On 2-Site Voronoi Diagrams under Geometric Distance Functions

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Abstract—We revisit a new type of a Voronoi diagram, in which distance is measured from a point to a *pair* of points. We consider a few more such distance functions, based on geometric primitives, and analyze the structure and complexity of the nearest- and furthest-neighbor Voronoi diagrams of a point set with respect to these distance functions.

Keywords-distance function; lower envelope; Davenport-Schinzel theory; crossing-number lemma

I. INTRODUCTION

The Voronoi diagram is one of the most fundamental concepts in computational geometry, which has plenty of applications in science and industry. Much information in this respect can be found in [4] and [13]; for important recent achievements, see [8].

The basic definition of the Voronoi diagram applies to a set S of n points (also called *sites*) in the plane: its *nearestneighbor* Voronoi diagram V(S) is a partition of the plane into n regions, each corresponding to a distinct site $s \in S$, and consisting of all the points being closer to s than to any other site from S. Similarly, the *furthest-neighbor* Voronoi diagram of S is obtained by assigning each point in the plane to the region of the most remote site. These notions can be generalized to higher-dimensional spaces, different types of sites, and in other ways.

One of the recent generalizations of this concept is a family of so-called 2-site Voronoi diagrams [5], which are based on distance functions that define a distance from a

point in the plane to a *pair* of sites from a given set S. Consequently, each Voronoi region corresponds to an (unordered) pair of sites from S. The original motivation for the study [5] was the famous Heilbronn's triangle problem [14]. Other motivations are mentioned therein.

For S being a set of points, Voronoi diagrams under a number of 2-site distance functions have been investigated, which include arithmetic combinations of point-to-point distances [5], [17] and certain geometric distance functions [5], [7], [9]. In this work, we develop further the latter direction.

Let $S \subset \mathbb{R}^2$, and consider $p, q \in S$ and a point v in the plane. We shall focus our attention on a few circle-based distance functions:

- radius of circumscribing circle: C(v, (p, q)) = Rad(o(v, p, q)), where o(v, p, q) is the circle defined by v, p, q and Rad(c) is the radius of the circle c;
- radius of containing circle: $\mathcal{K}(v, (p, q)) = \text{Rad}(C(v, p, q))$, where C(v, p, q) is the minimum circle containing v, p, q.¹
- view angle: V(v, (p, q)) = ∠pvq, or, equivalently, half of the angular measure of the arc of o(v, p, q) that the angle ∠pvq subtends;
- radius of inscribed circle: R(v, (p, q)) is the radius of the circle inscribed in △(v, p, q);
- center-of-circumscribing-circle-based functions: let o_{vpq} denote the center of the circle $\circ(v, p, q)$; then $\overset{\circ}{\mathcal{S}}(v, (p, q))$, $\overset{\circ}{\mathcal{A}}(v, (p, q))$, and $\overset{\circ}{\mathcal{P}}(v, (p, q))$ are the distance from o_{vpq} to the segment pq, the area of $\triangle o_{vpq}pq$, and the perimeter of $\triangle o_{vpq}pq$, respectively;

and on a parameterized perimeter distance function:

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¹ Obviously, $\circ(v, p, q) \neq C(v, p, q)$ if any of the three points is properly contained in the circle whose diameter is defined by the two other points.

• parameterized perimeter: $\mathcal{P}_c(v, (p, q)) = |vp| + |vq| + c \cdot |pq|$, where $c \ge -1$.

The first and third circle-based distance functions were first mentioned in [10]. The last function generalizes the perimeter distance function $\mathcal{P}(v, (p, q)) = \operatorname{Per}(\triangle(v, p, q))$ introduced in [5], and later addressed in [7], [9].

Since two points define a segment, any 2-point site distance function d(v, (p, q)) provides a distance between the point v and the segment pq, and vice versa. Consequently, geometric structures akin to 2-site Voronoi diagrams can arise as Voronoi diagrams of segments. This alternative approach was independently undertaken by Asano et al., and the "view angle" and "radius of circumscribing circle" distance functions reappeared in their works [2], [3] on Voronoi diagrams for segments soon after they had been proposed by Hodorkovsky [10] in the context of 2-site Voronoi diagrams. However, as Asano's et al. research was originally motivated by mesh generation and improvement tasks, they were mostly interested in sets of segments representing edges of a simple polygon, and thus, non-intersecting (except, possibly, at the endpoints), what significantly alters the essence of the problem.

In this paper, we analyze the structure and complexity of 2-site Voronoi diagrams under the distance functions listed above. Our obtained results are mostly of theoretical interest. The method used to derive an upper bound on the complexity of the nearest-neighbor 2-site Voronoi diagram under the "parameterized perimeter" distance function is first developed for the case of c = 1, yielding a much simpler proof for the "perimeter" function than the one developed in [9], and then generalized to any $c \ge 0$. We summarize our new results in Table I.

