

An efficient parallel algorithm for the longest path problem in meshes

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Abstract

In this paper, first we give a sequential linear-time algorithm for the longest path problem in meshes. This algorithm can be considered as an improvement of [13]. Then based on this sequential algorithm, we present a constant-time parallel algorithm for the problem which can be run on every parallel machine.

Keywords: grid graph, longest path, meshes, sequential and parallel algorithms.

MSC: 05C45; 05C85; 05C38.

1. Introduction

The longest path problem, i.e. the problem of finding a simple path with the maximum number of vertices, is one of the most important problems in graph theory. The well-known NP-complete Hamiltonian path problem, i.e. deciding whether there is a simple path that visits each vertex of the graph exactly once, is a special case of the longest path problem and has many applications [5, 7].

Only few polynomial-time algorithms are known for the longest path problem for special classes of graphs. This problem for trees began with the work of Dijkstra around 1960, and was followed by other people [2, 9,

16, 18, 22]. In the area of approximation algorithms it has been shown that the problem is not in APX, i.e. there is no polynomial-time approximation algorithm with constant factor for the problem unless $P=NP$ [9]. Also, it has been shown that finding a path of length $n - n^\epsilon$ is not possible in polynomial-time unless $P=NP$ [12]. For the background and some known result about approximation algorithms, we refer the reader to [1, 6, 24].

A grid graph is a graph in which vertices lie only on integer coordinates and edges connect vertices that are separated by a distance of once. A solid grid graph is a grid graph without holes. The rectangular grid graph $R(n, m)$ is the subgraph of G^∞ (infinite grid graph) induced by $V(m, n) = \{v \mid 1 \leq v_x \leq m, 1 \leq v_y \leq n\}$, where v_x and v_y are respectively x and y coordinates of v (see Figure 1). A mesh $M(m, n)$ is a rectangular grid graph $R(m, n)$. Grid graphs can be useful representation in many applications. Myers [19] suggests modeling city blocks in which street intersection are vertices and streets are edges. Luccio and Mugnia [17] suggest using a grid graph to represent a two-dimensional array type memory accessed by a read/write head moving up, down or across. The vertices correspond to the center of each cell and edges connect adjacent cells. Finding a path in the grid corresponds to accessing all the data.

Itai *et al.* [11] have shown that the Hamiltonian path problem for general grid graphs, with or without specified endpoints, is NP-complete. The problem for rectangular grid graphs, however, is in P requiring only linear-time. Later, Chen *et al.* [3] improved the algorithm of [11] and presented a parallel algorithm for the problem in mesh architecture. There is a polynomial-time algorithm for finding Hamiltonian cycle in solid grid graphs [15]. Also, the authors in [23] presented sufficient conditions for a grid graph to be Hamiltonian and proved that all finite grid graphs of positive width have Hamiltonian line graphs.

Recently the Hamiltonian cycle (path) and longest path problem of a grid graph has received much attention. Salman *et al.* [21] introduced a family of grid graphs, i.e. alphabet grid graphs, and determined classes of alphabet grid graphs that contain Hamiltonian cycles. Islam *et al.* [10] showed that the Hamiltonian cycle problem in hexagonal grid graphs is NP-complete. Also, Gordon *et al.* [8] proved that all connected, locally connected triangular grid graphs are Hamiltonian, and gave a sufficient condition for a connected graph to be fully cycle extendable and also showed that the Hamiltonian cycle problem for triangular grid graphs is NP-complete.

Moreover, Zhang and Liu [25] gave an approximation algorithm for the

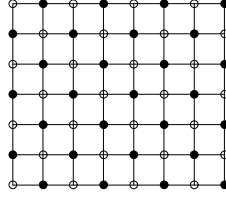


Figure 1: The rectangular grid graph $R(8, 7)$.

longest path problem in grid graphs and their algorithm runs in quadratic time. Also the authors in [13] has been studied the longest path problem for rectangular grid graphs and their algorithm is based on divide and conquer technique and runs in linear time. Some results of the grid graphs are investigated in [14, 20].

In this paper, we present a sequential and a parallel algorithms for finding longest paths between two given vertices in rectangular grid graphs (meshes). Our algorithm has improved the previous algorithm [13] by reducing the number of partition steps from $O(m + n)$ to only a constant.

The organization of the paper as follow: In Section 2, we review some necessary definitions and results that we will need. A sequential algorithm for the longest path problem is given in Section 3. In Section 4, a parallel algorithm for the problem is introduced which is based on the mentioned sequential algorithm. Conclusions is given in Section 5.

2. Preliminary results

In this section, we give a few definitions and introduce the corresponding notations. We then gather some previously established results on the Hamiltonian and the longest path problems in grid graphs which have been presented in [3, 11, 13].

The *two-dimensional integer grid* G^∞ is an infinite graph with vertex set of all the points of the Euclidean plane with integer coordinates. In this graph, there is an edge between any two vertices of unit distance. For a vertex v of this graph, let v_x and v_y denote x and y coordinates of its corresponding point (sometimes we use (v_x, v_y) instead of v). We color the vertices of the two-dimensional integer grid as black and white. A vertex v is colored *white* if $v_x + v_y$ is even, and it is colored *black* otherwise. A *grid graph* G_g is a finite vertex-induced subgraph of the two-dimensional integer grid. In a grid graph G_g , each vertex has degree at most four. Clearly, there is no

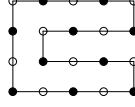


Figure 2: A Hamiltonian cycle for the rectangular grid graph $R(5,4)$.

edge between any two vertices of the same color. Therefore, G_g is a bipartite graph. Note that any cycle or path in a bipartite graph alternates between black and white vertices. A *rectangular grid graph* $R(m,n)$ (or R for short) is a grid graph whose vertex set is $V(R) = \{v \mid 1 \leq v_x \leq m, 1 \leq v_y \leq n\}$. In the figures we assume that $(1,1)$ is the coordinates of the vertex in the upper left corner. The size of $R(m,n)$ is defined to be mn . $R(m,n)$ is called *odd-sized* if mn is odd, and it is called *even-sized* otherwise. In this paper without loss of generality, we assume $m \geq n$ and all rectangular grid graphs considered here are odd \times odd, even \times odd and even \times even. $R(m,n)$ is called a *n-rectangle*.

