Harmonising Natural Deduction
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Dag Prawitz proved a theorem, formalising 'harmony' in Natural Deduction systems, which showed that, corresponding to any deduction there is one to the same effect but in which no formula occurrence is both the consequence of an application of an introduction rule and major premise of an application of the related elimination rule. As Gentzen ordered the rules, certain rules in Classical Logic had to be excepted, but if we see the appropriate rules instead as rules for Contradiction, then we can extend the theorem to the classical case. Properly arranged there is a thoroughgoing 'harmony', in the classical rules. Indeed, as we shall see, they are, all together, far more 'harmonious' in the general sense than has been commonly observed. As this chapter will show, the appearance of disharmony has only arisen because of the illogical way in which natural deduction rules for Classical Logic have been presented. Prawitz' treatment of Natural Deduction presents Intuitionistic Logic, and then Classical Logic as being developed from Minimal Logic, with the addition of rules for Absurdity ($\perp$). Thus in (Prawitz [1965], 20-21), he gives the 'and' rules as:

    &I: A, B / A&B
    &E: A&B / A   A&B / B,

the 'or' rules as

    vI: A / (A v B)   B / (A v B)
    vE: A v B, ((A) / C), ((B) / C) / C,

and the 'if' rules as

    $\supset$I: ((A) / B) / (A $\supset$ B)
    $\supset$E: A, (A $\supset$ B) / B.

From the latter, Gentzen's rules for Negation may be derived, upon defining '¬A' as 'A $\supset$ $\perp$', viz

    ¬I: ((A) / $\perp$) / ¬A
    ¬E: A, ¬A / $\perp$.

To get Intuitionistic Logic one adds the Intuitionistic Absurdity rule,

    $\perp$ / A,

and to get Classical Logic one adds the Classical Absurdity rule,

    ((¬A) / $\perp$) / A.

Adding the Classical Absurdity rule to the other rules is not conservative, since Peirce's Law, for instance, which is a purely implicational thesis, is not derivable without it, even though it does not itself contain '$\supset$'. But it is another exceptional feature of the Classical Absurdity

rule that is of concern here. Prawitz, in formalising a notion of 'harmony', proves his 'Inversion Theorem' (Prawtiz [1965], 34):

> If Γ / A then there is a deduction of A from Γ in which no formula occurrence is both the consequence of an application of an I-rule, and major premiss of an application of an E-rule.

This formalises 'the inversion principle' which says that nothing is 'gained' by inferring a formula through introduction for use as a major premises in an elimination. Prawitz then goes on to demonstrate that this principle fails in Classical Logic, showing specifically that, even if the Classical Absurdity Rule is replaced with a further pair of introduction and elimination rules for negation, then a proof of the Law of the Excluded Middle has to use a formula occurrence both as a consequence of an application of an I-rule and as a major premises of an application of an E-rule.

But with the Classical Absurdity Rule unamended, look again at the rules above involving '¬', namely the last one, and the ones labelled the introduction and elimination rules for '¬', according to Gentzen. There are three of these in all, not just two, and they are symmetric with respect to '¬A' and 'A', so they as much involve the introduction and elimination of '¬' as they involve the introduction and elimination of a bare space before 'A'. That means nothing distinctive about the presence rather than the absence of '¬' is displayed in these rules, mirroring the fact that not only is '¬A' the contradictory of 'A', but also 'A' is the contradictory of '¬A'. As a result, they are merely about the contradictory of a formula: they are introduction and elimination rules for Contradiction, and they show how forming the contradictory of a formula can be done in two ways: by adding, or, when appropriate, deleting the negation sign. On that understanding, the contradictory of a formula is introduced in both of

$$((A) / \perp) / \neg A \qquad ((\neg A) / \perp) / A,$$

while the contradictory of a formula is eliminated in

$$A, \neg A / \perp.$$

The consequence is that Prawitz's Inversion Theorem becomes exceptionless, and his various theorems about the normal forms of classical deductions become much more understandable, and symmetric.

