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A Symbolic-Numeric Approach for Parametrizing Ruled Surfaces*

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Abstract In this paper, we present symbolic algorithms to determine whether a given surface (implicitly or parametrically defined) is a rational ruled surface and find a proper parametrization of the ruled surface. However, in practical applications, one has to deal with numerical objects that are given approximately, probably because they proceed from an exact data that has been perturbed under some previous measuring process or manipulation. For these numerical objects, we adapt the symbolic algorithms presented by means of the use of numerical techniques. We develop numeric algorithms that allow to determine ruled surfaces "*close*" to an input (not necessarily ruled) surface, and the distance between the input and the output surface is computed.

Keywords ruled surface, standard parametrization, implicit representation, numeric algorithm.

1 Introduction

The ruled surface is an important surface widely used in computer aided geometric design (CAGD) and geometric modeling (see [1–18]). Thus, topics related to ruled surfaces are studied by many researchers. For instance, using the μ–bases method, Chen et al. (see [5]) give an implicitization algorithm for a rational ruled surface. The univariate resultant was also used to compute the implicit equations efficiently (see [12, 16]). In [4], for a given rational ruled surface, authors find a simplified reparametrization with the lowest possible degree which does not contain any non–generic base point. Busé and Dohm in [3, 7] study the ruled surface using

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μ-bases. Jia et al (see [8]) compute the self–intersection curves of a rational ruled surface based on μ-bases. In [10], Li et al. compute a proper reparametrization of an improper parametric ruled surface. Andradas et al. (see [1]) present an algorithm to decide whether a proper rational parametrization of a ruled surface can be properly reparametrized over a real field. The ruled surfaces have been used for geometric modeling of architectural freeform design (see e.g. [11]). The collision and intersection of the ruled surfaces were discussed in [6, 17]. In addition, S. Izumiya (see [9]) studies the cylindrical helices and Bertrand curves on ruled surfaces. In [14], authors present an algorithm that covers any given rational ruled surface with two rational parametrizations. In [18], authors develop algorithms to determine whether a given implicit or parametric algebraic surface is a rational ruled surface, and in the affirmative case, to compute a standard parametric representation for the surface. In [15], authors gave alternative way to characterize a rational ruled surfaces by μ-bases. Peternell et al. (see [13]), derive criterion for deciding the rationality of the cissoid of two given real affine rational surfaces.

Literature shows that several problems for ruled surfaces have been discussed in symbolic consideration. Nevertheless, in many practical applications, for instance in the frame of CAGD, these approaches tend to be insufficient, since most of the real data objects are given approximately. As a consequence, there has been an increasing interest in the development of hybrid symbolic–numeric algorithms, and approximate algorithms. We deal with the approximate parametrization problem for a given ruled surface (implicitly or parametrically defined) in this paper. More precisely, we consider two different problems: the first one, for a given polynomial $F(\overline{x}) \in \mathbb{C}[\overline{x}]$, $\overline{x} = (x_1, x_2, x_3)$ (with perturbed float coefficients) defining an algebraic surface $\mathcal{V}$, and we show how to find a rational parametrization $\mathcal{P}(\bar{t}) \in \mathbb{C}(\bar{t})^3$, $\bar{t} = (t_1, t_2)$ of an algebraic *ruled* surface $\mathcal{W}$ such that $\mathcal{V}$ and $\mathcal{W}$ are *close enough*. We refer to this problem as the numerical implicit ruled problem. For the second problem, for a given rational parametrization $\mathcal{M}(\bar{t}) \in \mathbb{C}(\bar{t})^3$ (with perturbed float coefficients) of an algebraic surface $\mathcal{V}$, and we find a rational parametrization $\mathcal{P}(\bar{t}) \in \mathbb{C}(\bar{t})^3$ of an algebraic *ruled* surface $\mathcal{W}$ such that $\mathcal{V}$ and $\mathcal{W}$ are *close enough*. We refer to this problem as the numerical parametric ruled problem. In both cases, we analyze and compute the distance between the input and the output surface.

We first present an algorithmic approach to symbolically parametric ruled surfaces. The method presented is new but it is based on the ideas developed in [18]. We show that this new algorithm can be easily be applied to objects given approximately. For this purpose, some numerical techniques have to be applied. In particular, one of the most important steps is the computation of proper rational parametrizations of several plane curves appearing in the algorithms. If one tries to apply these algorithms to input given approximately, one has to apply some numerical approaches to compute proper parametrizations of "aproximate" rational plane curves. We introduce the algorithm presented in [19] to compute these approximate parametrizations for the plane curves.

The paper is organized as follows. In Section 2, we present a symbolic approach for the ruled surfaces defined implicitly or parametrically. If the surface is a rational ruled surface, we compute a rational parametrization in standard form for it. In Section 3, we focus on the numerical problems with implicit and parametric surfaces. For both the cases, we look for a rational parametrization of a ruled surface "close" to the input surface, furthermore, we demonstrate how to measure the distance between the input and the output surface.

2 A Symbolic Approach for Parametrizing Ruled Surfaces

In this section we briefly review some notions and we present two algorithmic approaches (based on [18]) to symbolically parametric ruled surfaces.

2.1 Implicitly Rational Ruled Surfaces

Let $\mathcal{V}$ be a ruled surface defined by the polynomial $F(\overline{x}) \in \mathbb{C}[\overline{x}]$, $\overline{x} = (x_1, x_2, x_3)$, where $\mathbb{C}$ is the field of complex numbers. In the following, we analyze whether $\mathcal{V}$ is a rational ruled surface. In the affirmative case, we compute a proper rational parametrization of $\mathcal{V}$ in the standard reduced form (see (2)).

A standard parametrization of a rational ruled surface $\mathcal{V}$ is given by a parametrization of the form

$$\mathcal{Q}(\bar{t}) = (m_1(t_1) + t_2 n_1(t_1), m_2(t_1) + t_2 n_2(t_1), m_3(t_1) + t_2 n_3(t_1)) \in \mathbb{C}(\bar{t})^3, \tag{1}$$

where $\bar{t} = (t_1, t_2)$, and for at least one $i \in \{1, 2, 3\}$, it holds that $n_i \neq 0$ (otherwise, $\mathcal{V}$ degenerates to a space curve). We refer to $\mathcal{Q}$ as the standard form parametrization of $\mathcal{V}$.

Note that if $\mathcal{V}$ is defined by (1), and $n_3 \neq 0$, the surface $\mathcal{V}$ admits a parametrization of the form

$$\begin{aligned}\mathcal{P}(\bar{t}) &= (p_1(t_1) + t_2 q_1(t_1), p_2(t_1) + t_2 q_2(t_1), t_2) = \\ &\left(\frac{p_{11}(t_1) + t_2 q_{11}(t_1)}{q(t_1)}, \frac{p_{21}(t_1) + t_2 q_{21}(t_1)}{q(t_1)}, t_2\right) \in \mathbb{C}(\bar{t})^3,\end{aligned} \tag{2}$$

where $q_{k1}(t_1)/q(t_1) = n_k/n_3 \neq 0$, for some $k = 1, 2$. Such a parametrization is obtained by performing the birational transformation $(t_1, t_2) \to (t_1, (t_2 - m_3(t_1))/n_3(t_1))$. One may reason similarly as above, if $n_1 \neq 0$ or $n_2 \neq 0$. In the following, we refer to the parametrization $\mathcal{P}$ as the standard reduced form parametrization of $\mathcal{V}$. We assume w.l.o.g that $\mathcal{P}$ is proper (see [10]) and $\deg_{t_1}(p_{i1} + t_2 q_{i1}) = \deg(q) = d$, $i = 1, 2$ (otherwise, one considers a linear change of variables).

Under these conditions, the corresponding projective surface $\overline{\mathcal{V}}$ is defined by the projective parametrization

$$\overline{\mathcal{P}}(\bar{t}) = (q(t_1),\, p_{11}(t_1) + t_2 q_{11}(t_1),\, p_{21}(t_1) + t_2 q_{21}(t_1),\, t_2 q(t_1)) \in \mathbb{P}^3(\mathbb{C}(\bar{t})).$$

We observe that $\overline{\mathcal{V}}$ is implicitly defined by the homogenization $\overline{F}(x_0, x_1, x_2, x_3)$ of $F(x_1, x_2, x_3)$. Therefore, if we write

$$F(\overline{x}) = F_d(\overline{x}) + F_{d-1}(\overline{x}) + \cdots + F_0(\overline{x}), \quad \overline{x} = (x_1, x_2, x_3)$$

where $F_k(\overline{x})$ is a homogeneous polynomial of degree k, and $F_d \neq 0$, then

$$\overline{F}(x_0, \overline{x}) = F_d(\overline{x}) + F_{d-1}(\overline{x}) x_0 + \cdots + F_0(\overline{x}) x_0^d.$$

Finally, we represent by $\overline{\mathcal{P}}(t_0,t_1,t_2)$ the homogeneous parametrization of $\overline{\mathcal{V}}$ obtained from the homogenization of $\overline{\mathcal{P}}(t_1,t_2)$.

