Hitoshi Omori
Andreas Kapsner

Angell and McCall Meet Wansing

Abstract. In this paper, we introduce a new logic, which we call **AM3**. It is a connexive logic that has several interesting properties, among them being *strongly connexive* and validating the *Converse Boethius Thesis*. These two properties are rather characteristic of the difference between, on the one hand, Angell and McCall's **CC1** and, on the other, Wansing's **C**. We will show that in other aspects, as well, **AM3** combines what are, arguably, the strengths of both **CC1** and **C**. It also allows us an interesting look at how connexivity and the intuitionistic understanding of negation relate to each other. However, some problems remain, and we end by pointing to a large family of weaker logics that **AM3** invites us to further explore.

Keywords: Connexive logic, Strong connexivity, Three-valued logic.

1. Introduction

Connexive logic is a topic in non-classical logic that is receiving a lot of attention these days; however, that is not to say that it is a recent topic by any means. Some have claimed that it has its historical roots in antiquity, others have disputed this. Much less controversial is the important role of two connexive logical systems in the much more recent history of the subject. The first is the system **CC1** that is due to Richard Angell [1] and Storrs McCall [15] and marks, in many ways, the modern inception of connexive logic as a unified topic in logical research. While that topic never quite disappeared after the seminal work by Angell and McCall, it was only after Heinrich Wansing started making his contributions some forty years later that it truly started to blossom. For that reason alone, the first connexive logic Wansing introduced in [33], which is called **C**, has an indisputably important place in the history of connexivity. Also, it offered one of the most elegant semantics as well as proof systems for connexive logics to date (see also [22, p.178]).

[1] This is probably the impression one also gets from the overview in [21, §2].

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Beyond the connexive principles (see Section 2), the two logics seem to have little in common (see Section 3.3 for a summary of the differences).[1] However, it is the aim of this paper to bridge these differences and to try to preserve as many of the desirable properties of each of these logics by introducing a new logic. Conceptually, it is somewhat closer to **CC1** than to **C**, and for that reason we have chosen to call it **AM3**, abbreviating **A**ngell-and-**M**cCall-inspired **three**-valued logic. However, we will see that it has, in many ways, properties that are the best of the two worlds marked by **CC1** and **C**. One item that is of particular interest is the property of hyperconnexivity, a property that is rather distinctive of **C** and related systems in the connexive literature. The fact that **AM3** is hyperconnexive is one of the primary ways in which it fuses the influence of the two traditions.

Another way in which it does this concerns the relation between connexivity and constructivity. Wansing wrote that his work "suggests that connexive logic is *constructive*" ([33, p.367]). However, the account he gave utilizes a lesser known kind of constructive negation, namely Nelson-style strong negation. The better known constructive treatment of negation, i.e. viewing it as defined in terms of a conditional and a bottom constant, has so far not been investigated.[2] **AM3** naturally leads to an investigation of just that sort (see Section 6.2), and it will point the way to a much richer space of logics, as we will see in Section 6.3.

The paper is organized as follows. After recalling the key connexive principles in Section 2, we briefly recall the two basic systems **CC1** and **C** in connexive logic, and compare them in Section 3. This will be followed by Section 4 in which we introduce the new system **AM3**. Then, in Section 5, we observe some properties of **AM3**, and compare them to **CC1** and **C**. We then turn to reflect on **AM3** in Section 6 by looking at some old objections against **CC1**, and the arrow-bottom flavor of the negation in **AM3**. We will also specify a subsystem of **AM3** that we call **AMW** and that seems to be worth further investigations, and suggest some open problems. Finally, Section 7 concludes the paper with a brief summary of the paper.

2. Connexive Principles

First and foremost, connexive logic is characterized by two sets of contra-classical principles customarily called **Aristotle** and **Boethius**.[3]

[2] [26] mentions the topic in passing on p. 410.

[3] For an overview of connexive logic, see [16, 32].

ARISTOTLE: $\neg(A \to \neg A)$ and $\neg(\neg A \to A)$ are valid.

BOETHIUS: $(A \to B) \to \neg(A \to \neg B)$ and $(A \to \neg B) \to \neg(A \to B)$ are valid.

In [11], it was observed that some connexive logics (such as, indeed, Wansing's **C**, and John Cantwell's system **CN** introduced in [3]) allowed for satisfiable instances of $(A \to \neg A)$, as well as simultaneously satisfiable instances of $(A \to B)$ and $(A \to \neg B)$. Kapsner took that to go against the spirit of the connexive enterprise. To be able to judge those cases out of bounds, he suggested to add two unsatisfiability clauses:

UNSAT1: In no model, $(A \to \neg A)$ is satisfiable, and neither is $(\neg A \to A)$.

UNSAT2: In no model $(A \to B)$ and $(A \to \neg B)$ are satisfiable simultaneously (for any A and B).

He called logics that satisfy Aristotle, Boethius and the UnSat clauses *strongly connexive*, those that only satisfied Aristotle and Boethius *weakly connexive*. Though strongly connexive logics might be more appealing from certain philosophical perspectives, it has proven very difficult to find attractive systems that have this feature. As one of the benefits of this paper we will, by the end, be left with new directions to search for such systems.

Lastly, a connexive system is *hyperconnexive*[4] iff it is connexive and additionally satisfies the Converse of Boethius:

CONVERSE OF BOETHIUS: $\neg(A \to B) \to (A \to \neg B)$ and $\neg(A \to \neg B) \to (A \to B)$ are valid.

Though not necessarily part of the modern day conception of connexivity, some have argued that Boethius was actually committed to CBT, such as Susanne Bobzien, who writes of "Boethius' insistence that the negation of' If it is A, it is B' is 'If it is A, it is not B' ".[5] Others have found that the equivalence of $\neg(A \to B)$ and $(A \to \neg B)$ amounts to an attractive account of what it means to negate a conditional. For example, David Lewis conceded this attractiveness to Stalnaker, even though his own theory of counterfactuals did not bear it out (see [14, p.79-80]). We do not take a stand on whether satisfying CBT is a desirable property.[6] Our interest here is in its function as an important distinguishing feature between **CC1** and **C**.

[4] A terminology that is used by Richard Sylvan in [7, p.89].

[5] [2]. See also, e.g., [23] and [7, p.68].

[6] See also [16, p.446] for McCall arguing against CBT.

