# Null Controllability of Some Degenerate Wave Equations\*

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#### Abstract

This paper is devoted to a study of the null controllability problems for one-dimensional linear degenerate wave equations through a boundary controller. First, the well-posedness of linear degenerate wave equations is discussed. Then the null controllability of some degenerate wave equations is established, when a control acts on the non-degenerate boundary. Different from the known controllability results in the case that a control acts on the degenerate boundary, any initial value in state space is controllable in this case. Also, an explicit expression for the controllability time is given. Furthermore, a counterexample on the controllability is given for some other degenerate wave equations.

**Key Words.** Controllability, observability, Fourier expansion, degenerate wave equation

## 1 Introduction and main results

Let T > 0, L > 0 and  $\alpha > 0$ . Set  $Q = (0, L) \times (0, T)$ . Consider the following linear degenerate wave equation with a boundary controller:

$$\begin{cases} w_{tt} - (x^{\alpha} w_x)_x = 0 & (x, t) \in Q, \\ \begin{cases} w(0, t) = 0 & (0 < \alpha < 1) \\ (x^{\alpha} w_x)(0, t) = 0 & (\alpha \ge 1) \end{cases} & t \in (0, T), \\ w(L, t) = \theta(t) & t \in (0, T), \\ w(x, 0) = w_0(x), \ w_t(x, 0) = w_1(x) & x \in (0, L), \end{cases}$$

$$(1.1)$$

where  $\theta \in L^2(0,T)$  is the control variable,  $(w, w_t)$  is the state variable, and  $(w_0, w_1)$  is any given initial value.

In order to study the well-posedness of (1.1), we introduce a linear space  $H^1_{\alpha}(0,L)$ :

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(1) In the case of  $0 < \alpha < 1$ ,

$$H^1_\alpha(0,L) = \big\{u \in L^2(0,L) \ \big| \ u \text{ is absolutely continuous in } [0,L], \ x^{\frac{\alpha}{2}}u_x \in L^2(0,L) \\ \text{and } u(0) = u(L) = 0 \big\}.$$

(2) In the case of  $\alpha \geq 1$ ,

$$H^1_{\alpha}(0,L) = \{u \in L^2(0,L) \mid u \text{ is locally absolutely continuous in } (0,L], \ x^{\frac{\alpha}{2}}u_x \in L^2(0,L)$$
 and  $u(L) = 0\}.$ 

Then  $H^1_{\alpha}(0,L)$   $(\alpha > 0)$  is a Hilbert space with the inner product

$$(u,v)_{H^1_{\alpha}(0,L)} = \int_0^L (uv + x^{\alpha}u_xv_x)dx, \quad \forall \ u,v \in H^1_{\alpha}(0,L)$$

and associated norm  $||u||_{H^1_{\alpha}(0,L)} = ||u||_{L^2(0,L)} + ||x^{\frac{\alpha}{2}}u_x||_{L^2(0,L)}$ . By Hardy-poincaré inequality (see [1, Proposition 2.1]), it is easy to check that for  $\alpha \in (0,2)$ , there exists a constant C > 0 such that for any  $u \in H^1_{\alpha}(0,L)$  satisfies

$$||u||_{L^2(0,L)} \le C||x^{\frac{\alpha}{2}}u_x||_{L^2(0,L)}.$$

Therefore, when  $\alpha \in (0,2)$ ,  $||u||_{H^1_{\alpha}(0,L)}$  is equivalent to the norm  $||x^{\frac{\alpha}{2}}u_x||_{L^2(0,L)}$ . Also,  $H^*_{\alpha}(0,L)$  denotes the conjugate space of  $H^1_{\alpha}(0,L)$  and

$$||v||_{H^*_{\alpha}(0,L)} = \sup_{||u||_{H^1_{\alpha}(0,L)} = 1} \langle u, v \rangle_{H^1_{\alpha}(0,L), H^*_{\alpha}(0,L)}.$$

Moreover, for any  $0 \le \alpha < 2$ , set

$$T_{\alpha} = \frac{4}{2 - \alpha} L^{\frac{2 - \alpha}{2}}.\tag{1.2}$$

The degenerate wave equation (1.1) can describe the vibration problem of an elastic string. In a neighborhood of an endpoint x = 0 of this string, the elastic is sufficiently small or the linear density is large enough. First, we give the definition of solutions of the equation (1.1) in the sense of transposition.

**Definition 1.1** For any  $(w_0, w_1) \in L^2(0, L) \times H^*_{\alpha}(0, L)$  and  $\theta \in L^2(0, T)$ , a function  $w(\cdot) \in C([0, T]; L^2(0, L))$  is called a solution of the equation (1.1) in the sense of transposition, if for any  $\xi \in L^1(0, T; L^2(0, L))$ , it holds that

$$\int_{Q} w\xi dx dt = \langle w_1, v(0) \rangle_{H_{\alpha}^{*}(0,L), H_{\alpha}^{1}(0,L)} - \int_{0}^{L} w_0 v_t(0) dx - L^{\alpha} \int_{0}^{T} \theta(t) v_x(L,t) dt,$$
 (1.3)

where v is the weak solution of the following equation:

$$\begin{cases}
v_{tt} - (x^{\alpha}v_x)_x = \xi & (x,t) \in Q, \\
v(0,t) = 0 & (0 < \alpha < 1) \\
(x^{\alpha}v_x)(0,t) = 0 & (\alpha \ge 1)
\end{cases} t \in (0,T),$$

$$v(L,t) = 0 & t \in (0,T),$$

$$v(x,T) = 0, v_t(x,T) = 0 & x \in (0,L).$$
(1.4)

Then we have the following well-posedness result for the system (1.1).

