Grounding principles for (relevant) implication

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Abstract

Most of the logics of grounding that have so far been proposed contain grounding axioms, or grounding rules, for the connectives of conjunction, disjunction and negation, but little attention has been dedicated to the implication connective. The present paper aims at repairing this situation by proposing adequate grounding principles for relevant implication. Because of the interaction between negation and implication, new grounding principles concerning negation will also arise.

1 Introduction

In the last ten years the notion of grounding has become a vibrant area of research, with the concept being studied from several different perspectives: some papers retrace the history of grounding (e.g. see Betti (2010); Roski (2017); Rumberg (2013)), others deal with the metaphysics of grounding (e.g. see Fine (2012a); Schaffer (2009)), others analyse the properties enjoyed by the notion of grounding (e.g. Krämer (2013); de Rosset (2013)). Yet another approach concerns the logic of grounding: several different logics attempting to identify the structure underlying the notion of grounding have been developed. These logics differ in that they treat the notion of grounding either as a connective (e.g. see Correia (2014); Fine (2012b); Schnieder (2011)), or as a predicate (e.g. Korbmacher (2017)), or as a meta-linguistic relation (see Poggiolesi (2018)). They converge in that they present grounding axioms, or grounding rules, for the classical connectives of conjunction, disjunction and negation. However, perhaps unsurprisingly, little has been said about the connective of implication. More precisely, Schnieder (2011) is the only author who has explicitly formulated the grounding rules for implication, which are the following ones:

$$\frac{B}{A \to B \; because \; B} \to_1 \qquad \frac{\neg A}{A \to B \; because \; \neg A} \to_2$$

These rules should be read in the following way. Suppose that B is true, then $A \to B$ is true because of B (or B is the ground of $A \to B$); suppose that $\neg A$ is true, then $A \to B$ is true because of $\neg A$ (or $\neg A$ is the ground of $A \to B$). Thus the grounds of $A \to B$ are $\neg A$ or B.

Let us confront these formal grounds for implication with some intuitions that might naturally arise when considering ordinary-language conditionals. Let us for example take into account the following three sentences:

1. "If the glass is thrown, then it falls,"
2. "If the ball is pushed, it will roll,"
3. "If snow is white, it is not black."

What are the reasons for the truth of these sentences which all have the form "if A, then B"? For the first conditional the answer seems to be the law of gravity: it is *because* there is the law of gravity that if the glass is thrown, then it falls. For the second conditional, the ground seems to be "the ball is a sphere:" it is *because* the ball is a sphere that if it is pushed, then it rolls. And for the third and last condition the answer seems to be "snow only has one color:" it is *because* snow only has one color that if it is white, it is not black.

These examples suggest an intuitive pattern, that will be called *pattern**, concerning the grounds of conditionals: the ground of a conditional of the form "if A, then B" is neither $\neg A$ nor B but a sentence C such that from A and C, B follows[1]. These intuitions formulated in a completely non-formal way seem to be both natural and reasonable. Moreover, they only apply to a certain type of conditionals, namely indicative conditionals characterized by a connection between antecedent and consequent. This connection is very important since it precisely represents that which is grounded.

There is thus a contrast between, on the one hand, Schnieder's rules according to which the grounds of an implication are basically its truth-conditions, and, on the other hand, the intuitions that the grounds of an implication is a sentence C such that the consequent (of the implication) follows from the antecedent (of the implication) and C. Instead of arguing which interpretation of the grounds of a conditional is the most adequate – this might be interesting task for further research – we will follow an analysis recently put forward in Poggiolesi (2019) according to which there actually is a way to accommodate these two divergent approaches at the formal level: indeed while Schnieder's rules are suited for the material implication and thus find their natural habitat in classical logic, the insights that we have just illustrated are suited for relevant implication and should thus be further developed in the framework of relevance logic (e.g. see Anderson and Belnap (1975); Mares (2014); Poggiolesi (2020); Dunn and Restall (2002)). The aim of the present work is precisely this, namely to further develop the insights just exposed and to formulate adequate grounding principles for relevant implication. To develop our approach, we will rely on the work of Poggiolesi (2016b), who presents a definition of the notion of complete and immediate formal grounding, which naturally motivates and justifies grounding principles for the classical connectives. In this paper we will modify Poggiolesi's definition in order to develop natural and justified grounding principles for the relevant implication.

Another reason for starting from Poggiolesi's approach is that her way of treating the grounds of sentences in negative form is in sync with previous

[1] From now on, we use the term *following from* in Anderson and Belnap (1975)'s sense. B follows from A, C if there is a deduction of B from A, C which actually uses A, C (and A, C alone).

intuitions about the grounds of indicative conditionals being characterized by a connection between antecedent and consequent. Indeed, Poggiolesi's analysis is faithful to the intuition that the ground of a sentence like "it is not raining" is the sentence "it is sunny," which is such that from "it is sunny" and "it is raining" a contradiction *follows*. At the formal level, this is captured by grounding principles that all internalize the following schema: the ground of a formula $\neg A$ is a formula B (having a certain characteristic, called *complexity*) such that from A and B a contradiction is (relevantly) derivable.[2] This similarity suggests that Poggiolesi's framework may be a fruitful starting point for our project.

Note that as the approach of Poggiolesi (2016b) only provides complete[3] and immediate grounds for the classical connectives, our proposal, being based on it, will only provide complete and immediate grounds for (relevant) implication. The study of other types of grounds - complete and mediate, partial and immediate, partial and mediate - for implication is left for further research. Finally, our study being the first framework where negation and (relevant) implication interact, we will also enhance the complete and immediate grounding principles governing negation, notably adding principles regulating the complete and immediate grounds of negation of implication.

The paper is organized as follows. In *Section 2* we will briefly remind the reader the definition of the notion of complete and immediate grounding in the classical framework developed by Poggiolesi (2016b). In *Section 3* we will adapt this definition for the framework of relevant logic. While in *Section 4* we will discuss the grounding principles concerning implication which emerge from the account of Section 3, in *Section 5* we will discuss the grounding principles concerning the negation of implication which emerge from the account of Section 3. In *Section 6* we will draw some conclusions.

2 A definition of the notion of complete and immediate formal grounding in the classical framework

We use this section to briefly recall the definition proposed in Poggiolesi (2016b) of the notion of complete and immediate formal grounding, which will play an important role in the sequel. Two very simple ideas motivate it. The first consists in organizing all formulas of the propositional classical language in a grounding hierarchy: each level of the hierarchy contains formulas of different complexity, with complexity increasing from bottom to top. We will call this complexity *g-complexity* to differentiate it from the standard notion of logical complexity.

Once all formulas are organized into the hierarchy, the task is to identify

[2]Even if Poggiolesi's approach is developed in classical logic, the derivations she uses for negative formulas are clearly relevant.

[3]Note that complete grounds are different from full grounds, which are often advocated in the literature. On the difference, see Fine (2012b); Poggiolesi (2016a).

the formulas that stand in a dependence relation. But how is the dependence relation formally defined? Here it is where the second idea comes in: dependency is formalized by the two clauses of *positive* and *negative* derivability. Positive derivability states that the conclusion should be derivable from its grounds, while negative derivability states that the negation of the conclusion should be derivable from the negation of each ground. Thus, according to Poggiolesi's approach, a grounding relation is nothing but a dependence relation (given by positive and negative derivability) with an asymmetry or directionality, which is given by the increase of complexity from the grounds to the conclusion.

Note that the account put forward in Poggiolesi (2016b) involves a distinction between grounds and robust conditions, which can be described briefly on the example of a disjunction like $A \vee B$, in a situation where the formula A is true. In this case, A is certainly a ground for $A \vee B$; but in order for A to be the *complete* ground for $A \vee B$, it is necessary to specify that B is false (i.e. that B is not also a ground for $A \vee B$); in other terms, it is the falsity of B that ensures that, or is a *(robust) condition* for A to be the complete ground for $A \vee B$. Thus, A is the complete and immediate formal ground for $A \vee B$ under the robust condition that B is false.[4] The reader is referred to Poggiolesi (2016b) for a detailed explanation and discussion of the idea of robust conditions in a grounding framework. Robust conditions are denoted by square brackets and will be introduced in Proposition 2.8.

We now present the formalism inspired by these ideas. We refer the reader to Poggiolesi (2016b) for an even more detailed explanation of the notions introduced here.

Definition 2.1. The classical language $\mathcal{L}^c$ is composed of a denumerable stock of propositional atoms (p, q, r, ...), the logical operators $\neg$, $\wedge$ and $\vee$, the parentheses (,). The connectives $\rightarrow$ and $\leftrightarrow$ are defined as usual; the symbol $\perp$ is defined as $A \wedge \neg A$.

Once the classical language $\mathcal{L}^c$ is given, we can standardly define, by means of the classical Hilbert system **C** (e.g. see Troelstra and Schwichtenberg (1996)), the notion of classical derivability. We will write $M \vdash_C A$ to denote the fact that the formula A is derivable in the Hilbert system for classical logic **C** from the multiset[5] of formulas M.