The latter theorems are formulated about 'maximum formulas', which, in Prawitz's terminology, concern formula occurrences that are the consequence of an application of an introduction rule, or the Classical Absurdity rule, as well as being the major premise of an application of an elimination rule (Prawitz [1965], 34). But the Classical Absurdity rule, on the revised understanding, is not a deviant rule difficult to insert into the regular system of rules, but simply another introduction rule, which means, not only that the statement of the theorems is neater, but also that the reason why that rule is included in its given place is more evident. The Inversion Theorem likewise involved an anomaly which now vanishes. As before, that theorem shows that corresponding to any deduction there is one to the same effect but in which no formula occurrence is both the consequence of an application of an introduction rule and major premise of an application of the related elimination rule. So how

do the rules for Contradiction now fit in? Taking either premise in the elimination rule for Contradiction to be a major premise, here are the two cases in which a formula occurrence is both the consequence of an introduction rule, and a major premise of the elimination rule:

$$((A) / \perp) / \neg A \quad A / \perp , \quad ((\neg A) / \perp) / A \quad \neg A / \perp.$$

Clearly to the same effect are simply

$$A / \perp, \quad \neg A / \perp,$$

so the detours through the respective formula occurrences of '¬A' and 'A' can be removed, establishing the associated induction step in the proof of the Inversion Theorem (c.f. Prawitz [1965], 35-38). As a whole, therefore, the theory of classical logic becomes much more regular.

What Gentzen seems not to have noticed is that '¬', while commonly called a 'propositional connective', is very different from the other propositional connectives, especially in the classical case. Partly this is because it is a unary connective, but mainly it is because, on account of that, in Classical Logic it symbolises, through its presence and absence, a certain relation between propositions: the relation of Contradiction. In Intuitionistic Logic there is not that relation between '¬A' and 'A', since formulas such as these, there, all fall merely into the weaker relation of contrariety. But '¬' there does not symbolise this weaker relation, since not all contraries of the same thing are equivalent. As a result, when developing logic Gentzen's way one may even lose sight of the fact that relations between propositions are involved at all, other than those given by the deduction sign.

There is an even more fundamental reason, also, why this is so. Gentzen divided his inference rules according to whether they introduced, or eliminated propositional connectives. This corresponds, in most cases, with them introducing, or eliminating the associated formulas, namely formulas in which there are 'and's, 'or's, or 'if's between others. But it does not correspond to all cases where what is introduced, or eliminated, are formulas, since it does not cover all cases where the formulas are contradictories of others. The conjunction of two propositions can be expressed by using a symbol between them, and likewise the disjunction of them, and the implication of one by the other. But the same is not the case with respect to the contradiction between two propositions. For that is shown by the one possessing a sign that the other lacks — the negation sign. Gentzen's emphasis on the connectives, as a consequence, easily gives the impression that different sets of rules merely allow one to deduce less or more formulas involving the connectives, with, amongst them, the negation sign. But what is true instead, in the latter respect, is that the restricted rules in Minimal Logic, and Intuitionistic Logic do not allow one to deduce all formulas about contradictories. Certainly one cannot have, in Intuitionistic Logic, that A & ¬A, but that theorem applies to all contraries, and so does not single out just contradictories. For a contrary to be a contradictory it must also be the case that A v ¬A, i.e. that the two formulas are subcontraries. We shall see later that forgetting this kind of matter has had further consequences in connection with misunderstandings about classical logical truths like the Law of the Excluded Middle, and the Law of Non-Contradiction.

It is also involved in understanding why the subformula principle does not hold in the classical case. As Prawitz explains this (Prawitz [1965], 42), it is a corollary of his major theorem on the form of normal deductions:

> Every formula occurrence in a normal deduction … of A from Γ has the shape of a subformula of A or of some formula of Γ, except for assumptions discharged by applications of the (Classical Absurdity Rule), and for occurrences of ⊥ that stand immediately below such assumptions.