**Theorem 1.1** Let  $\alpha \in (0,2)$ . For any  $(w_0, w_1) \in L^2(0,L) \times H^*_{\alpha}(0,L)$  and  $\theta \in L^2(0,T)$ , the system (1.1) admits a unique solution  $w \in \mathcal{K} = C([0,T];L^2(0,L)) \cap C^1([0,T];H^*_{\alpha}(0,L))$  in the sense of transposition. Moreover,

$$||w||_{\mathscr{K}} \le C(||w_0||_{L^2(0,L)} + ||w_1||_{H^*_{\alpha}(0,L)} + ||\theta||_{L^2(0,T)}).$$

The purpose of this paper is to study the null controllability of the linear degenerate wave equation (1.1), and give an expression of the controllability time. The system (1.1) is null controllable in time T, if for any initial value  $(w_0, w_1) \in L^2(0, L) \times H^*_{\alpha}(0, L)$ , one can find a control  $\theta \in L^2(0,T)$  such that the corresponding solution w of (1.1) (in the sense of transposition) satisfies that  $w(T) = w_t(T) = 0$  in (0, L).

The main controllability results of this paper can be stated as follows.

**Theorem 1.2** (1) Let  $\alpha \in (0,2)$ . Then for any  $T > T_{\alpha}$ , the system (1.1) is null controllable in time T. For any  $T < T_{\alpha}$ , the system (1.1) is not null controllable.

(2) Let  $\alpha \in [2, +\infty)$ . Then for any T > 0, the system (1.1) is not null controllable.

**Remark 1.1** Notice that when  $\alpha = 0$ , the system (1.1) is a non-degenerate linear wave equation. By the known controllability result in [17], the controllability time  $T^* = 2L$  (for  $\alpha = 0$ ). Letting  $\alpha$  tend to zero in (1.2), one can find that  $\lim_{\alpha \to 0} T_{\alpha} = T^*$ .

**Remark 1.2** Since the linear degenerate wave equation (1.1) has time-reversibility, the null controllability of it is equivalent to the exact controllability.

Up to now, there are numerous works addressing the controllability problems of nondegenerate parabolic and hyperbolic equations (see e.g. [2], [6], [8], [12], [14], [16] and the references therein). The controllability of some degenerate parabolic equations was studied in the last decade (see, for instance, [3], [4] and the references therein). However, very little is known for the controllability of degenerate wave equations. In [9], for any  $0 < \alpha < 1$ , the null controllability of the following degenerate wave equation was considered:

$$\begin{cases} w_{tt} - (x^{\alpha} w_x)_x = 0 & (x, t) \in (0, 1) \times (0, T), \\ w(0, t) = \theta(t), \ w(1, t) = 0 & t \in (0, T), \\ w(x, 0) = w_0(x), \ w_t(x, 0) = w_1(x) & x \in (0, 1), \end{cases}$$

$$(1.5)$$

where  $\theta(\cdot)$  is the control variable and it acts on the degenerate boundary. Based on the Fourier expansion method, the null controllability of (1.5) was discussed in [9]. Indeed, for any  $T > \frac{4}{2-\alpha}$  and any initial value  $(w_0, w_1) \in \mathcal{H} = H_{\alpha}^{\frac{1-2\mu}{2}} \times H_{\alpha}^{\frac{-1-2\mu}{2}}$  with  $\mu = \frac{1-\alpha}{2-\alpha}$ , there exists a control  $\theta \in L^2(0,T)$  such that the solution w of (1.5) satisfies  $w(T) = w_t(T) = 0$  in (0,1). For any s > 0,

$$H_{\alpha}^{s}(0,1) = \left\{ u = \sum_{n \in \mathbb{N}^{*}} a_{n} \Phi_{n} \mid ||u||_{s}^{2} = \sum_{n \in \mathbb{N}^{*}} |a_{n}|^{2} \lambda_{n}^{s} < \infty \right\},$$

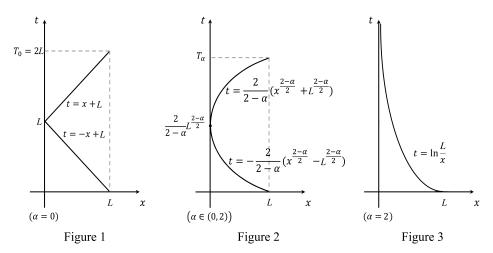
where  $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ , and  $\{\lambda_n\}_{n \in \mathbb{N}^*}$  and  $\{\Phi_n\}_{n \in \mathbb{N}^*}$  are eigenvalues and eigenvectors of a degenerate elliptic operator, respectively. Also, it is easy to check that  $\mathcal{H} \subset L^2(0,1) \times H^*_{\alpha}(0,1)$ , and any initial value in the space  $(L^2(0,1) \times H^*_{\alpha}(0,1)) \setminus \mathcal{H}$  is not controllable in time T (see [9]).

Borrowing some ideas from [9], we study the null controllability problems of one-dimensional degenerate wave equations. Different from [9], the control acts on the nondegenerate boundary in our problems. Notice that for the one-dimensional nondegenerate wave equations, the controllability results are same, no matter which endpoint of [0, L] a control acts on. However, there is a different new phenomena for degenerate wave equations. In fact, we find that when a control enters the system from the nondegenerate boundary, any initial value in the state space  $L^2(0, L) \times H^*_{\alpha}(0, L)$  is controllable. But, when a control acted on the degenerate boundary, only a subspace of the state space was controllable in [9]. Now, we explain it from the physical background of degenerate wave equations. Due to degeneracy, the propagation speed of waves is zero in the endpoint x = 0. If a control acts on this point, it has little effect on the whole waveform. But, when control acts on the nondegenerate boundary x = L, it will give a permanent influence on the whole waveform. Therefore, any initial value in the state function can be driven to zero.

Furthermore, by the method of characteristic lines, we explain controllability times for onedimensional nondegenerate and degenerate wave equations, respectively. Notice that a wave propagates along characteristic lines, and for  $\alpha \in [0,2)$ , its travel time from the endpoint x=L to the other x=0 is calculated as follows:

$$\int_0^L \frac{1}{x^{\frac{\alpha}{2}}} dx = \frac{T_\alpha}{2}.\tag{1.6}$$

For the degenerate wave equation  $(\alpha > 0)$ , though the propagation speed of the wave tends to zero, as it is sufficiently close to the endpoint x = 0, its distance to this point also tends to 0. Hence, the travel time may be finite by (1.6). When a control acts on the right endpoint x = L, it is easy to see that the controllability time is  $T_{\alpha}$  (resp.  $+\infty$ ) for  $\alpha \in [0, 2)$  (resp.  $\alpha \geq 2$ ) (from Figures 1-3).