We now introduce the key notion of g-complexity, which is a way of assigning a number to each formula of the language $\mathcal{L}^c$. The way that number is calculated reflects deep grounding-relevant features. As we will see, g-complexity leads to the identification of the relation of *being completely and immediately less g-complex*: if a multiset M is completely and immediately less g-complex than a formula A, then the sum of the g-complexity of its members is one less than the g-complexity of A.

Definition 2.2. As it is standard, we call atoms as well as negation of atoms *literals*. $l, l', ...$ denote literals.

[4] Even if it is not spelled out in these terms, a similar idea can be found in Fine (2010).

[5] We work with multisets of formulas rather than with sets of formulas because we need to take into account the number of occurrences of each formula of M.

Definition 2.3. The g-complexity of a formula $A \in \mathcal{L}^c$, $gcm(A)$, is defined in the following way:

- $gcm(l) = 0$,
- $gcm(\neg\neg A) = 1 +$ $gcm(A)$,
- $gcm(A \circ B) = gcm\neg(A \circ B) = 1 + gcm(A) + gcm(B)$.

where the symbol $\circ$ stands for either conjunction or disjunction.

To understand the notion of g-complexity, it must be kept in mind that grounding is concerned entirely with truths. Accordingly, the appropriate notion of complexity should track relationships among the truths expressed by the formulas if they were true. If A and B express truths, then the truth expressed by $A \wedge B$ or $A \vee B$ is obtained from the previous truths using a single operation, just as the formulas $A \wedge B$ and $A \vee B$ are constructed from the formulas A and B using a single connective. Counting the connective in this case is faithful to the relationship of interest among truths and indeed $gcm(A \circ B) = gcm(A) + gcm(B) + 1$.

By contrast, the negation is different, because there is no sense in which if a formula of the form $\neg A$ expresses a truth, then that truth is constructed from A itself. Consider for instance the formulas p and $\neg p$ (namely the literals). p is atomic thus has g-complexity 0, but does that mean that $\neg p$ should count as having g-complexity 1? That would be justified if the truth $\neg p$ (when it is a truth) was constructed from the truth p; but this is not the case in general, not least because when one of the formulas is a truth, the other (often) is not. From the point of view of grounding, which deals solely in truths, there is no truth from which $\neg p$ can be formally constructed, so, like p, it is atomic. Similar points hold for formulas of the form A, $\neg A$, where A is either a conjunction or a disjunction: the complexity of the latter cannot be counted as one more than the complexity of the former, since it is not reducible to it. Therefore in the formula $\neg A$ (where A does not itself start with a negation), the only g-complexity to count is that of A. This is precisely what Defintion 2.3 does, by setting the complexity of $A \circ B$ and $\neg(A \circ B)$ on the same level.

The case of the double negation, however, is different. A formula like $\neg\neg A$, if true, can be reduced to another, simpler truth, namely A. Moreover, such reduction is direct: there is no "intermediate" truth that one passes through to obtain the former from the latter. Thus, it makes sense to count the g-complexity of $\neg\neg A$ as equal to that of A plus one.

Let us now move to the key notion of *being completely and immediately less g-complex*. In order to define this notion, we first need to introduce other notions, namely that of *converse* of a formula, and the relations of *a-c* equivalence and $\cong$. (The notion of converse of a formula and the relation $\cong$ will be directly used to define the relation of "being completely and immediately less g-complex"; the relation of *a-c* equivalence serves to define the relation $\cong$).

Definition 2.4. Let D be a formula. The *converse* of D, written D^*, is defined in the following way

$$D^* = \begin{cases} \neg^{n-1}E, & if\ D = \neg^n E\ and\ \text{n}\ is\ odd \\ \neg^{n+1}E, & if\ D = \neg^n E\ and\ \text{n}\ is\ even \end{cases}$$

where the principal connective of E is not a negation, $n \geqslant 0$ and 0 is taken to be an even number.[6]

Note that the advantage of working with the notion of converse of a formula A rather than the negation of the formula A is that, while negation might increase the g-complexity of A, the converse of A is a formula B which has the same g-complexity has A. Let us provide some examples that help to clarify Definition 2.4. If $D = \neg\neg\neg\neg p$, then its converse, D^*, is $\neg\neg\neg\neg\neg p$. If $D = \neg(A \wedge B)$, then its converse, D^*, is $(A \wedge B)$; finally, if $D = (A \vee B)$, then its converse, D^*, is $\neg(A \vee B)$. From now on we will use capital letters to refer to formulas of the language $\mathcal{L}^c$ and their converse.

Definition 2.5. Consider a formula A. We will say that A is *a-c equiv* (for associatively and commutatively equivalent) to B, if, and only if, A can be obtained from B by applications of associativity and commutativity of conjunction and disjunction.

Let us provide some examples of formulas that are *a-c equiv*. If A is of the form $E \wedge F$, then the formula $F \wedge E$ is *a-c equiv* to it. To take another example, if A is of the form $\neg((B \vee C) \wedge (D \vee F))$ the formulas $\neg((C \vee B) \wedge (D \vee F))$, $\neg((B \vee C) \wedge (F \vee D))$, $\neg((C \vee B) \wedge (F \vee D))$ are *a-c equiv* to it.

Definition 2.6. For any two formulas A, B, $A \cong B$ if, and only if:

A is *a-c equiv* to B or A is *a-c equiv* to B^*

As extensively discussed in Poggiolesi (2016b), two formulas A and B stand in the relation denoted by $\cong$ when they *are about*, or *pertain to*, or *concern* the same issue. The relation $\cong$ is thus analogous (though not equivalent) to the notion of factual equivalence discussed in Correia (2014, 2016).

Definition 2.7. Given a multiset of formulas M and a formula C of the classical language $\mathcal{L}^c$, we say that M is *completely and immediately less g-complex* than C, if, and only if:

- $C \cong \neg\neg B$ and $M = \{B\}$ or $M = \{B^*\}$, or
- $C \cong (B \circ D)$ and $M = \{B, D\}$, or $M = \{B^*, D\}$, or $M = \{B, D^*\}$, or $M = \{B^*, D^*\}$.

The multiset M is completely and immediately less g-complex than the formula C since it contains *all* those 'subformulas'[7] of C which are such that the sum of their g-complexity is *one less than* that of C.

[6] Note that $\neg^0 E$ is just E. Also we keep the term *converse* for continuity with Poggiolesi's work. However, one should not confuse $*$ with an idempotent operator.

[7] For the rigorous definition of subformula in a grounding framework see Poggiolesi (2016b).

Definition 2.8. For any consistent multiset of formulas $C \cup M$ such that C and M are formulated in the classical language $\mathcal{L}^c$, we say that, under the robust condition C (that may be empty), M *completely and immediately formally grounds* A, in symbols $[C]\ M \ \mid\!\sim\ $ A, if and only if:

- $M \vdash_C A$
- $C, \neg(M) \vdash_C \neg A$
- $C \cup M$ is completely and immediately less g-complex than A in the sense of Definition 2.7.

where $\neg(M) := \{\neg B | B \in M\}$.[8]

Under the robust condition C, the multiset M completely and immediately formally grounds A if, and only if, (i) A is derivable from M – positive derivability; (ii) $\neg A$ is derivable from $\neg(M)$ plus C – negative derivability; (iii) $C \cup M$ is completely and immediately less g-complex than A.

Note that if this definition, together with the classical language on which it is based, was extended to cover the grounds of material implication, then such grounds would seemingly be similar to those for the disjunction connective and thus in line (though formulated in a different framework) with the grounds put forward by Schnieder (2011). However, as almost everyone who learns the principles of classical logic for the first time experiences a feeling of dissatisfaction with material implication and thus comes to be convinced that this connective does not really represent the conditional that we actually use, in an analogous way, when thinking of the grounds of conditionals in the natural language (see the Introduction), it seems that these are different from those that emerge in the classical setting. Because of this difference, the grounds for conditionals merits further examination and this is precisely what we will do in the rest of the paper, combining the relevance approach with Poggiolesi's account.

3 A definition of the notion of complete and immediate formal grounding in a relevant framework

We will use this section to provide a definition of the notion of complete and immediate grounding that conservatively extends Definition 2.8 and properly deals with relevant implication. First of all note that one of the main characteristics of Definition 2.8 lies in its flexibility: since grounding is relative to the

[8]Note that when the multiset M is composed of only one formula, then positive and negative derivability amounts to an equivalence relation. This is typically the case of the two formulas A and $\neg\neg A$, which although equivalent, are such that A completely and immediately grounds $\neg\neg A$ and not viceversa. While other accounts of grounding (e.g. Fine (2012b,a)) take this asymmetry as primitive, Definition 2.8 explains why A grounds $\neg\neg A$ and not viceversa: because only in one case the g-complexity increases from the grounds to the conclusion.

notion of derivability and g-complexity, grounding is also relative to the logic in which derivability and g-complexity are defined. Poggiolesi uses classical derivability and a notion of g-complexity conceived for classical connectives since she is interested in the grounding analysis of classical connectives. But if in the definition we use derivability in a relevance logic and a g-complexity conceived for relevant connectives, we will end up with a grounding relation appropriate for a relevant framework. In what follows, we will consider a grounding relation in terms of derivability in the relevant Hilbert system **R** (see Anderson and Belnap (1975); Mares (2014); Dunn and Restall (2002)); this will naturally provide grounding principles for the relevant conditional of **R**. We chose to work with **R** since amongst relevance logics it is probably the most well-known.