The problem which gives rise to the exception is that, in

$$((\neg A) / \perp) / A,$$

the '¬A' is not a subformula of 'A'. Of course, it is a subformula of '¬¬A', which is equivalent to 'A' classically. So the subformula principle obviously only arises in forms of 'logic' that concern themselves with signs and not with what those signs signify.

But what are we to make of the fact that there are three Contradiction rules? Does it not make Contradiction some kind of special case? Here we come to notice the always-evident fact that there are three natural deduction rules for every basic logical symbol in classical propositional logic. There is not just one elimination rule for 'and', but two, and not just one introduction rule for 'or' but two. The two elimination rules with 'and' are needed to harmonise with the fact that two premises are involved in its introduction rule, and so, given that a conjunction can only be arrived at from these two premises together, it must entail all they separately entail. The elimination rule for 'or' harmonises with its two introduction rules in a similar manner, since a disjunction can only be arrived at from one of its disjuncts, and so it must entail only what they both entail. Hence, in the elimination rule a common implication of the two further premises is required besides the 'or'.

It is not the case, of course, that there are three rules in the case of 'if … and only if …', for instance, or 'if … then … else …'. But these can be defined in terms of the elementary ones. The same holds in Predicate Logic, with complex quantifiers like 'for all … there exists…'. But for the basic quantifiers we can find three rules, so long as we rise above standard Predicate Logic and consider instead its conservative extension, the Epsilon Calculus. So there is nothing at all anomalous about three rules being involved for Contradiction.

It is already well known that the epsilon treatment of standard Predicate Logic enables considerable simplifications to be introduced. For while the rules of Universal Quantifier Elimination (UE), and Existential Quantifier Introduction (EI) are straightforward, the rules of Universal Quantifier Introduction (UI), and Existential Quantifier Elimination (EE), in Gentzen's and comparable formulations, are subject to several qualifications regarding the arbitrary names in them (c.f. Prawitz [1965], 20). But these qualifications can be completely removed if the appropriate epsilon terms are used in place of the arbitrary names (see, e.g. Hazen [1987]). This arises because, in the Epsilon Calculus individual terms of the form 'εxFx' are defined for all the predicates in the language, and 'εxFx' then refers to the F alluded to in the existential generalisation '(∃x)Fx', while 'εx¬Fx' refers to the strongest putative counterexample to the universal generalisation '(x)Fx'. To present the whole of quantification theory available when using epsilon terms, however, one must have enough

natural deduction rules to match a standard axiomatic base for the Epsilon Calculus, such as the axiom

$$Ay \supset A\epsilon xAx,$$

and the quantifier definition

$$(\exists x)Ax \equiv A\epsilon xAx.$$

That means we can take as the rules for the existential quantifier:

EI: Ay / AεxAx   AεxAx / (∃x)Ax
EE: (∃x)Ax / AεxAx,

the first two being introduction rules — using either the second alone, or the pair together — and the third being the elimination rule for it. The required, extra induction clauses in the proof of the Inversion Theorem for this case are again trivial (c.f. Prawitz [1965], 38), since a chain of inferences in which an existential generalisation is the conclusion of an introduction rule, and the premise of an elimination rule, viz

Ay / AεxAx / (∃x)Ax / AεxAx,

or simply

AεxAx / (∃x)Ax / AεxAx,

clearly allows the existential quantifier step to be removed. A full discussion of the theory of Epsilon calculi in Natural Deduction settings may be found in several other places (see Leisenring [1969] Ch V, Meyer-Viol [1995] Ch 2, Wessels [1977], and Yasuhara [1982]). But all that it is relevant to note here is that the elimination rule given above harmonises with the two introduction rules because, if the latter are together the only means by which '(∃x)Ax' may be introduced, then this must be via 'AεxAx' or something which implies it — and so neither by something which implies '(∃x)Ax' without implying 'AεxAx', or through some other, quite independent, route. Note also that what might seem sufficient natural deduction rules for the existential quantifier, namely

EI: Ay / (∃x)Ax,
EE: (∃x)Ax / AεxAx,

do not similarly harmonise, and in fact do not suffice as an explicit basis for the Epsilon Calculus, since there might be a subsidiary rule forbidding the substitution of epsilon terms for 'y', as, for instance, in Mints' Intuitionistic Epsilon Calculus (Mints [1982]). The situation is somewhat similar to omitting the second 'if' introduction rule to be discussed shortly.

