that either $w \Vdash^- p$ or that both $w \Vdash^+ p$ and $w \Vdash^- {\sim}p$, i.e., that $w \Vdash^+ p$. But the construction of the model rules both cases out, so $w \nVdash^+ {\sim}(p \to {\sim}p)$. ■

Let us turn to the connexive principles in the constellation of Boethius' Thesis, whose fates are heavily tied to the fate of Aristotle's Thesis.

Systems that rule out counterexamples to Aristotle's Thesis on grounds of anti-vacuity pose an *a priori* difficulty for the validity of Boethius' Thesis. Unlike Aristotle's Thesis, Boethius' Thesis is itself a conditional. As such, its status in our bilateral systems is subject to the constraints of Griss's executability, i.e., as a conditional, if its substitution instances include formulae with unsatisfiable antecedents, anti-vacuity stands in the way of the validity of Boethius' Thesis.

There clearly are instances of Boethius' Thesis whose antecedents are unsatisfiable in our systems. Most salient for our purposes is:

$$(\varphi \to {\sim}\varphi) \to {\sim}(\varphi \to {\sim}{\sim}\varphi)$$

By assuming executability, the validity of this instance requires the existence of a state verifying the antecedent $\varphi \to {\sim}\varphi$. As a counterexample to Aristotle's Thesis this contingency is shown to be impossible by Observation 2. Aristotle's Thesis and Boethius' Thesis are often considered to be reflections of a common intuition about reasoning. For this reason, it is curious that in e.g. $\mathsf{N_1PAI}$ those features that *ground* the validity of Aristotle's Thesis are precisely the features that *frustrate* the validity of Boethius' Thesis.

To be sure, we find some type of *proto-Boethian* phenomena in that $\varphi \to \psi$ and $\varphi \to {\sim}\psi$ are *contraries*:

Observation 5. In both $\mathsf{N_1PAI}$ and $\mathsf{N_2PAI}$, $\varphi \to \psi$ and $\varphi \to {\sim}\psi$ are not jointly satisfiable.

Proof. Suppose for contradiction that $w \Vdash^+ \varphi \to \psi$ and $w \Vdash^+ \varphi \to {\sim}\psi$. Then there exists a $w' \in w{\uparrow}$ such that $w' \Vdash^+ \varphi$. Because wRw', the verification of the two conditionals entails that $w' \Vdash^+ \psi$ and $w' \Vdash^+ \sim\psi$—and $w' \Vdash^- \psi$—respectively. But by prior observations, this is impossible.

■

This reflects a metatheoretic principle that if $\varphi \to \psi$ is true, $\varphi \to \sim\psi$ must fail. But, as the foregoing discussion suggests, the metatheoretic principle cannot be lifted to establish the validity of Boethius' Thesis itself:

Observation 6. In neither $\mathsf{N_1PAI}$ nor $\mathsf{N_2PAI}$ is $(\varphi \to \psi) \to \sim(\varphi \to \sim\psi)$ a theorem.

Proof. Consider the instance $(\varphi \to \sim\varphi) \to \sim(\varphi \to \sim\sim\varphi)$. In either $\mathsf{N_1PAI}$ or $\mathsf{N_2PAI}$, for this formula to be verified at w would require the existence of some $w' \in w\uparrow$ such that $w' \Vdash^+ \varphi \to \sim\varphi$. But we have observed that $\varphi \to \sim\varphi$ can be verified at no point w in any model on either option. Thus, there exist unsatisfiable instances of $(\varphi \to \psi) \to \sim(\varphi \to \sim\psi)$ and it is not a theorem. ■

Having surveyed some of the connexive features of the bilateral systems, we can turn to the particular consequences these features hold for our proposed nonprehensivist interpretation of Griss.

5.2. Anti-Vacuity and Mathematical Induction

The selection of the features of $\mathsf{N_1PAI}$ and $\mathsf{N_2PAI}$—and the intermediate systems from which they were incrementally defined—was guided by the observance of several criteria drawn from our interpretation of Griss's executable mathematics. As these systems' features correspond to features of Griss's intuitionism, problems for the systems are not merely logical problems, but serve as indicators of broader problems for executable mathematics in general.