Definition 3.1. The relevant language $\mathcal{L}^r$ is composed of a denumerable stock of propositional atoms (p, q, r, $\dots$), the logical operators $\neg$, $\wedge$, $\vee$ and $\rightarrow$, the parentheses (,). Propositional formulas are standardly constructed and the set of propositional formulas so defined is denoted by $\mathbb{PF}$.

Once the relevant language $\mathcal{L}^r$, together with a set of propositional formulas $\mathbb{PF}$, is given, we can introduce the Hilbert system **R**, whose axioms and rules are shown in Figure 2. We will write $M \vdash_R A$ to denote that "there is a proof in **R** that M *entails* A" in the sense of Anderson and Belnap (1975, p.277).[9] An important property of the relevant system **R** linked to this notion of entailment is the *Entailment theorem*, namely :

Theorem 3.2. *For any formula A and multiset $M \in \mathcal{L}^r$, we have that: $M \vdash_R A$ if, and only if, $\vdash_R \bigwedge M \rightarrow A$*

Proof. From left to right using Anderson and Belnap's theorem and the reasoning in its proof, in particular the relation with the natural deduction calculus for **R**, see (Anderson and Belnap, 1975, §23.6, §27.2). From right to left, an immediate application of the rules of the Hilbert calculus. $\square$

As explained by Dunn and Restall (2002), the system **R** has both an algebraic semantics and a frame-semantics. The former has been introduced by Dunn (1970), whilst the latter has been developed by Urquhart (1972); Fine (1974) and Routley and Meyer (1973). Routler and Meyer's approach is probably the most well-known (see Mares (2014)) and it is based on the idea of interpreting the conditional by means of a ternary relation $Rijz$ such that $i \models A \rightarrow B$ if, and only if, $\forall j, z \in W$ (if $Rijz$ and $j \models A$, then $z \models B$). In this approach, while the connectives of conjunction and disjunction are treated classically, the negation connective is treated by an unary operation $+$ on worlds, such that for each world i, there is a world i^+, $i \models \neg A$ if, and only if, $i^+ \not\models A$.[10] We will not

[9]This notion corresponds to a derivation from M to A with control indexes (namely M and A have the same set of indexes) in the natural deduction calculus for **R**, see Dunn and Restall (2002).

[10]Differently from worlds of Kripke semantics, worlds of the relevant approach can be either inconsistent or incomplete. This is due to fact that formulas as $p \wedge \neg p \rightarrow q$ or $p \rightarrow q \vee \neg q$ are not wanted to be proved to be valid. And for that matter, we need words to be able to satisfy both p and $\neg p$, or to not satisfy neither q nor $\neg q$.

Figure 1: Truth in a relevant model $\mathcal{M}$

T1	A is true in $\mathcal{M}$ if, and only if, $\neg\neg A$ is,
T2	$A \wedge B$ is true in $\mathcal{M}$ if, and only if, A is and B is,
T3	$A \vee B$ is true in $\mathcal{M}$ if, and only if, A is or B is.

dwell on details here; a clear and rigorous explanation can be found in Dunn and Restall (2002). Let us however underline two important points that will be useful later. First of all, by denoting with $\mathcal{R}^+$ the class of relevant frames of Routley and Meyer (1973), the soundness and completeness theorem is provable for the logic **R**.

Theorem 3.3. *For any formula $A \in \mathcal{L}^r$, we have that: A is provable in the relevant system* **R** *if, and only if, A is valid in the class of frames $\mathcal{R}^+$.*

Proof. See (Dunn and Restall, 2002, p.70-77). □

Secondly, let $\mathcal{R}^+$ be a relevant frame and $\mathcal{M}$ a model based on that frame. Then the facts listed in Figure 1 hold in a model $\mathcal{M}$.

In what follows we will formulate a definition of the notion of complete and immediate formal grounding in a relevant framework. In order to contain the complexity of the issue, we will restrict our attention on the grounds of those implicative formulas which contain conjunction, disjunction and negation, but are not in their turn composed of other implications. We leave the grounding analysis of these formulas for future research.

Definition 3.4. Given the set $\mathbb{PF}$ of all formulas of the language $\mathcal{L}^r$ we isolate the subset $\mathbb{PF}_{\rightarrow}$ that only contains those formulas that do not contain nested implications, i.e. if there is an implication, it does not itself contain another.

3.1 Positive and negative derivability in a relevant framework

The first two ingredients of Proposition 2.8 are positive and negative derivability, with derivability defined in classical logic. Now we are no longer interested in classical logic but in the relevant system **R**. As concerns positive derivability, in the classical framework, following Definition 2.8, the condition for a multiset of formulas M to completely and immediately ground a formula A, under a robust condition C (which may be empty), is that A is classically derivable from M. Now in the relevant framework, the condition becomes that there is a proof that A is entailed by the *conjunction* of the elements of M, denoted by $\bigwedge M \vdash_R A$. The need for the use of the entailment relation in **R** is motivated by Entailment theorem enjoyed by this relation, that will prove helpful later (e.g. see Section 4). The need for the use of the conjunction can be explained with an example.

Figure 2: Hilbert-style axiomatisation of relevant logic **R**

A1.1 $A \to A$	A1.2 $(A \to ((A \to B)) \to (A \to B)$
A1.3 $A \to ((A \to B) \to B)$	A1.4 $(A \to B) \to ((B \to C) \to (A \to C))$
A2.1 $A \wedge B \to A$	A2.2 $A \wedge B \to B$
A2.3 $((A \to B) \wedge (A \to C)) \to (A \to B \wedge C)$	
A3.1 $A \to A \vee B$	A3.2 $B \to A \vee B$
A3.3 $(A \vee B \to C) \leftrightarrow (A \to C) \wedge (B \to C)$	
A4.1 $\neg\neg A \to A$	A4.2 $(A \to \neg B) \to (B \to \neg A)$
A5 $((A \wedge B) \vee C) \to ((A \wedge B) \vee (A \wedge C))$	
MP $A \to B, A \vdash_R B$	
IC $A, B \vdash_R A \wedge B$	

Consider a true disjunction $A \vee B$ and suppose that both A and B are true. Then even in a relevant framework it seems plausible to say that both A and B completely and immediately ground $A \vee B$. This fits with the fact that $A \vee B$ is entailed by A, but also by B. However since the relevant logic **R** does not contain any weakening axiom, $A \vee B$ is not entailed by $\{A, B\}$. Therefore, to remedy the situation one simple solution is to use the conjunction $A \wedge B$; indeed we have that $A \wedge B \vdash_R A \vee B$.

For negative derivability, the situation is analogous. In the classical framework, following Definition 2.8, the condition for a multiset of formulas M to completely and immediately ground a formula A, under a robust condition C (which may be empty), is that $\neg A$ is classically derivable from $\neg(M), C$, where $\neg(M)$ stands for the negation of each element of M. In the relevant framework, the condition becomes that there is a proof that $\neg A$ is entailed by C and the conjunction of the negation of each element of M, denoted by $C, \bigwedge \neg(M) \vdash_R \neg A$. The motivation for this condition is analogous to that offered for positive derivability. The need for the use of the entailment relation in **R** is motivated by the Entailment theorem enjoyed by this relation, that will prove helpful later (e.g. see Section 4). The need for the use of the conjunction can be explained with an example. Consider a true conjunction $A \wedge B$. Even in a relevant framework it seems plausible to say that both A and B completely ground $A \wedge B$. This fit with the fact that $\neg(A \wedge B)$ is entailed by $\neg A$, but also from $\neg B$. However, since the relevant logic **R** does not contain any weakening, $\neg(A \wedge B)$ is not entailed by $\{\neg A, \neg B\}$. The solution to this situation is to use the conjunction $\neg A \wedge \neg B$; indeed we have that $\neg A \wedge \neg B \vdash_R \neg(A \wedge B)$.

The proposed versions of positive and negative derivability adapted for a relevant framework have been intuitively motivated. In Section 3, Theorem 3.12, we will prove that they are adequate to preserve their role in the definition of the grounding relation.

3.2 G-complexity in a relevant framework

The third ingredient of Definition 2.8 is the notion of being completely and immediately less g-complex, which provides the directionality or asymmetry of grounds: in a grounding relation the grounds must always be less g-complex than their conclusion. As we have seen, in order to define this notion, Poggiolesi proceeds into two steps. First, she defines the notion of g-complexity, which assigns to each formula a count of the complexity as appropriate for grounding. Successively and on the basis of g-complexity, she defines the relation of *being completely and immediately less g-complex* between a multiset of formulas M and a formula A: this relation is a central ingredient of the final account for the notion of complete and immediate formal grounding.

We will follow an analogous procedure to extend both the notion of g-complexity and the relation of being completely and immediately less g-complex, in such a way that also formulas containing (relevant) implications are taken into account.