The rest of this paper is organized as follows. In Section 2, the well-posedness results for the system (1.1) are given. In Section 3, we prove the main controllability results of this paper (Theorem 1.2).

## 2 Well-posedness of degenerate wave equations

In this section, we prove the well-posedness of the equation (1.1) (Theorem 1.1). First, consider the following linear degenerate wave equation:

$$\begin{cases} y_{tt} - (x^{\alpha} y_x)_x = f & (x, t) \in Q, \\ \begin{cases} y(0, t) = 0 & (0 < \alpha < 1) \\ (x^{\alpha} y_x)(0, t) = 0 & (\alpha \ge 1) \end{cases} & t \in (0, T), \\ y(L, t) = 0 & t \in (0, T), \\ y(x, 0) = y_0(x), \ y_t(x, 0) = y_1(x) & x \in (0, L), \end{cases}$$

$$(2.1)$$

where  $f \in L^1(0,T;L^2(0,L))$  and  $(y_0,y_1) \in H^1_{\alpha}(0,L) \times L^2(0,L)$ .

We are concerned with weak solutions of the system (2.1).

**Definition 2.1** A function  $y \in C([0,T]; H^1_{\alpha}(0,L)) \cap C^1([0,T]; L^2(0,L))$  is said to be a weak solution of the system (2.1), if for any  $\varphi \in L^2(0,T; H^1_{\alpha}(0,L))$  satisfying  $\varphi_t \in L^2(Q)$  and  $\varphi(\cdot,T) = 0$ , it holds that  $y(x,0) = y_0(x)$  in (0,L) and

$$\int_{Q} (-y_t \varphi_t + x^{\alpha} y_x \varphi_x) dx dt - \int_{0}^{L} y_1 \varphi(x, 0) dx = \int_{Q} f \varphi dx dt.$$
 (2.2)

We have the following well-posedness result for the equation (2.1).

**Theorem 2.1** For any  $f \in L^1(0,T;L^2(0,L))$  and  $(y_0,y_1) \in H^1_{\alpha}(0,L) \times L^2(0,L)$ , the system (2.1) has a unique weak solution  $y \in C([0,T];H^1_{\alpha}(0,L)) \cap C^1([0,T];L^2(0,L))$ . Moreover, there exists a constant  $C = C(T,L,\alpha)$  such that

$$||y||_{L^{\infty}(0,T;L^{2}(0,L))} + ||y_{t}||_{L^{\infty}(0,T;L^{2}(0,L))} + ||x^{\alpha}y_{x}^{2}||_{L^{\infty}(0,T;L^{1}(0,L))}$$

$$\leq C(||f||_{L^{1}(0,T;L^{2}(0,L))} + ||y_{0}||_{H^{1}_{\alpha}(0,L)} + ||y_{1}||_{L^{2}(0,L)}).$$

**Proof.** We borrow some ideas from [15]. First, suppose that f and  $(y_0, y_1)$  are sufficiently smooth. For any positive integer n, consider the following nondegenerate wave equation:

$$\begin{cases} y_{tt} - \left( \left( x^{\alpha} + \frac{1}{n} \right) y_x \right)_x = f, & (x, t) \in Q, \\ y(0, t) = y(L, t) = 0, & t \in (0, T), \\ y(x, 0) = y_0(x), \ y_t(x, 0) = y_1(x), \ x \in (0, L). \end{cases}$$
(2.3)

By the classical theory of wave equations, the system (2.3) admits a unique classical solution  $y^n$ . For simplicity, we denote by y the solution  $y^n$ . Multiplying both sides of the first equation of (2.3) by  $y_t$  and integrating it in  $Q_s = (0, s) \times (0, L)$ , by the Hölder inequality, we have that

$$\max_{s \in [0,T]} \int_0^L \left( y_t^2(x,s) + (x^{\alpha} + \frac{1}{n}) |y_x(x,s)|^2 \right) dx 
\leq \int_0^L \left( y_t^2(x,0) + (x^{\alpha} + \frac{1}{n}) |y_x(x,0)|^2 \right) dx + \max_{s \in [0,T]} ||y_t||_{L^2(0,L)} \int_0^T ||f||_{L^2(0,L)} dt.$$

This implies that

$$\int_{0}^{L} \left( y_{t}^{2}(x,s) + (x^{\alpha} + \frac{1}{n}) |y_{x}(x,s)|^{2} \right) dx 
\leq C \int_{0}^{L} \left( y_{t}^{2}(x,0) + (x^{\alpha} + \frac{1}{n}) |y_{x}(x,0)|^{2} \right) dx + C \left( \int_{0}^{T} ||f||_{L^{2}(0,L)} dt \right)^{2}.$$
(2.4)

On the other hand, notice that

$$y(x,s) = y(x,0) + \int_0^s y_t(x,t)dt.$$

By the Hölder inequality, we arrive at

$$\int_{0}^{L} y^{2}(x,s)dx \le 2 \int_{0}^{L} y^{2}(x,0)dx + 2s \int_{Q_{s}} y_{t}^{2}(x,t)dxdt.$$
 (2.5)

By (2.4) and (2.5), it follows that

$$\begin{split} & \int_0^L \left( y^2(x,s) + y_t^2(x,s) + (x^\alpha + \frac{1}{n}) |y_x(x,s)|^2 \right) dx \\ & \leq C \Big[ \int_0^L \left( y^2(x,0) + y_t^2(x,0) + (x^\alpha + \frac{1}{n}) |y_x(x,0)|^2 \right) dx \\ & + \int_{O_s} \left( y_t^2 + y^2 + (x^\alpha + \frac{1}{n}) |y_x|^2 \right) dx dt + \left( \int_0^T \|f\|_{L^2(0,L)} dt \right)^2 \Big]. \end{split}$$