Although we have worked in a propositional language so far, Griss's interest in intuitionistic reasoning is motivated by mathematical investigations. Consequently, a satisfactory account of the logic of Griss's executable mathematics will ultimately require an extension to include quantification theory. Prior to moving past the propositional basis to the first-order case, such a project will be well-served by noting that the connexive features we have observe portend several difficulties in the case of mathematical reasoning.

To identify such a difficulty, recall that the anti-vacuity of $\mathsf{N_1PAI}$ and $\mathsf{N_2PAI}$ establishes a kinship with Priest's connexive logic P_N introduced in [41]. By this relationship, difficulties detected in *connexive* mathematics may foretell analogous complications for a nonprehensivist executable mathematics as we have interpreted it.

Thus, the investigations into connexive mathematics carried out in [9] and [11] bear a great deal of relevance to the nonprehensivist interpretation of Griss. To be sure, difficulties in connexive mathematics are no surprise; [9] considered the prospects for theories of arithmetic based on first-order extensions of several connexive logics, concluding in each case that some pathology or other emerged. But among these pathologies, one concerning Priest's P_N stands out: A theorem that on any natural first-order extension of P_N, there exist instances of the axiom scheme of induction:

$$(\varphi(0) \wedge \forall n(\varphi(n) \to \varphi(n')) \to \forall n\varphi(n)$$

that are logically false.

Observation 3 of [9] placed the source of the conflict at the feet of Aristotle's Thesis. To see why, let $\varphi(x)$ be the formula $(x = x) \to {\sim}(x = x)$. As a counterexample to Aristotle's Thesis, $\varphi(n)$ is logically false for all n, whence both $\varphi(0)$ and $\forall n(\varphi(n) \to \varphi(n'))$ are logically false (on any natural interpretation of quantification). The logical falsehood of the antecedent of the corresponding instance of the induction schema follows, ensuring that the whole of this instance of the axiom scheme of induction is logically false as well.

That Peano arithmetic cannot be evaluated in any natural first-order extension of P_N suggests an apparent conflict between Aristotle's Thesis and the axiom scheme of induction. Furthermore, two observations make this especially concerning for our interpretation of Griss: First, the validity of Aristotle's Thesis in P_N is established by the property of anti-vacuity, suggesting a conflict between anti-vacuity and mathematical induction. Second, because anti-vacuity follows from the conjunction of nonprehensivism and Griss's executability, there is a risk that conflicts with mathematical induction follow from any nonprehensivist interpretation of executable mathematics.

One possible way to guard against this risk might be to exchange the axiom form of mathematical induction in favor of a *rule form*. Arguably, a rule form would push the connection between $\varphi(0)$ and $\forall n(\varphi(n) \to \varphi(n'))$ (on the one hand) and $\forall n\varphi(n)$ (on the other) outside of the scope of the executable conditional, thereby sidestepping the demands of anti-vacuity. In many settings—both classical and otherwise—the two are indeed interchangeable. Bob Meyer's presentation of the relevant arithmetic $\mathsf{R}^\sharp$ includes only a rule of mathematical induction; his discussion of the choice between axiom and rule forms in [31] makes a case that accepting a rule form is justified.

In the case of Griss's executable mathematics, though, this move seems like a dodge. Griss imparts a feature of executability to the intuitionistic conditional as a reflection of a more general requirement that mathematical reasoning *in toto* must be executable. Thus, to Griss, the activity of reasoning through a rule based version of induction should be subject to the same demands of anti-vacuity as the axiom form. Because applications of vacuous instances of the rule of induction (on e.g. a contradictory property) still requires that a reasoner mentally executes contradictory constructions, moving to rules does not avoid the issue in this case.