3. Preliminaries

The language $\mathcal{L}$ consists of a finite set $\{\neg, \wedge, \rightarrow\}$ of propositional connectives[7] and a countable set Prop of propositional variables which we denote by p, q, etc. Furthermore, we denote by Form the set of formulas defined as usual in $\mathcal{L}$. We denote a formula of $\mathcal{L}$ by A, B, C, etc. and a set of formulas of $\mathcal{L}$ by Γ, Δ, Σ, etc.

3.1. Angell and McCall

Let us first recall the four-valued logic **CC1**, introduced by Angell in [1] and explored in some detail by McCall in [15].

Definition 1. A **CC1**-interpretation of $\mathcal{L}$ is a function $v : \mathsf{Prop} \rightarrow \{1, 2, 3, 4\}$. Given a **CC1**-interpretation v, this is extended to a function I from Form to $\{1, 2, 3, 4\}$ by truth functions depicted in the form of truth tables as follows:

$\wedge$	1	2	3	4	$\neg$
1	1	2	3	4	4
2	2	1	4	3	3
3	3	4	3	4	2
4	4	3	4	3	1

$\rightarrow$	1	2	3	4
1	1	4	3	4
2	4	1	4	3
3	1	4	1	4
4	4	1	4	1

Definition 2. For all $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, $\Gamma \models_{\mathbf{CC1}} A$ iff for all **CC1**-interpretations v, $I(A) \in \mathcal{D}$ if $I(B) \in \mathcal{D}$ for all $B \in \Gamma$, where $\mathcal{D} = \{1, 2\}$.

Remark 3. Note that the above four-valued semantics was considered by Angell for the purpose of establishing the consistency of the main system he proposed in [1], known as **PA1**.[8] **CC1** itself was studied in [15]. The main results established by McCall include an axiomatization of **CC1**, as well as a proof of the Post-completeness of **CC1**. There were a number of criticisms against **CC1**, but we shall discuss them in some detail later (see Section 6.1).

3.2. Wansing

Let us now turn to Wansing's system of connexive logic **C**, introduced in [33]. Although the standard presentation of **C** will also include disjunction as a primitive connective, for our purpose, we will present it in the language $\mathcal{L}$ in

[7]Note that we are not much concerned with disjunction here, as the connexive principles don't involve it.

[8]A sound and complete semantics for **PA1** was suggested in [26].

which disjunction is not included. However, as one can easily see, disjunction can be defined in the standard manner in terms of negation and conjunction.

The semantics are very similar to the Kripke semantics for intuitionistic logic. The main difference is that negation is tied to an independent constructive notion of falsity, as opposed to being defined in terms of arrow and bottom.

Definition 4. A **C**-model for the language $\mathcal{L}$ is a triple $\langle W, \leq, V\rangle$, where W is a non-empty set (of states); $\leq$ is a partial order on W; and $V : W \times \mathsf{Prop} \longrightarrow \{\emptyset, \{0\}, \{1\}, \{0,1\}\}$ is an assignment of truth values to state-variable pairs with the condition that $i \in V(w_1, p)$ and $w_1 \leq w_2$ only if $i \in V(w_2, p)$ for all $p \in \mathsf{Prop}$, all $w_1, w_2 \in W$ and $i \in \{0,1\}$. Valuations V are then extended to interpretations I of state-formula pairs by the following conditions:

- $I(w, p){=}V(w, p)$,
- $1 \in I(w, \neg A)$ iff $0 \in I(w, A)$,
- $0 \in I(w, \neg A)$ iff $1 \in I(w, A)$,
- $1 \in I(w, A \wedge B)$ iff $1 \in I(w, A)$ and $1 \in I(w, B)$,
- $0 \in I(w, A \wedge B)$ iff $0 \in I(w, A)$ or $0 \in I(w, B)$,
- $1 \in I(w, A {\rightarrow} B)$ iff for all $w_1 \in W$: if $w \leq w_1$ and $1 \in I(w_1, A)$ then $1 \in I(w_1, B)$,
- $0 \in I(w, A {\rightarrow} B)$ iff for all $w_1 \in W$: if $w \leq w_1$ and $1 \in I(w_1, A)$ then $0 \in I(w_1, B)$.

Finally, semantic consequence is now defined as follows: $\Gamma \models_{\mathbf{C}} A$ iff for all **C**-models $\langle W, \leq, V\rangle$, and for all $w \in W$: $1 \in I(w, A)$ if $1 \in I(w, B)$ for all $B \in \Gamma$.

Remark 5. Historically, **C** is derived from Nelson's logic **N4** by replacing the falsity condition for implication, which in the case of **N4** is the following:

$$0 \in I(w, A {\rightarrow} B) \text{ iff } 1 \in I(w, A) \text{ and} 0 \in I(w, B).$$

N4 and its related systems are explored in great depth from a proof-theoretic perspective in [10]. Given the close relationship between **N4** and **C**, many of the results established for **N4** carry over to **C**.

3.3. A Quick Comparison

Let us now briefly compare **CC1** and **C**. Almost the only shared property for these two systems is that they are both connexive. We may also add that both systems enjoy double negation introduction and elimination as well as *modus ponens*, but beyond this, they are rather unlike each other. Let us point to some of the notable differences.

- **C** validates the Converse of Boethius' thesis, but **CC1** does not; thus, the former is hyperconnexive, while the latter is not.
- **C** isn't strongly connexive, while **CC1** is. In fact, in many ways **CC1** is one of the few systems that is satisfying strong connexivity in the literature so far (other examples include the approach to connexivity via relating semantics (cf. [8]) and the constructive approach of [5] which we will mention again below).
- **CC1** is consistent and not paraconsistent, whereas **C** is not only paraconsistent but also inconsistent (or contradictory) without being trivial. Indeed, we have both $\models_{\mathbf{C}} (A \wedge \neg A) \rightarrow A$ and $\models_{\mathbf{C}} \neg((A \wedge \neg A) \rightarrow A)$.
- **CC1** validates contraposition, namely $\models_{\mathbf{CC1}} (A{\rightarrow}B){\rightarrow}(\neg B{\rightarrow}\neg A)$, but we do not have contraposition in **C**.[9]
- Conjunction elimination fails in **CC1**, but holds in **C**. Indeed, given the Boethius' thesis, conjunction elimination would be the key to produce inconsistency in **CC1**.
- **C** has Importation and Exportation which shows that conjunction and the conditional stand in a residuation relation, but this is not the case in **CC1**.
- **C** validates the Weakening axiom, namely $\models_{\mathbf{C}} A \rightarrow (B \rightarrow A)$, but this fails in **CC1**.
- Finally, **C** enjoys the Deduction Theorem, but **CC1** does not.