For the rules for 'if', within Classical Logic, also display this same threefold nature — although that fact is hidden in the Gentzen-Prawitz approach based on Minimal Logic. Indeed Stephen Read has proposed a harmonization of the classical rules using a multiple

conclusion logic with this feature (Read [2000], 145), which, in place of the ⊃I rule before has ⊃I', viz

((A) B, X) / (A ⊃ B), X.

In single conclusion form, this adds to ⊃I the further introduction rule

((A) B v X) / (A ⊃ B) v X.

Multiple conclusions, of course, are not single propositions, and so the entailment of a proposition, in the first case, is not symbolised by '/'. The transcription into a single-conclusion form regularises this, but also demonstrates that Read's second 'if' rule is an impure introduction rule, since it essentially involves both of the connectives '⊃' and 'v'.
　　　There is a pure second 'if' introduction rule, however, besides the first one given before. That is, of course,

B / A ⊃ B.

This is a derived rule in Minimal Logic as presented before, because of additional, subsidiary rules about premise discharges, which allow chains of inferences like this:

A B / A&B,
A&B / B,

which give a deduction of 'B' from 'B' together with a redundant 'A', at the same time as allowing that redundant 'A' to be discharged in a further application of the 'if' introduction rule. So, in effect, one gets simply:

B / (A ⊃ B).

But there is no formal difficulty in adding this derived rule as a primitive rule — it introduces the 'if' connective, after all — or adding the appropriate further clause in the proof of the Inversion Theorem (c.f. Prawitz [1965], 37). More to the point, unless it is taken as a separate, explicit introduction rule, the full harmony of the introduction and elimination rules for 'if', in the classical case, is not fully apparent. For the single introduction rule given at the start makes 'A ⊃ B' invariably reflect a deduction of 'B' from 'A'. But, because of the additional rules about premises discharges, deductions of 'B' from 'A' include ones, like that above, where the truth of 'A' is irrelevant, hence 'B' might follow independently of 'A', since it was from 'B', essentially, that the conditional was derived. By contrast, once one embraces the second introduction rule, it is quite explicit that the truth of the conclusion of the elimination rule could arise independently of the truth of the first premise, 'A', and some derivation of 'B' from 'A'. In fact 'A' never combines with 'A ⊃ B' to generate 'B' (like 'A' and 'B' combine to generate 'A & B'), for if 'A ⊃ B' is present because there was a derivation of 'B' from 'A', in which 'A' was not redundant, then 'B' already follows from 'A' alone. On the other hand, if 'A ⊃ B' is present because there was a derivation of 'B' from 'A' in which 'A' is redundant, then 'B' follows entirely because it was on this, essentially, that 'A ⊃ B' was based. Incorporating the second introduction rule shows

explicitly that 'A ⊃ B' could have come about either directly from 'B' itself, or, alternatively from 'A' because there was a deduction of 'B' from 'A' in which 'A' was not redundant. Either way, of course, because 'A' is a further premise, 'B' follows as in the elimination rule. There is more to be said about the significance of this second 'if' introduction rule, however, since, as we shall see, Classical Logic, by incorporating this rule, recognises part of the natural language behaviour of 'if', while certain other logics, by not incorporating it, are the logics of imaginary languages. The 'harmony' of the above rules, though, after that emendation, we can now see, extends well beyond the 'harmony' defined in historical discussions of Natural Deduction. For there is, evidently, a remarkable threefold 'harmony', in a much more general sense, all the way through the whole structure of first-order logic, once we move over to the Epsilon Calculus for a formalisation of Predicate Logic. In this respect the above rules far surpass those that Milne obtained, for instance, which while 'harmonious' were, by his own admission, 'frankly unnatural' (Milne [2002], 499). Indeed, what is also most striking, surely, is how banal and obvious the full harmony is, once it is revealed.