Definition 3.5. The g-complexity of a formula A will take values in a set S such that each element of S consists of a natural number n and a multiset of ordered pairs of natural numbers $\phi_1, ..., \phi_n$, i.e. each element of S has the form $(n, \{\phi_1, \ldots, \phi_m\})$ and will be called *S-element*. Define the binary operation $+^{\cdot} : S \to S$, as follows:

$$(n, \{\phi_1, \ldots, \phi_m\}) +^{\cdot} (n', \{\phi'_1, \ldots, \phi'_{m'}\}) = (n + n', \{\phi_1, \ldots, \phi_m, \phi'_1, \ldots, \phi'_{m'}\})$$

Note that $(S, +^{\cdot})$ so defined is an algebra with $+^{\cdot}$ an associative and commutative operator, and $(0, \emptyset)$ the identity (or zero) element (i.e. for all $(n, \{\phi_1, \ldots, \phi_m\}) \in S$, $(n, \{\phi_1, \ldots, \phi_m\}) +^{\cdot} (0, \emptyset) = (n, \{\phi_1, \ldots, \phi_m\})$.

As a point of notation, we write

i. $n \in \mathbb{N}$ as shorthand for $(n, \emptyset)$,

ii. $(k, l) \in \mathbb{N}^2$ as shorthand for $(0, \{(k, l)\})$.

Finally, define the operator $< \bullet, \bullet >: (\mathbb{N} \times \emptyset)^2 \to S$ (where $(\mathbb{N} \times \emptyset)^2$ is the subset of S consisting of elements of the form $(n, \emptyset)$ for $n \in \mathbb{N}$) by:

$$< (n, \emptyset), (m, \emptyset) >= (0, \{(n, m)\})$$

Note that, using the notational convention introduced above, we have $< n, m >= (n, m)$. Indeed, by i., $< (n, \emptyset), (m, \emptyset) >$ can be written as $< n, m >$; by ii., $(0, \{(n, m)\})$ can be written as (n, m).

Definition 3.6. The g-complexity of a formula $A \in \mathbb{PF}_{\rightarrow}$, $gcm'(A)$, is defined in the following way:

- $gcm'(l) = (0, \emptyset)$,
- $gcm'(\neg\neg A) = (1, \emptyset) +^{\cdot} gcm'(A)$,
- $gcm'(A \rightarrow B) = gcm'(\neg(A \rightarrow B)) =< gcm'(A), gcm'(B) >$,
- $gcm'(A \circ B) = gcm'(\neg(A \circ B)) = (1, \emptyset) +^{\cdot} gcm'(A) +^{\cdot} gcm'(B)$.

where the symbol $\circ$ stands for either conjunction or disjunction. Note that, whenever $gcm'(A) = n$ and $gcm'(B) = m$, $gcm'(A \rightarrow B) = gcm'(\neg(A \rightarrow B)) = (n, m)$ thanks to the definition of the operator $< \bullet, \bullet >$ and notational conventions i. and ii.

Let us provide some examples of g-complexity before turning to the clarification of this notion.[11]

$$gcm'((p \wedge q) \wedge \neg\neg(q \vee (r \vee s))) = (5,\ \emptyset) = 5;$$

$$gcm'((\neg(s \vee (t \wedge r)) \rightarrow (p \wedge \neg q))) = (0,\ \{(2,1)\}) = (2,\ 1);$$

$$gcm'(\neg\neg(\neg(s \vee (t \wedge r)) \rightarrow (p \wedge \neg q))) = (1, \{(2,1)\}).$$

For formulas that do not contain the connective of implication, this new notion of g-complexity works analogously to that introduced in the classical framework, since it amounts to a natural number followed by an empty-set of ordered pairs of natural numbers, hence, i.e., by condition i., to a natural number. In view of principles T1-T3, when the implication does not occur in a formula A, there is no reason to change the way of calculating the g-complexity of A from the classical to the relevant framework.

Turning to the implication, let us begin by considering the case where antecedent and consequent are atoms, i.e. the implication of the form $p \rightarrow q$. Note that, if this implication is true, unlike say the conjunction $p \wedge q$, it is *not* constructed from the truths p and q. Rather, by the very idea of relevance, the truth is related to the link between p and q, and that link is not formally reducible to p nor to q. The use of an ordered pair allows us to capture this situation, whilst nevertheless keeping track of the fact that $p \rightarrow q$ involves p and q rather than other formulas. So, in this case, since the g-complexity of both p and q is 0, the g-complexity of $p \rightarrow q$ will be the ordered pair (0, 0). Generalizing this reflection to any implication of the form $A \rightarrow B$, we obtain as its g-complexity the ordered pair composed of the g-complexity of A and the g-complexity of B. Indeed, the truth of a relevant implication like $A \rightarrow B$ is not built up from the truth of its antecedent and the truth of its consequent

[11] Note that g-complexity does not respect logical equivalence, since the g-complexity of p, p is not the same as the g-complexity of $p \wedge p$. This fact is further developed in the diversified syntax introduced in Francez (2018).

(as it is the case for material implication!), yet continues to involve them. The g-complexity reflects this situation, insofar as it involves the g-complexities of the A and B without putting them together and adding one.

Note that this notion of g-complexity treats implication in an analogous way to negation. As we have seen in the previous section, the g-complexity of a formula of the type $\neg A$ (where A does not contain as main connective a negation) is equal to the g-complexity of A. This is because, on the one hand, if true, the truth $\neg A$ does not build upon the truth of A, so we cannot count its g-complexity as that of A plus one. And, on the other hand, $\neg A$ is still composed of the elements of A, and thus the elements of A should determine its g-complexity. The situation is the same for an implication, such as $A \to B$. On the one hand, if true, the truth is not constructed up from the truth of A and the truth of B; hence its g-complexity cannot be counted as the g-complexity of A plus that of B plus one. On the other hand, $A \to B$ is composed by the elements of A and the elements of B, so they should determine its g-complexity. Defining the g-complexity of $A \to B$ as the ordered pair of their g-complexities does just this.

Recall from the previous Section that beyond the notion of g-complexity, the definition of the relation of *being completely and immediately less g-complex* required the notion of converse of a formula, as well as the relations $\cong$ and *a-c equiv*. Whilst the former remains as in Definition 2.4, the relations $\cong$ and *a-c equiv* have to be adapted to the relevant framework, as follows.

Definition 3.7. Consider a formula $A \in \mathbb{PF}_{\to}$. We will say that A is *a-c equiv$'$* (for associatively and commutatively equivalent) to B, if, and only if, A can be obtained from B by applications of associativity and commutativity of conjunction and disjunction.

The notion of *a-c equiv$'$* naturally extends the notion *a-c equiv* by also taking into account formulas containing implications. For example, if A is of the form $G \to ((B \vee C) \vee (D \vee F))$, then the formulas $G \to ((B \vee D) \vee (C \vee F))$, $G \to ((D \vee B) \vee (F \vee C))$, $G \to ((B \vee F) \vee (D \vee C))$ are all *a-c equiv$'$* to it.

Definition 3.8. For any $A, B \in \mathbb{PF}_{\to}$, $A \cong' B$ if, and only if:

A is *a-c equiv$'$* to B or A is *a-c equiv$'$* to B^*

We finally have all the elements to introduce the notion of *completely and immediately less g-complex*, which will be a key notion in the account of the notion of complete and immediate formal grounding

Definition 3.9. Given a multiset of formulas M and a formula $C \in \mathbb{PF}_{\to}$, we say that M is *completely and immediately less g-complex* than C (in a relevant framework), if, and only if:

- $C \cong' \neg\neg B$ and $M = \{B\}$ or $M = \{B^*\}$, or
- $C \cong' B \circ D$ and $M = \{B, D\}$, or $M = \{B^*, D\}$, or $M = \{B, D^*\}$, or $M = \{B^*, D^*\}$, or

- $C \cong' \neg\neg B \to D$ or $C \cong' B \to \neg\neg D$, and $M = \{B \to D\}$ or $M = \{\neg(B \to D)\}$, or

- $C \cong' B \circ D \to E$ and $M = \{B \to E, D \to E\}$, or $M = \{\neg(B \to E), D \to E\}$, or $M = \{B \to E, \neg(D \to E)\}$, or $M = \{\neg(B \to E), \neg(D \to E)\}$, or

- $C \cong' \neg(B \circ D) \to E$ and $M = \{B^* \to E, D^* \to E\}$, or $M = \{\neg(B^* \to E), D^* \to E\}$, or $M = \{B^* \to E, \neg(D^* \to E)\}$, or $M = \{\neg(B^* \to E), \neg(D^* \to E)\}$, or

- $C \cong' B \to D \circ E$ and $M = \{B \to D, B \to E\}$, or $M = \{\neg(B \to D), B \to E\}$, or $M = \{B \to D, \neg(B \to E)\}$, or $M = \{\neg(B \to D), \neg(B \to E)\}$, or

- $C \cong' B \to \neg(D \circ E)$ and $M = \{B \to D^*, B \to E^*\}$, or $M = \{\neg(B \to D^*), B \to E^*\}$, or $M = \{B \to D^*, \neg(B \to E^*)\}$, or $M = \{\neg(B \to D^*), \neg(B \to E^*)\}$.

Let $gcm'(M)$ amount to the operation $+^{\cdot}$ applied to the g-complexity of each of the members of M. Both $gcm'(C)$ and $gcm'(M)$ are S-elements. Thanks to the form of S-elements, there are three ways in which $gcm'(M)$ can be reasonably compared to $gcm'(C)$ and seen as immediately lower:

(i) $gcm'(C)$ is equal to $(n, \{\phi_1, ..., \phi_n\})$ and $gcm'(M)$ is equal to $(n-1, \{\phi_1, ..., \phi_n\})$.