The differences we mentioned here can be summarized as follows in a table.

[9] That **C** does not have contraposition, just like **N4**, can be checked by observing that is it not even self-extensional by considering the equivalence $\models_{\mathbf{C}} \neg((A{\rightarrow}A){\rightarrow}\neg(A{\rightarrow}A)) \leftrightarrow ((A \wedge \neg A){\rightarrow}A)$ ($\leftrightarrow$ is defined in the usual way). Indeed, if **C** were self-extensional, then we obtain that $\models_{\mathbf{C}} ((A{\rightarrow}A){\rightarrow}\neg(A{\rightarrow}A)) \leftrightarrow \neg((A \wedge \neg A){\rightarrow}A)$. Given that the right hand side is valid in **C**, we obtain $\models_{\mathbf{C}} (A{\rightarrow}A){\rightarrow}\neg(A{\rightarrow}A)$, and thus $\models_{\mathbf{C}} \neg(A{\rightarrow}A)$, but we can check that $\neg(A{\rightarrow}A)$ is not valid in **C** by considering a classical extension of **C**, known as **MC** in the literature (cf. [32]). Therefore, **C** is not self-extensional and thus contraposition is not valid.

		CC1	C
Converse of BT	$\neg(A\to B)\to(A\to\neg B)$	$\times$	✓
Unsat 1	See Section 2.	✓	$\times$
Unsat 2	See Section 2.	✓	$\times$
Explosive	$A, \neg A \models B$	✓	$\times$
Paraconsistent	$A, \neg A \not\models B$	$\times$	✓
Contraposition	$(A \to B) \to (\neg B \to \neg A)$	✓	$\times$
Conjunction elimination	$(A \wedge B) \to A$	$\times$	✓
Exportation/Importation	$((A \wedge B)\to C) \leftrightarrow (A\to(B\to C))$	$\times$	✓
Weakening	$A\to(B\to A)$	$\times$	✓
Deduction theorem	$\Gamma, A \models B$ only if $\Gamma \models A \to B$	$\times$	✓

4. Semantics for AM3

We will now introduce a three-valued logic, called **AM3**, that can be obtained by making some changes to **CC1**. We will first introduce the new system, and explain the way we are making some modifications to **CC1**.

Definition 6. An **AM3**-interpretation of $\mathcal{L}$ is a function $v : \mathsf{Prop} \to \{\mathbf{1}, \mathbf{i}, \mathbf{0}\}$. Given an **AM3**-interpretation v, this is extended to a function I that assigns every formula a truth value by truth functions depicted in the form of truth tables as follows:

	$\neg$
1	**0**
i	**0**
0	**1**

$\wedge$	**1**	**i**	**0**
1	**1**	**i**	**0**
i	**i**	**i**	**0**
0	**0**	**0**	**i**

$\to$	**1**	**i**	**0**
1	**1**	**i**	**0**
i	**1**	**1**	**0**
0	**0**	**0**	**1**

Then, the semantic consequence relation for **AM3** (notation: $\models$) is defined as follows.

Definition 7. For all $\Gamma\cup\{A\} \subseteq \mathsf{Form}$, $\Gamma \models A$ iff for all **AM3**-interpretations v, $I(A) \in \mathcal{D}$ if $I(B) \in \mathcal{D}$ for all $B \in \Gamma$ where $\mathcal{D} = \{\mathbf{1}\}$.

Let us now explain how **AM3** is obtained from **CC1**. To this end, let us manipulate the four-valued tables for **CC1**, and remove the value 2 by eliminating rows and columns with 2 as the input value. Then, we obtain the following truth tables.

$\wedge$	1	3	4	$\neg$
1	1	3	4	4
3	3	3	4	2
4	4	4	3	1

$\to$	1	3	4
1	1	3	4
3	1	1	4
4	4	4	1

Here, though, we still have an occurrence of 2 for the truth table for negation when we look at input 3. In order to eliminate 2 from our tables completely, we reconsider the truth table for negation in **CC1**. More specifically, we consider negation as arrow-bottom, where bottom is defined as $\neg_o(p \to p)$ for some $p \in \mathsf{Prop}$ in **CC1**, and $\neg_o$ is the old (or original) negation in **CC1**. Note that the bottom defined here will always take the value 4. Then, the resulting truth tables will be as follows, where $\neg_n A := A \to \bot$, and n stands for new.

$\wedge$	1	2	3	4	$\neg_n$
1	1	2	3	4	4
2	2	1	4	3	3
3	3	4	3	4	4
4	4	3	4	3	1

$\to$	1	2	3	4
1	1	4	3	4
2	4	1	4	3
3	1	4	1	4
4	4	1	4	1

Now, we can safely remove the value 2 (as well as the subscript n for negation) and we obtain the following truth tables.

$\wedge$	1	3	4	$\neg$
1	1	3	4	4
3	3	3	4	4
4	4	4	3	1

$\to$	1	3	4
1	1	3	4
3	1	1	4
4	4	4	1

And these are the truth tables for **AM3** written in terms of a different set of truth values. Note again that $\bot$ is defined as $\neg(p \to p)$ for some $p \in \mathsf{Prop}$ in **AM3**, and $\bot$ will always take the value 4, or the value 0 in the new notation above. In brief, for conjunction and conditional, we took the submatrix of **CC1**, and for negation, given that the set $\{1, 3, 4\}$ is not closed under the original negation, we replaced the truth table by considering the arrow-falsum operator, and by combining these, we obtain **AM3** from **CC1**.

Remark 8. Interestingly, the truth tables for **AM3** were independently discovered and discussed by Davide Fazio, Antonio Ledda, and Francesco Paoli in their [5], and they also observe that it appears in [29]. Note also that in their paper, they define a new conditional within intuitionistic logic as $(A \to B) \wedge (\neg A \to \neg B)$, and their results imply that there are uncountably many strongly connexive logics.

5. Observations

Let us offer a brief comparison of the three systems **CC1**, **C** and **AM3**.

5.1. Connexive Principles (I)

Let us first compare the three systems in view of the first set of connexive principles. We will not only discuss Aristotle's and Boethius' theses, but also the additional principles discussed in the literature (see §2).