Throughout the paper we use the notation $V_{\mathcal{F}}^{(n)}(S)$ (resp., $V_{\mathcal{F}}^{(f)}(S)$) for denoting the nearest- (resp., furthest-) 2-site Voronoi diagram, under the distance function \mathcal{F} , of a point set S. The set S is always assumed to contain n points.

II. CIRCUMSCRIBING CIRCLE

Let $\circ(p, q, r)$ denote the unique circle defined by three distinct points p, q, and r in the plane. We now define the 2-site circumscribing-circle distance function:

Definition 1: Given two points p,q in the plane, the "circumcircle distance" C from a point v in the plane to the unordered pair (p,q) is defined as $C(v,(p,q)) = \text{Rad}(\circ(v,p,q))$.

For a fixed pair of points p and q, the curve $C(v, (p, q)) = \infty$ is the line \overline{pq} . This implies that all the points on \overline{pq} belong to the region of (p,q) in $V_C^{(f)}(S)$. In this section we assume that the points in S are in general position, i.e., there are no three collinear points, and no three pairs of points define three distinct lines that intersect at one point. The given sites are singular points, that is, for any two sites p, q, the function C(v, (p, q)) is not defined at v = p or v = q. Theorem 1: Let S be a set of n points in the plane. The combinatorial complexity of $V_c^{(f)}(S)$ is $\Omega(n^4)$.

Proof: The *n* points of \tilde{S} define $\Theta(n^2)$ lines, which always have $\Theta(n^4)$ intersection points. All these intersection points are features of $V_{\mathcal{C}}^{(f)}(S)$, and hence the lower bound.

Theorem 2: Let S be a set of n points in the plane. The combinatorial complexity of both $V_{\mathcal{C}}^{(n)}(S)$ and $V_{\mathcal{C}}^{(f)}(S)$ is $O(n^{4+\varepsilon})$ (for any $\varepsilon > 0$).

Proof: Clearly, the combinatorial complexity of $V_{\mathcal{C}}^{(n)}(S)$ or $V_{\mathcal{C}}^{(f)}(S)$ is identical to that of the respective diagram of the 2-site distance function $\mathcal{C}^2(v, (p, q)) = \operatorname{Rad}^2(\circ(v, p, q))$. It is known that $\operatorname{Rad}^2(\circ(v, p, q)) = ((|vp||vq||pq|)/(4|\Delta vpq|))^2 = (((v_x - p_x)^2 + (v_y - p_y)^2)((v_x - q_x)^2 + (v_y - q_y)^2)((p_x - q_x)^2 + (p_y - q_y)^2))/(4(v_x(p_y - q_y) - p_x(v_y - q_y) + q_x(v_y - p_y))^2).$ The respective collection of $\Theta(n^2)$ Voronoi surfaces fulfills Assumptions 7.1 of [16, p. 188]:

- 1) Each surface is an algebraic surface of maximum constant degree;
- 2) Each surface is totally defined (this is stronger than needed); and
- 3) Each triple of surfaces intersects in at most a constant number of points.

Hence, we may apply Theorem 7.7 of [ibid., p. 191] and obtain the claimed bound on the complexity of $V_{\mathcal{C}}^{(n|f)}(S)$.

III. CONTAINING CIRCLE

Let C(p, q, r) denote the minimum-radius circle containing three points p, q, and r in the plane. (That it, C(p, q, r)is the minimum circle containing the triangle $\triangle pqr$.) We now define the 2-site containing-circle distance function:

Definition 2: Given two points p, q in the plane, the "containing-circle distance" \mathcal{K} from a point v in the plane to the unordered pair (p,q) is defined as $\mathcal{K}(v,(p,q)) = \text{Rad}(C(v,p,q))$.

In our context we have that $p \neq q$. Assume first that $v \neq p, q$. Observe that if all angles of $\triangle pqr$ are acute (or $\triangle pqr$ is right-angled), then C(p, q, r) is identical to $\circ(p, q, r)$. Otherwise, if one of the angles of $\triangle pqr$ is obtuse, then C(p,q,r) is the circle whose diameter is the longest edge of $\triangle pqr$, that is, the edge opposite to the obtuse angle. If v coincides with either p or q, then C(v, p, q) is the circle whose diameter is the longest of Δpqr .

Theorem 3: Let S be a set of n points in the plane. The combinatorial complexity of $V_{\mathcal{K}}^{(n)}(S)$ is $\Omega(n)$.