(ii) $gcm'(C)$ is equal to $(n, \{\phi_1, ..(m, p).., \phi_n\})$ and $gcm'(M)$ is either equal to $(n, \{\phi_1, ..(m-1, p).., \phi_n\})$ or to $(n, \{\phi_1, ..(m, p-1).., \phi_n\})$.

(iii) $gcm'(C)$ is equal to $(n, \{\phi_1, ..(m, p).., \phi_n\})$ and $gcm'(M)$ is either equal to $(n, \{\phi_1, ..(m', p), (m'', p).., \phi_n\})$, where $m = m' + m'' + 1$, or to $(n, \{\phi_1, ..\{(m, p'), (m, p'')\}.., \phi_n\})$, where $p = p' + p'' + 1$.

In each of the cases where the multiset M is completely and immediately less g-complex than C, see Definition 3.9, $gcm'(M)$ is immediately lower than $gcm'(C)$ according to one of (i)-(iii).[12] Hence, as it was the case in the classical framework, the multiset M is completely and immediately less g-complex than the formula C since it contains all those 'subformulas' of C which are such that the sum of their g-complexity is immediately lower that of C.

Let us provide some examples that help to clarify this point.

- the multisets $\{(p \wedge q) \wedge r\}$, $\{\neg((q \wedge p) \wedge r)\}$ are both completely and immediately less g-complex than the formulas $\neg\neg(r \wedge (q \wedge p))$ and $\neg\neg\neg(r \wedge (q \wedge p))$;

- the multisets $\{(p \vee q), r\}$, $\{\neg(p \vee q), r\}$, $\{(p \vee q), \neg r\}$, $\{\neg(p \vee q), \neg r\}$ are all completely and immediately less g-complex than the formulas $(p \vee q) \wedge r$ and $(q \vee p) \wedge r$, as well as than the formulas $\neg((p \vee q) \wedge r)$, $\neg((q \vee p) \wedge r)$.

[12] Note that in the last five items of Definition 3.9, the natural number in the S-element corresponding to the g-complexity of C is 0.

- the multisets $\{s \to p, s \to r\}$, $\{\neg(s \to p), s \to r\}$, $\{s \to p, \neg(s \to r)\}$, $\{\neg(s \to p), \neg(s \to r)\}$ are all completely and immediately less g-complex than the formulas $s \to p \circ r$, $\neg(s \to p \circ r)$, as well as the formulas $s \to r \circ p$, $\neg(s \to r \circ p)$, where $\circ = \{\wedge, \vee\}$.

3.3 The definition of complete and immediate formal grounding in a relevant framework

We now have all the elements to introduce a definition for the notion of complete and immediate grounding in a relevant framework.

Definition 3.10. For any formula $A \in \mathbb{PF}_{\to}$, and for any consistent multiset of formulas $C \cup M$ such that C and $M \in \mathbb{PF}_{\to}$, we say that, under the robust condition C (that may be empty), M *completely and immediately formally grounds* A in a relevant framework, in symbols $[C]\ M\ \mid\!\sim_R$ A, if and only if:

- $\bigwedge M \vdash_R A$,
- $C, \bigwedge \neg(M) \vdash_R \neg A$,
- $\{C, M\}$ is completely and immediately less g-complex than A according to Definition 3.9.

This definition of complete and immediate formal grounding adapts Poggiolesi's Definition 2.8 to the relevant framework. The notion of g-complexity describes the grounding hierarchy in which formulas, including relevant implications, are organized, while the notions of positive and negative derivability tell us which formulas in each step of this hierarchy enter into a dependence relation. The fact that in both positive and negative derivability the premises are in conjunctive form is motivated by the issue of relevance as discussed in the previous section.

We will now prove that Definition 3.10 conservatively extends Definition 2.8: the principles concerning negation, conjunction and disjunction in the classical framework remain the same in the relevant framework. In the next section we will study the grounding principles governing relevant implication and negation of relevant implication that emerge from Definition 3.10.

Lemma 3.11. *Given a multiset M and a formula A such that both only contain the connectives of conjunction, disjunction and negation, we have that M is completely and immediately less g-complex than A according to Definition 2.7 if, an only if, M is completely and immediately less g-complex than A according to Definition 3.9.*

Proof. By a straightforward analysis of cases. □

Theorem 3.12. *For any consistent multiset of formulas $M \cup C$ and a formula A, both $M \cup C$ and A only containing the connectives of conjunction, disjunction and negation, we have that, under the robust condition C (which may be empty), M completely and immediately grounds A according to Definition 2.8 if, and only if, under the robust condition C (which may be empty), M completely and immediately grounds A according to Definition 3.10.*

Proof. Let us consider first the right-to-left direction. Suppose that under the robust condition C, M completely and immediately grounds A according to Definition 3.10. Then we have that: (i) $\bigwedge M \vdash_R A$; (ii) $C, \bigwedge \neg(M) \vdash_R \neg(A)$, and (iii) $\{C, M\}$ is completely and immediately less g-complex than A according to Definition 3.9. If $\bigwedge M \vdash_R A$, then by logic we also have $\bigwedge M \vdash_C A$. From that and $M \vdash_C \bigwedge M$, by transitivity we have (i)$'$ $M \vdash_C A$. From (ii) by logic we have that $C, \bigwedge \neg(M) \vdash_C \neg(A)$; from such result and $M \vdash_C \bigwedge M$, by transitivity, we have that (ii)$'$ $C, \neg(M) \vdash_C \neg(A)$. From (iii) by Lemma 3.11, we have that (iii)$'$ $\{C, M\}$ is completely and immediately less g-complex than A according to Definition 2.7. From (i)$'$, (ii)$'$ and (iii)$'$, we have that, under the robust condition C, M completely and immediately grounds A according to Definition 2.8.

Let us now consider the left-to-right direction. Suppose that under the robust condition C, M completely and immediately grounds A according to Defintion 2.8. Then we have that: (i) $M \vdash_C A$; (ii) $C, \neg(M) \vdash_C \neg(A)$, and (iii) $\{C, M\}$ is completely and immediately less g-complex than A, according to Definition 2.7. Consider (i); by analysis of cases, we have two possible situations: either $M \vdash_R A$ or there exist two disjoint subsets M', M'' such that $M' \cup M'' = M$ and $M' \vdash_R A$ and $M'' \vdash_R A$. In either of these cases, given that $\bigwedge M \vdash_R B$, where $B \in M$, by transitivity that by inspection of cases can be applied, we get (i)$'$ $\bigwedge M \vdash_R A$.

By applying an analogous procedure on (ii), we get (ii)$'$ $C, \bigwedge \neg M \vdash_R A$. From (iii), by Lemma 3.11, we have that (iii)$'$ $\{C, M\}$ is completely and immediately less g-complex than A according to Definition 3.9. From (i)$'$, (ii)$'$ and (iii), we have that, under the robust condition C, M completely and immediately ground A according to Definition 3.10.

□

4 Grounding principles for relevant implication

The goal of this section is to examine the grounding principles for relevant implication that arise from Definition 3.10.

Recall that, beyond the account of grounding for implication contained in Definition 3.10, we also have the informal intuition, called *pattern**, with which this paper began: the ground of an ordinary-language conditional of the form "if A, then B" is a sentence C such that from A and C, B follows. It may be seen as a test of any account of grounding for implication – and hence for the formal account developed in the previous sections – that it matches *pattern**. The aim will thus be to check whether the grounds M for formulas containing the relevant implication of the form $A \to B$ emerging from Definition 3.10 are such that from M and A, B follows. Of course, such a match can only corroborate the adequacy of the account.

Recall furthermore that, under Definition 3.10 (just as for Definition 2.8), the complete and immediate grounds of a formula of the form $\neg A$ depends on the form of the formula A. For instance, $\neg p$, the negation of an atom, has no *formal* grounds, since $\neg p$ has g-complexity 0. By contrast, negations of conjunctions or disjunctions – formulas of the form $\neg(A \vee B)$ or $\neg(A \wedge B)$ – have different formal grounds. It seems safe to claim that this way of treating the grounds of formulas whose main connective is a negation is accepted in the contemporary literature, e.g. see Correia (2010); Fine (2012a).

According to Definition 3.10, what holds for formulas of the form $\neg A$ also holds for implications of the form $A \to B$. The complete and immediate formal grounds of these formulas will depend on what type of formulas A and B are. In cases where both A and B are literals, which is to say $A \to B$ might have the form $p \to q$, then no *formal* ground can be formulated; $p \to q$ has g-complexity $(0,\{0,0\})$. All the other cases are of one of the following types:

1. $\neg\neg A \to B$	**2.** $A \to \neg\neg B$		
3. $A \to B \wedge C$	**4.** $A \to \neg(B \vee C)$	**5.** $A \vee B \to C$	**6.** $\neg(A \wedge B) \to C$
7. $A \wedge B \to C$	**8.** $\neg(A \vee B) \to C$	**9.** $A \to B \vee C$	**10.** $A \to \neg(B \wedge C)$

To analyze the complete and immediate formal grounds that emerge from Definition 3.10 for each of the types of implication **1-10**, we begin with the following Lemma.

