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SI: FOUNDATIONS OF MATHEMATICS

Introduction

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1 The idea of the series

The *Symposia on the Foundations of Mathematics* (SOTFOM), whose first edition was held in 2014 in Vienna, were conceived of and organised by the editors of this special issue with the goal of fostering scholarly interaction and exchange on a wide range of topics relating both to the *foundations* and the *philosophy* of mathematics. The last few years have witnessed a tremendous boost of activity in this area, and so the organisers felt that the time was ripe to bring together researchers in order to help spread novel ideas and suggest new directions of inquiry.

Two distinctive features of the SOTFOM events were (1) their focus on blending both philosophical and mathematical considerations, and (2) the active and numerous participation of many young researchers (including doctoral and post-doctoral scholars). This was mixed with talks from experienced experts in the field presenting

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their perspective on the state of the art concerning topics central to the foundations and philosophy of mathematics. Among the topics covered were: (i) the set-theoretic multiverse (including the universism/multiversism dichotomy), (ii) new mathematical axioms and their justification, (iii) foundational theories alternative to set theory (in particular, *category theory* and *homotopy type theory*), (iv) reflection principles in model theory and set theory, (v) mathematical naturalism, and (vi) philosophical issues in reverse mathematics.

Given the quality of both the papers presented and of the ensuing discussion, we thought that the scientific community would welcome the proceedings of the first three events. Two were held at the Kurt Gödel Research Center for Mathematical Logic in Vienna (July 2014 and September 2015) and one was held at the Institute of Philosophy in London (January 2015). The readers of this special issue of *Synthese* will, therefore, find here a selected collection of papers given and thoroughly discussed at the first three SOTFOMs.

As said before, we particularly valued the work of young scholars, and we think that a very positive feature of the collection is precisely the fact that almost all the papers appearing here have been produced by early career researchers, who have already proved to be able to make significant contributions to these areas, despite their very young age.

In what follows, we first provide a few more details concerning each of the three conferences, and then proceed to briefly describe the structure and the contents of this special issue.

2 SOTFOMs I–III

The first conference (SOTFOM I, 7–8 July 2014) was held at the Kurt Gödel Research Center for Mathematical Logic at the University of Vienna on the theme of different approaches to *ontology* in the philosophy of set theory. In particular, the conference focussed on the division between multiversism and universism, that is, on whether there are, respectively, *many* equally legitimate universes of sets, or there is a *unique* maximal universe of sets.

SOTFOM II was held in London at the Institute of Philosophy (12–13 January 2015) and examined the possibility of competing foundations. The talks presented addressed both the *plurality* of different possible set-theoretic backgrounds and the possibility of different languages for examining the foundations of mathematics. In particular, many talks and much discussion focussed on contrasting foundational frameworks (e.g. category theory, homotopy type theory, and set theory).

SOTFOM III (21–23 September 2015) was a closing conference for the Templeton-funded *Hyperuniverse Programme* based at the Kurt Gödel Research Center. It examined several issues pertaining to this programme for addressing independence in set theory, in particular issues regarding maximality axioms for V, and the use of *extensions* in formulating such axioms.

Since then, one further conference hosted by the Munich Center for Mathematical Philosophy (not covered by the present collection) on the relationship between reverse mathematics and philosophy (SOTFOM IV, 9–11 October 2017) has been held, and we hope to have further conferences in the series in the future.

3 Structure of the issue

The issue has three parts, each of which addresses what one could view as the three main topical strands of the three conferences.

The first part (‘New Perspectives on Old Problems’) is philosophical in character, but also contains some technical work (which can be found, in particular, in Sam Sanders’ paper). It provides a new perspective on central issues that have historically been seen to belong to the philosophy of logic and mathematics (*incompleteness*, *indeterminacy* and the issue of what axioms are needed for proving what theorems) by addressing recently emerged methodologies and programmes, such as *reverse mathematics*.

The second one (‘Issues in the Foundations of Set Theory’) is of a more definite set-theoretic character. The section offers a suggestive picture of how research in the philosophy of set theory has been conducted in the last years, that is, by blending very technical set-theoretic work and sophisticated philosophical analysis. Most papers revolve around the core issue of *set-theoretic indeterminacy*, examine different perspectives concerning ways to address it, and assess the mathematical and philosophical potential of various alternatives.

Finally, the third section (‘Homotopy Type Theory and Its Applications’) addresses an approach which has recently garnered much interest, namely Homotopy Type Theory (HoTT). The two papers appearing here focus on the way HoTT provides a kind of invariance and how the approach might be underpinned philosophically.

4 The papers

4.1 Part 1: New perspectives on old problems

There has been much discussion recently concerning the alleged *indeterminacy* of mathematics. Now, while incompleteness is a robustly justified notion in the context of the foundations of mathematics, indeterminacy is a less understood claim which requires an accurate philosophical analysis. While it is well-known that any consistent axiomatic system extending elementary arithmetic is incomplete, we should not *automatically* view mathematics as inherently indeterminate. It is a further step to join the meaningful and relevant connections between incompleteness and indeterminacy.