Proof: For simplicity assume that each point from S has a unique closest neighbor in S. For each point $p \in S$, consider its closest neighbor q. Then, the points on the line segment pq lying sufficiently close to p belong to the region of (p,q) in $V_{\mathcal{K}}^{(n)}(S)$, which is thus non-empty. Since no

	F C		\mathcal{K}		\mathcal{V}		\mathcal{R}			
1	$V_{\mathcal{F}}^{(n)}(S) = O(n^{4+\varepsilon})$		$O(n^{4+\varepsilon})$	Ω	$\Omega(n), O(n^{2+\varepsilon})$		$\Omega(n^4), O(n^{4+\varepsilon})$		$O(n^{4+\varepsilon})$	
1	$ V_{\mathcal{F}}^{(f)}(S) = \Omega(n$		$\Omega(n^4), O(n^{4+\varepsilon})$		$O(n^{4+\varepsilon})$	$\Omega(n), O(n^{4+\varepsilon})$		$\Omega(n^4), O(n^{4+\varepsilon})$		ε)
	\mathcal{F}		$\overset{\circ}{\mathcal{S}}$		$\overset{\circ}{\mathcal{A}}$		$\overset{\circ}{\mathcal{P}}$		\mathcal{P}	Γ
	$ V_{\mathcal{F}}^{(n)}(S) $		$\Omega(n^4), O(n^{4+\varepsilon})$		$\Omega(n^4), O(n^{4+\varepsilon})$		$\Omega(n), O(n^{4+\varepsilon})$		$O(n^{2+\varepsilon})$	
	$ V_{\mathcal{F}}^{(f)}(S) $		$\Omega(n^4), O(n^{4+\varepsilon})$		$\Omega(n^4), O(n^{4+\varepsilon})$		$\Omega(n^4), O(n^{4+\varepsilon})$]

Table I

Our results: Worst-case combinatorial complexities of 2-site Voronoi diagrams of a set S of n points with respect to different distance functions



Figure 1. If p, q have a non-empty region in $V_{\mathcal{K}}^{(n)}(S)$, then pq is an edge in DT(S).

region is thereby encountered more than twice, $V_{\mathcal{K}}^{(n)}(S)$ has at least $\lceil n/2 \rceil$ non-empty regions. The claim follows.

Theorem 4: Let S be a set of n points in the plane. The combinatorial complexity of $V_{\mathcal{K}}^{(n)}(S)$ is $O(n^{2+\varepsilon})$ (for any $\varepsilon > 0$).

Proof: Let a point v belong to a non-empty region of (p,q). No matter if the triangle $\triangle vpq$ is acute (Figure 1(a)), $\triangle vpq$ is obtuse with v being the obtuse vertex (Figure 1(b)), or $\triangle vpq$ is obtuse with p or q being the obtuse vertex (Figure 1(c)), the circle C(v, p, q) cannot contain any other point $x \in S$. Otherwise, regardless of the location of x in C(v, p, q), we will always have $\mathcal{K}(v, (p, q)) > \mathcal{K}(v, (x, q))$, which is a contradiction. This follows from the fact (see [6, Lemma 4.14]) that given a point set K and its minimum enclosing circle C, where C is defined by three points $a, b, c \in K$ (resp., two diametrical points $s, t \in K$), removing from K one of a, b, c (resp., one of s, t) will result in a point set with a smaller minimum enclosing circle. Thus, there is a circle containing p, q that is empty of any other site from S. This immediately implies that pq is an edge of the Delaunay triangulation of S. Consequently, there are O(n) pairs of sites in S that have non-empty regions

in $V_{\mathcal{K}}^{(n)}(S)$. Furthermore, it follows from the definition of $\mathcal{K}(v,(p,q))$ that the respective Voronoi surface of (p,q) is made of a constant number of patches, each of which is a "well-behaved" function in the sense discussed in the proof of Theorem 2. Again, by standard Davenport-Schinzel machinery, the combinatorial complexity of the lower envelope of these O(n) surfaces is $O(n^{2+\varepsilon})$ (for any $\varepsilon > 0$), and the claim follows.

Theorem 5: Let S be a set of n points in the plane. The combinatorial complexity of $V_{\mathcal{K}}^{(f)}(S)$ is $O(n^{4+\varepsilon})$ (for any $\varepsilon > 0$).

Proof: As in the proof of Theorem 2, we prove this claim by using the upper envelope of $\Theta(n^2)$ "well-behaved" Voronoi surfaces.

IV. VIEW ANGLE

We now define the 2-site view-angle distance function:

Definition 3: Given two points p,q in the plane, the "view-angle distance" \mathcal{V} from a point v in the plane to the unordered pair (p,q) is defined as $\mathcal{V}(v,(p,q)) = \measuredangle pvq$. Similarly to the circumcircle-radius distance function, the view-angle function is undefined at the n given points. For a fixed pair of points p and q, the curve $\mathcal{V}(v,(p,q)) = \pi$ is the open line segment connecting the two points p and q, while the curve $\mathcal{V}(v,(p,q)) = 0$ is the line \overline{pq} excluding the closed line segment pq. The curve $\mathcal{V}(v,(p,q)) = \pi/2$ is the circle whose diameter is the line segment pq (excluding, again, p and q).