The following lemma states a result about the Hamiltonicity of even-sized rectangular graphs.

Lemma 2.1. [3] *$R(m,n)$ has a Hamiltonian cycle if and only if it is even-sized and $m, n > 1$.*

Figure 2 shows a Hamiltonian cycle for an even-sized rectangular grid graph, found by Lemma 2.1. Every Hamiltonian cycle found by this lemma contains all the boundary edges on the three sides of the rectangular grid graph. This shows that for an even-sized rectangular graph R , we can always find a Hamiltonian cycle, such that it contains all the boundary edges, except of exactly one side of R which contains an even number of vertices.

Two different vertices v and v' in $R(m,n)$ are called *color-compatible* if either both v and v' are white and $R(m,n)$ is odd-sized, or v and v' have different colors and $R(m,n)$ is even-sized. Let $(R(m,n), s, t)$ denote the rectangular grid graph $R(m,n)$ with two specified distinct vertices s and t . Without loss of generality, we assume $s_x \leq t_x$.

$(R(m,n), s, t)$ is called *Hamiltonian* if there exists a Hamiltonian path between s and t in $R(m,n)$. An even-sized rectangular grid graph contains the same number of black and white vertices. Hence, the two end-vertices of any Hamiltonian path in the graph must have different colors. Similarly, in an odd-sized rectangular grid graph the number of white vertices is one more than the number of black vertices. Therefore, the two end-vertices of

any Hamiltonian path in such a graph must be white. Hence, the color-compatibility of s and t is a necessary condition for $(R(m, n), s, t)$ to be Hamiltonian. Furthermore, Itai *et al.* [11] showed that if one of the following conditions hold, then $(R(m, n), s, t)$ is not Hamiltonian:

- (F1) $R(m, n)$ is a 1-rectangle and either s or t is not a corner vertex (Figure 3(a))
- (F2) $R(m, n)$ is a 2-rectangle and (s, t) is a nonboundary edge, i.e. (s, t) is an edge and it is not on the outer face (Figure 3(b)).
- (F3) $R(m, n)$ is isomorphic to a 3-rectangle grid graph $R'(m, n)$ such that s and t is mapped to s' and t' and all of the following three conditions hold:
 1. m is even,
 2. s' is black, t' is white,
 3. $s'_y = 2$ and $s'_x < t'_x$ (Figure 3(c)) or $s'_y \neq 2$ and $s'_x < t'_x - 1$ (Figure 3(d)).

Also by [11] for a rectangular graph $R(m, n)$ with two distinct vertices s and t , $(R(m, n), s, t)$ is Hamiltonian if and only if s and t are color-compatible and $R(m, n)$, s and t do not satisfy any of conditions (F1), (F2) and (F3). In the following we use $P(R(m, n), s, t)$ to indicate the problem of finding a longest path between vertices s and t in a rectangular grid graph $R(m, n)$, $L(R(m, n), s, t)$ to show the length of longest paths between s and t and $U(R(m, n), s, t)$ to indicate the upper bound on the length of longest paths between s and t .

The authors in [13] showed that the longest path problem between any two given vertices s and t in rectangular grid graphs satisfies one of the following conditions:

- (C0) s and t are color-compatible and none of (F1)- (F3) hold.
- (C1) Neither (F1) nor (F2*) holds and either
 1. $R(m, n)$ is even-sized and s and t are same-colored or
 2. $R(m, n)$ is odd-sized and s and t are different-colored.
- (C2) 1. $R(m, n)$ is odd-sized and s and t are black-colored and neither (F1) nor (F2*) holds, or

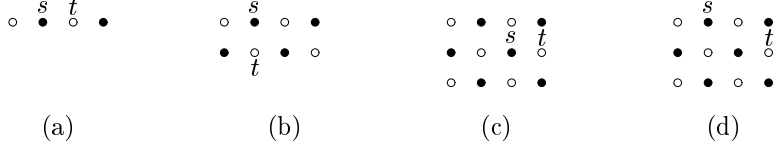


Figure 3: Rectangular grid graph in which there is no Hamiltonian path between s and t .

2. s and t are color-compatible and (F3) holds.

Where (F2*) is defined as follows:

(F2*) $R(m, n)$ is a 2-rectangle and $s_x = t_x$ or ($s_x = t_x - 1$ and $s_y \neq t_y$).

They also proved some upper bounds on the length of longest paths as following:

$$U(R(m, n), s, t) = \begin{cases} t_x - s_x + 1, & \text{if (F1),} \\ \max(t_x + s_x, 2m - t_x - s_x + 2), & \text{if (F2*),} \\ mn, & \text{if (C0),} \\ mn - 1, & \text{if (C1),} \\ mn - 2, & \text{if (C2).} \end{cases}$$

Theorem 2.1. [13] Let $U(R(m, n), s, t)$ be the upper bound on the length of longest paths between s and t in $R(m, n)$ and let $L(R(m, n), s, t)$ be the length of longest paths between s and t . In a rectangular grid graph $R(m, n)$, a longest path between any two vertices s and t can be found in linear time and its length (i.e., $L(R(m, n), s, t)$) is equal to $U(R(m, n), s, t)$.

3. The sequential algorithm

In this section, we present a sequential algorithm for finding a longest path between two vertices in rectangular grid graphs. This algorithm is the base of our parallel algorithm which is introduced in Section 4. First, we solve the problem for 1-rectangles and 2-rectangles.