A more attractive alternative, I think, lies in adopting a restricted scheme of induction. [9] identifies imposing restrictions on induction as a possible solution to the problem for P_N:

> [S]everal philosophical standpoints *anticipate* that inclusion of the induction axioms with the other Peano axioms *should* yield a trivial result. From the perspective [*e.g.*] of strict finitism... the true pathology is found not in the *failure* of induction but in the supposition that it *holds*. [9, p. 373]

While the induction scheme of Peano arithmetic is *unrestricted*—allowing induction over arbitrary open formulae—one may frequently encounter arithmetics with an induction scheme restricted to open formulae of a particular type. Such restrictions can be made on philosophical grounds, e.g., Edward Nelson's predicative arithmetic restricts induction to only predicative $\varphi(x)$. Restrictions can be made on computational or feasibility grounds, including a whole hierarchy of bounded arithmetics due to Sam Buss. In other words, there is a great deal of precedent for taking such a position.

Further investigation is needed—until first-order extensions of $\mathsf{N}_1\mathsf{PAI}$ and $\mathsf{N}_2\mathsf{PAI}$ are developed, it is unclear whether the pathology greeting arithmetic in P_N will visit our interpretations of Griss. In case the above phenomenon *does* emerge, the development of a nonprehensivist interpretation of Griss's executable mathematics will likely have to place restrictions on mathematical induction.

What restrictions might be required, then, for a nonprehensivist interpretation of arithmetic? Likely, this will necessitate a notion of *practicable induction* that codifies the validity of induction restricted to *satisfiable formulae*. One very preliminary understanding of a practicable induction scheme, e.g., would admit instances of the induction scheme only in case $\varphi(n)$ has a model for some n.

Whether such a restriction is realizable in practice is left open. There are risks of circularity (perhaps a fixed-point theorem would establish a $\varphi(x)$ that satisfiable iff the corresponding instance of the axiom scheme is true). There are risks of undecidability (if the requirement that $\varphi(n)$ has a model presupposes that arithmetic has a model, then the condition is likely undecidable or inconsistent). This problem can be tackled in earnest only once first-order extensions to $\mathsf{N_1PAI}$ and $\mathsf{N_2PAI}$ have been developed.

Nevertheless, it is interesting that the conflicts between connexivity and arithmetic noted in [9] should potentially bear upon systems whose motivations are entirely distinct from connexive logic.

6. Concluding Remarks

During this paper, we have identified several criteria that a propositional logic corresponding to Griss's program understood as *executable mathematics* and have iteratively developed a sequence of systems that satisfy each criterion. This resulted in a pair of propositional logics that appear to satisfactorily model the features expected of a propositional basis for a nonprehensivist interpretation of Griss.

In targeting an interpretation of executable mathematics that is maximally harmonious with Griss's philosophy, we have privileged a nonprehensivist picture in guiding the shape of our propositional logics. But following Griss in demanding executability does not necessitate that one accept the nonprehensivist picture. One could be convinced that constructivity requires executability while accepting a maximally permissive picture of propositional simulation. There is inarguably merit in investigating propositional logics suitable to prehensivist interpretations of Griss.

This could be as simple as relaxing the condition that $v^+(p) \cap v^-(p) = \varnothing$ from Definition 21 and evaluating the structures under the $\mathsf{N_2PAI}$ truth conditions. It may require more severe departures from the bilateral systems' truth and falsification conditions.

Noting that this paper is just an intermediate step in a longer goal of providing formalizations of Griss's project understood as a project of executable mathematics, the next stage in the investigation is clear. As we have noted, a propositional language is inadequate for expressing the types of statements crucial to mathematics. To provide a formal logic suitable to evaluating e.g. arithmetic requires the development of first-order extensions for the systems described in this paper.

In some ways, both the model theoretic and proof theoretic accounts of quantification in bilateral contexts described in [33] or [44] can be more-or-less directly imported to provide quantification theory for e.g. $\mathsf{N_1PAI}$ and $\mathsf{N_2PAI}$. The greatest barrier to their integration is likely the matter of providing an adequate theory of the topics of quantified formulae, i.e., updating the conditions on topic assignment functions t_w. Adequate accounts of subject-matter in quantified contexts are notoriously tricky, as recognized by Fine in [15].

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