Theorem 6: Let S be a set of n points in the plane. The combinatorial complexity of $V_{\mathcal{V}}^{(n)}(S)$ is $\Omega(n^4)$. *Proof:* Consider a set S of n points in the plane.

Proof: Consider a set S of n points in the plane. An example of the intersection of the complements of two segments defined by two pairs of points (with respect to the supporting lines) is shown in Figure 2(a). These intersection points are features of $V_{\mathcal{V}}^{(n)}(S)$; we show that there are $\Omega(n^4)$ such points. To this aim we create a geometric graph G whose vertices are the given points, in which each segment's complement defines two edges. We add one additional point far away from the convex hull of S, and connect it (without adding intersections) to all the rays as shown in Figure 2(b). We can now use the crossing-number lemma for bounding from below the number of intersections of the original rays. The lemma tells us that every drawing of a graph with



Figure 2. The graph of $V_{\mathcal{V}}^{(n)}(S)$.

n vertices and $m \geq 4n$ edges (without self or parallel edges) has $\Omega(m^3/n^2)$ crossing points [1], [12]. In our case $m = 2\binom{n}{2} = n(n-1)$, so the number of intersection points in *G* is $\Omega(n^6/n^2) = \Omega(n^4)$. All these intersection points are features of $V_{\mathcal{V}}^{(n)}(S)$, and hence the lower bound.

Theorem 7: Let S be a set of n points in the plane. The combinatorial complexity of both $V_{\mathcal{V}}^{(n)}(S)$ and $V_{\mathcal{V}}^{(f)}(S)$ is $O(n^{4+\varepsilon})$ (for any $\varepsilon > 0$).

Proof: For analyzing $V_{\mathcal{V}}^{(n)}(\mathbf{S})$ and $V_{\mathcal{V}}^{(f)}(S)$ we consider the function $(-\cos \measuredangle pvq)$ instead of that of $\measuredangle pvq$. This is permissible since the cosine function is strictly decreasing in the range $[0, \pi]$. By the cosine law, we have $-\cos \measuredangle pvq = (|pq|^2 - |vp|^2 - |vq|^2)/(2|vp||vq|)$. As we have already seen more than once in this paper, this means that the respective collection of $\Theta(n^2)$ Voronoi surfaces fulfills Assumptions 7.1 of [16, p. 188]. Hence, we may apply Theorem 7.7 of [ibid., p. 191] and obtain the claimed bound on the complexity of $V_{\mathcal{V}}^{(n|f)}(S)$.

Theorem 8: Let S be a set of n points in the plane. The combinatorial complexity of $V_{\mathcal{V}}^{(f)}(S)$ is $\Omega(n^4)$.

Proof: Given a set S of n points in the plane, we count the intersections of pairs of line segments, where each segment is defined by points of S (see Figure 3(a)). We create a geometric graph whose vertices are the given points, and the edges are the line segments connecting every pair of points (see Figure 3(b)). The intersections of the segments defined by all pairs of points define features of $V_{\mathcal{V}}^{(f)}(S)$, because along these segments the view-angle



Figure 4. $\mathcal{R}(v, (p, q))$ is the radius of the circle inscribed in $\triangle vpq$.

function assumes its maximum possible value, π . We can now use the crossing-number lemma for counting these intersections. The graph with n vertices and $m \ge 4n$ edges (without self or parallel edges) has $\Omega(m^3/n^2)$ crossing points [1], [12]. In this case $m = \binom{n}{2} = n(n-1)/2$, hence $\Omega(n^4)$ is a lower bound on the complexity of $V_{\mathcal{V}}^{(f)}(S)$.

Results by Asano et al. [2] immediately imply that the edges of $V_{\mathcal{V}}^{(n|f)}(S)$ represent pieces of polynomial curves of degree at most three. However, the structure of the part of $V_{\mathcal{V}}^{(f)}(S)$ that lies outside the convex hull $\mathcal{CH}(S)$ of S is fairly simple: it is given by the arrangement of lines supporting the edges of $\mathcal{CH}(S)$. This arrangement can be computed by a standard incremental algorithm in optimal $\Theta(k^2)$ time and space, where k denotes the number of vertices of $\mathcal{CH}(S)$. Each cell of the arrangement should then be labeled with a pair of sites from S, to the Voronoi region of which it belongs; this extra task can be completed within the same complexity bounds.

V. RADIUS OF INSCRIBED CIRCLE

We now define the 2-site "radius-of-inscribed-circle" distance function:

Definition 4: Given two points p,q in the plane, the "inscribed radius distance" \mathcal{R} from a point v in the plane to the unordered pair (p,q), denoted by $\mathcal{R}(v,(p,q))$, is defined as the radius of the circle inscribed in the triangle $\triangle vpq$ (Figure 4).