That raises the further question of why the extensive, yet simple, threefold structure revealed above has not been noticed before. This is a very important question, I think, and it is not just a historical or sociological one, but also a formal one. For countering the lines of thought that have led people to doubt the harmony of Classical Logic is itself a logical matter. There are a number of lines of thought that need to be countered, ranging from ignorance of the Epsilon Calculus through to misconceptions about the relation between Contradiction and 'not', and Implication and 'if'.

In connection with ignorance of the Epsilon Calculus, for instance, there is obviously the weight of the entrenched text-book tradition to contend with, setting out, again and again, just Fregean Predicate Logic. That is a hindrance to perceiving the common, threefold, and harmonic form of the above rules, since, for one thing, it encourages tolerance of the messy restrictions there have to be on Fregean Predicate Logic's Universal Introduction and Existential Elimination rules. In addition, the further applications of epsilon terms are in areas of language not dealt with in this text-book tradition, namely, the formulation of certain anaphors, and their related, purely referential definite descriptions. But even amongst scholarly experts on the epsilon calculus, its larger illumination of logical theory is not commonly recognised, so those people who might have been expected to see the harmonisation this calculus provides, in the areas highlighted above, have not woken up to its central virtues. This point may be made with respect to advanced classical logicians themselves, but there is also the extent of the pre-occupation, within learned circles, of Intuitionistic Logic, which does not sit easily with the Epsilon Calculus. There are major problems with adding epsilon terms, and the epsilon axioms, to Intuitionistic Propositional Logic (Meyer-Viol [1995], Ch3). The natural consequence of such incompatibilities, given the tradition of academic interest there has been in Intuitionistic Logic, is no doubt one reason why the purity and simplicity of the Epsilon Calculus, within its original Classical Logic setting, has tended to get overlooked, and its general harmony with other parts of Logic has been left uninvestigated.

Likewise also, of course, with the internal harmony of the classical Negation rules, once these are all set out, and seen to be presented better as rules about Contradiction. Intuitionistic Logic lacks the Classical Absurdity rule, and so, directly against Dummett (Dummett [1991], 291f) there is a clear sense in which both the absurdity and the negation in that logic remain 'disharmonic'. Specifically, as we saw in part before, Intuitionistic Logic

perpetuates the Conventionalist idea that there have to be rules about the negation sign rather than the relation of Contradiction. This is even present in its claim to have denied the Law of the Excluded Middle, by not having a formula of the form 'A v ¬A' as a thesis, when that law (which is therefore necessarily true) says instead that either a proposition or its contradictory is true. There is no way to symbolise this law simply as a formula involving an undifferentiated negation sign, since it is not that sign, but the relation it classically symbolises, that is involved. So denial of that law cannot consist in finding something merely of the symbolic form 'A v ¬A' which is not a thesis. As a result, many proponents of Intuitionistic Logic even need to be questioned about whether they know what it is that is classically true.

It is common in describing different logical systems to indicate by means of a suffix on the inference sign which system of inferences is involved. The same kind of relativisation should properly be employed with the propositional connectives if the conditions on their use are significantly different. Then it would be clear that what is necessarily true is that A v_c $\neg_c$A, while what is not necessarily true is something different, namely that A v_i $\neg_i$A. Alternatively, using the Gödelian interpretation of the Intuitionistic connectives (c.f. for instance, Beziau [2005]), what is necessary is simply that A or ¬A, while what is not necessary is that LA or L¬LA, for a certain modal operator sometimes read 'it is assertable that' or 'it is provable that'. Either way that shows that the Law of the Excluded Middle is not denied in Intuitionistic Logic. It also shows that the Classical Absurdity Rule, with '¬' as '¬c' does not fail in Intuitionistic Logic. Indeed it cannot fail at all. What fails is merely of the same form, but with '¬' as '¬i'.