By Gronwall's inequality, we get that

$$||y||_{L^{\infty}(0,T;L^{2}(0,L))} + ||y_{t}||_{L^{\infty}(0,T;L^{2}(0,L))} + ||(x^{\alpha} + \frac{1}{n})|y_{x}|^{2}||_{L^{\infty}(0,T;L^{1}(0,L))}$$

$$\leq C\Big(||f||_{L^{1}(0,T;L^{2}(0,L))} + ||y_{0}||_{H^{1}(0,L)} + ||y_{1}||_{L^{2}(0,L)}\Big). \tag{2.6}$$

Hence, there exist a subsequence  $\{y^{n_j}\}$  of  $\{y^n\}$ , and a function  $y \in L^{\infty}(0,T;L^2(0,L))$  satisfying  $y_t \in L^{\infty}(0,T;L^2(0,L))$ , such that as  $j \to \infty$ ,

$$y^{n_j} \to y$$
 weakly in  $L^2(Q)$ ;  $y_t^{n_j} \to y_t$  weakly in  $L^2(Q)$ ; 
$$\sqrt{x^{\alpha} + \frac{1}{n_j}} y_x^{n_j} \to \sqrt{x^{\alpha}} y_x$$
 weakly in  $L^2(Q)$ . (2.7)

Since  $\{y^{n_j}\}$  is the classical solution of (2.3), we have that  $y^{n_j}(x,0) = y_0(x)$ , and for any  $\varphi \in C^{\infty}(\overline{Q})$ , with  $\varphi = 0$  in some neighborhood of  $\{0\} \times (0,T)$ ,  $\{L\} \times (0,T)$  and  $\{0,L\} \times \{T\}$ ,

$$\int_{Q} \left[ -y_t^{n_j} \varphi_t + (x^{\alpha} + \frac{1}{n_j}) y_x \varphi_x \right] dx dt - \int_{Q}^{L} y_1(x) \varphi(x, 0) dx = \int_{Q} f \varphi dx dt.$$

By (2.7), taking a limit in the above equality and noticing that  $C_0^{\infty}(Q)$  is dense in  $L^2(0,T;H^1_{\alpha}(0,L))$ , we obtain that y satisfies (2.2) for any  $\varphi \in L^2(0,T;H^1_{\alpha}(0,L))$  with  $\varphi_t \in L^2(Q)$  and  $\varphi(\cdot,T)=0$ . Also, it is easy to show that  $\{y^{n_j}\}$  is a Cauchy sequence in  $C([0,T];H^1_{\alpha}(0,L)) \cap C^1([0,T];L^2(0,L))$ . This

implies  $y \in C([0,T]; H^1_{\alpha}(0,L)) \cap C^1([0,T]; L^2(0,L))$ . Hence, y is the weak solution of the system (2.1) associated to smooth functions  $(y_0, y_1)$  and f.

Next, we prove the existence of weak solutions of (2.1) for any  $(y_0, y_1) \in H^1_{\alpha}(0, L) \times L^2(0, L)$  and  $f \in L^1(0, T; L^2(0, L))$ . Let  $\{y_0^m\}$ ,  $\{y_1^m\}$  and  $\{f^m\}$  be sequences of smooth functions, respectively, such that as  $m \to \infty$ ,

$$y_0^m \to y_0 \text{ in } H^1_\alpha(0, L), \quad y_1^m \to y_1 \text{ in } L^2(0, L) \quad \text{ and } \quad f^m \to f \text{ in } L^1(0, T; L^2(0, L)).$$
 (2.8)

Denote by  $y^m$  the solution of (2.1) associated to  $(y_0^m, y_1^m)$  and  $f^m$ . Similarly, we can obtain that

$$||y^m - y^n||_{C([0,T]:H^1_{\sigma}(0,L))} + ||y_t^m - y_t^n||_{C([0,T]:L^2(0,L))}$$

$$\leq C\Big(\|f^m - f^n\|_{L^1(0,T;L^2(0,L))} + \|y_0^m - y_0^n\|_{H^1_\alpha(0,L)} + \|y_1^m - y_1^n\|_{L^2(0,L)}\Big).$$
(2.9)

Therefore, there exists  $y \in C([0,T]; H^1_\alpha(0,L)) \cap C^1([0,T]; L^2(0,L))$ , such that as  $m \to \infty$ ,

$$y^m \to y$$
 in  $C([0,T]; H^1_{\alpha}(0,L))$  and  $y^m_t \to y_t$  in  $C([0,T]; L^2(0,L))$ .

Moreover, it is easy to show that y is the weak solution of (2.1) for  $(y_0, y_1) \in H^1_\alpha(0, L) \times L^2(0, L)$  and  $f \in L^1(0, T; L^2(0, L))$ .

Finally, we prove the uniqueness of weak solutions. Let  $\widetilde{y}$  and  $\overline{y}$  be two weak solutions of the system (2.1), and set  $\widehat{y} = \widetilde{y} - \overline{y}$ . Then  $\widehat{y} \in C([0,T]; H^1_{\alpha}(0,L)) \cap C^1([0,T]; L^2(0,L))$  satisfies

$$\int_{Q} (-\widehat{y}_t \varphi_t + x^{\alpha} \widehat{y}_x \varphi_x) dx dt = 0, \ \forall \ \varphi \in L^2(0, T; H^1_{\alpha}(0, L)) \text{ with } \varphi_t \in L^2(Q) \text{ and } \varphi(\cdot, T) = 0. \ (2.10)$$

For any  $\ell \in C_0^{\infty}(Q)$ , consider the following degenerate wave equation:

$$\begin{cases}
\psi_{tt} - (x^{\alpha}\psi_{x})_{x} = \ell & (x,t) \in Q, \\
\begin{cases}
\psi(0,t) = 0 & (0 < \alpha < 1) \\
(x^{\alpha}\psi_{x})(0,t) = 0 & (\alpha \ge 1)
\end{cases} & t \in (0,T), \\
\psi(L,t) = 0 & t \in (0,T), \\
\psi(x,T) = 0, \ \psi_{t}(x,T) = 0 & x \in (0,L).
\end{cases}$$
(2.11)

By the above argument on the existence of weak solutions, one can find a weak solution  $\psi \in C([0,T]; H^1_\alpha(0,L)) \cap C^1([0,T]; L^2(0,L))$  of (2.11). This implies that

$$\int_{Q} (-\psi_{t}\varphi_{t} + x^{\alpha}\psi_{x}\varphi_{x})dxdt = \int_{Q} \ell\varphi dxdt,$$

$$\forall \varphi \in L^{2}(0, T; H^{1}_{\alpha}(0, L)) \text{ with } \varphi_{t} \in L^{2}(Q) \text{ and } \varphi(\cdot, 0) = 0 \text{ in } (0, L).$$
(2.12)

Choosing  $\varphi = \psi$  in (2.10), and  $\varphi = \hat{y}$  in (2.12), we have that

$$\int_{Q} \ell \widehat{y} dx dt = 0, \ \forall \ \ell \in C_0^{\infty}(Q).$$

Hence,  $\hat{y}(x,t) = 0$  a.e. in Q. The proof is completed.