Lemma 4.1. *The following formula $A \to \neg\neg A$ is provable in the Hilbert system **R**.*

Proof. The proof is the following.

a.	$(\neg\neg\neg A \to \neg A) \to (A \to \neg\neg\neg\neg A)$	A4.2
b.	$(\neg\neg\neg A \to \neg A)$	A4.1
c.	$(A \to \neg\neg\neg\neg A)$	MP(a, b)
d.	$\neg\neg\neg\neg A \to \neg\neg A$	A4.1
e.	$A \to \neg\neg A$	A1.4 + MP(c, d)

□

Now let us consider each of the types of implication **1-10** in turn.

1-2. We only examine **1**; the analysis of **2** is analogous. Consider the sentence "if it is not the case that it is not raining, then the road will be wet." It seems intuitively reasonable - it fits *pattern** - to claim that the complete and immediate ground of this conditional is the sentence "if it is raining, the road will be wet." Indeed from this ground and "it is not the case that it is not raining" (namely the antecedent of the implication), "the road will be wet" (namely the consequent of the implication) follows. But this is precisely what emerges from Definition 3.10. Indeed, formalizing “if it is not the case that it is not raining, the road will be wet” by $\neg\neg A \to B$, and “if it is raining, the road will be wet” by $A \to B$, we now show that the multiset $\{A \to B\}$ and the formula $\neg\neg A \to B$ enjoys positive and negative derivability,[13] as well as the relation of being completely and immediately less g-complex.

Positive derivability. We need to prove that $A \to B \vdash_R \neg\neg A \to B$. Assume $A \to B$. Then by applying MP to axiom A4.1, $\neg\neg A \to A$, and axiom A1.4 $(\neg\neg A \to A) \to ((A \to B) \to (\neg\neg A \to B))$, we obtain $(A \to B) \to (\neg\neg A \to B)$. By applying again MP to $(A \to B) \to (\neg\neg A \to B)$ and the assumption $A \to B$, we obtain $\neg\neg A \to B$.

Negative derivability. We need to prove $\neg(A \to B) \vdash_R \neg(\neg\neg A \to B)$. By the Entailment theorem, i.e. Theorem 3.2, and axiom A4.2, this is the same as proving $\neg\neg A \to B \vdash_R A \to B$. Assume $\neg\neg A \to B$. Then by applying MP to the formula $A \to \neg\neg A$ (see Lemma 4.1) and axiom A1.4, $(A \to \neg\neg A) \to ((\neg\neg A \to B) \to (A \to B))$, we obtain $(\neg\neg A \to B) \to (A \to B)$. By applying again MP to the assumption $\neg\neg A \to B$ and $(\neg\neg A \to B) \to (A \to B)$, we obtain $A \to B$.

G-complexity. It is straightforward to verify that $\{A \to B\}$ is completely and immediately less g-complex than $\neg\neg A \to B$ according to Definition 3.9.

Hence the multiset $\{A \to B\}$ is the complete and immediate formal ground of the formula $\neg\neg A \to B$ according to Definition 3.10. This matches our intuitions.

3-4. We only examine **3**; the analysis of **4** is analogous. Consider the sentence “if it is raining, then the road will be wet and slippery.” It sounds intuitively acceptable - it fits *pattern** - to claim that the complete and immediate formal grounds of this implication are the sentences “if it is raining, then the road will be wet” and “if it is raining, then the road will be slippery.” Indeed, from these grounds and “it is raining” (namely the antecedent of the implication),“the road will be wet and slippery” (namely the consequent of the implication) follows. But this is precisely what emerges from Definition 3.10. Let us indeed formalize the sentence “if it is raining, then the road will be wet and slippery” with the formula $A \to B \wedge C$; then the sentence “if it is raining, then the road will be wet” with the formula $A \to B$, and finally the sentence “if it is raining, then the road will be slippery” with the sentence $A \to C$. We now show that the multiset $\{A \to B, A \to C\}$ and the formula $A \to B \wedge C$ enjoy positive and negative derivability as well as the relation of being completely and immediately less g-complex.

Positive derivability. We need to show that $A \to B \wedge A \to C \vdash_R A \to B \wedge C$. Assume $A \to B \wedge A \to C$. By applying MP to this assumption and the axiom

[13]For each proof of positive and negative derivability of this and the next section, it is straightforward to check that they are proofs of entailment in the sense of (Anderson and Belnap, 1975, p.277), that is that they correspond to derivations with control indexes in the natural deduction calculus for **R**, or that is that stars may be prefixed to the steps of the proofs, satisfying the conditions set up by Anderson and Belnap. Not to burden the presentation, we omit to prefix formulas with stars.

A2.3, $(A \to B \wedge A \to C) \to (A \to B \wedge C)$, we obtain $A \to B \wedge C$.

Negative derivability. We need to show that $\neg(A \to B) \wedge \neg(A \to C) \vdash_R \neg(A \to B \wedge C)$. By the Entailment theorem, i.e. Theorem 3.2, and axiom A4.2, this is the same as proving $A \to B \wedge C \vdash_R A \to B \vee A \to C$. Assume $A \to B \wedge C$. By applying MP to this assumption and axiom A1.4, $(A \to B \wedge C) \to ((B \wedge C \to B) \to (A \to B))$, we obtain $(B \wedge C \to B) \to (A \to B)$. By applying again MP to axiom 2.1, $B \wedge C \to B$, and $(B \wedge C \to B) \to (A \to B)$, we obtain $A \to B$. By applying MP to $A \to B$ and axiom A3.1, $(A \to B) \to ((A \to B) \vee (A \to C))$, we obtain $(A \to B) \vee (A \to C)$.

G-complexity. It is straightforward to verify that $\{A \to B, A \to C\}$ is completely and immediately less g-complex than $A \to B \wedge C$ according to Definition 3.9.

Hence the multiset $\{A \to B, A \to C\}$ is the complete and immediate formal ground of the formula $A \to B \wedge C$ according to Definition 3.10. This matches our intuitions.

5-6. We only examine **5**; the analysis of **6** is analogous. Consider the sentence "if it is raining or it is snowing, then the road will be wet." It seems intuitively acceptable - it fits *pattern** - to claim that the complete and immediate grounds of this implication are "if it is raining, then the road will be wet" and "if it is snowing, then the road will be wet." Indeed, from these grounds and "it is raining or it is snowing" (namely the antecedent of the implication), "the road will be wet" (namely the consequent of the implication) follows. But this is precisely what emerges from Definition 3.10. Let us indeed formalize the sentence "if it is raining or it is snowing, then the road will be wet" with the formula $A \vee B \to C$; the sentence "if it is raining, then the road will be wet" with the formula $A \to C$, and finally the sentence "if it is snowing, then the road will be slippery" with the formula $B \to C$. We now show that the multiset $\{A \to C, B \to C\}$ and the formula $A \vee B \to C$ enjoy positive and negative derivability as well as the relation of being completely and immediately less g-complex.

Positive derivability. We need to show that $A \to C \wedge B \to C \vdash_R A \vee B \to C$. Assume $A \to C \wedge B \to C$. By applying MP to this assumption and (one side of) the axiom A3.3, $(A \to C \wedge B \to C) \to (A \vee B \to C)$, we obtain $A \vee B \to C$.

Negative derivability. We need to show that $\neg(A \to C) \wedge \neg(B \to C) \vdash_R \neg(A \vee B \to C)$. By the Entailment theorem, i.e. Theorem 3.2, and axiom A4.2, this is the same as proving $A \vee B \to C \vdash_R A \to C \vee B \to C$. Assume $A \vee B \to C$. By applying MP to this assumption and (one side of) axiom A3.3, $(A \vee B \to C) \to ((A \to C) \wedge (B \to C))$, we obtain $(A \to C) \wedge (B \to C)$. By applying MP to this and the formula $((A \to C) \wedge (B \to C)) \to ((A \to C) \vee (B \to C))$ (which is easily shown to be provable in $\mathbf{R}$), we get the desired result.

G-complexity. It is straightforward to verify that $\{A \to C, B \to C\}$ is completely and immediately less g-complex than $A \vee B \to C$ according to Definition 3.9.

Hence the multiset $\{A \to C, B \to C\}$ is the complete and immediate formal ground of the formula $A \vee B \to C$ according to Definition 3.10. This matches our intuitions.

Figure 3: Complete and immediate formal grounds for relevant implication

Conclusion	Complete and Immediate Formal Grounds
$\neg\neg A \to B$	$A \to B$
$A \to \neg\neg B$	$A \to B$
$A \to B \wedge C$	$\{A \to B, A \to C\}$
$A \to \neg(B \vee C)$	$\{A \to B^*, A \to C^*\}$
$A \vee B \to C$	$\{A \to C, B \to C\}$
$\neg(A \wedge B) \to C$	$\{A^* \to C, B^* \to C\}$

For cases **1-6**, the grounding principles emerging from Definition 3.10 correspond to pre-theoretical intuitions; moreover, this is also the case for formulas that are *a-c equiv*$'$ to those in **1-6**. So Definition 3.10 captures adequately the grounds for a large spectrum of cases. We now consider the more subtle **7-10**.