From the rational parametrization $\mathcal{P}(\bar{t}\,)$, we introduce an auxiliary parametrization of a space curve defined over $\overline{\mathbb{C}(t_1)}$. More precisely, we consider the partial parametrization associated to $\mathcal{P}$,

$$\mathcal{P}^*_{t_1}(t_2) := (p_1(t_1)+t_2q_1(t_1), p_2(t_1)+t_2q_2(t_1), t_2) \in \mathbb{C}(t_1)[t_2]^2$$

(that is, $\mathcal{P}^*_{t_1}(t_2)$ is defined over $\mathbb{C}(t_1)$). We denote by $\mathcal{C}^*_{t_1}$ the space curve defined by $\mathcal{P}^*_{t_1}(t_2)$ (note that $\mathcal{C}^*_{t_1}$ is a line over $\overline{\mathbb{C}(t_1)}$). Observe that the corresponding projective curve $\overline{\mathcal{C}}^*_{t_1}$ is defined by the projective parametrization

$$\overline{\mathcal{P}}^*_{t_1}(t_2) = (q(t_1),\, p_{11}(t_1)+t_2q_{11}(t_1),\, p_{21}(t_1)+t_2q_{21}(t_1), t_2q(t_1)) \in \mathbb{P}^3((\mathbb{C}(t_1))(t_2)).$$

We represent by

$$\overline{\mathcal{P}}^*_{t_1}(t_0,t_2) = (t_0q(t_1),\, t_0p_{11}(t_1)+t_2q_{11}(t_1),\, t_0p_{21}(t_1)+t_2q_{21}(t_1), t_2q(t_1))$$

the homogeneous parametrization of $\overline{\mathcal{C}}^*_{t_1}$ obtained from the homogenization of $\overline{\mathcal{P}}^*_{t_1}(t_2)$. We observe that $\overline{\mathcal{C}}^*_{t_1} \subset \overline{\mathcal{V}}$ and then $\overline{F}(\overline{\mathcal{P}}^*_{t_1}(t_0,t_2)) = 0$.

Under these conditions, and taking into account that $\overline{\mathcal{P}}(t_0,t_1,t_2)$ parametrizes the surface $\overline{\mathcal{V}}$ which is implicitly defined by the polynomial $\overline{F}(x_0,x_1,x_2,x_3)$ (this implies that $\overline{F}(\overline{\mathcal{P}}(t_0,t_1,t_2)) = 0$, and that $\overline{F}(\overline{\mathcal{P}}^*_{t_1}(t_0,t_2)) = 0$), we get that:

- $\overline{\mathcal{P}}_1(t_1) := (q(t_1),\, p_{11}(t_1),\, p_{21}(t_1))$ (which is obtained from $\overline{\mathcal{P}}(t_1,0)$) parametrizes a rational plane curve $\overline{\mathcal{C}}_1$. Let $F_1(x_0,x_1,x_2)$ be the implicit equation defining $\overline{\mathcal{C}}_1$. Thus, $F_1(x_0,x_1,x_2)$ is a factor of the polynomial $\overline{F}(x_0,x_1,x_2,0)$. The associated affine curve $\mathcal{C}_1$ is defined by the rational affine parametrization $\mathcal{P}_1(t_1) := (p_1(t_1),\, p_2(t_1))$ and the affine polynomial $f_1(x_1,x_2) = F_1(1,x_1,x_2)$.

- $\overline{\mathcal{P}}_2(t_1) = (q_{11}(t_1),\, q_{21}(t_1),\, q(t_1))$ (which is obtained from $\overline{\mathcal{P}}^*_{t_1}(0,t_2)$) parametrizes a rational plane curve $\overline{\mathcal{C}}_2$. Let $F_2(x_1,x_2,x_3)$ be the implicit equation defining $\overline{\mathcal{C}}_2$. Then $F_2(x_1,x_2,x_3)$ is a factor of the polynomial $\overline{F}(0,x_1,x_2,x_3)$. The curve $\mathcal{C}_2$ is defined by the rational affine parametrization $\mathcal{P}_2(t_1) := (q_1(t_1),\, q_2(t_1))$ and the affine polynomial $f_2(x_1,x_2) = F_2(x_1,x_2,1)$.

Under these conditions, we have the following theorem.

Theorem 2.1 *A surface $\mathcal{V}$ defined by a polynomial $F(\overline{x}\,) \in \mathbb{C}[\,\overline{x}\,]$ is a rational ruled surface if and only if the following statements hold:*

1. *There exist two plane curves, $\mathcal{C}_1$ and $\mathcal{C}_2$, defined by a factor of the polynomials $\overline{F}(x_0,x_1,x_2,0)$ and $\overline{F}(0,x_1,x_2,x_3)$, respectively, that are rational. Let $\mathcal{P}_1 = (p_1,p_2) \in \mathbb{C}(t_1)^2$ and $\mathcal{P}_2 = (q_1,q_2) \in \mathbb{C}(t_1)^2$ be proper rational parametrizations of $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively.*

2. *Let $g(x_1,x_2,t_2) = \mathrm{numer}(F(p_1(x_1)+t_2q_1(x_2), p_2(x_1)+t_2q_2(x_2), t_2))$. There exists $R(t_1) := (r_1(t_1), r_2(t_1)) \in (\mathbb{C}(t_1)\setminus\mathbb{C})^2$ proper such that and $g(R(t_1),t_2) = 0$. In this case,*

$$\mathcal{P}(\bar{t}\,) = (p_1(r_1(t_1))+t_2q_1(r_2(t_1)), p_2(r_1(t_1))+t_2q_2(r_2(t_1)), t_2)$$

is a proper rational parametrization of $\mathcal{V}$.

Proof It is clear that if statements 1 and 2 hold, then $\mathcal{V}$ is a rational ruled surface. Reciprocally, let $\mathcal{V}$ be a rational ruled surface. Then, a parametrization of $\mathcal{V}$ is given by a standard form parametrization (see (1))

$$\mathcal{Q}(\bar{t}) = (m_1(t_1) + t_2 n_1(t_1), m_2(t_1) + t_2 n_2(t_1), m_3(t_1) + t_2 n_3(t_1)) \in \mathbb{C}(\bar{t})^3.$$

We assume that $n_3 \neq 0$. Thus, reasoning as in the paragraph before to Theorem 2.1, we get that statement 1 holds. Let $\mathcal{P}_1 = (p_1, p_2) \in \mathbb{C}(t_1)^2$ and $\mathcal{P}_2 = (q_1, q_2) \in \mathbb{C}(t_1)^2$ be proper rational parametrizations of the rational plane curves $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively.

In order to prove statement 2, we observe that since $n_3 \neq 0$, $\mathcal{V}$ admits a standard reduced form parametrization (see (2))

$$\mathcal{M}(\bar{t}) = (u_1(t_1) + t_2 v_1(t_1), u_2(t_1) + t_2 v_2(t_1), t_2) \in \mathbb{C}(\bar{t})^3.$$

We assume w.l.o.g. that $\mathcal{M}$ is proper (otherwise, it can be easily reparametrized using the results in [10]). Observe that $\mathcal{M}(t_1, 0) = (u_1, u_2) \in \mathbb{C}(t_1)^2$ is a rational parametrization of $\mathcal{C}_1$. Then, since $\mathcal{P}_1$ is a proper parametrization of $\mathcal{C}_1$, there exists $r_1 \in \mathbb{C}(t_1) \setminus \mathbb{C}$ such that $\mathcal{P}_1(r_1) = (p_1(r_1), p_2(r_1)) = (u_1, u_2)$. Reasoning similarly with $\mathcal{P}_2$, we get that there exists $r_2 \in \mathbb{C}(t_1) \setminus \mathbb{C}$ such that $\mathcal{P}_2(r_2) = (q_1(r_2), q_2(r_2)) = (v_1, v_2)$. Thus, since $\mathcal{M}$ parametrizes properly $\mathcal{V}$, we have that

$$\mathcal{P}(\bar{t}) := (p_1(r_1(t_1)) + t_2 q_1(r_2(t_1)), p_2(r_1(t_1)) + t_2 q_2(r_2(t_1)), t_2) \in \mathbb{C}(\bar{t})^3,$$

is a proper parametrization of $\mathcal{V}$ (note that $\mathcal{P} = \mathcal{M}$), and

$$R(t_1) = (r_1(t_1), r_2(t_1)) \in (\mathbb{C}(t_1) \setminus \mathbb{C})^2,$$

satisfies that

$$g(R(t_1), t_2) = \mathrm{numer}(F(p_1(r_1(t_1)) + t_2 q_1(r_2(t_1)), p_2(r_1(t_1)) + t_2 q_2(r_2(t_1)), t_2)) = 0.$$

Observe that since $\mathcal{P}$ is proper, R is proper. Indeed: if R is not proper, there exists $\alpha(s_1) \in \overline{\mathbb{C}(s_1)}$, $\alpha(s_1) \neq s_1$ such that $R(\alpha(s_1)) = R(s_1)$ ($\overline{\mathbb{C}(s_1)}$ is the algebraic closure of $\mathbb{C}(s_1)$, and s_1 is a new variable). Then, $\mathcal{P}(\alpha(s_1), s_2) = \mathcal{P}(s_1, s_2)$ and $\alpha(s_1) \neq s_1$, which is impossible since $\mathcal{P}$ is proper. ∎

Remark 2.2 In the following, we denote by $\mathcal{D}$ the rational plane curve parametrized by $R(t_1)$. We observe that, if $h(x_1, x_2)$ denotes the implicit polynomial defining $\mathcal{D}$, then $h(x_1, x_2)$ divides the polynomial $g(x_1, x_2, t_2)$ introduced in statement 2 in Theorem 2.1.