Lemma 3.1. [13] Let $P(R(m, n), s, t)$ be a longest path problem with $n = 1$ or $n = 2$, then $L(R(m, n), s, t) = U(R(m, n), s, t)$.

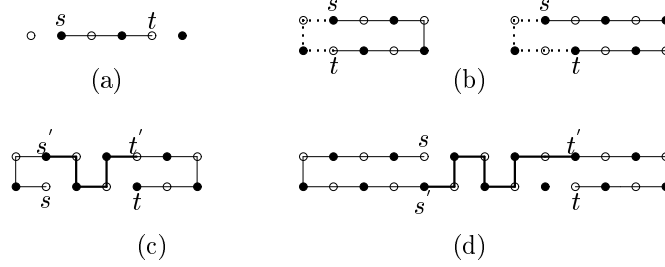


Figure 4: (a) Longest path between s and t in a 1-rectangle, (b) Longest path between s and t in a 2-rectangle, (c) and (d) A path with length $2m$ and $2m - 1$ for a 2-rectangle, respectively.

Proof. For a 1-rectangle obviously the lemma holds for the single possible path between s and t (see Figure 4(a)). For a 2-rectangle, if removing s and t splits the graph into two components, then the path going through all vertices of the larger component has the length equal to $U(R(m, n), s, t)$ (see Figure 5(b)). Otherwise, let s' be the vertex adjacent to s and t' be the vertex adjacent to t such that $s'_y \neq s_y$ and $t'_y \neq t_y$. Then we make a path from s to s' and a path from t to t' as shown in Figure 4(c), (d), and connect s' to t' by a path such that at most one vertex remains out of the path as depicted in this figure. \square

From now on, we assume that $m \geq n > 2$, so one of conditions (C0), (C1) and (C2) should hold. Following the technique used in [3] we develop an algorithm for finding longest paths.

Definition 3.1. [13] A *separation* of a rectangular grid graph R is a partition of R into two disjoint rectangular grid graphs R_1 and R_2 , i.e. $V(R) = V(R_1) \cup V(R_2)$, and $V(R_1) \cap V(R_2) = \emptyset$.

Definition 3.2. [11] Let v and v' be two distinct vertices in R . If $v_x \leq 2$ and $v'_x \geq m - 1$, then v and v' are called *antipodes*.

Definition 3.3. [3] Partitioning a rectangular grid graph R into five disjoint rectangular grid subgraphs $R_1 - R_5$ that is done by two horizontal and two vertical separations are called *peeling operation*, if the following two conditions hold:

1. $s, t \in R_5$ and s and t are antipodes.

- Each of four rectangular grid subgraphs $R_1 - R_4$ is an even-sized rectangular grid graph whose boundary sizes are both greater than one, or is empty.

Generally the two vertical separation of a peeling are done before the two horizontal separation. However, for an odd \times odd or odd \times even rectangular grid graph with $s_x = t_x$, this order is reversed in order to guarantee that the boundary sizes of R_3 and R_4 are greater than one. Figure 5 shows a peeling on $R(15, 11)$ where s is $(6, 5)$ and t is $(8, 9)$.

The following lemma can be obtained directly from Definition 3.3.

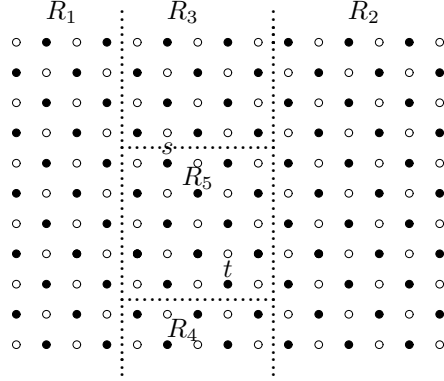


Figure 5: A peeling on $R(15, 11)$.

Lemma 3.2. [3] Let $R_5(n_5, m_5)$ be the resulting rectangular grid subgraph of a peeling on $R(n, m)$, where $s, t \in V(R_5)$. Then

- s, t remain the same color in R_5 as in R ; and
- R_5 has the same parity as R , that is, $m_5 \bmod 2 = m \bmod 2$, and $n_5 \bmod 2 = n \bmod 2$.

Definition 3.4. A peeling operation on R is called *proper* if $|R_1| + |R_2| + |R_3| + |R_4| + U(R_5, s, t) = U(R(n, m), s, t)$, where $|R_i|$ denotes the number of vertices of R_i .

Lemma 3.3. For the longest path problem $P(R(n, m), s, t)$, any peeling on $R(m, n)$ is proper if:

- The condition (C0) holds and $m \bmod 2 = n \bmod 2$ (i.e. $R(m, n)$ is even \times even or odd \times odd), or

2. One of the conditions (C1) and (C2) hold and $R(m, n)$ is $even \times odd$ or $odd \times odd$.

Proof. The lemma has been proved for the case that (C0) holds (see [3]). So, we consider conditions (C1) and (C2). From Lemma 3.2, we know that s and t are still color-compatible, and we are going to prove that $P(R_5(m_5, n_5), s, t)$ is not in cases $F1$ and $F2^*$.

By Lemma 3.2, when $R(m, n)$ is an $odd \times odd$ rectangular grid graph, $R_5(m_5, n_5)$ is also an $odd \times odd$ rectangular grid graph, s and t have the same color as in R , and hence $R_5(m_5, n_5)$ is not a 2-rectangle. If R_5 is a 1-rectangle, then $s_y = t_y$ or $s_x = t_x$ and then we have the two following cases:

Case1. (C1) holds and both s and t are different color. In this case, one of s_x and t_x (s_y and t_y) is even and the other is odd. Considering that s and t are antipodes and R_5 is $odd \times odd$, one of s and t must be at the corner and exactly one of the vertex goes out of the path.