		CC1	**AM3**	**C**
(AT)	$\neg(A\to\neg A)$	✓	✓	✓
(BT)	$(A\to\neg B)\to\neg(A\to B)$	✓	✓	✓
(CBT)	$\neg(A\to B)\to(A\to\neg B)$	×	✓	✓
Unsat 1	See §2.	✓	✓	×
Unsat 2	See §2.	✓	✓	×

Our comparison shows that with respect to the basic connexive principles, namely Aristotle and Boethius' theses, all three systems are on a par. However, beyond that, there are interesting differences. Indeed, the converse of Boethius' thesis, and thus hyperconnexivity, is shared by **AM3** and **C**, but not with **CC1**, whereas strong connexivity (i.e., UnSat 1 and 2) is shared by **AM3** and **CC1**, but not with **C**.

We would like to highlight and emphasize here that all the systems of connexive logic which enjoy hyperconnexivity, to the best of our knowledge, have been obtained along Wansing's idea of replacing the falsity condition of the conditional in **N4** (see Section 3), and thus the negation was always paraconsistent.[10] In contrast, we are here achieving hyperconnexivity with Explosion (cf. Section 5.6). Thus, given that hyperconnexivity is one of the key features of Wansing's system **C**, **AM3** occupies an interesting place bridging two traditions in connexive logic that are, as we've seen in Section 3.3, very different in their approach as well as their properties.

There are more connexive principles we will discuss in Sections 5.4 and 5.5, but before turning to them, it makes sense to first clarify the properties of the implication and conjunction in **AM3**, which we will do in the next two subsections.

5.2. The Implicational Fragment

We now turn our attention to the implicational fragments of the logics we are concerned with. Here, we see that **AM3** lies somewhere in between **CC1** and **C** when it comes to the strength of the conditional.

[10] The same move of changing the falsity condition of conditionals can be done with weaker conditionals, such as those with conditional logics (cf. [12,31]) as well as relevant logics (cf. [6,20,35]).

		CC1	**AM3**	**C**
(**B**)	$(B \to C) \to ((A \to B) \to (A \to C))$	✓	✓	✓
(**C**)	$(A \to (B \to C)) \to (B \to (A \to C))$	×	✓	✓
(**I**)	$A \to A$	✓	✓	✓
(**K**)	$A \to (B \to A)$	×	×	✓
(**W**)	$(A \to (A \to B)) \to (A \to B)$	×	×	✓

Our observation shows that the conditional of **AM3** lies somewhere between **BCI** and **BCK**, and this seems to let us conclude that the conditional of **AM3** enjoys a reasonable strength, if one is happy to grant that the conditional of **BCI** is reasonably strong.

In contrast, **C** has the full strength of the intuitionistic conditional, and therefore enjoys the deduction theorem. This is of course not the case with **CC1** nor **AM3**, and one may dismiss **CC1** and **AM3** on this ground, but we shall remain neutral with respect to this concern.

Given that the conditional of **AM3** is below **BCK**, one may wonder about the variable sharing property for **AM3**. This, unfortunately, does not hold since we have $\models (A\to A)\to(B\to B)$.

Note finally that, in general, we may observe that if an implicational formula is either valid in **CC1** or valid in **AM3**, then the corresponding formula obtained by replacing $\to$ by the biconditional is valid in classical logic (indeed, we only need to consider the submatrix obtained by eliminating the middle values). As a corollary, we obtain that none of the implicational formulas with an odd number of occurrences of variables are valid in both **CC1** and **AM3**.

5.3. The Conjunction-Conditional Fragment

We now turn to see some differences with respect to conjunction which are known to be problematic, and thus heavily criticized for **CC1**. To this end, we focus on the conjunction-conditional fragment.

		CC1	**AM3**	**C**
Simplification	$(A \wedge B) \to A$, $(A \wedge B) \to B$	×	×	✓
Idempotence	$A \to (A \wedge A)$	×	×	✓
Importation	$(A \to (B \to C)) \to ((A \wedge B) \to C))$	×	✓	✓
Exportation	$((A \wedge B) \to C)) \to (A \to (B \to C))$	×	✓	✓
Adjunction	$A \to (B \to (A \wedge B))$	×	✓	✓

Our observation shows that unlike the case with **CC1**, there is a clear sense in which $\wedge$ indeed deserves to be called conjunction in **AM3**, if the so-called *intensional* conjunction, or fusion, in relevant logic and linear logic are justified to be called conjunction. It therefore seems that **AM3** scores far better for both implication and conjunction compared to **CC1**.

In contrast, **C** again enjoys all the listed properties, but those that are not enjoyed by **AM3** can be explained in terms of the weakness of the conditional of **AM3**, and thus the matter to some extent boils down to the strength of the conditional. We shall return to this point later when we revisit some old criticisms of **CC1**.

5.4. Secondary Connexive Principles: Abelard and Aristotle's Second

Now that we have an idea of conjunction in **AM3**, we move on to see the connexive formulas that involve conjunction in their formulations.

		CC1	**AM3**	**C**
Abelard	$\neg((A \to B) \wedge (A \to \neg B))$	✓	✓	×
Aristotle's second	$\neg((A \to B) \wedge (\neg A \to B))$	✓	✓	×

Our observation shows that **AM3** inherits the validity of both Abelard's thesis as well as Aristotle's second thesis from **CC1**. Note that in **AM3**, we have the equivalence of Boethius's thesis and Abelard's thesis in view of Importation and Exportation, as well as the arrow-falsum account of negation.

Note also that although both theses fail for **C**, there is an extension of **C** by the law of excluded middle, called **C3**, that is worth noting in this context. More specifically, the conditional of **C3** will validate Abelard's thesis but not Aristotle's second thesis. Moreover, if we consider the strong implication, defined in terms of $(A \to B) \wedge (\neg B \to \neg A)$, then the strong implication of **C3** will validate both theses (for the details, see [19]).

5.5. Yet More Connexive Principles: Superconnexivity and Super-Bot-Connexivity

When he introduced the notion of strong connexivity, Kapsner suggested that there might be a way to push the requirement into the object language. The idea is to draw an analogy to the use of Explosion, $(A \wedge \neg A) \to B$, as a normative bar from satisfying contradictions. He wrote:

> In analogy to this use of explosion to express the unsatisfiability of any contradiction, we might try to ask that $(A \to \neg A) \to B$ should be valid, in order to express in the object language that $A \to \neg A$ is unsatisfiable (and similarly for the rest of the connexive theses). Call a logic that validates all of these schemata and satisfies all the requirements for strong connexivity *superconnexive*. ([11, p.143])

However adding this to a system with substitutivity of logical equivalents quickly leads to triviality (cf. [13]).