Remark 2.3 Theorem 2.1 improves Theorem 1 in [18] in the following sense: in statement 1, only two rational curves are considered (in Theorem 1 in [18], one has to consider three plane curves). In statement 2, a rational plane curve has to be parametrized (in Theorem 1 in [18], one has to parametrize a rational space curve).

In Theorem 2 in [18], a different approach is presented. There, only two plane curves are considered but in statement 2 of this theorem, several cases have to be analyzed.

We will see that Theorem 2.1 can be easily be applied to objects that are given approximate (see Subsection 3.1).

In the following, we present the algorithm obtained from Theorem 2.1, and we illustrate it with an example.

Algorithm 1: Symbolic computation of a parametrization from an implicit (ruled) surface.
[Step 1] Compute the polynomials $F(x_1,x_2,0)$ and $\overline{F}(0,x_1,x_2,1)$, and check whether there exist two rational plane curves $\mathcal{C}_1$ and $\mathcal{C}_2$ defined by a factor of the above polynomials, respectively. In the affirmative case, go to Step 2. Otherwise, RETURN "`$\mathcal{V}$ is not a rational ruled surface`".
[Step 2] Compute $\mathcal{P}_1=(p_1,p_2)\in\mathbb{C}(t_1)^2$ and $\mathcal{P}_2=(q_1,q_2)\in\mathbb{C}(t_1)^2$ proper rational parametrizations of the plane curves $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively.
[Step 3] Let $g(x_1,x_2,t_2)=\mathrm{numer}(F(p_1(x_1)+t_2q_1(x_2),p_2(x_1)+t_2q_2(x_2),t_2))$. Check whether there exists a rational plane curve $\mathcal{D}$ defined by a factor, $h(x_1,x_2)$, of the above polynomial (see Remark 2.2). In the affirmative case, go to Step 4. Otherwise, RETURN "`$\mathcal{V}$ is not a rational ruled surface`".
[Step 4] Compute a proper rational parametrization, $R(t_1):=(r_1(t_1),r_2(t_1))\in(\mathbb{C}(t_1)\setminus\mathbb{C})^2$, of the curve $\mathcal{D}$.
[Step 5] RETURN

$$\mathcal{P}(\bar{t}\,)=(p_1(r_1(t_1))+t_2q_1(r_2(t_1)),p_2(r_1(t_1))+t_2q_2(r_2(t_1)),t_2)$$

"`is a proper rational parametrization of $\mathcal{V}$`".

Example 2.4 Let $\mathcal{V}$ be the surface over $\mathbb{C}$ implicitly defined by the polynomial

$F(x_1,x_2,x_3)=4-25x_1+87x_2+12x_3+173x_3x_2+12x_3^2-75x_1x_2-56x_1x_3+78x_3^2x_2-36x_1x_3^2+16x_3^2x_2^2+12x_3^3x_2+60x_3x_2^2-10x_3^3x_1+25x_1^2+55x_1^2x_3+25x_1^2x_3^2+9x_2^2-154x_3x_2x_1-44x_3^2x_2x_1+5x_3^3+x_3^4\in\mathbb{R}[x_1,x_2,x_3]$.

We apply Algorithm 1 and in Step 1, we get that the polynomials

$$F(x_1,x_2,0)=4-25x_1+87x_2-75x_1x_2+25x_1^2+9x_2^2,$$

$$\overline{F}(0,x_1,x_2,1)=16x_2^2+12x_2-10x_1+25x_1^2-44x_1x_2+1,$$

define two rational plane curves $\mathcal{C}_1$ and $\mathcal{C}_2$. Thus, in Step 2, we compute

$$\mathcal{P}_1(t_1)=(p_1(t_1),p_2(t_1))=\left(\frac{-450-75t_1+t_1^2}{225-75t_1+t_1^2},\frac{-2t_1^2-450-75t_1}{675-225t_1+3t_1^2}\right)\in\mathbb{C}(t_1)^2,$$

$$\mathcal{P}_2(t_1)=(q_1(t_1),q_2(t_1))=\left(\frac{656-220t_1+5t_1^2}{8400-924t_1+21t_1^2},\frac{-t_1^2-40t_1-400}{8400-924t_1+21t_1^2}\right)\in\mathbb{C}(t_1)^2$$

proper rational parametrizations of the curves $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively.

Now, we compute the polynomial $g(x_1, x_2, t_2) = \text{numer}(F(p_1(x_1)+t_2q_1(x_2), p_2(x_1)+t_2q_2(x_2), t_2))$, and we get that the factor

$$h(x_1, x_2) = -120 - 36x_1 + 15x_2 + x_1x_2$$

of $g(x_1, x_2, t_2)$, defines implicitly a rational plane curve $\mathcal{D}$. In Step 4, we compute a proper rational parametrization of $\mathcal{D}$, and we get

$$R(t_1) := (r_1(t_1), r_2(t_1)) = (t_1, 12(10 + 3t_1)/(15 + t_1)) \in (\mathbb{C}(t_1) \setminus \mathbb{C})^2.$$

Finally, we return the proper rational parametrization of $\mathcal{V}$,

$$\mathcal{P}(\bar{t}\,) = (p_1(r_1(t_1)) + t_2q_1(r_2(t_1)), p_2(r_1(t_1)) + t_2q_2(r_2(t_1)), t_2) =$$

$$\left(\frac{225t_2 + 105t_2t_1 + t_1^2t_2 + 1350 + 225t_1 - 3t_1^2}{-3(225 - 75t_1 + t_1^2)}, \frac{(2t_1 + 15)(2t_2t_1 + t_1 + 15t_2 + 30)}{-3(225 - 75t_1 + t_1^2)}, t_2\right).$$

2.2 Parametrically Ruled Surfaces

In the following, we consider a surface $\mathcal{V}$ defined by a rational parametrization (not necessarily proper) over $\mathbb{C}$,

$$\mathcal{M}(\bar{t}\,) = (m_1(\bar{t}\,), m_2(\bar{t}\,), m_3(\bar{t}\,)) \in \mathbb{C}(\bar{t}\,)^3.$$

We will check whether there exists a proper parametrization of the form obtained in Theorem 2.1,

$$\mathcal{P}(\bar{t}\,) = (p_1(r_1(t_1)) + t_2q_1(r_2(t_1)), p_2(r_1(t_1)) + t_2q_2(r_2(t_1)), t_2) \in \mathbb{C}(\bar{t}\,)^3,$$

and $(U, V) \in (\mathbb{C}(\bar{t}\,) \setminus \mathbb{C})^2$ such that $\mathcal{P}(U, V) = \mathcal{M}$.

To start with the problem, we first assume that $\mathcal{V}$ is not a plane. Note that this assumption is not a loss of generality, because one can easily deduce whether a parametrically given surface is a plane.

Now, we deal with the cylinder case. In order to analyze whether $\mathcal{V}$ is a cylinder over any of the coordinate planes of $\mathbb{C}^3$, we apply the following result presented in [12].

Theorem 2.5 *Let $H_i(\bar{t}\,, \bar{s}\,) = \text{numer}(m_i(\bar{t}\,) - m_i(\bar{s}\,))$, where $\bar{s} = (s_1, s_2)$ are new variables, and $i \in \{1, 2, 3\}$. Then, $\mathcal{V}$ is a cylinder over the x_ix_j–plane if and only if $\gcd(H_i, H_j) \neq 1$.*

Remark 2.6 If $\mathcal{V}$ is a cylinder over the x_1x_2-plane and $F(x_1, x_2) \in \mathbb{C}[x_1, x_2]$ is the implicit equation defining $\mathcal{V}$, we consider $a \in \mathbb{K}$ such that $(m_1, m_2)(a, t_2) \notin \mathbb{C}^2$, and we get that, up to multiplication by non-zero constants,

$$F(x_1, x_2)^r = \text{Res}_{t_2}(G_1(a, t_2, x_2), G_2(a, t_2, x_2)),$$

where $r \in \mathbb{N}$, and $G_i(\bar{t}\,, x_i) = \text{numer}(m_i(\bar{t}\,) - x_i)$, $i = 1, 2$ (see Theorem 8 in [12]). Then, one computes a proper parametrization $(p_1(t_1), p_2(t_1)) \in \mathbb{C}(t_1)^2$ of the plane curve defined by the equation $F(x_1, x_2) = 0$ (see e.g. Chapter 4 in [20]), and we get that $\mathcal{P}(\bar{t}\,) = (p_1(t_1), p_2(t_1), t_2)$ is a proper parametrization of $\mathcal{V}$. One reasons similarly if $\mathcal{V}$ is a cylinder over a different plane.