7-8. We analyze in detail case **7**; case **8** can be treated analogously. Consider first the sentence: "if Andrew is a man and Andrew is unmarried, then Andrew is a bachelor." Note that the sentences "if Andrew is a man, then Andrew is a bachelor" and "if Andrew is unmarried, then Andrew is a bachelor" cannot serve as its grounds, since they are quite simply not true: Andrew is a bachelor if both conditions – being a man and being unmarried – are realized, not if only one is. Indeed, the ground of "if Andrew is a man and Andrew is unmarried, then Andrew is a bachelor" does not seem to be expressible by one of its subsentences; rather, it would seem to depend on the meaning of the word 'bachelor' itself.

Now consider the sentence: "if John recycles and uses public transport, he acts ecologically." By contrast with the previous case, "if John recycles, he acts ecologically" and "if John uses public transport, he acts ecologically" do intuitively appear to be the complete and immediate grounds of this sentence.

Both of the sentences in these examples can be formalized in the same way in the language of relevance logic, namely by $A \wedge B \to C$. But only in the second example do $A \to C, B \to C$ seem to be grounds; in the first example, they are not. Thus it would seem that whether or not they are the grounds of a specific instance of $A \wedge B \to C$ depends not only on the form of the formula, but on the content of the particular formula in question. Hence $A \to C, B \to C$ *cannot* be *formal* grounds of $A \wedge B \to C$ since, although in specific cases, these formulas are the grounds, in other examples, they are not. Definition 3.10 faithfully reflects this conclusion. Although $\{A \to C, B \to C\}$ is completely and immediately less g-complex than $A \wedge B \to C$, and $A \wedge B \to C$ is relevantly derivable from $A \to C \wedge B \to C$, $\neg(A \wedge B \to C)$ is *not* relevantly derivable from $\neg(A \to C) \wedge \neg(B \to C)$.[14] Thus negative derivability is not respected and $\{A \to C, B \to C\}$ is not the complete and immediate ground for $A \wedge B \to C$ according to Definition 3.10.

Note in passing that one might have the intuition that the two examples involve two different readings of conjunction: in the first case, intensional, and in the second, extensional (e.g. see Došen (1993)). This leaves open the possibility

[14] It is classically derivable but in the derivation weakening is used.

that in a richer logic (see e.g. Francez (2019)), with a richer language containing two conjunctions, with sentences involving different readings being formalized with different conjunctions, $\{A \to C, B \to C\}$ may be the grounds for $A \wedge B \to C$ for one of the conjunctions but not for the other. We leave the exploration of such a possibility for future research, focussing in this paper on the standard relevant logic **R**, for a language containing a single conjunction.

9-10. We analyze in detail case **9**; case **10** can be treated analogously. Consider first the sentence "if John is your grandfather, then he is the father of your father or the father of your mother." Note that the sentences "if John is your grandfather, then he is the father of your father," and "if John is your grandfather, then he is the father of your mother" cannot serve as its grounds, since they are quite simply not true: if John is your grandfather, then the whole disjunction - he is the father of your father or the father of your mother - is a consequence of it, not just one disjunct. Indeed, the ground of "if John is your grandfather, then he is the father of your father or the father of your mother" does not seem to be expressible by one of its subsentences; rather it would seem to depend on the meaning of the word 'grandfather' itself.

Now consider the sentence "if it rains, then the road will be slippery or wet." By contrast with the previous case, "if it rains, the road will be slippery" and "if it rains, the road will be wet" do intuitively appear to be the complete and immediate grounds of this sentence.

Both of the sentences in these examples can be formalized in the same way in the language of relevance logic, namely by $A \to B \vee C$. But only in the second example do $A \to B, A \to C$ seem to be its grounds; in the first example, they are not. Thus it would seem that whether or not they are the grounds of a specific instance of $A \to B \vee C$ depends not only on the form of the formula, but on the content of the particular formula in question. Hence $A \to B, A \to C$ *cannot* be *formal* grounds of $A \to B \vee C$ (since, although in specific cases, these formulas are the grounds, in other examples, they are not). Definition 3.10 faithfully reflects this conclusion. Although $\{A \to B, A \to C\}$ is completely and immediately less g-complex than $A \to B \vee C$, and $A \to B \vee C$ is relevantly derivable from $A \to B \wedge A \to C$, $\neg(A \to B \vee C)$ is *not* relevantly derivable from $\neg(A \to B) \wedge \neg(A \to C)$.[15] Thus negative derivability is not respected and $\{A \to B, A \to C\}$ is not the complete and immediate ground for $A \to B \vee C$ according to Definition 3.10.

Note again that one might have the intuition that the two examples involve two different readings of disjunction: in the first case, intensional, and in the second, extensional. This leaves open the possibility that in a richer logic, with a richer language containing two disjunctions, with sentences involving different readings being formalized with different disjunctions, $\{A \to B, A \to C\}$ may be the grounds for $A \to B \vee C$ for one of the disjunctions but not for the other. We leave the exploration of such a possibility for future research, focussing in this paper on the standard relevant logic **R**, for a language containing a single disjunction.

[15] It is classically derivable but in the derivation weakening is used.

5 Grounding principles for negated relevant implication

As pointed out at the beginning of the previous section, according to Definition 3.10 (but also to Definition 2.8), the complete and immediate grounds of a formula of the form $\neg A$ depends on the form of the formula A. The existing literature on grounding, when considering the grounds of $\neg A$, focus on the cases of A being an atom, a negation, a conjunction or a disjunction. Since implication is generally not examined, neither is its negation. By contrast, here, given that we treat the grounds of relevant implication, we also need to treat the grounds of the negation of a relevant implication. But since, as we have seen in the previous section, the grounds of a relevant implication depend in their turn on the form of the antecedent and the consequent, the same goes for the complete and immediate grounds of the negation of a relevant implication. The cases are analogous to those analyzed previously. If both antecedent and consequent are literals, so the implication might have the form $\neg(p \to q)$, then it has no *formal* ground, since $\neg(p \to q)$ has g-complexity $(0,\{0,0\})$. All the other cases are of one of the following types:

$1'$. $\neg(\neg\neg A \to B)$	$\mathbf{2'}$. $\neg(A \to \neg\neg B)$
$\mathbf{3'}$. $\neg(A \to B \wedge C)$	$\mathbf{4'}$. $\neg(A \to \neg(B \vee C))$
$\mathbf{5'}$. $\neg(A \vee B \to C)$	$\mathbf{6'}$. $\neg(\neg(A \wedge B) \to C)$
$\mathbf{7'}$. $\neg(A \wedge B \to C)$	$\mathbf{8'}$. $\neg(\neg(A \vee B) \to C)$
$\mathbf{9'}$. $\neg(A \to (B \vee C))$	$\mathbf{10'}$. $\neg(A \to \neg(B \wedge C)$

As we have underlined at the beginning of this paper, according to Poggiolesi's account, when we analyze the grounds of ordinary-language sentences featuring the negation, an intuitive pattern emerges: the ground of a formula of the form "it is not the case that A" is a sentence B such that from A and B, a contradiction follows. Our approach extends that of Poggiolesi; hence, the complete and immediate grounds M for a negative implication of the form $\neg(A \to B)$ emerging from Definition 3.10 will be such that from M and $A \to B$ a contradiction follows.[16] This is actually the case but not to burden the paper, we omit an extended discussion.

Let us now analyze in detail cases **1′-10′**.

1′-2′. We only examine **1′**; the analysis of **2′** is analogous. Consider the sentence "it is not the case that if it is not the case that it is not sunny, then the road will be slippery." It seems intuitively reasonable to claim that the complete and immediate ground of this conditional is the sentence "it is not the case that if

[16] We still use *follow* in Anderson and Belnap (1975)'s sense. A contradiction follows from $M, A \to B$ if there is a deduction of the contradiction from $M, A \to B$ which actually uses $M, A \to B$ (and $M, A \to B$ alone).

it is sunny, then the road will be slippery." This is precisely what emerges from Definition 3.10. Indeed, formalizing "it is not the case that if it is not the case that it is not sunny, then the road will be slippery" by $\neg(\neg\neg A \to B)$, and "it is not the case that if it is sunny, then the road will be slippery" by $\neg(A \to B)$, we can show that $\{\neg(A \to B)\}$ and $\neg(\neg\neg A \to B)$ enjoys positive and negative derivability, as well as the relation of being completely and immediately less g-complex. The proof of such fact can be developed in the same way as the proof of case **1** of the previous section. Hence the multiset $\{\neg(A \to B)\}$ is the complete and immediate formal ground of the formula $\neg(\neg\neg A \to B)$ according to Definition 3.10 and this seems to match our intuitions.

3′-4′. We only examine **3′**; the analysis of **4′** is analogous. Consider the following three sentences:

> (i) "it is not the case that if it is snowing, then the road will be dry and safe,"
>
> (ii) "it is not the case that if it is snowing, then the road will be wet and safe,"
>
> (iii) "it is not the case that if it is snowing, then the road will be dry and dangerous."