**Remark 2.1** The well-posedness of the system (2.1) was proved by the semigroup theory (for  $\alpha \in (0,1)$ ) in [9, Proposition 4.2]. Here we use another method to prove it for any  $\alpha \in (0,\infty)$ . Our method is also applicable to the more general multi-dimensional degenerate wave operator:

$$\mathcal{L}y = y_{tt} - div(b(x, t)\nabla y),$$

where  $b \in C(\overline{\Omega} \times [0,T])$  satisfies  $\frac{b_t}{b} \in L^{\infty}(\Omega \times (0,T))$  and b > 0 in  $\Omega \times (0,T)$ , with  $\Omega$  being a nonempty bounded domain of  $\mathbb{R}^n$ .

In the following, we prove a hidden regularity result for solutions of the following degenerate wave equation:

$$\begin{cases}
z_{tt} - (x^{\alpha} z_x)_x = h & (x, t) \in Q, \\
z(0, t) = 0 & (0 < \alpha < 1) \\
(x^{\alpha} z_x)(0, t) = 0 & (\alpha \ge 1) & t \in (0, T), \\
z(L, t) = 0 & t \in (0, T), \\
z(x, 0) = z_0(x), z_t(x, 0) = z_1(x) & x \in (0, L),
\end{cases}$$
(2.13)

where  $h \in L^1(0,T;L^2(0,L))$  and  $(z_0,z_1) \in H^1_\alpha(0,L) \times L^2(0,L)$ . Define the following energy functional:

$$E(t) = \frac{1}{2} \int_0^L \left[ z_t(x,t)^2 + x^{\alpha} z_x^2(x,t) \right] dx.$$

As a preliminary, we have the following energy estimate.

**Lemma 2.1** There exists a constant C > 0 such that  $E(t) \leq C\Big(E(0) + \Big(\int_0^T \|h\|_{L^2(0,L)} dt\Big)^2\Big)$ ,  $\forall t \in [0,T]$ .

**Sketch of the proof.** For any  $h \in L^1(0,T;L^2(0,L))$ , multiplying both sides of the first equation in (2.13) by  $z_t$  and integrating it on (0,L), by Gronwall's inequality and a simple calculation, we can obtain the desired estimate.

By Lemma 2.1, we have the following hidden regularity result for (2.13).

**Proposition 2.1** For any  $\alpha \in (0,2)$ , any solution z of (2.13) satisfies  $z_x(L,\cdot) \in L^2(0,T)$ . Moreover,

$$\int_0^T z_x^2(L,t)dt \le C\Big(E(0) + \Big(\int_0^T \|h\|_{L^2(0,L)}dt\Big)^2\Big). \tag{2.14}$$

**Proof.** It suffices to prove (2.14) for classical solutions of (2.13). Hence, we assume that h and  $(z_0, z_1)$  are sufficiently smooth. Then the system (2.13) admits a unique classical solution  $z \in C([0,T]; H^2_{\alpha}(0,L)) \cap C^1([0,T]; H^1_{\alpha}(0,L))$  (see [9, Proposition 4.2]), where  $H^2_{\alpha}(0,L) = \{u \in C([0,T]; H^2_{\alpha}(0,L)) \cap C^1([0,T]; H^2_{\alpha}(0,L)) \in \{u \in C([0,T]; H^2_{\alpha}(0,L)) \cap C^1([0,T]; H^2_{\alpha}(0,L)) \}$ 

 $H^1_{\alpha}(0,L) | x^{\alpha}u_x \in H^1(0,L)$ . Choose  $q(x) = \begin{cases} x & x \in [0,\frac{L}{2}), \\ -2x + \frac{3}{2}L & x \in [\frac{L}{2},L], \end{cases}$  multiplying both sides of the first equation in (2.13) by  $qz_x$  and integrating it on Q, we obtain that

$$\frac{1}{2} \int_{0}^{T} x^{\alpha} q z_{x}^{2} dt \Big|_{0}^{L} = \int_{0}^{L} q z_{t} z_{x} dx \Big|_{0}^{T} + \frac{1}{2} \int_{Q} q_{x} (z_{t}^{2} + x^{\alpha} z_{x}^{2}) dx dt - \frac{1}{2} \int_{Q} \alpha q x^{\alpha - 1} z_{x}^{2} dx dt - \int_{Q} q h z_{x} dx dt - \frac{1}{2} \int_{0}^{T} q z_{t}^{2} dt \Big|_{0}^{L}.$$
(2.15)

(2.15) implies that

$$\begin{split} &\frac{L^{\alpha+1}}{2} \int_{0}^{T} z_{x}^{2}(L,t)dt + \int_{0}^{T} (x^{\alpha+1}z_{x}^{2})(0,t)dt \\ &= -2 \int_{0}^{L} qz_{t}z_{x}dx \Big|_{0}^{T} - \int_{Q} q_{x} \Big(z_{t}^{2} + x^{\alpha}z_{x}^{2}\Big)dxdt + \int_{Q} \alpha qx^{\alpha-1}z_{x}^{2}dxdt \\ &+ 2 \int_{Q} qhz_{x}dxdt + \int_{0}^{T} qz_{t}^{2}dt \Big|_{0}^{L}. \end{split} \tag{2.16}$$