Once the plane case and the cylinder case are analyzed, we assume that $\mathcal{V}$ is neither a cylinder nor a plane. As we stated above, we are interested in applying Theorem 2.1. For this purpose, first we need to compute a proper rational parametrization of $\mathcal{C}_i$ for $i = 1, 2$ (see statement 1 of Theorem 2.1).

Since we do not have the implicit equation defining the surface $\mathcal{V}$, we have to compute the polynomials $f_i(x_1, x_2)$ defining implicitly the plane curves $\mathcal{C}_i$, for $i \in \{1, 2\}$, using the input parametrization $\mathcal{M}$. For this purpose, we use Theorem 10 in [12], and the fact that if $\bar{t}_0 \in \mathbb{C}^2$ is such that $m_1(\bar{t}_0) - x_1^0 = m_2(\bar{t}_0) - x_2^0 = m_3(\bar{t}_0) = 0$, then $(x_1^0, x_2^0) \in \mathcal{C}_1$. Similarly, if $\bar{t}_0 \in \mathbb{C}^2$ is such that $m_1(\bar{t}_0)/m_3(\bar{t}_0) - x_1^0 = m_2(\bar{t}_0)/m_3(\bar{t}_0) - x_2^0 = 1/m_3(\bar{t}_0) = 0$, then $(x_1^0, x_2^0) \in \mathcal{C}_2$ (observe that, in this case, we are using the dehomogenization of $\mathcal{M}$ w.r.t. the 3-component).

In order to apply Theorem 10 in [12], we need to assume that none of the projective curves defined by each numerator and denominator of m_i, $i = 1, 2, 3$ passes through the points at infinity $(0 : 0 : 1)$ and $(0 : 1 : 0)$, where the homogeneous variables are (t_0, t_1, t_2). Note that this requirement can always be achieved by applying a linear change of variables to $\mathcal{M}$. This assumption implies that each numerator and denominator of m_i has positive degree w.r.t. t_i, and then its leading coefficient w.r.t. t_i does not depend on $t_j, i \neq j$, $i, j \in 1, 2$. Thus, for $k = 1, 2, 3$, and $i \neq j$, $i, j \in 1, 2$, $\deg_{t_i}(G_k(\bar{t}, x_k)) > 0$, and the leading coefficient of $G_k(\bar{t}, x_k)$ w.r.t. t_i does not depend on t_j, where $G_k(\bar{t}, x_i) = \mathrm{numer}(m_k(\bar{t}) - x_k)$.

Finally, since $\mathcal{V}$ is neither a cylinder nor a plane, we may assume that for every $i, j \in \{1, 2, 3\}$, with $i < j$, the gradients $\{\nabla m_i(\bar{t}), \nabla m_j(\bar{t})\}$ are linearly independent.

Under these conditions, Theorem 2.7 shows how to compute the polynomials $F(x_1, x_2, 0)$ and $\overline{F}(0, x_1, x_2, 1)$. The theorem is obtained from Lemmas 12, 13, 14, 15, and Theorem 10 in [12].

Theorem 2.7 *It holds that*

$$F(x_1, x_2, 0)^r = \mathrm{pp}_{x_1}(\mathrm{Content}_{\{Z,W\}}(\mathrm{Res}_{t_2}(T(t_2, x_2),\ K(t_2, Z, W, x_1, x_2)))) \in \mathbb{C}[x_1, x_2],$$

where $r \in \mathbb{N}$, *and*

1. $K(t_2, Z, W, x_1, x_2) = \mathrm{Res}_{t_1}(S(t_1, x_2),\ G_{Z,W}(\bar{t}, Z, W, x_1, x_2))$,
2. $G_{Z,W}(\bar{t}, Z, W, x_1, x_2) = G_1(\bar{t}, x_k) + ZG_3(\bar{t}, 0) + WG_2(\bar{t}, x_1)$,
3. $S(t_1, x_2) = \mathrm{pp}_{x_2}(\mathrm{Res}_{t_2}(G_3(\bar{t}, 0), G_2(\bar{t}, x_2)))$,
4. $T(t_2, x_2) = \mathrm{pp}_{x_2}(\mathrm{Res}_{t_1}(G_3(\bar{t}, 0), G_2(\bar{t}, x_2)))$.

Remark 2.8 1. In order to compute $\overline{F}(0, x_1, x_2, 1)$, we reason similarly as in Theorem 2.7 using the polynomials

$$G_1(\bar{t}, x_1) = \mathrm{numer}(m_1/m_3 - x_1), \quad G_2(\bar{t}, x_2) = \mathrm{numer}(m_2/m_3 - x_2),$$

$$G_3(\bar{t}, x_3) = \mathrm{numer}(1/m_3 - x_3).$$

2. For almost all values of $(Z, W) = (Z_i, W_i) \in \mathbb{C}^2$, $i = 1, 2$, it holds that

$$\text{Content}_{\{Z,W\}}(R(Z, W, x_1, x_2)) = \gcd(R(Z_0, W_0, x_1, x_2), R(Z_1, W_1, x_1, x_2)),$$

where $R(Z, W, x_1, x_2) = \text{Res}_{t_2}(T(t_2, x_2),\ K(t_2, Z, W, x_1, x_2))$.

Under these conditions, we apply Theorem 2.1 to compute a proper reparametrization of a given parametrization $\mathcal{M}$, if $\mathcal{V}$ is a ruled surface. More precisely, if $\mathcal{V}$ is a rational ruled surface, there exists a proper parametrization given in standard reduced form

$$\mathcal{P}(\bar{t}) = (p_1(r_1(t_1)) + t_2 q_1(r_2(t_1)), p_2(r_1(t_1)) + t_2 q_2(r_2(t_1)), t_2) \in \mathbb{C}(\bar{t})^3$$

where p_j, q_j, r_j are given in Theorem 2.1 (note that, in this case, r_1, r_2 can not be computed since $F(\overline{x})$ is unknown). Thus, we only have to check whether there exists $(U, V) \in (\mathbb{C}(\bar{t}) \setminus \mathbb{C})^2$ such that $\mathcal{P}(U, V) = \mathcal{M}$. Observe that from this equality, we get that $V = m_3$, and thus, we have to decide whether there exists $U \in (\mathbb{C}(\bar{t}) \setminus \mathbb{C})^2$ such that

$$p_i(r_1(U(\bar{t}))) + m_3 q_i(r_2(U(\bar{t}))) = m_i(\bar{t}),\ i = 1, 2.$$

Note that this equality is equivalent to check whether there exists $(L_1, L_2) \in (\mathbb{C}(\bar{t}) \setminus \mathbb{C})^2$ such that

$$p_i(L_1(\bar{t})) + m_3 q_i(L_2(\bar{t})) = m_i(\bar{t}),\ i = 1, 2.$$

Note that $L_i(\bar{t}) = r_i(U(\bar{t}))$, $i = 1, 2$, and thus, $L = (L_1, L_2)$ parametrizes the curve defined by $R = (r_1, r_2)$.

Taking into account the above reasoning, we prove the following theorem that is equivalent to Theorem 2.1 but for the parametric case. Similarly as in Theorem 2.1, Theorem 2.9 involves the computation of two planar parametrizations (see statement 1) that will be used to determine a rational planar base curve of the ruled surface $\mathcal{V}$, and to compute the ruling direction of $\mathcal{V}$ (see statement 2).