Recently, the present authors revived the idea by slightly modifying it in [13]. We called our idea *Super-Bot-Connexivity*, because it involves a bottom constant. The Super-Bot versions of Aristotle and Boethius (or rather, UnSat1 and UnSat2) then become:

Super-Bot-Aristotle: $(A \to \neg A) \to \perp$ and $(\neg A \to A) \to \perp$ are valid.

Super-Bot-Boethius: $(A \to B) \to ((A \to \neg B) \to \perp)$ and $(A \to \neg B) \to ((A \to B) \to \perp)$ are valid.

We then argued that these conditions are not as problematic as the idea of superconnexivity, and, in particular, are no threat to having substitutivity. Moreover, by making sure that $\perp$ is never satisfiable, we argued that the philosophical impact of the UnSat principles is preserved.

We also considered versions of this idea that can capture the unsatisfiability intuitions that might make us adopt Abelard and Aristotle's Second Thesis[11]:

Super-Bot-Abelard: $((A \to B) \wedge (A \to \neg B)) \to \perp$ is valid;

Super-Bot-Aristotle2: $((A \to B) \wedge (\neg A \to B)) \to \perp$ is valid.

Let us see how this notion is displayed by the three systems we are concerned with in this paper. Recall from Section 4 that we can define a bottom constant in **CC1** and **AM3** as $\neg(p \to p)$ for some $p \in \mathsf{Prop}$ such that it always takes value 4, and this is the constant we will consider in this context, as well. However, given that we cannot define $\perp$ in **C**, we will consider an expansion of **C** by the falsum constant $\perp$, with the following truth and falsity condition following the approach considered by Sergei Odintsov in [18].

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[11] In case of Ableard, this just coincides with UnSat2, in case of Aristotle's Second Thesis we suggested:

UnSat3: In no model $(A \to B)$ and $(\neg A \to B)$ are satisfiable simultaneously (for any A and B).

- $1 \notin I(w, \perp)$,
- $0 \in I(w, \perp)$.

Following the naming convention for the case with **N4**, we refer to the resulting expansion of **C** by $\perp$ as $\mathbf{C}^\perp$.

Once this extension is made, we obtain the following table as a result of the comparison.

		CC1	**AM3**	$\mathbf{C}^\perp$
(S⊥A1)	$(A \to \neg A) \to \perp$	✓	✓	×
(S⊥A2)	$(\neg A \to A) \to \perp$	✓	✓	×
(S⊥B1)	$(A \to B) \to ((A \to \neg B) \to \perp)$	✓	✓	×
(S⊥B2)	$(A \to \neg B) \to ((A \to B) \to \perp)$	✓	✓	×
Super-Bot-Abelard	$((A \to B) \wedge (A \to \neg B)) \to \perp$	✓	✓	×
Super-Bot-AT2	$((A \to B) \wedge (\neg A \to B)) \to \perp$	✓	✓	×

Though we will not argue for this here, we believe that the super-bot versions of the connexive principles are an attractive way of capturing the intuitions that might draw one towards strong connexivity (we deliver this argument and all technical details in [13]). Given that **C** isn't strongly connexive, it is no surprise that $\mathbf{C}^\perp$ fails to obey these features, nor is it surprising that **CC1** and **AM3**, both being strongly connexive, do. Rather, these examples might be seen as a small-scale proof of concept of the super-bot idea.

5.6. Negation

Finally, we add a few more comparisons of the three systems by focusing on some properties related to negation.

		CC1	**AM3**	$\mathbf{C}^\perp$
(DNI)	$A \to \neg\neg A$	✓	✓	✓
(DNE)	$\neg\neg A \to A$	✓	×	✓
(Contra.1)	$(A \to B) \to (\neg B \to \neg A)$	✓	✓	×
(Contra.2)	$A \to B \models \neg B \to \neg A$	✓	✓	×
(ECQ1)	$\neg A \to (A \to B)$	×	×	×
(ECQ2)	$\neg A \to (A \to \perp)$	×	✓	×
(ECQ3)	$\neg A, A \models B$	✓	✓	×
Consistency	In all models, either A or $\neg A$ doesn't hold.	✓	✓	×

First, observe that this table shows that **AM3** and **CC1** are independent of each other. Turning to the specific principles, note that (**DNE**) is the only property we are discussing that is shared by **CC1** and **C**, but missing for **AM3**. Given this observation, one may wonder if (**DNE**) can be added to **AM3** or not without a collapse. One possible approach, though definitely not the best nor the only approach, is to address this question in view of the three-valued semantics. Then, the validity of (**DNE**) will require the elimination of the third value **i**. If we consider this submatrix, then the truth table for the conditional becomes the truth table for the biconditional in classical logic. This is not showing that the resulting extension will be trivial, but still it is a kind of collapse (we will point to some other approaches to this question later in our conclusion).

For contraposition, **AM3** has it in both forms, like in **CC1**, but as we observed earlier, even the rule form fails for **C**.

For the explosion related results, note that all three systems share the result concerning the failure of (**ECQ1**), and thus if paraconsistency is understood in terms of the failure of **ECQ1**, then they are all paraconsistent. Moreover, for **CC1** and **AM3**, the results related to (**ECQ1**) and (**ECQ3**) imply that the deduction theorem fails in both systems. Furthermore, the difference between (**ECQ1**) and (**ECQ2**) for **AM3** reflects our approach to negation in terms of arrow-falsum, and we shall return to this point in more details later. Finally, in view of (**ECQ3**), we obtain the consistency of **CC1** and **AM3**, but **C** will be standing out for having some contradictory formulas being valid (recall Section 3.3).

6. Reflections

6.1. Revisiting Old Objections

Let us begin by revisiting the two old objections against **CC1** in view of **AM3**. These objections were raised almost immediately after the publication of [15] by John Woods in [36] and by Richard Routley and Hugh Montgomery in [27].