When  $\alpha \in (0,2), (xz^2)(0,t) = 0$  (see [1, Proposition 2.4]). Therefore,  $\int_0^T qz_t^2 dt \Big|_0^L = 0$ . Further, by the Hölder inequality and Lemma 2.1, we arrive at

$$-2\int_{0}^{L} qz_{t}z_{x}dx\Big|_{0}^{T} = -2\int_{0}^{\frac{L}{2}} xz_{t}z_{x}dx\Big|_{0}^{T} - 2\int_{\frac{L}{2}}^{L} z_{t}(-2x + \frac{3}{2}L)z_{x}dx\Big|_{0}^{T}$$

$$\leq C\Big(E(0) + \Big(\int_{0}^{T} \|h\|_{L^{2}(0,L)}dt\Big)^{2}\Big). \tag{2.17}$$

Similarly,

$$-\int_{Q} q_{x} \left(z_{t}^{2} + x^{\alpha} z_{x}^{2}\right) dx dt + \int_{Q} \alpha q x^{\alpha - 1} z_{x}^{2} dx dt + 2 \int_{Q} q h z_{x} dx dt$$

$$\leq C \left(E(0) + \left(\int_{0}^{T} \|h\|_{L^{2}(0,L)} dt\right)^{2}\right). \tag{2.18}$$

Combining (2.17) and (2.18) with (2.16), we get the desired estimate (2.14). 

Now, we give a proof of Theorem 1.1.

**Proof of Theorem 1.1.** First, by Theorem 2.1 and Proposition 2.1, the equation (1.4) admits a unique weak solution  $v \in C([0,T]; H^1_\alpha(0,L)) \cap C^1([0,T]; L^2(0,L))$  satisfying  $v_x(L,\cdot) \in L^2(0,T)$ . Next, define a linear functional on  $L^1(0,T;L^2(0,L))$ :

$$\mathscr{L}(\xi) = \langle w_1, v(0) \rangle_{H^*_{\alpha}(0,L), H^1_{\alpha}(0,L)} - \int_0^L w_0 v_t(0) dx - L^{\alpha} \int_0^T \theta(t) v_x(L,t) dt, \quad \forall \ \xi \in L^1(0,T; L^2(0,L)).$$

Then we have

$$|\mathcal{L}(\xi)| \leq |(w_0, w_1)|_{L^2(0, L) \times H_{\alpha}^*(0, L)} \cdot |(v(0), v_t(0))|_{H_{\alpha}^1(0, L) \times L^2(0, L)} + C \|\theta\|_{L^2(0, T)} \cdot \|v_x(L, \cdot)\|_{L^2(0, T)}.$$

By Proposition 2.1 and Lemma 2.1,  $\mathcal{L}$  is a bounded linear functional on  $L^1(0,T;L^2(0,L))$ . Therefore, there exists a function  $w \in L^{\infty}(0,T;L^{2}(0,L))$ , such that (1.3) holds. By the standard smoothing technique, we can get  $w \in C([0,T]; L^2(0,L))$ .

Similarly, consider the following degenerate wave equation:

$$\begin{cases} v_{tt} - (x^{\alpha}v_x)_x = g_t & (x,t) \in Q, \\ \begin{cases} v(0,t) = 0 & (0 < \alpha < 1) \\ (x^{\alpha}v_x)(0,t) = 0 & (\alpha \ge 1) \end{cases} & t \in (0,T), \\ v(L,t) = 0 & t \in (0,T), \\ v(x,T) = 0, v_t(x,T) = 0 & x \in (0,L), \end{cases}$$

where  $g \in C_0^{\infty}(0,T;H^1_{\alpha}(0,L))$ . For this system, we can obtain the conclusions similar to Proposition 2.1 and Lemma 2.1. Notice that the density of  $C_0^{\infty}(0,T;H_{\alpha}^1(0,L))$  in  $L^1(0,T;H_{\alpha}^1(0,L))$ , we can also prove  $w \in C^1([0,T]; H^*_{\alpha}(0,L))$ . This completes the proof.

#### 3 Controllability of degenerate wave equations

In this section, we study the boundary controllability of the degenerate wave equation (1.1). First, consider the following linear degenerate wave equation:

$$\begin{cases}
v_{tt} - (x^{\alpha}v_x)_x = 0 & (x,t) \in Q, \\
v(0,t) = 0 & (0 < \alpha < 1) \\
(x^{\alpha}v_x)(0,t) = 0 & (\alpha \ge 1) \\
v(L,t) = 0 & t \in (0,T), \\
v(x,0) = v_0(x), v_t(x,0) = v_1(x) & x \in (0,L),
\end{cases}$$
(3.1)

where  $(v_0, v_1) \in H^1_{\alpha}(0, L) \times L^2(0, L)$ .

By the duality technique, it is easily seen that, the null controllability of (1.1) can be reduced to an observability estimate for the system (3.1).

**Proposition 3.1** The system (1.1) is null controllable in time T with a control  $\theta \in L^2(0,T)$ , if and only if there exists a constant C > 0, such that any solution v of (3.1) satisfies

$$||v_0||_{H^1_\alpha(0,L)}^2 + ||v_1||_{L^2(0,L)}^2 \le C \int_0^T v_x^2(L,t)dt, \quad \forall \ (v_0,v_1) \in H^1_\alpha(0,L) \times L^2(0,L). \tag{3.2}$$

First, introduce some notations. We write  $A \approx B$ , if there exist two constants  $C_1, C_2 > 0$ , such that  $C_1A \leq B \leq C_2A$ . Set  $l^2(\mathbb{Z}) = \left\{ \{a_n\}_{n \in \mathbb{Z}} \mid a_n \in \mathbb{C}, \sum_{n \in \mathbb{Z}} |a_n|^2 < +\infty \right\}$  and i denotes the imaginary unit.

In order to prove the observability inequality (3.2) for the system (3.1), we adopt the similar method used in [9, Theorem 3.1], based on the following known Ingham inequality.