Theorem 2.9 *A surface $\mathcal{V}$ defined by the parametrization*

$$\mathcal{M}(\bar{t}) = (m_1(\bar{t}), m_2(\bar{t}), m_3(\bar{t})) \in \mathbb{C}(\bar{t})^3$$

is a rational ruled surface if and only if the following statements hold:

1. *The plane curves $\mathcal{C}_i$, $i \in \{1, 2\}$, are rational. Let $\mathcal{P}_1 = (p_1, p_2) \in \mathbb{C}(t_1)^2$ and $\mathcal{P}_2 = (q_1, q_2) \in \mathbb{C}(t_1)^2$ be proper rational parametrizations of $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively.*

2. *There exists $L = (L_1, L_2) \in (\mathbb{C}(\bar{t}) \setminus \mathbb{C})^2$ such that*

$$p_i(L_1(\bar{t})) + m_3 q_i(L_2(\bar{t})) = m_i(\bar{t}),\ i = 1, 2,$$

and L parametrizes the rational plane curve $\mathcal{D}$ (see Remark 2.2). In this case,

$$\mathcal{P}(\bar{t}) = (p_1(r_1(t_1)) + t_2 q_1(r_2(t_1)), p_2(r_1(t_1)) + t_2 q_2(r_2(t_1)), t_2) \in \mathbb{C}(\bar{t})^3,$$

where $R(t_1) := (r_1(t_1), r_2(t_1)) \in (\mathbb{C}(t_1) \setminus \mathbb{C})^2$ is a proper rational parametrization of $\mathcal{D}$, is a rational proper reparametrization of $\mathcal{M}$.

Proof It is clear that if statements 1 and 2 hold, then $\mathcal{V}$ is a rational ruled surface. Reciprocally, let $\mathcal{V}$ be a rational ruled surface. Then, statement 1 holds (see statement 1 in Theorem 2.1), and statement 2 of Theorem 2.1 holds. That is,

$$\mathcal{P}^*(\bar{t}) = (p_1(r_1^*(t_1)) + t_2q_1(r_2^*(t_1)), p_2(r_1^*(t_1)) + t_2q_2(r_2^*(t_1)), t_2) \in \mathbb{C}(\bar{t})^3$$

is a proper parametrization of $\mathcal{V}$, where $(r_1^*, r_2^*) \in (\mathbb{C}(t_1) \setminus \mathbb{C})^2$ is a proper rational parametrization of a curve $\mathcal{D}$. Since $\mathcal{M}$ is also a parametrization of $\mathcal{V}$, there exists $(U, V) \in (\mathbb{C}(\bar{t}) \setminus \mathbb{C})^2$ such that $\mathcal{P}^*(U, V) = \mathcal{M}$. From this equality, we get that $V = m_3$, and

$$p_i(r_1^*(U(\bar{t}))) + m_3q_i(r_2^*(U(\bar{t}))) = m_i(\bar{t}),\ i = 1, 2.$$

That is,

$$p_i(L_1(\bar{t})) + m_3q_i(L_2(\bar{t})) = m_i(\bar{t}),\ i = 1, 2,$$

where $L_i(\bar{t}) = r_i^*(U(\bar{t}))$, $i = 1, 2$. Observe that since (r_1^*, r_2^*) is a rational parametrization of $\mathcal{D}$, then (L_1, L_2) also parametrizes $\mathcal{D}$.
Now, we consider $(r_1(t_1), r_2(t_1)) \in (\mathbb{C}(t_1) \setminus \mathbb{C})^2$ a new proper rational parametrization of $\mathcal{D}$, and

$$\mathcal{P}(\bar{t}) = (p_1(r_1(t_1)) + t_2q_1(r_2(t_1)), p_2(r_1(t_1)) + t_2q_2(r_2(t_1)), t_2) \in \mathbb{C}(\bar{t})^3.$$

Since (r_1^*, r_2^*) and (r_1, r_2) are both proper rational parametrizations of $\mathcal{D}$, there exists $r \in \mathbb{C}(t_1) \setminus \mathbb{C}$, $\deg(r) = 1$, such that $(r_1^*, r_2^*) = (r_1(r), r_2(r))$. Hence, $\mathcal{P}(r(t_1), t_2) = \mathcal{P}^*(t_1, t_2)$ which implies that $\mathcal{P}$ is a proper rational reparametrization of $\mathcal{M}$ (note that $(r(t_1), t_2)$ and $\mathcal{P}^*(\bar{t})$ are both proper, and thus $\mathcal{P}(\bar{t})$ is also proper). ▮

The rational parametrization $R(t_1) := (r_1(t_1), r_2(t_1)) \in (\mathbb{C}(t_1) \setminus \mathbb{C})^2$ of statement 2 of Theorem 2.9 (see also Remark 2.2) can be computed from the implicit equation of the plane curve $\mathcal{D}$. This implicit equation can be determined using the following corollary.

Corollary 2.10 *Let $h(x_1, x_2)$ be the irreducible polynomial defining the curve $\mathcal{D}$ of statement 2 of Theorem 2.9. Let $e_i(x_1, x_2, t_1, t_2) = \mathrm{numer}(p_i(x_1) + m_3q_i(x_2) - m_i(\bar{t}))$, $i = 1, 2$. It holds that $h(x_1, x_2)$ divides $R(x_1, x_2, t_k) = \mathrm{Res}_{t_j}(e_1, e_2)$ for $j, k \in \{1, 2\}$ and $j \neq k$.*

Proof From $e_i(L_1, L_2, \bar{t}) = 0$, and using the properties of the resultants (see e.g., [20]), we get that $R(L_1, L_2, t_k) = 0$. In addition, since $L = (L_1, L_2)$ parametrizes the rational curve $\mathcal{D}$ (see Theorem 2.9), and $R(L_1(t_j, a_k), L_2(t_j, a_k), a_k) = 0$, we get that for almost all values of $t_k = a_k \in \mathbb{C}$, the polynomial $h(x_1, x_2)$ divides $R(x_1, x_2, t_k)$. ▮

Remark 2.11 Theorem 2.9 and Corollary 2.10 are new but they are based on the ideas developed in [18]. In particular, Theorem 2.9 improves Theorem 5 in [18] in the following sense: in statement 1, only two rational curves are considered (in Theorem 5 in [18], one has to consider three plane curves). In statement 2 (see also Corollary 2.10), a rational plane curve has to be parametrized (in Theorem 5 in [18], several cases have to be analyzed and finally, one has to solve an algebraic system).

We will see that Theorem 2.9 can be easily be applied to objects that are given approximate (see Subsection 2.2).

In the following, we present the algorithm obtained from Theorem 2.9, and we illustrate it with an example.

Algorithm 2: Symbolic computation of a reparametrization of a parametric (ruled) surface.
[Step 1] Check whether $\mathcal{V}$ defines a plane. In the affirmative case, compute a proper parametrization of $\mathcal{V}$. Otherwise, go to Step 2.
[Step 2] Check whether $\mathcal{V}$ defines a cylinder (apply Theorem 2.5). In the affirmative case, compute a proper parametrization of $\mathcal{V}$ (apply Remark 2.6). Otherwise, go to Step 3.
[Step 3] Compute the polynomials $F(x_1, x_2, 0)$ and $\overline{F}(0, x_1, x_2, 1)$ by applying Theorem 2.7 and Remark 2.8. Check whether there exist two rational plane curves $\mathcal{C}_1$ and $\mathcal{C}_2$ defined by a factor of the above polynomials, respectively. In the affirmative case, let $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$ be these polynomials and go to Step 4. Otherwise, RETURN "`$\mathcal{V}$ is not a rational ruled surface`".
[Step 4] Compute $\mathcal{P}_1 = (p_1, p_2) \in \mathbb{C}(t_1)^2$ and $\mathcal{P}_2 = (q_1, q_2) \in \mathbb{C}(t_1)^2$ proper rational parametrizations of the curves $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively.
[Step 5] Check whether there exists a rational plane curve $\mathcal{D}$ defined by a factor of the polynomial $R(x_1, x_2, t_1) = \mathrm{Res}_{t_2}(e_1, e_2)$, where $e_i(x_1, x_2, t_1, t_2) = \mathrm{numer}(p_i(x_1) + m_3 q_i(x_2) - m_i(\bar{t}\,))$, $i = 1, 2$. In the affirmative case, compute $R(t_1) := (r_1(t_1), r_2(t_1)) \in (\mathbb{C}(t_1) \setminus \mathbb{C})^2$ a proper rational parametrization of the curve $\mathcal{D}$ and RETURN

$$\mathcal{P}(\bar{t}\,) = (p_1(r_1(t_1)) + t_2 q_1(r_2(t_1)), p_2(r_1(t_1)) + t_2 q_2(r_2(t_1)), t_2) \in \mathbb{C}(\bar{t}\,)^3$$

"`is a proper rational parametrization of` $\mathcal{V}$". Otherwise, RETURN "`$\mathcal{V}$ is not a rational ruled surface`".