In his short note [36], Woods raised two worries. First, given that we have

- $\not\models_{\mathbf{CC1}} p \to (p \wedge p)$,
- $\not\models_{\mathbf{CC1}} (p \wedge p) \to p$, but
- $\models_{\mathbf{CC1}} p \to (p \wedge (p \wedge p))$,

he wrote that

> The upshot would appear to be that p connexively implies only odd-numbered conjunctions of occurrences of itself, and never even-numbered ones. Simplification is, therefore, not unrestrictedly valid.

Second, as a somewhat more general objection that can be seen as directed to connexive logic in general, Woods observed that Aristotle's thesis, transitivity of the conditional (in the rule form), conjunction elimination, the rule of contraposition, and substitution will prove negation inconsistency[12], and asks

> With respect to [these assumptions], is there anything which you would be willing to give up?

For **AM3**, we can now respond to these worries by repeating that the conjunction of **AM3** is an intensional one (cf. Section 5.3), and thus the failure of conjunction elimination is to be expected (this is not necessarily to say that the failure is not a problem, but we refer to the literature on intensional conjunction for further discussion). Moreover, and more specifically to the second worry, we may point exactly to the conjunction elimination from the perspective of **AM3** that needs to be given up. All other principles that Woods listed are kept in **AM3**.[13]

For the worries raised by Routley and Montgomery in their [27], they extensively consider some weaker systems below **PA1**, in terms of axiomatic proof systems, and establish a number of interesting results that will be of direct importance for **PA1** as well as **CC1**. However, given that our interest is to see how **AM3** removes the odd features of **CC1**, we focus on a result reported in [27] that applies specifically to **CC1**.

The result we would like to mention is their Theorem 21 of [27] which shows that if A is a classically valid formula in the fragment of classical logic only with conjunction and the conditional, then $\tau(A)$ is valid in **CC1**, where τ is defined as follows:

- $\tau(p) = p \wedge p$ for $p \in \mathsf{Prop}$ and
- $\tau(A * B) = \tau(A) * \tau(B)$ where $* \in \{\wedge, \rightarrow\}$.

Since this result is established by considering the submatrix of **CC1** in the conjunction-conditional fragment by restricting the carrier set to $\{1, 3\}$, their observation will carry over to **AM3**. In view of this result, we obtain the following in **AM3**:

- $\not\models p \rightarrow (q \rightarrow p)$, but $\models (p \wedge p) \rightarrow ((q \wedge q) \rightarrow (p \wedge p))$,
- $\not\models (p \wedge q) \rightarrow p$, but $\models ((p \wedge p) \wedge (q \wedge q)) \rightarrow (p \wedge p)$.

Now, given that we can view the conjunction in **AM3** as an intensional conjunction, the failure of Weakening as well as conjunction elimination can

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[12] Note that one may wonder if modus ponens is necessary or not, as addressed by one of the referees, but if transitivity of the conditional is understood in the rule form, then the derivation observed by Woods does *not* require modus ponens.

[13] Alternatively, one can dispute that negation inconsistency is bad in the first place.

be explained on that ground. However, it is far from obvious how we can explain the validity of the versions with conjunction everywhere. At least, we are not able to offer an explanation of this phenomenon. It rather seems to be a byproduct of working with a three-valued semantics, that is a special case of a more general semantics. Therefore, this will give us some reasons to generalize **AM3**, and we shall discuss some possible directions in the conclusion.

A last point that has been raised against **CC1** by Routley and Montgomery is that the values in the characteristic matrices have no intuitively plausible interpretation. While **AM3** is no worse off than **CC1** here, we can not claim that the problem went away by removing one of the four values. This is still a strong point for **C**, which has a much more compelling story that backs its semantics up.

6.2. Intuitionistic Handling of Negation and Connexivity

Though we explained in Section 4 that the truth tables for **AM3** were arrived at by thinking of an quasi-intuitionistic understanding of negation, in fact we have so far treated it as primitive, and defined $\perp$ in terms of it. Let us now push this line of constructive thinking a bit further: What if we adopt that view of negation wholesale and take $\perp$ as primitive and consider $\neg A$ as *defined* as $A \rightarrow \perp$?

It is interesting to note that connexivity has been investigated in relation to many kinds of negation[14], but no discussion we are aware of relates the connexive principles to the intuitionistic understanding of negation. This is surprising, especially given Wansing's suggestion that connexivity and constructivity are closely related (see the quote in the introductory section of this paper). Maybe this lacuna exists because the natural habitat of that account of negation, viz., intuitionistic logic, is more than hostile to the connexive principles. As the proof in proposition 1 below shows, intuitionistic logic becomes trivial once **AT** is added.

One move is to consider giving up **EFQ**, namely $\perp \rightarrow A$, to avoid such calamity. And of course, that is an idea that has been considered in intuitionistic circles, as well (and for a long time, at that). The move famously leads to minimal logic (cf. [9]).

Minimal logic achieves to avoid **EFQ** by making $\perp$ a proposition that does not necessarily take an undesignated value. We argue in [13] that this would take the normative sting out of the superconnexive idea. What is worse,

[14]E.g. cancellation accounts of negation ([23,28,34]), strong constructive negation ([33]), a 'reversal' theory of negation ([30]) etc.

however, is the following: While the connexive principles can be added to minimal logic without triviality, we obtain "half-triviality", namely that $\neg A$ can be derived for any A:

Proposition 1. *Aristotle's thesis, Weakening (i.e. $A \to (B \to A)$) and modus ponens proves $\bot$, and thus $\neg A$ for any A where negation is defined as $A \to \bot$.*

Proof. The proof runs as follows.

1	$\neg(A \to \neg A) \to ((A \to \neg A) \to \neg(A \to \neg A))$	[Weakening]
2	$((A \to \neg A) \to \neg(A \to \neg A))$	[Aristotle's thesis, 1, MP]
3	$\neg((A \to \neg A) \to \neg(A \to \neg A))$	[Aristotle's thesis]
4	$\bot$	[2, 3, def. of $\neg$, MP]

Moreover, we obtain $\neg A$ for any A by final applications of Weakening and MP. ■

Remark 9. If one tries to make sense of Aristotle's thesis in terms of arrow-bottom, and keep modus ponens, then we need to tame Weakening. One of the many ways will be to go subintuitionistic in the sense that we strip off some of the frame conditions for the intuitionistic conditional. For this purpose, there are roughly two approaches, depending on whether we have a base point in the Kripke semantics. More specifically, Giovanna Corsi in [4] formulates the semantics without the base point, whereas Greg Restall in [24] formulates the semantics with the base point. Then, interestingly, there is a crucial difference in these two approaches. Indeed, in Corsi's approach, the rule from $\vdash A$ to $\vdash B \to A$ still preserves validities, even in the weakest system, but this is not the case with Restall's approach (in which the validity of Weakening corresponds to the heredity condition). Noting that the above proof will go through even with the rule form, Restall's approach is the only reasonable way, if we aim at a conditional as strict implication. Deeper investigations into this matter, however, need to be left for interested readers.