**Lemma 3.1** ([10, Theorem 9.2]) Suppose that  $\{\eta_n\}_{n\in\mathbb{Z}}$  is a sequence of real numbers satisfying the following gap condition for some constant  $\delta > 0$ ,

$$\eta_{n+1} - \eta_n > \delta, \quad \forall \ n \in \mathbb{Z}.$$

Let  $n^+(r)$  denote the largest number of terms of the sequence  $\{\eta_n\}_{n\in\mathbb{Z}}$  contained in an interval of length r, and  $D^+ = \lim_{r \to \infty} \frac{n^+(r)}{r}$ . Then, it holds that
(1) For any  $T > 2\pi D^+$ ,

$$\int_0^T \left| \sum_{n \in \mathbb{Z}} a_n e^{i\eta_n t} \right|^2 dt \approx \sum_{n \in \mathbb{Z}} \left| a_n \right|^2, \quad \text{for any } \{a_n\}_{n \in \mathbb{Z}} \in l^2(\mathbb{Z}). \tag{3.3}$$

(2) For any  $T < 2\pi D^+$ , (3.3) does not hold.

The main result of this section is the following observability inequality for the system (3.1).

**Theorem 3.1** Suppose that  $\alpha \in (0,2)$ . Then for any  $T > T_{\alpha}$  (defined in (1.2)), the estimate (3.2) holds for (3.1). For any  $T < T_{\alpha}$ , the estimate (3.2) does not hold.

**Proof.** (1) For any  $\alpha \in (0,1)$ , we consider the eigenvalue problem:

$$\begin{cases} -(x^{\alpha}\Phi'(x))' = \lambda\Phi(x) & x \in (0, L), \\ \Phi(0) = \Phi(L) = 0. \end{cases}$$
(3.4)

Similar to [9], for any  $n \ge 1$ , the solutions of (3.4) are given as follows:

$$\lambda_n = \left(\frac{\rho j_{\mu,n}}{L^{\rho}}\right)^2 \quad \text{and} \quad \Phi_n(x) = \frac{(2\rho)^{\frac{1}{2}}}{L^{\rho} |J'_{\mu}(j_{\mu,n})|} x^{\frac{1-\alpha}{2}} J_{\mu} \left(j_{\mu,n} \frac{x^{\rho}}{L^{\rho}}\right), \ \forall \ x \in (0,L),$$
 (3.5)

where

$$\mu = \frac{1-\alpha}{2-\alpha}, \ \rho = \frac{2-\alpha}{2}, \ J_{\mu}(x) = \sum_{m>0} \frac{(-1)^m}{m! \cdot \Gamma(m+\mu+1)} \left(\frac{x}{2}\right)^{2m+\mu}, \ x \ge 0, \tag{3.6}$$

 $\Gamma(\cdot)$  is the Gamma function, and  $\{j_{\mu,n}\}_{n\geq 1}$  are the positive zeros of the Bessel function  $J_{\mu}$ . Moreover, for any  $\mu \geq -\frac{1}{2}$  and  $\beta > 0$ ,

$$\int_{0}^{L} x^{2\beta - 1} J_{\mu} \left( j_{\mu,n} \left( \frac{x}{L} \right)^{\beta} \right) J_{\mu} \left( j_{\mu,m} \left( \frac{x}{L} \right)^{\beta} \right) dx = L^{2\beta} \frac{\delta_{n}^{m}}{2\beta} [J'_{\mu} (j_{\mu,n})]^{2}, \tag{3.7}$$

where  $\delta_n^m$  is the Kronecker symbol (see [9, (4.26)]). Therefore,  $\{\Phi_n\}_{n\in\mathbb{N}^*}$  is an orthonormal basis of  $L^2(0,L)$ .

Next, for any  $(v_0, v_1) \in H^1_\alpha(0, L) \times L^2(0, L)$ , set

$$v_0(x) = \sum_{n \in \mathbb{N}^*} v_0^n \Phi_n(x)$$
 and  $v_1(x) = \sum_{n \in \mathbb{N}^*} v_1^n \Phi_n(x)$ .

Then it is easy to check that the solution v of (3.1) is as follows:

$$v(x,t) = \sum_{n \in \mathbb{N}^*} v_n(t) \Phi_n(x)$$
 with  $v_n(t) = c_n e^{i\sqrt{\lambda_n}t} + c_{-n} e^{-i\sqrt{\lambda_n}t}$ ,

where

$$c_n = \frac{1}{2} \left( v_0^n + \frac{v_1^n}{i\sqrt{\lambda_n}} \right)$$
 and  $c_{-n} = \frac{1}{2} \left( v_0^n - \frac{v_1^n}{i\sqrt{\lambda_n}} \right)$ .

By a simple calculation, we have that

$$\Phi'_n(x) = \frac{(2\rho)^{\frac{1}{2}}}{L^{\rho}|J'_{\mu}(j_{\mu,n})|} \left(\frac{1-\alpha}{2} x^{\frac{-1-\alpha}{2}} J_{\mu}(j_{\mu,n} \frac{x^{\rho}}{L^{\rho}}) + x^{\frac{1-2\alpha}{2}} J'_{\mu}(j_{\mu,n} \frac{x^{\rho}}{L^{\rho}}) \frac{j_{\mu,n}\rho}{L^{\rho}}\right).$$

It follows that

$$\Phi'_n(L) = L^{-\frac{3}{2}} \frac{\sqrt{2\rho}}{|J'_n(j_{\mu,n})|} \rho j_{\mu,n} J'_{\mu}(j_{\mu,n}).$$

This implies that

$$v_x(L,t) = L^{-\frac{3}{2}} \rho \sqrt{2\rho} \sum_{n \in \mathbb{N}^*} j_{\mu,n} \frac{J'_{\mu}(j_{\mu,n})}{|J'_{\mu}(j_{\mu,n})|} (c_n e^{i\sqrt{\lambda_n}t} + c_{-n} e^{-i\sqrt{\lambda_n}t}).$$

By Lemma 3.1, we have that for any  $T > 2\pi D^+$ ,

$$\int_{0}^{T} v_{x}^{2}(L,t)dt \simeq \sum_{n \in \mathbb{N}^{*}} (j_{\mu,n})^{2} \left( |v_{0}^{n}|^{2} + \frac{|v_{1}^{n}|^{2}}{(\rho j_{\mu,n})^{2}} \right)$$

$$\simeq \sum_{n \in \mathbb{N}^{*}} (\lambda_{n} |v_{0}^{n}|^{2} + |v_{1}^{n}|^{2}) \simeq ||v_{0}||_{H_{\alpha}^{1}(0,L)}^{2} + ||v_{1}||_{L^{2}(0,L)}^{2}.$$
(3.8)