Theorem 9: Let S be a set of n points in the plane. The combinatorial complexity of $V_{\mathcal{R}}^{(n)}(S)$ is $\Omega(n^4)$.

Proof: The intersection point of any two lines defined by the points from S is a distinct feature of the Voronoi diagram under discussion. Thus, n points in S define $\Theta(n^2)$ lines, which have $\Theta(n^4)$ intersection points.

Theorem 10: Let S be a set of n points in the plane. The combinatorial complexity of both $V_{\mathcal{R}}^{(n)}(S)$ and $V_{\mathcal{R}}^{(f)}(S)$ is $O(n^{4+\varepsilon})$ (for any $\varepsilon > 0$).

Proof: Let p,q be two points in S, and v a point in the plane. It is a well-known fact that $\mathcal{R}(v,(p,q)) = 2\mathcal{A}(v,(p,q))/\mathcal{P}(v,(p,q))$, where $\mathcal{A}(v,(p,q))$ and $\mathcal{P}(v,(p,q))$ are the area and perimeter, respectively, of the triangle $\triangle vpq$. Both the numerator and denominator of this fraction can be written as algebraic expressions using the coordinates of the points v, p, q. Hence, as above, the standard Davenport-Schinzel machinery can be applied for obtaining the claim bounds.

Theorem 11: Let S be a set of n points in the plane. The combinatorial complexity of $V_{\mathcal{R}}^{(f)}(S)$ is $\Omega(n)$ in the worst case.

Proof: The complexity of $V_{\mathcal{R}}^{(f)}(S)$ can be as high as $\Omega(n)$. Let S be a set of n point in convex position with no three collinear points. Let p and q be two antipodal vertices of $\mathcal{CH}(S)$, the convex hull of S, and consider two parallel lines $\ell_p \ni p$ and $\ell_q \ni q$ tangent to $\mathcal{CH}(S)$ only at p and q, respectively. Next, consider any point $v \in \ell_p$, and let it move along ℓ_p in either direction. In the limit, the distance from v to any pair (s, t) of sites in S equals the width of the infinite strip bounded by two lines parallel to ℓ_p and passing through s and t, respectively. Consequently, the points of ℓ_p lying sufficiently far from p belong to the Voronoi region of (p,q). Since the number of pairs of antipodal vertices of $\mathcal{CH}(S)$ is $\Theta(n)$, the bound follows.

A similar reasoning leads to a conclusion that $V_{\mathcal{R}}^{(f)}(S)$ has at most a linear number of unbounded regions. To demonstrate this, consider any point u in the plane, and a line $\ell \ni u$. Observe that the points of ℓ lying sufficiently far from u belong to the Voronoi region of the pair(s) of points from S that define the width of S in the direction orthogonal to ℓ , and, thus, represent a pair (pairs) of antipodal vertices of $\mathcal{CH}(S)$. Since the union of all such lines gives the whole plane, and the number of antipodal vertices of $\mathcal{CH}(S)$ is at most linear, the claim follows.

VI. DISTANCES BASED ON THE CENTER OF THE CIRCUMSCRIBING CIRCLE

Let v, p, q be three points in the plane. Consider the circle $\circ(v, p, q)$ passing through v, p, q with center o_{vpq} . We now define three more distance functions based on the above notation:

Definition 5: Given two points p,q in the plane, the three distances, denoted by $\overset{\circ}{\mathcal{S}}(v,(p,q))$, $\overset{\circ}{\mathcal{A}}(v,(p,q))$, and



Figure 5. The circle $\circ(v, p, q)$ is defined by the points v, p, q, and has the center at o_{vpq} . $\overset{\circ}{\mathcal{S}}(v, (p, q))$ is the distance from o_{vpq} to the segment pq (or, equivalently, the height of $\triangle o_{vpq}pq$ perpendicular to pq), and $\overset{\circ}{\mathcal{A}}(v, (p, q))$, and $\overset{\circ}{\mathcal{P}}(v, (p, q))$ are the area and the perimeter of $\triangle o_{vpq}pq$, respectively.

 $\mathcal{P}(v, (p, q))$, respectively, are the distance from o_{vpq} to the line segment pq, the area of the triangle $\triangle o_{vpq}pq$, and the perimeter of $\triangle o_{vpq}pq$, respectively (Figure 5).

The upper bound of $O(n^{4+\varepsilon})$ (for any $\varepsilon > 0$) on the complexity of the nearest- and furthest-neighbor Voronoi diagrams under each of these distance functions can be, again, derived by means of Davenport-Schinzel machinery. Below we provide some lower bounds. First, we address the nearest-neighbor case.

Theorem 12: Let S be a set of n points in the plane. The combinatorial complexity of $V^{(n)}_{\overset{\circ}{\mathcal{S}}}(S)$ and $V^{(n)}_{\overset{\circ}{\mathcal{A}}}(S)$ is $\Omega(n^4)$ in the worst case.