Compared to these two unsatisfactory cases, **AM3** is free of triviality/half-triviality problems, while it handles negation in arrow-bottom style. Note here that we are now taking the bottom as primitive, so we are assuming the following truth tables for **AM3**.

	$\bot$
1	**0**
i	**0**
0	**0**

$\wedge$	**1**	**i**	**0**
1	**1**	**i**	**0**
i	**i**	**i**	**0**
0	**0**	**0**	**i**

$\to$	**1**	**i**	**0**
1	**1**	**i**	**0**
i	**1**	**1**	**0**
0	**0**	**0**	**1**

A further interesting fact is that the double negation laws behave just like they do for intuitionistic logic, i.e., **DNI** holds while **DNE** does not.[15] This does not seem to track any intuitive constructive reading of the semantics, as arguably, the Kripke semantics for intuitionistic logic allow. But as we will point out in the conclusion, **AM3** is only the strongest of a large family of systems, and maybe for some of the weaker members such an interpretation might be possible.

More generally, **AM3** is not without problems, as we mentioned above in light of the old observations due to Routley and Montgomery. Still, we believe that as the first step towards a full understanding of the intuitionistic handling of negation in the context of connexivity, **AM3** seems to play an important role, as opening up the view of a fuller picture.

6.3. Looking Ahead

Before wrapping up the paper, let us make a concrete proposal for the purpose of further investigations. We submit a subsystem of **AM3**, called **AMW** after Angell, McCall and Wansing, as an interesting system of connexive logic. To this end, we make a small change to the language by taking $\perp$ as primitive, and take negation to be defined in terms of $\rightarrow$ and $\perp$, as motivated in the last section.

Definition 10. Let **AMW** be the system with the following axioms and a rule of inference.

$$(B \rightarrow C) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C)) \tag{\textbf{B}}$$

$$(A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C)) \tag{\textbf{C}}$$

$$A \rightarrow A \tag{\textbf{I}}$$

$$(A \rightarrow (B \rightarrow C)) \rightarrow ((A \wedge B) \rightarrow C) \tag{Importation}$$

$$((A \wedge B) \rightarrow C) \rightarrow (A \rightarrow (B \rightarrow C)) \tag{Exportation}$$

$$((A \wedge B) \rightarrow \perp) \rightarrow ((A \rightarrow B) \rightarrow \perp) \tag{$\wedge{\rightarrow}\perp1$}$$

$$((A \rightarrow B) \rightarrow \perp) \rightarrow ((A \wedge B) \rightarrow \perp) \tag{$\wedge{\rightarrow}\perp2$}$$

[15] Not just for the arrow forms we mention above, but also for the entailment versions (not a completely trivial fact because we don't have the deduction theorem).

$$\frac{A \qquad A \to B}{B} \tag{MP}$$

Remark 11. Here are some reasons why we chose these axioms. First, for the implicational fragment, we observed that **BCI** is included by **AM3**, which is a reasonably strong arrow, unlike in the case of **CC1**. For the purpose of making sure that the arrow is an arrow, we keep the strength of **BCI**. Second, for conjunction, we observed that we can make sense of conjunction in **AM3** as an intensional conjunction, standing in the residuation relation to $\to$. Therefore, we kept Importation/Exportation axioms. Finally, for negation, we kept two axioms that require conjunction and implication to be equivalent under the scope of negation, which is defined in terms of $\to$ and $\perp$. These two axioms will also allow us to derive (CBT), and thus we added 'W' in the name of the system for Wansing.

Remark 12. Note that **AMW** is not trivial thanks to the fact that **AM3** is not trivial.

In other words, **AMW** is an expansion of a fragment of linear logic expanded by two axioms that carry the connexive flavor. Let us then observe some derivable formulas in **AMW**, before listing a number of questions for further research.

Proposition 2. *The following formulas are derivable in* **AMW**, *where* $\neg A$ *abbreviates* $A \to \perp$.

$$(A \to B) \to (\neg B \to \neg A) \tag{Contra.}$$

$$A \to \neg\neg A \tag{DNI}$$

$$\neg(\neg A \to A) \tag{AT1}$$

$$\neg(A \to \neg A) \tag{AT2}$$

$$(A \to \neg B) \to \neg(A \to B) \tag{BT1}$$

$$(A \to B) \to \neg(A \to \neg B) \tag{BT2}$$

$$\neg(A \to B) \to (A \to \neg B) \tag{CBT}$$

$$\neg((A \to B) \land (A \to \neg B)) \tag{Abelard}$$

Proof. For (Contra.), it suffices to confirm that $(A \to B) \to ((B \to C) \to (A \to C))$ is derivable, and this is indeed the case in view of (**B**), (**C**) and (MP). For (DNI), it suffices to confirm that $A \to ((A \to B) \to B)$ is derivable, and this is indeed the case in view of (**C**), (**I**) and (MP). For (AT1), the proof runs as follows.

1 $((\neg A \wedge A) \to \bot) \to ((\neg A \to A) \to \bot)$ [$(\wedge{\to}\bot 1)$]
2 $(\neg A \to (A \to \bot)) \to ((\neg A \wedge A) \to \bot)$ [(Importation)]
3 $\neg A \to (A \to \bot)$ [(I) and def. of $\neg$]
4 $(\neg A \to A) \to \bot$ [1, 2, 3, (MP)]
5 $\neg(\neg A \to A)$ [4, def. of $\neg$]

For (AT2), the proof runs as follows.

1 $((A \wedge \neg A) \to \bot) \to ((A \to \neg A) \to \bot)$ [$(\wedge{\to}\bot 1)$]
2 $(A \to (\neg A \to \bot)) \to ((A \wedge \neg A) \to \bot)$ [(Importation)]
3 $A \to (\neg A \to \bot)$ [(DNI) and def. of $\neg$]
4 $(A \to \neg A) \to \bot$ [1, 2, 3, (MP)]
5 $\neg(A \to \neg A)$ [4, def. of $\neg$]

For (BT1), the proof runs as follows.