Case2. (C2) holds and both s and t are black color. In this case, all s_x , s_y , t_x and t_y are even. Hence, vertices s and t are before corner vertices and exactly two vertices go out of the path.

In the similar way, when $R(m, n)$ is an $even \times odd$ rectangular grid graph ((C1) holds), $R_5(m_5, n_5)$ is also an $even \times odd$ rectangular grid graph, and hence $R_5(m_5, n_5)$ is not a 2-rectangle. If R_5 is a 1-rectangle, then $s_y = t_y$. In this case, s_x and t_x are both odd or even. Hence, s or t are at the corner and exactly one vertex goes out of the path.

Therefore by Theorem 2.1, $U(R_5(m_5, n_5), s, t) = U(R(m, n), s, t)$ and any peeling of $R(m, n)$ is always proper. \square

Nevertheless, a peeling operation in an $even \times even$ rectangular grid graph $R(m, n)$ may not be proper, and $U(R_5(m_5, n_5), s, t) \neq U(R(m, n), s, t)$, see Figure 6 where the dotted-lines represent a peeling operation. In the two following cases a peeling operation is not proper:

- (F1') s is black, s_x is even (or odd), $t_y = s_y + 1$ and $s_x \neq t_x$
- (F2') s is white, s_x is even (or odd), $t_y = s_y - 1$ and $s_x \neq t_x$;

Lemma 3.4. *For the longest path problem $P(R(n, m), s, t)$, where $R(m, n)$ is an $even \times even$ rectangular grid graph, a peeling operation on $R(m, n)$ is proper if and only if $P(R(n, m), s, t)$ is not cases in (F1') and (F2').*

When a peeling operation is not proper it can be made proper by adjustment the peeling boundaries. In that case, if R_1 , R_2 , R_3 and R_4 are empty, then

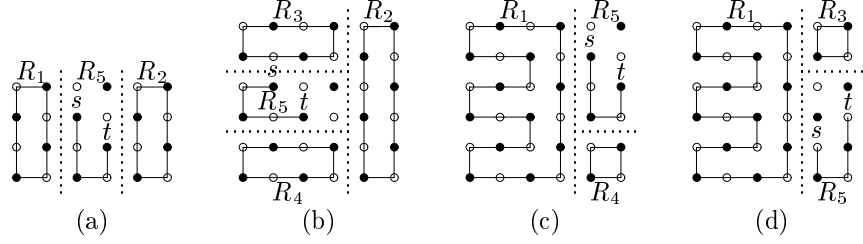


Figure 6: Rectangular grid graph in which a peeling operation is not proper.

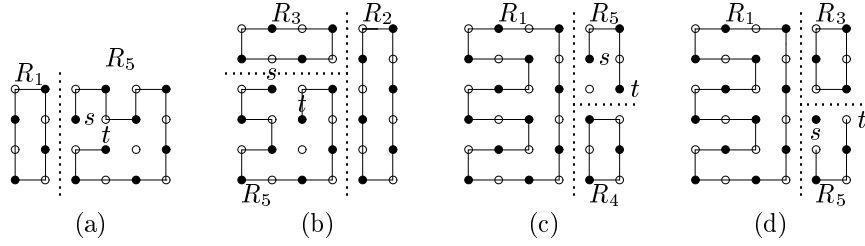


Figure 7:

R_5 is 2-rectangle that is in case (F2*). Therefore, without loss of generality, we assume R_1 , R_2 , R_3 or R_4 is not empty. If rectangular grid subgraphs R_3 and R_4 are empty, then we move one column (or two columns when R_1 or R_2 is a 2-rectangle) from R_1 or R_2 to R_5 such that R_1 or R_2 is still even-sized rectangular grid graphs; see Figure 7(a). If R_1 , R_2 , R_3 and R_4 (or R_3 and R_4) is not empty, then we move one row (or two rows when R_3 or R_4 is 2-rectangle) from R_3 or R_4 to R_5 (Figure 7(b)), or move the bottom row to R_4 (Figure 7(c)) or move the upper row to R_3 (Figure 7(d)), such that R_3 or R_4 is still even-sized rectangular grid graphs.

After a peeling operation on $R(m, n)$, we construct longest paths in $R_5(m_5, n_5)$. Consider the following cases for $R_5(m_5, n_5)$:

- (a) $m_5, n_5 \leq 3$.
- (b) m_5, n_5 are even, and either $m_5 \geq 4$ or $n_5 \geq 4$;
- (c) m_5, n_5 are odd, and either $m_5 \geq 5$ or $n_5 \geq 5$;
- (d) m_5 is even and n_5 is odd, and either $m_5 \geq 4$ or $n_5 \geq 5$.

For case (a), we showed that when $n = 1, 2$ the problem can be solved easily.

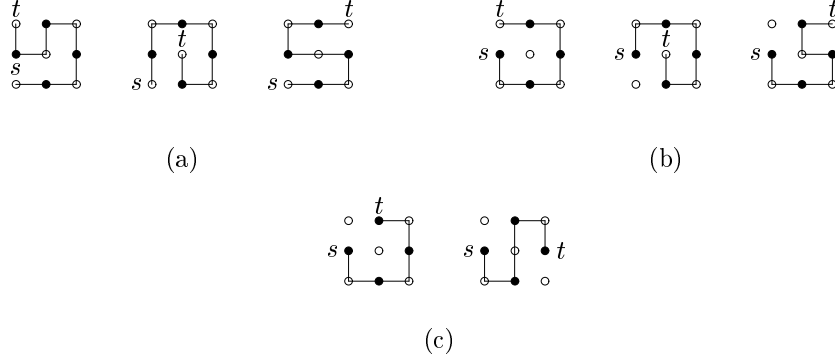


Figure 8: For $n = m = 3$, (a) s and t are white, then there is Hamiltonian path, (b) s and t have different colors, then there is a path with $U(R, s, t) = mn - 1$ and (c) s and t are black, then there is a path with $U(R, s, t) = mn - 2$.