Example 2.12 Let $\mathcal{V}$ be the surface defined by the parametrization $\mathcal{M}(\bar{t}\,) = (m_1(\bar{t}\,), m_2(\bar{t}\,), m_3(\bar{t}\,)) \in \mathbb{C}(\bar{t}\,)^3$, where

$$m_1(\bar{t}\,) = \frac{-2 - 9t_1 + 9t_2 - 2t_1^2 t_2 + 4t_2 t_1 + t_1^2 + 2t_1^3 - 4t_2^2 + t_1^3 t_2 - 2t_2^2 t_1^2 + t_2^3 t_1}{(t_1 - 1)(t_1^2 - 2t_2 t_1 + 2t_1 + t_2^2 - 2t_2 + 2)},$$

$$m_2(\bar{t}\,) = \frac{t_1^3 - t_1^2 - 7t_1^2 t_2 + 9t_1 + 6t_2^2 t_1 - 3t_2^2 - 8t_2 + 21 + t_1^3 t_2 - 2t_2^2 t_1^2 - 7t_2 t_1 + t_2^3 t_1}{(t_1 - 1)(t_1^2 - 2t_2 t_1 + 2t_1 + t_2^2 - 2t_2 + 2)},$$

$$m_3(\bar{t}\,) = \frac{t_2 t_1 - 2}{t_1 - 1}.$$

We apply Algorithm 2 and from Steps 1 and 2, we get $\mathcal{V}$ is not a plane neither a cylinder. In Step 3, we compute the polynomials $F(x_1, x_2, 0)$ and $\overline{F}(0, x_1, x_2, 1)$ by applying Theorem 2.7 and Remark 2.8. We get that

$$F(x_1, x_2, 0) = 9 + 6x_1 - 31x_2 - 10x_1 x_2 + x_1^2 + 26x_2^2,$$

$$\overline{F}(0, x_1, x_2, 1) = 85x_1^2 - 109x_1 + 10x_2 + x_2^2 - 12x_1x_2 + 25,$$

define two rational plane curves $\mathcal{C}_1$ and $\mathcal{C}_2$. Thus, in Step 4, we compute

$$\mathcal{P}_1(t_1) = (p_1(t_1), p_2(t_1)) = \left(\frac{-78 + 31t_1 - 3t_1^2}{26 - 10t_1 + t_1^2}, \frac{t_1^2}{676 - 260t_1 + 26t_1^2}\right) \in \mathbb{C}(t_1)^2,$$

$$\mathcal{P}_2(t_1) = (q_1(t_1), q_2(t_1)) = \left(\frac{49}{85 - 12t_1 + t_1^2}, \frac{-425 + 109t_1 - 5t_1^2}{85 - 12t_1 + t_1^2}\right) \in \mathbb{C}(t_1)^2$$

proper rational parametrizations of $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively.

Finally, in Step 5, we check whether there exist a rational plane curve $\mathcal{D}$ defined by a factor of the polynomial

$$R(x_1, x_2, t_1) = \mathrm{Res}_{t_2}(e_1, e_2) = (-6 + x_2)(85 - 12x_2 + x_2^2)^3(26 - 10x_1 + x_1^2)^3(x_1x_2 - 41x_1 + 182),$$

where $e_i(x_1, x_2, t_1, t_2) = \mathrm{numer}(p_i(x_1) + m_3q_i(x_2) - m_i(\bar{t}\,))$, $i = 1, 2$. We consider the curve $\mathcal{D}$ defined by the polynomial $h(x_1, x_2) = x_1x_2 - 41x_1 + 182$, and we compute a proper rational parametrization of $\mathcal{D}$. We get

$$R(t_1) := (r_1(t_1), r_2(t_1)) = (t_1, (41t_1 - 182)/t_1) \in (\mathbb{C}(t_1) \setminus \mathbb{C})^2.$$

Finally, we return the proper rational parametrization of $\mathcal{V}$,

$$\mathcal{P}(\bar{t}\,) = (p_1(r_1(t_1)) + t_2q_1(r_2(t_1)), p_2(r_1(t_1)) + t_2q_2(r_2(t_1)), t_2) =$$

$$\left(\frac{t_1^2t_2 - 2028 + 806t_1 - 78t_1^2}{26(26 - 10t_1 + t_1^2)}, \frac{89t_1^2t_2 - 1118t_2t_1 + 3380t_2 - t_1^2}{-26(26 - 10t_1 + t_1^2)}, t_2\right).$$

3 A Numeric Approach for Parametrizing Ruled Surfaces

The problem of numerical reparametrization for (ruled) surfaces can be stated from two different points of view, the implicit and the parametric one. More precisely,

[Numerical Implicit Ruled Problem]: Given a polynomial $F(\overline{x}\,) \in \mathbb{C}[\overline{x}\,]$ (with perturbed float coefficients) defining an algebraic surface $\mathcal{V}$, find a rational parametrization $\mathcal{Q}(\bar{t}\,) \in \mathbb{C}(\bar{t}\,)^3$ of an algebraic *ruled* surface $\mathcal{W}$ such that $\mathcal{V}$ and $\mathcal{W}$ are *close enough*.

[Numerical Parametric Ruled Problem]: Given a rational parametrization $\mathcal{M}(\bar{t}\,) \in \mathbb{C}(\bar{t}\,)^3$ (with perturbed float coefficients) of an algebraic surface $\mathcal{V}$, find a rational parametrization $\mathcal{P}(\bar{t}\,) \in \mathbb{C}(\bar{t}\,)^3$ of an algebraic *ruled* surface $\mathcal{W}$ such that $\mathcal{V}$ and $\mathcal{W}$ are *close enough*.

In this section, we deal with mathematical objects that are given approximately. Our idea is to adapt the algorithms obtained in Section 2. For this purpose, we observe that, in these methods, one of the key steps is the computation of proper rational parametrizations of several plane curves appearing in the algorithms. Fortunately, this problem is partially solved for instance in [19]. Here, given an algebraic plane curve $\mathcal{C}$ (we refer to $\mathcal{C}$ as the *aproximate rational plane curve*), authors develop an algorithm that computes a proper parametrization of a new curve, $\mathcal{D}$, that is rational. We refer to this parametrization as the *aproximate rational* *parametrization* of $\mathcal{C}$, and it parametrizes (exactly) the curve $\mathcal{D}$. Furthermore, a bound on the distance between the input and the output curve is provided.

We here need to recall the notion of *ε–irreducible polynomial*. More precisely, a polynomial $f(x_1, x_2) \in \mathbb{C}[x_1, x_2]$ is ε–irreducible (over $\mathbb{C}$) if it can not be expressed as $f(x_1, x_2) = g(x_1, x_2)h(x_1, x_2) + \mathcal{E}(x_1, x_2)$ where $g, h, \mathcal{E} \in \mathbb{C}[x_1, x_2]$ and $\|\mathcal{E}(x_1, x_2)\| < \varepsilon\|f(x_1, x_2)\|$. We refer to g and h as *approximate factors* of the polynomial $f(x_1, x_2)$. In the following, we will need to decide whether a polynomial is not ε–irreducible and in the affirmative case, compute the factors g, h. For this purpose, one may apply, for instance, the results in [21], [22], [23], [24] or [25].

3.1 Numerical Implicit Ruled Problem

In the following, we deal with the Numerical Implicit Ruled Problem. More precisely, given a surface $\mathcal{V}$ defined implicitly by a polynomial $F(\overline{x}) \in \mathbb{C}[\overline{x}]$ with perturbed float coefficients, we present an algorithm that returns a rational parametrization $\mathcal{P}(\bar{t}) = (p_1(r_1(t_1)) + t_2q_1(r_2(t_1)), p_2(r_1(t_1)) + t_2q_2(r_2(t_1)), t_2)$ that defines a ruled surface $\mathcal{W}$. In this case, we say that $\mathcal{V}$ is an approximate rational ruled surface. In Theorem 3.3 and Corollary 3.4, we show how to compute the distance between the input surface $\mathcal{V}$ and the output surface $\mathcal{W}$.

The algorithm presented is obtained from Algorithm 1. We illustrate this algorithm with an example.

<u>Algorithm 3:</u> Computation of a rational ruled surface from an approximate implicit surface

[Step 1] Compute the polynomials $F(x_1, x_2, 0)$ and $\overline{F}(0, x_1, x_2, 1)$, and check whether there exist two approximate rational plane curves $\mathcal{C}_1$ and $\mathcal{C}_2$ defined by an approximate factor of the above polynomials, respectively (see [19]). In the affirmative case, go to Step 2. Otherwise, RETURN "`$\mathcal{V}$ is not an approximate rational ruled surface`".

[Step 2] Compute $\mathcal{P}_1 = (p_1, p_2) \in \mathbb{C}(t_1)^2$ and $\mathcal{P}_2 = (q_1, q_2) \in \mathbb{C}(t_1)^2$ approximate proper rational parametrizations of $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively (see [19]).

[Step 3] Let $g(x_1, x_2, t_2) = \mathrm{numer}(F(p_1(x_1) + t_2q_1(x_2), p_2(x_1) + t_2q_2(x_2), t_2))$. Check whether there exists an approximate rational plane curve $\mathcal{D}$ defined by an approximate factor $h(x_1, x_2)$ of the above polynomial. In the affirmative case, go to Step 4. Otherwise, RETURN "`$\mathcal{V}$ is not an approximate rational ruled surface`".