We therefore see that the root cause of the seeming lack of harmony in the classical natural deduction rules was a misunderstanding about logical truth itself. Conventionalism does not give a correct account of what is logically necessary, i.e. what cannot be otherwise, and progressive series of rules, such as Gentzen's, should therefore not be seen as simply accumulating rules about the same signs, but as developing different concepts, which need different expressions to avoid ambiguity. This does not only happen with the rules for negation, since, without the second 'if' introduction rule, the one given by Gentzen is ambiguous. For that single rule on its own does not explicitly define a concept of 'if', and needs subsidiary rules about premise discharges to resolve its intended sense, and the major problem has been understanding the classical sense. In that regard, calling the associated rules 'implication' rules is what is primarily misleading, since what is involved in the classical case are merely rules for 'if', and 'if' does not invariably indicate an implication, as is well known. Thus we can say 'There is jam in the cupboard, if you want some', and also 'If there is jam in the cupboard, then I am a Dutchman', and the like, and in neither case is there any claim that the consequent is implicated in the antecdent.

In fact, by far the biggest area of thought which needs correction, in coming to see the fully harmonious formal structure there is in Classical Logic must be the set of traditions that found difficulty with the second 'if' introduction rule unearthed before. That applies both with regard to the narrow notion of 'harmony' connected with the Inversion Theorem, and also the broader notion that we have seen also arises. Without the second 'if' introduction rule in its natural place — and in place not just as a consequence of a definition of 'if' in terms of 'or', but directly because of a feature of 'if' itself — the broader notion of harmony displayed above can hardly be suspected, and the Classical Absurdity Rule is more likely to be seen as anomalous. For the idea that there is a threefold structure within the rules of Natural Deduction is then not very apparent even at the propositional level, since the fact that

it is present in the 'and' rules and 'or' rules easily gets dismissed, given their repetitive form, and also the two-fold structure of the Fregean quantification rules.

The traditions that have had this more basic difficulty have all taken off from what have been called the Paradoxes of Material and Strict Implication, and perhaps the most substantial is that formalised within Relevance Logic, where the rules about premise discharges are tightened up in various ways. In particular, there may no longer be any derivations of 'B' from 'A' in which 'A' is redundant. The idea is that an 'if' or 'only if' statement, such as 'A ⊃ B', should give conditions under which its consequent is true. Asserting this merely on the basis of its consequent, 'B', makes no attempt at presenting such conditions, since then, with an arbitrary antecedent, should one remember it, the 'if' or 'only if' statement is part of an affirmation '(A ⊃ B) & (¬A ⊃ B)' saying that its consequent is true come what may, i.e. true whether or not its antecedent is true. Likewise in the reverse case, when such an expression is uttered on the basis that its antecedent is false. To say 'A ⊃ B' then, with a quite arbitrary consequent, is to view it as irrelevant to the current point what might follow from the antecedent, since all that matters, in that case, is that that antecedent is false, i.e. emphatically and unconditionally false, and so false whatever else might be the case. An 'if' statement, in other words, can be used to express certainty, and assert that there are no conditions.

In the discussion of C. I. Lewis' proof that a contradiction entails everything, therefore, relevance logicians (e.g. Read [1988], 31f) see an ambiguity that is not there in natural language. From

A & ¬A

one can derive

A,

and

¬A;

while from

A

one can derive

A v B;

but can one then use Disjunctive Syllogism to derive

B

from

¬A

together with

A v B?