For $m, n = 3$ the longest paths of all the possible problems are depicted in Figure 8 (the isomorphic cases are omitted).

For cases (b), (c) and (d) we use the definition of trisecting.

Definition 3.5. [3] Two separations of R_5 that partition it into three rectangular grid subgraphs R_5^s , R_5^t and R_5^m is called *trisecting*, if

- (i). R_5^s and R_5^t are a 2-rectangle, and
- (ii). $s \in V(R_5^s)$ and $t \in V(R_5^t)$.

A trisecting can be done by two ways horizontally and vertically. If $m_5 < 4$ or $m_5, n_5 \geq 4$, then trisecting is done horizontally, if $n_5 < 4$, then trisecting is done vertically.

Definition 3.6. A corner vertex on the boundary of R_5^s (resp., R_5^t) facing R_5^m is called a *junction vertex* of R_5^s (resp., R_5^t) if either

- (i) The condition (C0) holds and it has different color from s and t , or
- (ii) One of the conditions (C1) or (C2) hold and $U(R_5^s, s, p) + U(R_5^m, m, m') + U(R_5^t, q, t) = U(R(m, n), s, t)$. Where p is one of the corner vertices of R_5^s , q is one of the corner vertices of R_5^t , and m and m' are two of the corner vertices of R_5^m facing R_5^s and R_5^t , respectively.

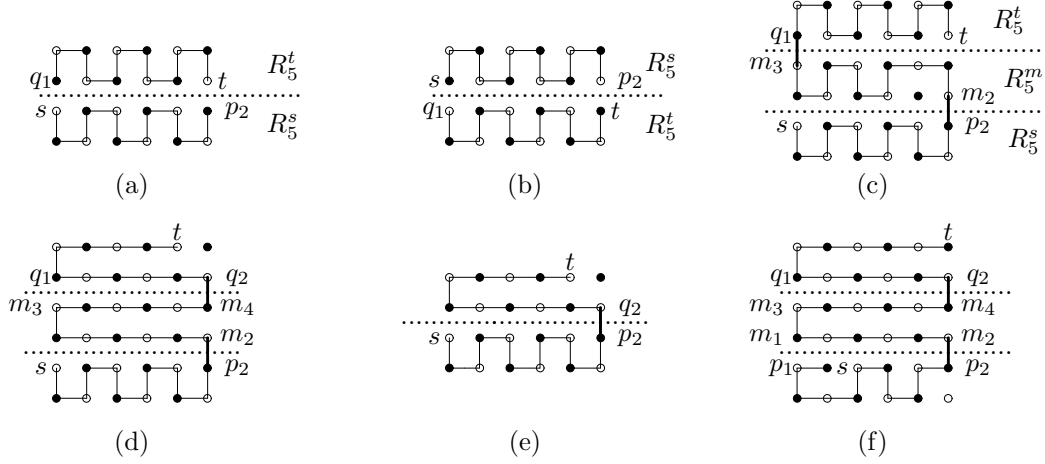


Figure 10: A trisection on $R(6, 6)$, $R(6, 4)$.

does not pass through a vertex. Therefore, $U(R_5^s, s, p_2) + U(R_5^m, m_3, m_2) + U(R_5^t, q_1, t) = U(R(m, n), s, t)$ and $U(R_5^s, s, p_2) + U(R_5^m, m_2, m_4) + U(R_5^t, q_2, t) = U(R(m, n), s, t)$ and hence R_5^s has a unique junction vertex and R_5^t have two junction vertices (the same argument is also applied to t). In this case, where $n_5 = 4$, both R_5^s and R_5^t have a unique junction vertex; see Figure 10(e).

Case 3. s and t are not the corner vertices on the boundary of R_5^s and R_5^t facing R_5^m ; see Figure 10(f). By Theorem 2.1, there exists a Hamiltonian path from s to p_1 , m_1 to m_3 and a path from q_1 to t which does not pass through a vertex, or a Hamiltonian path from s to p_1 , from q_2 to t and a path from m_1 to m_4 which does not pass through a vertex, or a Hamiltonian path from m_2 to m_4 , from q_2 to t and a path from s to p_2 which does not pass through a vertex. Therefore, $U(R_5^s, s, p_1) + U(R_5^m, m_1, m_3) + U(R_5^t, q_1, t) = U(R(m, n), s, t)$, $U(R_5^s, s, p_1) + U(R_5^m, m_1, m_4) + U(R_5^t, q_2, t) = U(R(m, n), s, t)$ and $U(R_5^s, s, p_2) + U(R_5^m, m_2, m_4) + U(R_5^t, q_2, t) = U(R(m, n), s, t)$ and hence both R_5^s and R_5^t have two junction vertices.

In case (c), s and t are black or different color, and two corner vertices on the boundary of R_5^s (resp., R_5^t) facing R_5^m are black and also R_5^m is a k -rectangle that $k \geq 1$ and odd. There are three cases for s and t :

Case 1. Both s and t are the corner vertices on the boundary of R_5^s and R_5^t facing R_5^m ; see Figure 11(a). Then s and t are black. By Theorem 2.1, there exists a path from s to p_2 , from q_1 to t which does not pass through a vertex in R_5^s and R_5^t , respectively, and a Hamiltonian path from m_3 to m_2 in

R_5^m . Therefore, $U(R_5^s, s, p_2) + U(R_5^m, m_3, m_2) + U(R_5^t, q_1, t) = U(R(m, n), s, t)$ and hence both R_5^s and R_5^t have a unique junction vertex.