[Step 4] Compute an approximate proper rational parametrization $R(t_1) := (r_1(t_1), r_2(t_1)) \in (\mathbb{C}(t_1) \setminus \mathbb{C})^2$ of $\mathcal{D}$ (see [19]).

[Step 5] RETURN "`$\mathcal{W}$ is a ruled surface parametrized by`

$$\mathcal{P}(\bar{t}) = (p_1(r_1(t_1)) + t_2q_1(r_2(t_1)), p_2(r_1(t_1)) + t_2q_2(r_2(t_1)), t_2).\text{"}$$

Remark 3.1 We observe that, in Step 3, in order to compute the approximate factor $h(x_1, x_2)$, one may compute the ε–gcd of the polynomials $g(x_1, x_2, a_i)$, $i = 1, \ldots$, for random values of the parameter $t_2 = a_i \in \mathbb{C}$. For this purpose, one may apply for instance, the results in [24], [26], [27] or [28].

Example 3.2 Let $\mathcal{V}$ be the surface over $\mathbb{C}$ implicitly defined by the polynomial

$F(x_1,x_2,x_3) = -x_2 - 5x_3 + 8x_3x_2 - 6x_3^2 - 2.x_1x_2 - 5.9999x_1x_3 + 31x_3^2x_2 - 42.x_1x_3^2 + 10x_3^2x_2^2 + 22x_3^3x_2 + 8x_3x_2^2 - 36x_3^3x_1 + 1.001x_1^2 + 12x_1^2x_3 + 36.0001x_1^2x_3^2 + 2x_2^2 - 18x_3x_2x_1 - 36x_3^2x_2x_1 + 3x_3^3 + 4x_3^4 - 0.001x_2^3 + 0.001.$

By applying Algorithm 1, one gets that $\mathcal{V}$ is not a ruled surface. Let us apply Algorithm 3 to check whether $\mathcal{V}$ is an approximate rational ruled surface and, in the affirmative case, we compute a parametrization of a ruled surface $\mathcal{W}$. Afterwards, we measure the distance between $\mathcal{V}$ and $\mathcal{W}$ (see Theorem 3.3 and Corollary 3.4).

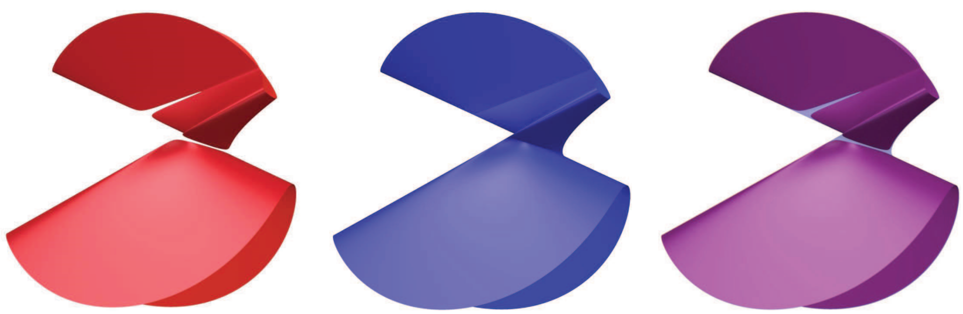

Figure 1: Input surface $\mathcal{V}$ (left), Output surface $\mathcal{W}$ (center), and both surfaces (right)

In Step 1 of Algorithm 3, we get that the polynomials

$F(x_1,x_2,0) = -0.5x_2 - x_1x_2 + 0.5005x_1^2 + x_2^2 - 0.0005x_2^3 + 0.0005,$

$\overline{F}(0,x_1,x_2,1) = 0.2777770062x_2^2 + 0.6111094136x_2 - 0.9999972222x_1 + x_1^2 - 0.9999972222x_1x_2 + 0.1111108025,$

define two approximate rational plane curves $\mathcal{C}_1$ and $\mathcal{C}_2$. Thus, in Step 2, we compute $\mathcal{P}_1(t_1) = (p_1(t_1), p_2(t_1)) =$

$$\left(\frac{-5.0012\cdot 10^5 - 1.52\cdot 10^2 t_1 + 1.06t_1^2}{2.94294\cdot 10^6 - 2.94\cdot 10^3 t_1 + 1.47t_1^2}, \frac{3.97t_1^2 + 2.31946\cdot 10^6 - 6.0632\cdot 10^3 t_1}{8.4084\cdot 10^6 - 8.4\cdot 10^3 t_1 + 4.2t_1^2}\right),$$

$\mathcal{P}_2(t_1) = (q_1(t_1), q_2(t_1)) =$

$$\left(\frac{-2.449581840\cdot 10^{11} + 1.696014\cdot 10^6 t_1 - 2.1600t_1^2}{1.743592843\cdot 10^{11} - 1.743588\cdot 10^6 t_1 + 4.8433t_1^2}, \frac{-2.31178t_1^2 + 1.64939616\cdot 10^6 t_1 - 2.485392904\cdot 10^{11}}{8.717964216\cdot 10^{10} - 8.717940000\cdot 10^5 t_1 + 2.42165t_1^2}\right)$$

approximate proper parametrizations of the curves $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. For this purpose, we apply the algorithm presented in [19].

Now, we compute the polynomial $g(x_1,x_2,t_2) = \mathrm{numer}(F(p_1(x_1)+t_2q_1(x_2), p_2(x_1)+t_2q_2(x_2), t_2))$, and we get that there exists an approximate factor $h(x_1,x_2) = x_1x_2 + x_2 - 15.0001x_1$ defining an approximate rational plane curve $\mathcal{D}$. In Step 4, we compute an approximate proper parametrization of $\mathcal{D}$. We get

$$R(t_1) := (r_1(t_1), r_2(t_1)) = (t_1, 15.0001t_1/(t_1+1)) \in (\mathbb{C}(t_1)\setminus\mathbb{C})^2.$$

Finally, in Step 5, we return the ruled surface $\mathcal{W}$ defined by the proper rational parametrization

$$\mathcal{P}(\bar{t}\,) = (p_1(r_1(t_1)) + t_2q_1(r_2(t_1)), p_2(r_1(t_1)) + t_2q_2(r_2(t_1)), t_2) =$$

$p_1(r_1(t_1)) + t_2q_1(r_2(t_1)) = -0.1966388521(2.055378432\cdot 10^7t_2t_1^2 - 2.055377510\cdot 10^4t_2t_1^3 + 1.028717526\cdot 10t_2t_1^4 + 4.117141228\cdot 10^7t_2t_1 + 2.059706394\cdot 10^7t_2 + 2.492580063\cdot 10^6t_1^2 + 4.983272946\cdot 10^6t_1 + 2.491444722\cdot 10^6 + 7.465434870\cdot 10^2t_1^3 - 5.279803414t_1^4)/((2.002\cdot 10^6 - 2\cdot 10^3t_1 + t_1^2)(1.439788009t_1^2 + 2.879792t_1 + 1.440004))$,

$p_2(r_1(t_1)) + t_2q_2(r_2(t_1)) = -3.441179912(2.383354895\cdot 10^6t_2t_1^2 - 2.383353821\cdot 10^3t_2t_1^3 + 1.19286984t_2t_1^4 + 4.774102602\cdot 10^6t_2t_1 + 2.388363165\cdot 10^6t_2 - 0.3954871614t_1^4 + 6.032184769\cdot 10^2t_1^3 - 2.298544147\cdot 10^5t_1^2 - 4.61554819\cdot 10^5t_1 - 2.310967918\cdot 10^5)/((2.002\cdot 10^6 - 2000t_1 + t_1^2)(1.439788009t_1^2 + 2.879792t_1 + 1.440004))$.

In Figure 1, we plot the input surface and the output surface. One may check that both surfaces are very "close".