As for (i), the complete and immediate grounds seem to be "it is not the case that if it is snowing, then the road will be dry" and "it is not the case that if it is snowing, then the road will be safe;" as for (ii), under the robust condition that "if it is snowing, then the road will be wet," "it is not the case that if it is snowing, the road will be safe" is the complete and immediate ground for "it is not the case that if it is snowing, then the road will be wet and safe." Finally as for (iii), under the robust condition that "if it is snowing, the road will be dangerous", "it is not the case that if it is snowing, the road will be dry" is the complete and immediate ground for "it is not the case that if it is snowing, the road will be dry and dangerous." At the formal level, the formula to be grounded is an implication of the form $\neg(A \to B \wedge C)$: in (i) the complete and immediate ground is the multiset $\{\neg(A \to B), \neg(A \to C)\}$; in (ii) the ground is the multiset $\{\neg(A \to C)\}$ under the robust condition $A \to B$; in (iii) the ground is the multiset $\{\neg(A \to B)\}$ under the robust condition $A \to C$. We will verify that in case (ii) the ground $\{\neg(A \to C)\}$ together with the robust condition $A \to B$ satisfy positive and negative derivability, but also the relation of being completely and immediately less g-complex with the conclusion $\neg(A \to B \wedge C)$. The analysis of cases (i) and (iii) is analogous.

Positive derivability. We need to show that $\neg(A \to B) \vdash_R \neg(A \to B \wedge C)$. By the Entailment theorem, i.e. Theorem 3.2, and axiom A4.2, this is equivalent to showing $A \to B \wedge C \vdash_R A \to B$. Assume $A \to B \wedge C$. By applying MP to this assumption and axiom A1.4, $(A \to B \wedge C) \to ((B \wedge C \to B) \to (A \to B))$, we obtain $(B \wedge C \to B) \to (A \to B)$. By applying MP to this and axiom A.2.1, $B \wedge C \to B$, we obtain $A \to B$.

Negative derivability. We need to show that $A \to C, \neg\neg(A \to B) \vdash_R \neg\neg(A \to B \wedge C)$. Because in relevant logic **R**, we have that $A \leftrightarrow \neg\neg A$, this is equivalent to show $A \to C, A \to B \vdash_R A \to B \wedge C$. Assume $A \to C, A \to B$; by applying the rule IC, we get $A \to C \wedge A \to B$. By applying modus ponens to this formula and the axiom A2.3, $((A \to C) \wedge (A \to B)) \to (A \to B \wedge C)$ we obtain $A \to B \wedge C$.

G-complexity. It is straightforward to verify that $\{\neg(A \to B), A \to C\}$ is completely and immediately less g-complex than $\{\neg(A \to B \wedge C)\}$ according to Definition 3.9.

Hence, under the robust condition $A \to C$, the multiset $\{\neg(A \to B)\}$ is the complete and immediate formal ground of the formula $\neg(A \to B \wedge C)$ according to Definition 3.10. It is straightforward to verify that this matches our intuitions.

5′-6′. We only examine **5′**; the analysis of **6′** is analogous. Consider the following three sentences:

> (i) "it is not the case that if it is raining or cold, then we will go to the sea,"
>
> (ii) "it is not the case that if it is sunny or cold, then we will go to the sea,"
>
> (iii) "it is not the case that if it is raining or hot, then we will go to the sea."

As for (i), the complete and immediate grounds seem to be "it is not the case that if it is raining, then we will go to the sea" and "it is not the case that if it is cold, then we will go to the sea;" as for (ii), under the robust condition that "if it is sunny, then we will go to the sea," "it is not the case that if it is cold, we will go to the sea" is the complete and immediate ground for "it is not the case that if it is sunny or cold, then we will go to the sea." Finally, as for (iii), under the robust condition that "if it is hot, we will go to the sea", "it is not the case that if it is raining, we will go to the sea" is the complete and immediate ground for "it is not the case that if it is raining or hot, we will go to the sea." At the formal level, in (i) - (iii) the formula to be grounded is an implication of the form $\neg(A \vee B \to C)$: in (i) the complete and immediate ground is the multiset $\{\neg(A \to C), \neg(B \to C)\}$; in (ii) the complete and immediate ground is the multiset $\{\neg(A \to C)\}$ under the robust condition $B \to C$; in (iii) the complete and immediate ground is the multiset $\{\neg(B \to C)\}$ under the robust condition $A \to C$. We will verify that in case (ii) the ground $\{\neg(A \to C)\}$ together with the robust condition $B \to C$ satisfy positive and negative derivability, but also the relation of being completely and immediately less g-complex with the conclusion $\neg(A \vee B \to C)$. The analysis of cases (i) and (iii) is analogous.

Positive derivability. We need to show that $\neg(A \to C) \vdash_R \neg(A \vee B \to C)$. By the Entailment theorem, i.e. Theorem 3.2, and axiom A4.2, this is the same as proving $A \vee B \to C \vdash_R A \to C$. Assume $A \vee B \to C$. By applying MP to this

Figure 4: Complete and immediate formal grounds for negation of relevant implication

Conclusion	Complete and Immediate Formal Grounds
$\neg(\neg\neg A \to B)$	$\neg(A \to B)$
$\neg(A \to \neg\neg B)$	$\neg(A \to B)$
$\neg(A \to B \wedge C)$	$\{\neg(A \to B), \neg(A \to C)\}$ $\{\neg(A \to B)\}[A \to C]$ $\{\neg(A \to C)\}[A \to B]$
$\neg(\neg(A \wedge B) \to C)$	$\{\neg(A^* \to C), \neg(B^* \to C)\}$ $\{\neg(A^* \to C)\}[B^* \to C]$ $\{\neg(B^* \to C)\}[A^* \to C]$
$\neg(A \vee B \to C)$	$\{\neg(A \to C), \neg(B \to C)\}$ $\{\neg(A \to C)\}[B \to C]$ $\{\neg(B \to C)\}[A \to C]$
$A \to \neg(B \vee C)$	$\{\neg(A \to B^*), \neg(A \to C^*)\}$ $\{\neg(A \to B^*)\}[A \to C^*]$ $\{\neg(A \to C^*)\}[A \to B^*]$

assumption and (one side of) axiom A3.3, $(A \vee B \to C) \to ((A \to C) \wedge (B \to C))$, we obtain $A \to C \wedge B \to C$. By applying modus ponens to this and axiom A2.1, $((A \to C) \wedge (B \to C)) \to (A \to C)$, we obtain $A \to C$.

Negative derivability. We need to show that $B \to C, \neg\neg(A \to C) \vdash_R \neg\neg(A \vee B \to C)$. Because in relevant logic **R**, we have that $A \leftrightarrow \neg\neg A$, this is equivalent to show $A \to C, B \to C \vdash_R A \vee B \to C$. Assume $A \to C, B \to C$; by applying the rule IC, we get $A \to C \wedge B \to C$. By applying MP to this formula and (one side of) the axiom A3.3, $((A \to C) \wedge (B \to C)) \to (A \vee B \to C)$, we obtain $A \vee B \to C$.

G-complexity. It is straightforward to verify that $\{\neg(A \to C), B \to C\}$ is completely and immediately less g-complex than $\{\neg(A \vee B \to C)\}$ according to Definition 3.9.

Hence, under the robust condition $B \to C$, the multiset $\{\neg(A \to C)\}$ is the complete and immediate formal ground of the formula $\neg(A \to B \wedge C)$ according to Definition 3.10. It is straightforward to verify that this matches our intuitions.

For cases $\mathbf{1'}$-$\mathbf{6'}$, the grounding principles emerging from Definition 3.10 correspond to pre-theoretical intuitions; moreover, this is also the case for formulas that are *a-c equiv′* to those in $\mathbf{1'}$-$\mathbf{6'}$. So Definition 3.10 captures adequately the grounds for a large spectrum of cases. Items $7'$-$10'$ are more difficult to treat and present the same problems as the corresponding cases 7-10.[17] On

[17] We omit here all the details that we have developed for cases 7-10, since they are quite similar to them. The interested reader can straightforwardly reconstruct them on her own.

the one hand, at the intuitive level, it seems hard to settle the questions of what grounds formulas of the form 7-10 since the connectives of conjunction and disjunction can received an extensional as well as an intentional readings, and this double reading leads to the identification of different grounds. On the other hand, following Definition 3.10, there is no formula which is completely and immediately less g-complex than formulas of the form $7'$-$10'$ and also enjoys positive and negative derivability with them. Therefore, at the intuitive level, as well as according to Definition 3.10, it does not seem possible to formulate *formal* grounds for this type of negative relevant implications.

6 Conclusions

This paper tackles an issue that has been largely ignored in the literature on grounding, namely that of the grounds of non-material implications. We have put forward grounding principles for relevant implications and negations of relevant implications, which faithfully capture some basic intuitions concerning natural-language conditionals. Moreover, these principles naturally follow from a definition of the notion of complete and immediate formal grounding which conservatively extends the notion put forward by Poggiolesi (2016b) for classical logic without implication. Therefore, the grounding principles are not only justified by the formal context provided by the Definition, but they also reflect some natural insights.

It seems to us that this work opens up some interesting questions. Let us just mention two. On the one hand it calls for a logic validating the principles for relevant implication and negation of relevant implication that we have put forward in this paper. In this direction, on the semantic side, interesting links could emerge with truth-making semantics for relevant logics, as in Jago (2019). On the other hand, it calls for a wider reflection on the way the grounds of the main connectives vary from one logic to another.

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