The question raised is whether 'A v B' involves the right sense of 'or'. Supposedly, given the way it was introduced, it's 'v' is just an 'extensional' connective, and not the 'intensional' one, 'fission', which would enable one to derive, from the 'or', the conditional 'A → B' with relevant '→' (note the distinctive sign). Only 'fission', we are told, can figure in valid Disjunctive Syllogisms, on account of this. Where '→', 'fission', and particularly its conjunctive mate 'fusion' are to be found in natural language is the main problem. But that is doubly a problem, since it is not first recognised as relevant, within this tradition, that these connectives should even be looked for there. Natural language, it seems, is only there as the meta-language in which to talk about these connectives, which are in some other, distant, object language. Inspection of the banal piece of natural language that halts this tradition in its tracks is thus inhibited, and by far the larger percentage of time and attention is given to the admitted-to-be dreamt-up semantics, and the consequent need to present the associated soundness and completeness proofs. Is it all a fiction, or does it deal with facts? That is not a worry, so long as one fills one's mind with the mathematics of telling the fine details of the story. Contrariwise, semantics, completeness, and soundness are not a worry if one attends to natural language, since then 'object' and 'meta-' language are exactly the same. We say, in natural language, to focus on a fact, things like 'B is true, whether or not A is', and ironically the 'whether or not' is there to remove irrelevancies to the focus, which is not on why, or how, or because of what B is true, but simply on the certainty that B is true. Hence, if the logic is not to be the logic of some imaginary 'object language', but instead of the 'meta-language' in which it is talked about, namely the logician's Mother Tongue, then the second 'if' introduction rule must be incorporated.

Without a recognition that 'if' and 'only if' statements can express certainty, and do not always present necessary or sufficient conditions, there are, in addition, ambiguities about whether Logic might be an empirical science, where such conditions have a central, causal place. One telling sign of this is that the previously discussed derived rule, where an arbitrary formula 'B' is deduced from a contradiction 'A & ¬A', is commonly given the name 'Explosion'. This rule is abandoned not just within the Relevance Logic tradition, but also in the general study of Paraconsistent Logics. As the name of the rule indicates, supposedly large-scale devastation would follow, if a contradiction were true. But that description embraces totally the wrong image for the logical sense of 'following', since contradictions are not elements in this world, and so they can have no causal or temporal consequences. Given 'A & ¬A', in its classical sense, one cannot obtain 'B' in the manner of some chemical experiment, or the like. Rather, 'B', for arbitrary 'B' would have to be already true, for 'A & ¬A' to be true. As before, it is not a matter of why, or how, or because of what 'B' is true, but simply, now, the certainty that 'A & ¬A' is not true. What the derived rule implies, and what is made explicit in the elimination rule for Contradiction, is that Absurdity can only arise *per impossibile*, so Absurdity is something that, while it may be envisioned, cannot in fact arise. The two Contradiction introduction rules fully harmonise with this elimination rule in the same sense: they each prevent a contradiction from arising in fact. For each introduction rule also shows that if Absurdity would follow, given a formula, then one can deduce the contradictory of that formula. So, by denying the respective

premises from which Absurdity would follow, the two introduction rules ensure that Absurdity does not arise amongst the possibilities. The elimination rule for Contradiction, therefore, does not state some condition under which something will in fact occur, but rather states that under no actual condition can its absurd consequence occur. So there is no bomb set to go off, in the related rule 'Explosion', and hence no need to remove its fuse. In the formal terms described before, '⊥' cannot occur at the end of a deduction without there being undischarged assumptions; indeed, it is on this basis that the consistency of Classical Logic can be established.

In Paraconsistent Logic these facts about Contradiction are not expressed, but that does not show they are not facts. For the crucial point made before about the Law of the Excluded Middle has to be made about the Law of Non-Contradiction, 'Explosion', and the like: they only fail in a nominal sense in Paraconsistent Logic, through the illicit use of the same sign as in the case of Classical Logic. Logic is not about signs, but about the concepts expressed by those signs. The negation sign in Paraconsistent Logic is used so that 'A & ¬A' is not a logical falsehood, and the move from 'A & ¬A' to 'B' is not a valid inference. Hence it does not represent the relation of Contradiction, and should be symbolised differently. In fact the '¬A' in Paraconsistent Logic is merely subcontrary to 'A', rather than contradictory to it, and again its '¬' does not symbolise the weaker relation then involved, since not all subcontraries of the same thing are equivalent. For total clarity (see Beziau [2005] again) the negation in Paraconsistent Logic should have a modal expression dual to that appropriate for Intuitionistic Logic. So the failure of neither of the above verbal forms goes against it being the case that a contradiction cannot be true, which also means, because of the previous points about conditionals, that a contradiction entails anything. One cannot escape the force of logical necessity.

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