One additional historical issue in the foundations of mathematics is that of whether we can (and should) rely on the methods of non-computational mathematics. This looks like an especially pressing issue, insofar as it would seem that several mathematical disciplines which have a non-computable content lie at the heart of the current mathematical enterprise.

The papers in this section address the aforementioned questions, and both challenge what one might call the ‘received views’.

In the first paper, 'A Metasemantic Challenge for Mathematical Determinacy', Daniel Waxman and Jared Warren address the issue of *indeterminacy*. They begin by showing that a simplistic association of indeterminacy to independence is misguided, but then proceed to challenge the claim that mathematics is *determinate*, by posing what they call a 'metasemantic' challenge for the claim. This challenge is metasemantic in that it comes from the outside of logical semantics (from epistemology). By following this lead, the authors pursue an altogether alternative route to provide an explanation of indeterminacy.

First, they set out two basic constraints, the 'metaphysical' and the 'cognitive' constraint, which overall serve the purpose of describing a *naturalistically-minded* view of the role and essence of mathematics. Afterwards, they show how indeterminacy is plausible given their constraints. In particular, the metaphysical constraint states that abstract objects should not be used to explain epistemological facts about mathematics, whereas the cognitive constraint asserts that humans cannot be attributed non-computational causal powers. The conjunction of such statements will then entail the fact that mathematics is indeterminate. This is because any commonly used strategy to justify the determinacy of mathematics (in fact, already of *arithmetic*), such as moving to second-order languages, or adding further rules, will have to violate the metaphysical and/or the cognitive constraint. A major upshot of the authors' treatment of the issue is that simply adding new axioms is not enough to compensate for the lack of determinacy of even basic theories such as first-order arithmetic, as doing so will, again, result in a violation of either or both constraints.

In 'Reverse Formalism 16', Sam Sanders presents an interesting case study for the debate on whether we should believe mathematical theories which lack computational content, that is, Robinson's Non-Standard Analysis (**NSA**). Errett Bishop and Alain Connes, in different ways and coming from different philosophical backgrounds, have argued that Robinson's **NSA** lacks meaning, on the grounds that *meaning* in mathematics is exclusively conveyed by 'computational content', a requirement that **NSA**, a theory of infinitesimals, allegedly fails to meet.

Now, Sanders explains how this position can be successfully challenged, by showing that the theorems of **NSA** are *provably equivalent* to the theorems of different versions of second-order arithmetic which are taken into account within the programme of *reverse mathematics* (**RM**). **RM** deals with the search for axioms which are needed to prove statements of *ordinary* mathematics. Now, the **RM** programme shows that most ordinary mathematical statements are proved from axioms which deal with, at most, countable objects and, thus, via the proved equivalence, and contrary to Bishop's and Connes' expectations, **NSA** may, in fact, be seen as having computational content.

One further goal of Sanders' examination of the topic is that of showing that, through using this strategy, Robinson's formalism (as discussed in the latter's article *Formalism 64*) about such (ideal) entities as *infinitesimals* can be fully vindicated.

4.2 Part 2: Issues in the foundations of set theory

As is known, there are fundamental set-theoretic statements which are independent from the **ZFC** axioms (i.e. the current standard axiomatisation of set theory). Independence is, in most cases, established by showing that there are different models of the axioms which, respectively, satisfy a statement or its negation (e.g., the Continuum Hypothesis). In ontological terms, the overall outcome of this is frequently parsed as follows: our discourse concerning the realm of sets and the cumulative hierarchy is indeterminate. Some have pushed the point further, by conjecturing that there is no preferred universe of sets (no *preferred* picture of V, so to speak), but that the subject matter of set theory is constituted by a *plurality* of different universes. The latter is the *multiversist* point of view, which has recently been taken up and defended by some set-theorists and philosophers. Much present work proceeds from this framework, addressing questions such as: is it possible for the believer in just one universe of sets to counteract the multiversist claims? Are there philosophical problems with the positions? How does this bear on how we should think of the project of justifying new axioms for set theory?

Whilst much of the literature focusses on the dialectic between the proponent of the view that there is one determinate universe and those who think that our discourse is indeterminate between many universes, Chris Scambler offers a third option in his paper ‘An Indeterminate Universe of Sets’, which he calls ‘Universe Indeterminism’. He argues that while there is a unique universe that the axioms of set theory describe, such a universe is inherently indeterminate. In order to substantiate such a view, Scambler makes appeal to Feferman’s semi-constructive axiomatisation of set theory (**SCS**), which partly uses an intuitionistic approach, whereby bivalence holds only for statements whose status is conceptually determinate. Thus, he argues, there is no fact of the matter about whether independent statements like the Continuum Hypothesis are true or false, as the Bivalence Principle and Law Of Excluded Middle do not generally hold for such statements. As a consequence, the universe-indeterminist may use a theory similar to **SCS** as a way to explore the determinacy of set-theoretic statements, while rejecting the claim that there are multiple equally legitimate universes of set theory.