1 $((A \wedge B) \to \bot) \to ((A \to B) \to \bot)$ [$(\wedge{\to}\bot 1)$]
2 $(A \to (B \to \bot)) \to ((A \wedge B) \to \bot)$ [(Importation)]
3 $(A \to \neg B) \to \neg(A \to B)$ [1, 2, (Trans), def. of $\neg$]

For (BT2), the proof runs as follows.

1 $\neg\neg(A \to B) \to \neg(A \to \neg B)$ [(BT1), (Contra.), (MP)]
2 $(A \to B) \to \neg\neg(A \to B)$ [(DNI)]
3 $(A \to B) \to \neg(A \to \neg B)$ [1, 2, (Trans)]

For (CBT), the proof runs as follows.

1 $((A \to B) \to \bot) \to ((A \wedge B) \to \bot)$ [$(\wedge{\to}\bot 2)$]
2 $((A \wedge B) \to \bot) \to (A \to (B \to \bot))$ [(Exportation)]
3 $\neg(A \to B) \to (A \to \neg B)$ [1, 2, (Trans), def. of $\neg$]

Finally, for (Abelard), the proof runs as follows.

1 $((A \to B) \to \neg(A \to \neg B)) \to (((A \to B) \wedge (A \to \neg B)) \to \bot)$ [Importation]
2 $\neg((A \to B) \wedge (A \to \neg B))$ [1, (BT2), (MP), def. of $\neg$]

This completes the proof. ■

Remark 13. Let us make some remarks about the intuitionistic understanding of negation and the notion of Super-Bot-Connexivity we talked about above. Take another look at Aristotle and Super-Bot-Aristotle:

Aristotle: $\neg(A \to \neg A)$

Super-Bot-Aristotle: $(A \to \neg A) \to \bot$

Here it becomes clear that Super-Bot-Aristotle just *is* Aristotle when the outer negation is understood along intuitionistic lines. Similar observations can be made about the other super-bot-principles and their more traditional counterparts.

As there is more than one negation involved in these principles, the intuitionistic understanding of negation invites us to consider other ways of unpacking these negations, which might in a context where negation is not defined be independent and interesting objects of investigation in themselves:

- $\neg(A \to \neg A)$
- $(A \to \neg A) \to \bot$
- $\neg(A \to (A \to \bot))$
- $(A \to (A \to \bot)) \to \bot$

Note, in particular, that the last formula can be seen as an instance of Super contraction in [25, p.423], namely $(A \to (A \to B)) \to B$.

Then, we can formulate a number of open problems for **AMW**. Here is a small list.

Open problem 1. Given that **AMW** is an expansion of **BCI**, we may ask how we can devise the algebraic semantics for **AMW** along Robert Meyer and Hiroakira Ono in [17]?

Open problem 2. We may also ask if we can devise other kinds of semantics by borrowing insights from linear logic, as well as combinatory logic.

Open problem 3. Given that there is a cut-free sequent calculus presentation of **BCI**, can we also devise one for **AMW**?

Open problem 4. A systematic investigation into extensions and expansions of **AMW** will be particularly interesting, giving us a clear understanding of **AM3** as well. For example, the bottom constant in **AMW** is like that in minimal logic in the sense that we do not have **EFQ**. We already know that we cannot add $\bot \to A$, but we can add $\bot \vdash A$, and thanks to **AM3**, such extension is not trivial. What then is the effect of this additional rule to the semantics and proof systems, and are there other interesting ways to constrain $\bot$? Moreover, how many systems are there between **AMW** with **EFQ** and **AM3**?

Open problem 5. A systematic investigation into expansions will be also interesting. For example, how should we add disjunction? Moreover, are there technical/philosophical reasons to add extensional conjunction?

We can even continue more to list some questions, but we will stop here. Instead, we would like to invite the readers to add more and answer them.

7. Concluding Remarks

We started out by considering two iconic connexive systems, Angell and McCall's **CC1** and Wansing's **C**. We pointed out that these systems have little in common beyond satisfying Aristotle's Thesis and Boethius' Thesis. For example, we noted that **CC1** is strongly connexive, but not hyperconnexive, whereas it is the other way around for **C**. We found a system that broadly succeeded in bridging the distance between these logics and called it **AM3**. Due to some interpretational problems we discussed, we ended by conjecturing that a weaker system we called **AMW** or some system in between the two might ultimately turn out to be a more interesting logic than **AM3** itself.[16]

Nonetheless, **AM3** has several pleasant properties and serves to make several interesting observations worthy of further investigation, among them:

- The problem with strong connexivity has been to find any attractive logics beyond **CC1** that satisfy the UnSat principles. **AM3** points the way to a realm of logics that might instantiate strong connexivity in a more intuitively satisfying way.
- Hyperconnexivity has, to our knowledge, been only observed in logics that are paraconsistent (such as **C**). **AM3** is not paraconsistent, and thus shows that there is no necessary connection between these features.
- **AM3** has invited us to consider how connexivity and the intuitionistic handling of negation go together. We pointed out that this fills a curious lacuna in the connexive literature.
- **AM3** serves as a showcase of our recently developed idea of super-bot-connexivity. Indeed, we found that considering the negation defined as arrow-bottom allows us to see the super-bot versions as mere notational variants of the connexive theses. The connection will not be quite as tight in other environment, but we take this as a small scale proof of concept of our idea.

[16]Due to the problems of **AM3** we discussed earlier in Section 6.1, we did not consider the problem of axiomatizing **AM3**. We will therefore leave the problem of axiomaitzation to interested readers.

Again, **AM3** is surely not the final word on the understanding of connexivity, but rather a hopefully fruitful starting point. Indeed, given the problems pointed out by Routley and Montgomery for the system **CC1**, we need to seek for some interesting subsystems that will be free of those problems. As a starting point for that project, we submitted the system **AMW** as the basic system to continue the investigation. Given the close connection to linear logics, we may expect some new insights into connexivity from the techniques developed in linear logics. It remains to be seen how rich this new field is, and we hope some readers will be motivated to join the authors to continue with the search for interesting connexive logics along these lines.

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H. Omori
Graduate School of Information Sciences
Tohoku University
Sendai
Japan
hitoshiomori@gmail.com

A. Kapsner
Munich Center for Mathematical Philosophy, LMU
Munich
Germany
Andreas.Kapsner@lrz.uni-muenchen.de