Case 2. s is the corner vertex on the boundary of R_5^s facing R_5^m , then s is black and t is black or white; see Figure 11(b). By Theorem 2.1, there exists a path from s to p_2 and from q_1 (or q_2) to t , where t is black, which does not pass through a vertex, and a Hamiltonian path from m_2 to m_4 (or from m_3 to m_2), or a Hamiltonian path from q_1 (or q_2) to t , where t is white and m_2 to m_4 (or from m_3 to m_2) and a path from s to p_2 which does not pass through a vertex. Therefore, $U(R_5^s, s, p_2) + U(R_5^m, m_3, m_2) + U(R_5^t, q_1, t) = U(R(m, n), s, t)$ and $U(R_5^s, s, p_2) + U(R_5^m, m_2, m_4) + U(R_5^t, q_2, t) = U(R(m, n), s, t)$ and hence R_5^s has a unique junction vertex and R_5^t have two junction vertices (the same argument is also applied to t). In this case, where $n_5 = 5$, both R_5^s and R_5^t a unique junction vertex; see Figure 11(c).

Case 3. s and t are not the corner vertices on the boundary of R_5^s and R_5^t facing R_5^m ; see Figure 11(d). By Theorem 2.1, there exists a Hamiltonian path from s to p , m to m' and q to t , where s (or t) is white, and a path from s to p and q to t which does not pass through a vertex where s (or t) is black, p is p_1 or p_2 , q is q_1 or q_2 , m is m_1 or m_2 and m' is m_3 or m_4 . Therefore, $U(R_5^s, s, p) + U(R_5^m, m, m') + U(R_5^t, q, t) = U(R(m, n), s, t)$ and hence both R_5^s and R_5^t have two junction vertices.

In case (d), if $n_5 > 3$, the trisecting is performed horizontally, and the claim is proved by applying the same argument for case (b); see Figure 11(e). If $n_5 = 3$, the trisecting is performed vertically and also two corner vertices on the boundary of R_5^s facing R_5^m are black. Therefore, the claim is proved by applying the same argument for case (c). \square

After trisecting, we construct a longest path in R_5^s , R_5^m and R_5^t between s and p , m and m' and q and t , respectively. In the case that none of R_5^s and R_5^t have junction vertices (when $n_5 = 4$ and both s and t facing the common border R_5^s and R_5^t), we construct a longest path in R_5^s (resp., R_5^t) between s (resp., t) and a none-corner vertex of the boundary facing R_5^t (resp., R_5^s); see Figure 12. At the end, the longest paths in R_5 are combined through the junction vertices; see Figures 10 and 11.

Then we construct Hamiltonian cycles in rectangular grid subgraphs R_1 to R_4 , by Lemma 2.1; see Figure 13. Then combine all Hamiltonian cycles to a single Hamiltonian cycle.

Two non-incident edges e_1 and e_2 are parallel, if each end vertex of e_1 is adjacent to some end vertex of e_2 . Using two parallel edges e_1 and e_2 of

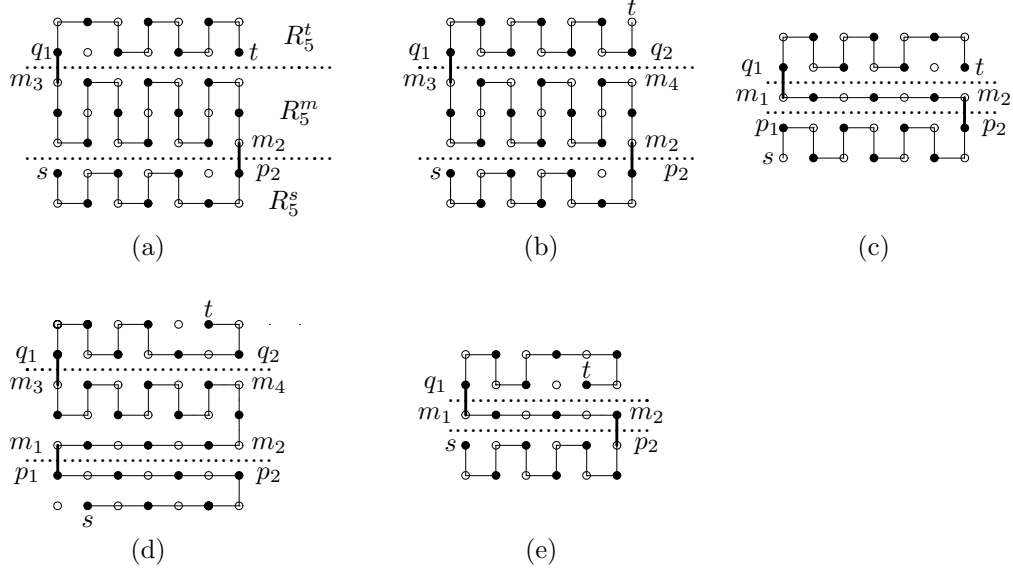


Figure 11: A trisection on $R(7, 7)$, $R(7, 5)$ and $R(6, 5)$.

two Hamiltonian cycles (or a Hamiltonian cycle and a longest path), such as two darkened edges of Figure 14(a), we can combine them as illustrated in Figure 14(b) and obtain a large Hamiltonian cycle.

Combining the resulted Hamiltonian cycle with the longest path of R_5 is done as in Figure 15.

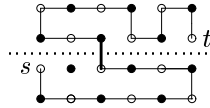


Figure 12: the Longest path in $R(6, 4)$.

Considering all of the above, we get the algorithm of finding a longest path in rectangular grid graphs, as shown in Algorithm 3.1.