Moreover, by the definition of  $n^+(r)$ .

$$\lim_{r \to \infty} \left[ \frac{r}{n^+(r)} - \frac{1}{n^+(r)} \sum_{n=1}^{n^+(r)-1} \frac{\rho}{L^{\rho}} \int_{j_{\mu,n}}^{j_{\mu,n+1}} dx \right] = 0.$$

Since  $(j_{\mu,n+1} - j_{\mu,n})$  converges to  $\pi$  as  $n \to +\infty$  ([9, Lemma 1]), we get that  $D^+ = \frac{L^{\rho}}{\rho \pi}$ . Hence, (3.2) holds for any  $T > 2\pi D^+ = T_{\alpha}$ , but it fails for any  $T < T_{\alpha}$ .

(2) For any  $\alpha \in [1, 2)$ , we consider the eigenvalue problem:

$$\begin{cases} -(x^{\alpha}\widehat{\Phi}'(x))' = \lambda \widehat{\Phi}(x) & x \in (0, L), \\ (x^{\alpha}\widehat{\Phi}')(0) = \widehat{\Phi}(L) = 0. \end{cases}$$
(3.9)

Similar to the case of  $\alpha \in (0,1)$ , the solutions of (3.9) are as follows:

$$\widehat{\lambda}_n = \left(\frac{\rho j_{\widehat{\mu},n}}{L^{\rho}}\right)^2, \ n \ge 1 \quad \text{and} \quad \widehat{\Phi}_n(x) = \frac{(2\rho)^{\frac{1}{2}}}{L^{\rho}|J_{\widehat{\mu}}(j_{\widehat{\mu},n})|} x^{\frac{1-\alpha}{2}} J_{\widehat{\mu}}\left(j_{\widehat{\mu},n} \frac{x^{\rho}}{L^{\rho}}\right), \ \forall \ x \in (0,L),$$

where  $\widehat{\mu} = \frac{\alpha - 1}{2 - \alpha}$  and  $\rho = \frac{2 - \alpha}{2}$ . Also, it is easy to check that  $\{\widehat{\Phi}_n\}_{n \in \mathbb{N}^*}$  is an orthonormal basis of  $L^2(0, L)$ . The remainder of the proof is similar to that in the case (1) and we omit it here.

**Remark 3.1** It is well known that the controllability time T = 2L is optimal for the classical onedimensional wave equation (with constant coefficients) in (0, L). However, the controllability of the system (1.1) in time  $T = T_{\alpha}$  is still an open problem.

**Proof of (2) in Theorem 1.2.** For any  $\alpha > 2$ , let

$$X = \int_x^L y^{-\frac{\alpha}{2}} dy$$
 and  $W(X,t) = x^{\frac{\alpha}{4}} w(x,t)$  (see [5]).

Then the null controllability of the equation (1.1) is transformed into a null controllability problem for the following nondegenerate wave equation on the half-line:

$$\begin{cases}
W_{tt}(X,t) - W_{XX}(X,t) + M(X)W(X,t) = 0 & (X,t) \in \mathbb{R}^+ \times (0,T), \\
W(0,t) = \theta(t) & t \in (0,T),
\end{cases}$$
(3.10)

where 
$$M(X) = \frac{\alpha(3\alpha-4)}{\left[4L^{\frac{2-\alpha}{2}} - 2(2-\alpha)X\right]^2}$$
.

On the other hand, by the duality, the null controllability of (3.10) is equivalent to the following observability estimate:

$$|\psi_0|_{H_0^1(\mathbb{R}^+)}^2 + |\psi_1|_{L^2(\mathbb{R}^+)}^2 \le C \int_0^T \psi_X^2(0, t) dt, \tag{3.11}$$

where  $\psi$  is the solution of the nondegenerate wave equation:

$$\begin{cases} \psi_{tt} - \psi_{XX} + M(X)\psi = 0 & (X,t) \in \mathbb{R}^+ \times (0,T), \\ \psi(0,t) = 0 & t \in (0,T), \\ \psi(X,0) = \psi_0(X), \ \psi_t(X,0) = \psi_1(X) & X \in \mathbb{R}^+. \end{cases}$$
(3.12)

For any given  $\widehat{\psi}_0$ ,  $\widehat{\psi}_1 \in C_0^{\infty}(\mathbb{R}^+)$ , we choose  $\psi_0^n(X) = \widehat{\psi}_0(X - n)$  and  $\psi_1^n(X) = \widehat{\psi}_1(X - n)$  with sufficiently large n (see [13]). Let  $\psi^n$  be the solution of (3.12) with initial value  $(\psi_0^n, \psi_1^n)$ . It is easy

to see that

$$\frac{|\psi_0^n|_{H_0^1(\mathbb{R}^+)}^2 + |\psi_1^n|_{L^2(\mathbb{R}^+)}^2}{\int_0^T \psi_X^n(0,t)^2 dt} \to \infty, \text{ as } n \to \infty.$$

This implies that the system (3.10) is not null controllable.

For 
$$\alpha = 2$$
, we can also prove a similar result for  $M(X) = \frac{1}{4}$ .

Note. After completion of this work, we learned that Professor F. Alabau-Boussouira, Professor P. Cannarsa and Professor G. Leugering proved the null controllability for some one-dimensional degenerate wave equations with a more general diffusion coefficient, using the multiplier method in the paper [F. Alabau-Boussouira, P. Cannarsa, G. Leugering, Control and stabilization of degenerate wave equations, arXiv:1505.05720]. Parts results of our work were contained in this paper. But we prove the null controllability results of degenerate wave equations by a (different) spectral method. Also, the controllability time  $T_{\alpha}$  is sharp in our work, i.e., if  $T > T_{\alpha}$ , the controllability holds and it is false if  $T < T_{\alpha}$ . Our work was reported by Professor Hang Gao on the 9th Workshop on Control of Distributed Parameter Systems on July 2, 2015 in Beijing.

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