Proof: The key observation is the following. Consider a pair (p,q) of sites, and let $\circ(p,q)$ denote the circle with the diameter pq. Then, for any point $v \in \circ(p,q) \setminus \{p,q\}$, we have $\overset{\circ}{\mathcal{S}}(v,(p,q)) = \overset{\circ}{\mathcal{A}}(v,(p,q)) = 0$.

Consider two parallel lines l_1 and l_2 , and let d denote the distance between them. For a given $n \ge 2$, let us construct a set S of n points as a union of two sets $S_1 \subset l_1$ and $S_2 \subset l_2$ consisting of $\lceil n/2 \rceil$ and $\lfloor n/2 \rfloor$ points, respectively, in the following way. The sets S_1 and S_2 are constructed iteratively; at each odd step, a new point is added to S_1 , and at each even one—to S_2 . For any $i: 2 \leq i \leq n$, let S_1^i and S_2^i denote the two sets constructed so far, and let $M^i = \{ \circ(p,q) | p \in S_1^i, q \in S_2^i \}$ denote the set of circles defined by pairs of points from different sets. We want each circle from M^n to pass through precisely two points from S (those defining it), each two circles from M^n to intersect, and no three of them to pass through the same point not contained in S. Then $\Theta(n^2)$ circles composing M^n will give rise to $\Theta(n^4)$ distinct intersection points, each belonging to a separate feature of either Voronoi diagram under consideration, and the claim will follow.

To ensure the first property, we select the points so that the distance between each two points contained in the same set S_i is much smaller than d, where i = 1, 2. To guarantee the second property, at each step $j: 3 \le j \le n$, when adding a new point s to the respective set, we make sure that for any point t from the other set, the circle $\circ(s, t)$ passes neither through any point from $S_1^{j-1} \cup S_2^{j-1} \setminus \{t\}$ nor through any intersection point of the circles from M^{j-1} . This completes the proof.

Theorem 13: Let S be a set of n points in the plane. The combinatorial complexity of $V^{(n)}_{\mathcal{P}}(S)$ is $\Omega(n)$ in the worst case.

Proof: A linear lower bound in the worst case for $V_{\hat{\mathcal{P}}}^{(n)}(S)$ can be obtained in the following way. Choose the set S of points to lie on some line ℓ , so that the distance between any two consecutive points is 1. Then, the minimum possible value for the distance function $\hat{\mathcal{P}}$ is obviously 2, and can be achieved only for a pair (p,q) of consecutive points. For each such pair (p,q), consider the circle $\circ(p,q)$ with the diameter pq. Evidently, for any point $v \in \circ(p,q) \setminus \{p,q\}$, we have $\hat{\mathcal{P}}(v,(p,q)) = 2$, and for any other pair (s,t) of sites, $\hat{\mathcal{P}}(v,(s,t)) > 2$. We conclude that each pair of consecutive points along ℓ has a non-empty region in $V_{\hat{\mathcal{P}}}^{(n)}(S)$. Since there are n-1 pairs of consecutive points, the bound follows.

Second, we address the furthest-neighbor Voronoi diagrams.

Theorem 14: Let S be a set of n points in the plane. The combinatorial complexity of all of $V^{(f)}_{S}(S)$, $V^{(f)}_{A}(S)$, and $V^{(f)}_{P}(S)$ is $\Omega(n^4)$.

In each case, the proof is identical to that of Theorem 9.

VII. PARAMETERIZED PERIMETER

Finally, we define the 2-site parameterized perimeter distance function:

Definition 6: Given two points p, q in the plane and a real constant $c \ge -1$, the "parameterized perimeter distance" \mathcal{P}_c from a point v in the plane to the unordered pair (p, q) is defined as $\mathcal{P}_c(v, (p, q)) = |vp| + |vq| + c \cdot |pq|$.

We require that c be greater than or equal to -1 since allowing c < -1 would result in negative distances. Letting c = -1 results in a distance function that equals 0 for all the points on the line segment pq. If c = 0, we deal with the "sum of distances" distance function introduced in [5] and recently revisited in [17]. For c = 1, the above definition yields the "perimeter" distance function $\mathcal{P}(v, (p, q)) = \operatorname{Per}(\Delta vpq)$.

In [9] it was proven that the combinatorial complexity of the nearest-neighbor 2-site perimeter Voronoi diagram of a set of n points is slightly superquadratic in n. In a nutshell, the proof was based on the observation that any pair of sites that has a non-empty region in the perimeter diagram also has a non-empty region in the sum-of-distances diagram. This immediately implies that the number of such pairs is linear in n. (However, unlike in the sum-of-distances diagram, a region in the perimeter diagram is not necessarily continuous. We were able to construct examples in which the number of connected components of a *single* region is



Figure 6. An empty circle containing sites in \mathcal{P} .

comparable to the number of points!) Again, one can apply the standard Davenport-Schinzel machinery and conclude the claimed upper bound on the complexity of the diagram. It remains unclear whether the worst-case complexity of the diagram is linear, quadratic, or in between. The proof in [9] of the main observation was extremely complex. We provide here an alternative and much simpler proof of the same bound, which generalizes to the case of "parameterized perimeter" distance function for any $c \ge 0$.