Consider the pseudo-code of our algorithm in Algorithm 3.1. The step 1 dose only a constant number of partitioning, during the peeling operation, which is done in constant time. The step 2 trisects R_5 which requires also a constant number of partitioning. Then finds a longest path in R_5 by merging paths of the partitions which can be done in linear time. The step 3 finds Hamiltonian cycles of R_1 to R_4 which is done in linear time. The step 4

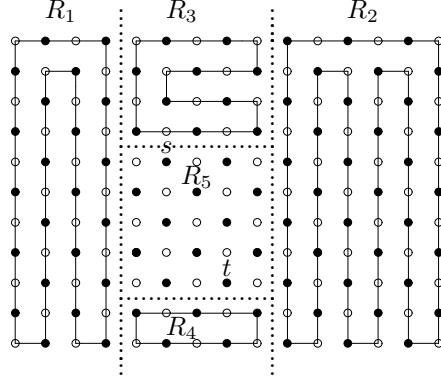


Figure 13: Hamiltonian cycles in R_1 to R_4 .

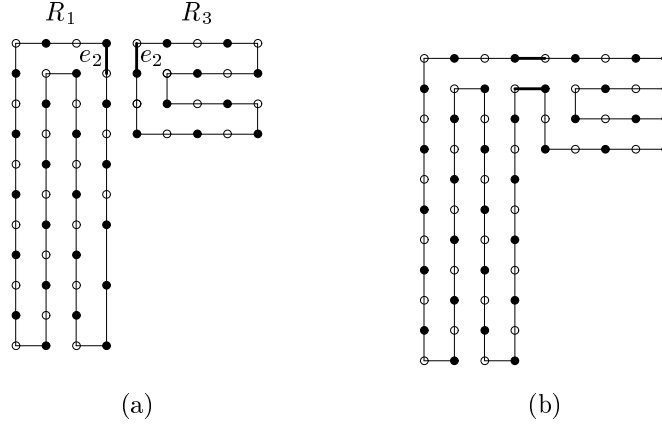


Figure 14: Combining two Hamiltonian cycles.

which combines the Hamiltonian cycles and the longest path requires only constant time. Therefore, in total our sequential algorithm has linear-time complexity.

4. The parallel algorithm

In this section, we present a parallel algorithm for the longest path problem. This algorithm is based on the sequential algorithm presented in the previous sections. Our parallel algorithm runs on every parallel machine, we do not need any inter-processor connection in our algorithm. We assume there are nm processors and they work in SIMD mode. For simplicity, we use a two-dimensional indexing scheme. Each vertex v of the given rectangular

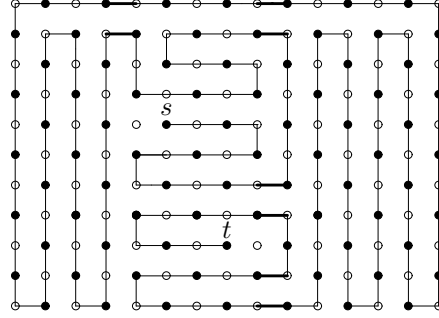


Figure 15: The longest path between s and t .

Algorithm 3.1 The longest path algorithm

procedure LongestPath($R(m, n), s, t$)

Step 1. By a peeling operation, $R(n, m)$ partitions into five disjoint rectangular grid subgraphs R_1 to R_5 , such that $s, t \in R_5$

Step 2. Finding longest path between s and t in R_5 .

Step 3. Construct Hamiltonian cycles in rectangular grid subgraphs R_1 to R_4 .

Step 4. Construct a longest path between s and t by combine all Hamiltonian cycles and a longest path.

grid graph $R(m, n)$ is mapped to processor (v_x, v_y) . Each processor knows its index, coordinates s and t , and m and n .

The peeling phase is parallelized easily, every processor calculates the following four variables, in parallel [3]:

$$\begin{aligned}
 r_1 &= \begin{cases} s_x - 2; & s_x \bmod 2 = 0 \\ s_x - 1; & \text{otherwise} \end{cases} \\
 r_2 &= \begin{cases} t_x + 1; & t_x \bmod 2 = m \bmod 2 \\ t_x + 2; & \text{otherwise} \end{cases} \\
 r_3 &= \begin{cases} \min(s_y; t_y) - 2; & \min(s_y; t_y) \bmod 2 = 0 \\ \min(s_y; t_y) - 1; & \text{otherwise} \end{cases} \\
 r_4 &= \begin{cases} \max(s_y; t_y) + 1; & \max(s_y; t_y) \bmod 2 = n \bmod 2 \\ \max(s_y; t_y) + 2; & \text{otherwise} \end{cases}
 \end{aligned}$$

Where variables r_1 , r_2 , r_3 and r_4 correspond to the right-most column number of R_1 , the left-most column number of R_2 , the bottom row number of

R_3 , and the top row number of R_4 , respectively. Then a processor can identify its subrectangular by comparing its coordinates with these four variables. In case (F1') and (F2'), the boundary adjustment can be done by simply decrementing R_1 , R_2 , R_3 or R_4 or incrementing R_3 or R_4 .

The trisecting phase is also parallelized in a similar manner. In the following we describe how we parallelized the horizontal trisecting, in two cases when $R(m, n)$ is even \times odd (or odd \times odd) and when it is even \times even. In case $R(m, n)$ is even \times odd or odd \times odd, every processor simultaneously calculate the following two variables:

$$l = \begin{cases} \min(s_y; t_y) & \min(s_y; t_y) \bmod 2 = 0 \\ \min(s_y; t_y) + 1 & \min(s_y; t_y) \bmod 2 \neq 0 \end{cases}$$

$$r = \begin{cases} \max(s_y; t_y) & \max(s_y; t_y) \bmod 2 = 0 \\ \max(s_y; t_y) - 1 & \max(s_y; t_y) \bmod 2 \neq 0 \end{cases}$$

Where variables l and r correspond to the bottom row number of R_5^s (resp. R_5^t), and the top row number of R_5^t (resp. R_5^s), respectively.