In her paper, ‘Why Is the Universe of Sets Not A Set?’, Zeynep Soysal addresses the crucial issue of providing an explanation for the fact that there are collections, such as V (the universe of sets), or Ω (the class of all ordinals) which fail to be sets. A common explanation for this (that she calls ‘the minimal explanation’) is that assuming that they are sets leads to a contradiction (in **ZFC**). As Soysal explains in the paper, though, this is far from constituting a fully satisfactory response, as it would seem that the axioms have been formulated precisely to avoid such contradictions and, thus, that they represent the solution rather than the problem. Therefore, the author chooses to tackle the issue from a broader perspective.

There are two viewpoints concerning the universe of sets, the *actualist* and the *potentialist*. Soysal shows that both conceptions are not fully adequate to provide a response to the question of why the universe of sets is not a set. As for actualism, the most serious drawback is that the Limitation of Size Doctrine supported by actualists (which locates the problem in the fact that the ‘size’ of some collections may be too big to be measurable in set-theoretic terms) ultimately does not provide a good explanation. On the other hand, the main problem with a potentialist account is represented by the fact that the latter re-construes set theory and set-theoretic practice in *modal* terms, something which seems to be at odds with the way the discipline is standardly construed. Therefore, as her preferred alternative to the explanations provided by the two conceptions, Soysal identifies the ‘conception-based’ explanation, which consists in reformulating the minimal explanation (‘V is not a set, as this would entail a contradiction in **ZFC**’), by further specifying that the axioms of **ZFC** are motivated by the *iterative concept of set* and, thus, that viewing V as a set would not only contradict **ZFC** but, what counts more, the iterative concept of set.

Shivaram Lingamneni’s ‘Can We Resolve the Continuum Hypothesis?’ instead examines some extant proposals for justifying new axioms of set theory that might resolve the Continuum Hypothesis. In particular, he argues that none of several contemporary axiom candidates supports a *realist* solution (in the sense that every set-theoretic sentence should be seen as either true or false, and not both). The author begins with a survey of set-theoretic independence. Next, he examines various programmes for independence: (i) the idea of maximizing structures (including Maddy’s proposal for analysing MAXIMIZE and its derivatives), (ii) maximizing sets (as with forcing axioms), (iii) maximizing interpretive power (as with the inner model programme and generic-multiverse truth), (iv) the hyperuniverse programme, and (v) non-set-theoretic foundations. Each he argues, as things stand, does not provide a solution to CH acceptable for the realist.

Neil Barton and Sy-David Friedman’s ‘Maximality and Ontology: How Axiom Content Varies Across Philosophical Frameworks’ argues that the same axiom can express a different content depending upon background philosophical assumptions. In particular, they analyse the *Inner Model Hypothesis* and a reflection axiom ($\sharp$-*generation*) both of which appear to require extensions of universes with more subsets in order to be expressed. They can, however, be coded in infinitary logic when extensions are not available, and so the axioms either express higher-order relationships between universes, or statements about how *ontology* relates to *expressibility* (in higher-order logic), depending on whether or not one thinks such extensions are available in certain contexts.

4.3 Part 3. Homotopy type theory and its applications

Part 3 of the proceedings deals with issues discussed primarily at the second conference in London (entitled ‘Competing Foundations?’) and comprises a pair of papers dealing with homotopy type theory and its role in foundations. This represents a new and interesting approach in the foundations of mathematics, proceeding by endowing intensional type theory with homotopical interpretations. In particular, it provides a new way of looking at several debates in the foundations of mathematics, including notions of *structure*, *constructivism*, and *proof checking*.

The first paper, Dimitris Tsementzis’ ‘A Meaning Explanation for HoTT’, provides a way of intuitively thinking about the subject matter of homotopy type theory. After clarifying the notion of a ‘meaning explanation’ for a theory **T** (i.e. a genuine philosophical or phenomenological account of **T**’s content), he goes on to try and provide one for HoTT, in much the same way as the iterative conception of set does for **ZFC**. He does this by appealing to a notion of *shape* as composed of *points*, and how they may be *observed* from certain viewpoints, combined with a notion of how two such observations may be *symmetric*. These, in turn, are used to interpret *types*, *terms*, *contexts*, and *judgment equality* from HoTT. The various rules of HoTT can then be understood as certain kinds of visualisations from within this framework. He then gives a discussion of some core mathematical structures and univalence, before closing with some remarks on the difference between a *heuristic* and a *justification*.

The second, written by David Corfield, is entitled 'Expressing 'the structure of' in Homotopy Type Theory', and deals with an application of HoTT to definite descriptions (in particular apparent definite descriptions to structures). In the first part of his paper Corfield provides a sketch of dependent type theory, and examines the use of definite description terms (e.g. "the") for dependent types. He applies this analysis to Max Black's spheres and some related problems. He then argues that the ability of his framework to express uniqueness up to canonical equivalence by definite description eliminates the need to find 'canonical' representatives for certain structures (as in set theory). Finally, he provides further application of these observations to structuralism and the case of the complex numbers.

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