Theorem 15: Let S be a set of n points in the plane. The combinatorial complexity of $V_{\mathcal{P}}^{(n)}(S)$ is $O(n^{2+\varepsilon})$ (for any $\varepsilon > 0$).

Proof: Refer to Figure 6. Let $p, q \in S$ be two sites which have a non-empty region in $V_{\mathcal{P}}^{(n)}(S)$, and let v be a point in this region, noncollinear with p and q. In addition, let ℓ be the perpendicular bisector of the line segment pq. Assume, without loss of generality, that $|vp| \leq |vq|$.

Consider the ellipse O_{vpq} passing through q with v and p as foci. By definition, for any point s inside this ellipse we have |vs| + |ps| < |vq| + |pq|. Therefore,

$$\mathcal{P}(v, (p, s)) = |vs| + |ps| + |vp|$$
(1)
$$< |vq| + |pq| + |vp| = \mathcal{P}(v, (p, q)).$$

This means that s cannot be a site in S, for otherwise v would belong to the region of (p, s) instead of to the region of (p, q). It follows that the ellipse O_{vpq} is empty of any sites other than p and q.

Now consider the line ℓ' that is tangent to O_{vpq} at q, and the ray \vec{r} perpendicular to ℓ' at q and passing through O_{vpq} . It is a known property of ellipses that this ray bisects the angle $\measuredangle vqp$, and, thus, it intersects the line segment vp, say, at point o. The circle C centered at o and passing through q is tangent to O_{vpq} at q (as well as at another point), and is entirely contained in O_{vpq} . Since v is closer to p than to q (by our assumption), it follows that the circle C also contains p. (If p were on the extension of vp in the shaded area, a contradiction would easily be obtained by using the triangle inequality: |op| > |oq|, hence |vp| = |ov| + |op| > |ov| + |oq| > |vq|, contradicting the assumption that $|vp| \le |vq|$.) Since O_{vpq} is empty of sites (except p and q), so is the circle C. Therefore, pq is an edge of the Delaunay triangulation of S. The number of such edges is linear in n, the cardinality of S.

Hence, there are $\Theta(n)$ respective surfaces of these pairs of sites. One can now apply the standard Davenport-Schinzel machinery (as in the proof of Theorem 2). The claim follows.

Finally, we state the following theorem.

Theorem 16: Let S be a set of n points in the plane.

- (a) The combinatorial complexity of $V_{\mathcal{P}_{-1}}^{(n)}(S)$ is $\Omega(n^4)$ and $O(n^{4+\varepsilon})$ (for any $\varepsilon > 0$).
- (b) If there is a unique closest pair p, q ∈ S, then when c → ∞, the combinatorial complexity of V⁽ⁿ⁾_{P_c}(S) is asymptotically 1.
- (c) For c ≥ 0, the combinatorial complexity of V⁽ⁿ⁾_{Pc}(S) is O(n^{2+ε}) (for any ε > 0).
 Proof:
- (a) To see the lower bound on the complexity of V⁽ⁿ⁾_{P-1}(S), note that every point on the segment pq has P₋₁-distance zero to the pair (p, q), and therefore, the intersection of any pair of segments p₁q₁ and p₂q₂ defined by sites p₁, q₁, p₂, q₂ ∈ S is a feature of V⁽ⁿ⁾_{P-1}(S). As is demonstrated in the proof of Theorem 8, the number of these features is Ω(n⁴). The upper bound is obtained by using the usual Davenport-Schinzel machinery, as in the proof of Theorem 2.
- (b) It is easy to verify that as c → ∞, the term c · |pq| dominates the distance P_c(v, (p, q)), and, hence, every point v in the plane is closer to the unique closest pair of sites p, q ∈ S than to any other pair in S. Hence, the asymptotic diagram contains zero vertices, zero edges, and one face (the entire plane).
- (c) The proof is a generalized version of the proof of the special case c = 1. Refer to Figure 7. As in the proof of Theorem 15, we assume that there is a point v in the region of (p,q), such that $|vp| \leq |vq|$, and v is noncollinear with p and q. Our goal is to show that for any $c \geq 0$ there exists a circle having q on its boundary and containing p, which is empty of any other site s, implying that p, q are Delaunay neighbors.