In case $R(m, n)$ is even \times even, every processor simultaneously calculate the following two variables:

$$l = \begin{cases} \min(s_y; t_y) & \min(s_y; t_y) \bmod 2 = 0 \\ \min(s_y; t_y) + 1 & \min(s_y; t_y) \bmod 2 \neq 0 \end{cases}$$

$$r = \begin{cases} \max(s_y; t_y) & \max(s_y; t_y) \bmod 2 \neq 0 \\ \max(s_y; t_y) - 1 & \max(s_y; t_y) \bmod 2 = 0 \end{cases}$$

A similar method can be used to parallelize the vertically trisecting. After peeling and trisecting, all processors in the same subrectangles simultaneously construct either a longest path, Hamiltonian path or cycle according to the pattern associated with the subrectangle. For constructing a Hamiltonian path in a rectangular grid graph, we use the constant-time algorithm of [3]. For constructing a Hamiltonian cycle in an even-sized rectangle, we use the constant-time algorithm of [4] in which every processor computes its successor in the cycle. This algorithm is given in Algorithm 4.1; see Figure 16(a) .

For constructing a longest path, parallel algorithms can be easily developed for each different pattern shown in Figure 16(b), (c). As two examples, for constructing a longest path between vertices $(2, 1)$ and $(m - 1, n)$ in an odd \times odd rectangular grid graph $R(m, n)$, and vertices $(m - 1, 1)$ and $(n - 1, 1)$ in an even \times even rectangular grid graph $R(m, n)$. We have developed the sim-

ple algorithms Algorithm 4.2 and 4.3, respectively. The algorithms for other patterns can be derived in the similar way.

Then combining phase is parallelized as follows. The two processors at the two endpoints of a corner edge in a Hamiltonian cycle c_1 check whether a neighboring Hamiltonian cycle c_2 exists or not. If c_2 exists, then their successors are changed to the adjacent processors in c_2 . Similarly, the two processors at the endpoints of a corner edge in the longest path P in R_5 also check the existence of the adjacent edge in the Hamiltonian cycle C , and change their successors. Thus, the combining phase can be parallelized in constant steps without inter-processor communication.

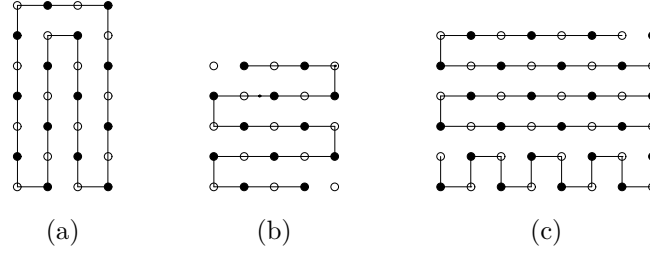


Figure 16: (a) A Hamiltonian cycle in $R(4, 7)$, (b) and (c) two patterns of longest path in $R_5(5, 5)$ and $R_5(8, 6)$.

Algorithm 4.1 The Hamiltonian cycle parallel algorithm for an even-sized rectangular grid graphs

procedure LongestPath $R(m, n)$

- 1: **for** each processor (x, y) in $R(m, n)$ **do** in parallel
 - 2: **if** $y = 1$, **then** successor $(x, y) \leftarrow (x + 1, y)$
 - 3: **elseif** ($y = 2$, x is odd and $x \neq 1$) or ($y = n$ and x even), **then** successor $(x, y) \leftarrow (x - 1, y)$
 - 4: **slesif** x is even and $y < n$, **then** successor $(x, y) \leftarrow (x, y + 1)$
 - 5: **else** x is odd and $y \leq n$, **then** successor $(x, y) \leftarrow (x, y - 1)$
-

5. Conclusion and future work

We presented a linear-time sequential algorithm for finding a longest path in a rectangular grid graph between any two given vertices. Since the longest path problem is NP-hard in general grid graphs [11], it remains open if the

Algorithm 4.2 The longest path parallel algorithm for odd \times odd rectangular grid graphs

procedure LongestPath($R_5(m_5, n_5), s, t$)

- 1: **for** each processor (x, y) in $R_5(m_5, n_5)$ **do** in parallel
 - 2: **if** $x = 1$ and $y = 1$ or $x = m$ and $y = n$, **then** successor $(x, y) \leftarrow \text{null}$
 - 3: **elseif** y is odd and $x < m$, **then** successor $(x, y) \leftarrow (x + 1, y)$
 - 4: **slesif** y is odd and $x = m$, **then** successor $(x, y) \leftarrow (x, y + 1)$
 - 5: **elseif** y is even and $x > 1$, **then** successor $(x, y) \leftarrow (x - 1, y)$
 - 6: **else** y is even and $x = 1$, **then** successor $(x, y) \leftarrow (x, y + 1)$
-

Algorithm 4.3 The longest path parallel algorithm for even \times even rectangular grid graphs

procedure LongestPath($R_5(m_5, n_5), s, t$)

- 1: **for** each processor (x, y) in $R_5(m_5, n_5)$ **do** in parallel
 - 2: **if** $x = m$ and $y = 1$, **then** successor $(x, y) \leftarrow \text{null}$
 - 3: **elseif** (y is odd and $x = m$), (y is even and $x = 1$) or ($y = n$ and x is even) **then** successor $(x, y) \leftarrow (x, y - 1)$
 - 4: **slesif** (y is odd and $x < m$), ($y = n - 1$ and x is even), ($y = n$ and x is odd) **then** successor $(x, y) \leftarrow (x + 1, y)$
 - 5: **elseif** y is even and $x > 1$ **then** successor $(x, y) \leftarrow (x - 1, y)$
 - 6: **elseif** $y = n - 1$ and x is odd **then** successor $(x, y) \leftarrow (x, y + 1)$
-

problem is polynomially solvable in solid grid graphs. Based on the sequential algorithm a constant-time parallel algorithm is introduced for the problem, which can be run on every parallel machine.

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