As in the proof of Theorem 15, let $O_{vpq}^{(c)}$ be the locus of points q' for which $\mathcal{P}_c(v, (p, q')) = \mathcal{P}_c(v, (p, q))$. Thus, $O_{vpq}^{(c)}$ is the *Cartesian oval* (v, p, c, k) consisting of all points q' that satisfy |vq'| + c|pq'| = k, where k = |vq| + c|pq| is constant. (Unless c = 1, this oval has exactly one axis of symmetry: the line joining the two foci v, p.) Then, if there were a site s within $O_{vpq}^{(c)}$, it would lead to a smaller value of \mathcal{P}_c , so $O_{vpq}^{(c)}$ must be empty of sites other than p.

As before, let \vec{r} be the ray emanating from q perpendicular to and pointing into $O_{vpq}^{(c)}$, and let o be the point



Figure 7. The Cartesian oval $O_{vpq}^{(c)}$ is the locus of points q', for which $|vq'| + c \cdot |pq'| = |vq| + c \cdot |pq|$. The ray \vec{r} passes through q and is perpendicular to $O_{vpq}^{(c)}$, and intersects the axis of symmetry of $O_{vpq}^{(c)}$ at the point o. The circle C is centered at o, and is tangent to $O_{vpq}^{(c)}$ at q. For any point x on the axis of abscissas residing inside C, t(x) denotes the point o.

where \vec{r} crosses the line pv.

Let us further suppose that $c \neq 1$. Without loss of generality, assume that O_{vpq}^c is symmetric with respect to the axis of abscissas (see Figure 7); consequently, the points p, v, and o belong to the latter. Let x_p , x_v , x_o , and x_q denote the corresponding coordinate of p, v, o, and q, respectively.

Consider a circle C centered at o of the radius R = |oq|. By construction, C is tangent to $O_{vpq}^{(c)}$ at q.

For any $x \in \mathbb{R}$, such that the point (x, 0) lies inside C, let t(x) denote the point of C lying above (x, 0). For any such x, let

$$f_v(x) = d(v, t(x))$$

= $\sqrt{R^2 - (x - x_o)^2 + (x - x_v)^2}$
= $\sqrt{2(x_o - x_v) \cdot x + x_o^2 + x_v^2 + R^2}.$

Since $f_v(x)$ represents a square root of a linear function, it is concave on its domain. The same will hold for a function $f_p(x) = d(p, t(x))$. Consequently, their weighted combination $f(x) = f_v(x) + c \cdot f_p(x)$ is also concave on the same domain, and, thus, has a single local maximum.

Recall that the circle C is tangent to O_{vpq}^c at q by construction. It is easy to see that C is tangent to O_{vpq}^c from the inside: otherwise, x_q would be a local minimum of f(x) achieved at an inner point of the domain, contradicting the concavity of f(x). It follows that f(x) has a local maximum at x_q . Together with the previous observation, this implies that f(x) has a global maximum at x_q . This means that q is the only common point of $O_{vpq}^{(c)}$ and the upper half of C. By symmetry, we conclude that C lies inside $O_{vpq}^{(c)}$ and touches it at q and the point symmetric to q. Thus, Cmust be empty of sites other than p.

It remains to demonstrate that p lies inside C. To

this end, it is sufficient to show that the point o lies between v and p; then, as in the case of c = 1, the needed property can be easily derived using the triangle inequality.

Let us argue as follows. The above reasoning can be carried out for any point $q' \in O^c_{vpq}$ noncollinear with v and p, providing us with a maximum empty circle inscribed in O_{vpq}^c , and tangent to it at precisely two points—namely, at q' and its symmetric point. It follows that the medial axis of O_{vpq}^c is a segment of the line \overline{vp} through v and p. Let v' and p' be the intersection points of \overline{vp} and $O_{vpq}^{(c)}$ being closer to v and p, respectively (see Figure 7). Consider the circle C_v with radius |vv'| centered at v. Obviously, v' is a common point of C_v and $O_{vpq}^{(c)}$, but any other point z of C_v lies strictly inside O_{vpa}^c , since for any such point z, we have |zv| = |v'v| and |zp| < |v'p|. This implies that the radius of curvature of $O_{vpq}^{(c)}$ at v'is greater than |vv'|. A similar statement holds for p'. Consequently, the two endpoints of the medial axis must lie between v and p, and the same must hold for the point o.

We conclude that C is a circle containing both pand q and otherwise empty of sites, so p and q are Delaunay neighbors. Hence, there are $\Theta(n)$ pairs of sites that generate regions in the Voronoi diagram, and the claim follows from the standard Davenport-Schinzel machinery.

VIII. CONCLUSION

In this paper, we have investigated 2-site Voronoi diagrams of point sets with respect to a few geometric distance functions. The Voronoi structures obtained in this way cannot be explained in terms of the previously known kinds of Voronoi diagrams (which is the case for the 2-site distance functions thoroughly analyzed in [5]), what makes them particularly interesting. On the other hand, our results can be exploited to advance research on Voronoi diagram for segments. Potential directions for future work include consideration of other distance functions, and generalizations to higher dimensions and to k-site Voronoi diagrams.

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