Analysis of the Error

Let $\mathcal{V}$ and $\mathcal{W}$ be the input and output surfaces, respectively, of Algorithm 3. In addition, let $F(\bar{x}\,)$ and $G(\bar{x}\,)$ be the defining polynomials of $\mathcal{V}$ and $\mathcal{W}$, respectively, and let $\mathcal{P}(\bar{t}\,) \in \mathbb{C}(\bar{t}\,)^3$ be the parametrization of $\mathcal{W}$ outputs by the algorithm. In the following, we study the distance between both surfaces. For this purpose, we first consider the normal line to $\mathcal{W}$ at the generic point $\mathcal{P}(\bar{t}\,)$:

$$\mathcal{L}_1(\bar{t}\,, s) = \mathcal{P}(\bar{t}\,) + s\,\mathcal{T}(\bar{t}\,), \quad \text{where} \quad \mathcal{T}(\bar{t}\,) = \frac{\frac{\partial \mathcal{P}}{\partial t_1}\times\frac{\partial \mathcal{P}}{\partial t_2}}{\|\frac{\partial \mathcal{P}}{\partial t_1}\times\frac{\partial \mathcal{P}}{\partial t_2}\|_2}$$

as well as the normal line to $\mathcal{V}$ at the generic point $(a, b, c) \in \mathcal{V}$:

$$\mathcal{L}_2(a, b, c, s) = (a, b, c) + s\,\mathcal{N}(a, b, c), \quad \text{where} \quad \mathcal{N}(a, b, c) = \frac{\nabla F(a, b, c)}{\|\nabla F(a, b, c)\|_2}.$$

Moreover, we introduce the polynomials

$$\mathcal{D}_1(\bar{t}\,, s) = F(\mathcal{L}_1(\bar{t}\,, s)) \in \overline{\mathbb{R}(\bar{t}\,)}[s], \quad \mathcal{D}_2(a, b, c, s) = G(\mathcal{L}_2(a, b, c, s)) \in \mathbb{C}(\mathcal{V})[s],$$

where $\overline{\mathbb{R}(\bar{t}\,)}$ denotes the algebraic closure of $\mathbb{R}(\bar{t}\,)$ and $\mathbb{C}(\mathcal{V})$ the field of rational functions over $\mathcal{V}$. We may write $\mathcal{D}_1$ and $\mathcal{D}_2$ as

$$\mathcal{D}_1(\bar{t}\,, s) = A_n(\bar{t}\,)s^n + \cdots + A_0(\bar{t}\,), \quad \mathcal{D}_2(a, b, c, s) = B_n(a, b, c)s^n + \cdots + B_0(a, b, c)$$

(note that $\deg_s(\mathcal{D}_1) = \deg_s(\mathcal{D}_2) = \deg(\mathcal{V}) = \deg(\mathcal{W}) = n$). Reasoning similarly as in [19], we get the following results.

Theorem 3.3 *Let $\bar{t}_0 \in \mathbb{C}$, and $(a_0, b_0, c_0) \in \mathcal{V}$ be such that $\mathcal{D}_1(\bar{t}_0, s)$ and $\mathcal{D}_2(a_0, b_0, c_0, s)$ are well defined. Then,*

1. $d(\mathcal{P}(\bar{t}_0), \mathcal{V}) \leq \min\left\{ \binom{n}{i} \left| \frac{A_0(\bar{t}_0)}{A_i(\bar{t}_0)} \right|^{\frac{1}{i}} \text{ where } A_i(\bar{t}_0) \neq 0 \text{ and } 1 \leq i \leq n \right\}$.
2. $d((a_0, b_0, c_0), \mathcal{W}) \leq \min\left\{ \binom{n}{i} \left| \frac{B_0(a_0,b_0,c_0)}{B_i(a_0,b_0,c_0)} \right|^{\frac{1}{i}} \text{ where } B_i(a_0, b_0, c_0) \neq 0 \text{ and } 1 \leq i \leq n \right\}$.

In addition, using the expression of the coefficients given by the Taylor expansion, the next corollary also holds.

Corollary 3.4 *Let $\bar{t}_0 \in \mathbb{C}$, and $(a_0, b_0, c_0) \in \mathcal{V}$ such that $\mathcal{D}_1(\bar{t}_0, s)$ and $\mathcal{D}_2(a_0, b_0, c_0, s)$ are well defined. Then,*

1. *if $\nabla F(\mathcal{P}(\bar{t}_0))$ and $\mathcal{T}(\bar{t}_0)$ are not orthogonal, then*
$$d(\mathcal{P}(\bar{t}_0), \mathcal{V}) \leq n \left| \frac{A_0(\bar{t}_0)}{A_1(\bar{t}_0)} \right| = n \left| \frac{F(\mathcal{P}(t_0))}{\nabla F(\mathcal{P}(\bar{t}_0)) \cdot \mathcal{T}(\bar{t}_0)} \right|,$$
2. *if $\nabla G(a_0, b_0, c_0)$ and $\mathcal{N}(a_0, b_0, c_0)$ are not orthogonal, then*
$$d((a_0, b_0, c_0), \mathcal{W}) \leq n \left| \frac{B_0(a_0, b_0, c_0)}{B_1(a_0, b_0, c_0)} \right| = n \left| \frac{G(a_0, b_0, c_0)}{\nabla G(a_0, b_0, c_0) \cdot \mathcal{N}(a_0, b_0, c_0)} \right|.$$

In the example below we apply the above results, and we look for empirical evidences indicating that the input surface and the output surface are very "close".

Example 3.5 Let $\mathcal{V}$ be the surface considered in Example 3.2, and the output parametrization $\mathcal{P}(\bar{t})$ obtained from Algorithm 3. First, we compute

$$\mathcal{D}_1(\bar{t}, s) = F(\mathcal{L}_1(\bar{t}, s)) = A_4(\bar{t})s^4 + \cdots + A_0(\bar{t}),$$

and the functions

$$e_1(\bar{t}_0) := 4 \left| \frac{A_0(\bar{t}_0)}{A_1(\bar{t}_0)} \right|, \qquad e_2(\bar{t}_0) := 6 \left| \frac{A_0(\bar{t}_0)}{A_2(\bar{t}_0)} \right|^{\frac{1}{2}},$$
$$e_3(\bar{t}_0) := 4 \left| \frac{A_0(\bar{t}_0)}{A_3(\bar{t}_0)} \right|^{\frac{1}{3}}, \qquad e_4(\bar{t}_0) := \left| \frac{A_0(\bar{t}_0)}{A_4(\bar{t}_0)} \right|^{\frac{1}{4}}.$$

Using Theorem 3.3 (statement 1), we estimate $d(\mathcal{P}(\bar{t}_0), \mathcal{V})$. For this purpose, for instance, we consider $t_2 = 5$, and we compute $\min\{e_j(t_1, 5),\, j = 1, \ldots, 4\}$. Let us maximize the function $e_1(t_1, 5)$ (see statement 1 in Corollary 3.4). We observe that $e_1(t_1, 5)$ is continuous in $\mathbb{R} \setminus \{\alpha_1, \alpha_2\}$, where $\alpha_1 = -9992.124992$, $\alpha_2 = 1614.847856$ are real zeros of the denominator. If we are far away of these roots we have that

$$d(\mathcal{P}(\bar{t}_0), \mathcal{V}) \leq e_1(t_1, 5) \leq 0.042$$

(see Figure 2). For $t_1 \in \mathbb{C}$ in an interval close of these roots, one gets that

$$d(\mathcal{P}(\bar{t}_0), \mathcal{V}) \leq \min\{e_j(t_1, 5),\, j = 1, \ldots, 4\} \leq 3.42$$

(see Figure 3).

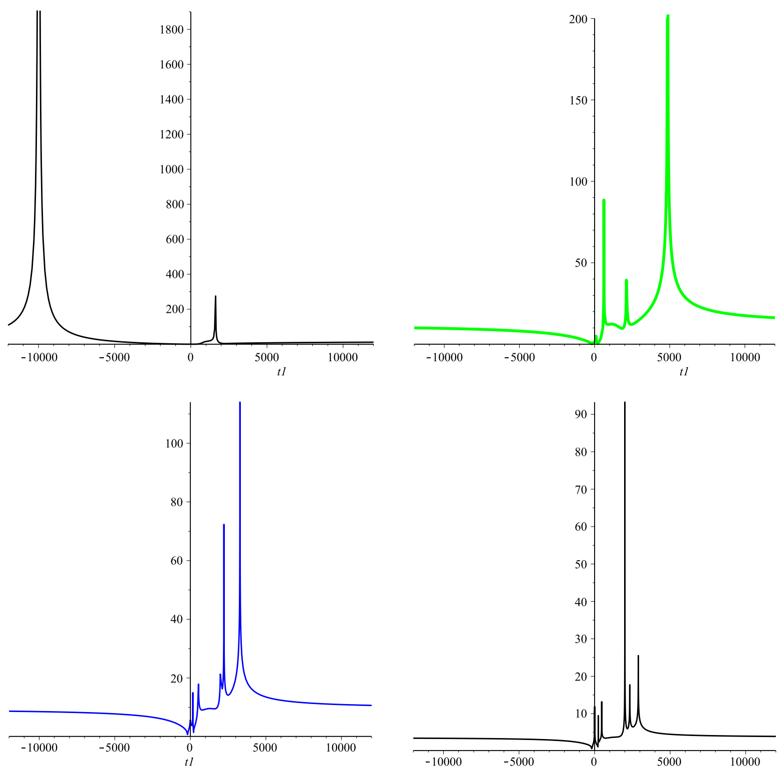

Figure 2: Functions $e_j(t_1, 5)$, $j = 1, \ldots, 4$ for $t_1 \in (-10000,\, 10000)$

In Figure 2, we plot the functions $e_j(t_1, 5)$, $j = 1, \ldots, 4$, for $t_1 \in (-10000,\, 10000)$. In Figure 3, we plot the functions $e_j(t_1, 5)$, $j = 1, \ldots, 4$, for $t_1 \in (0,\, 100)$.

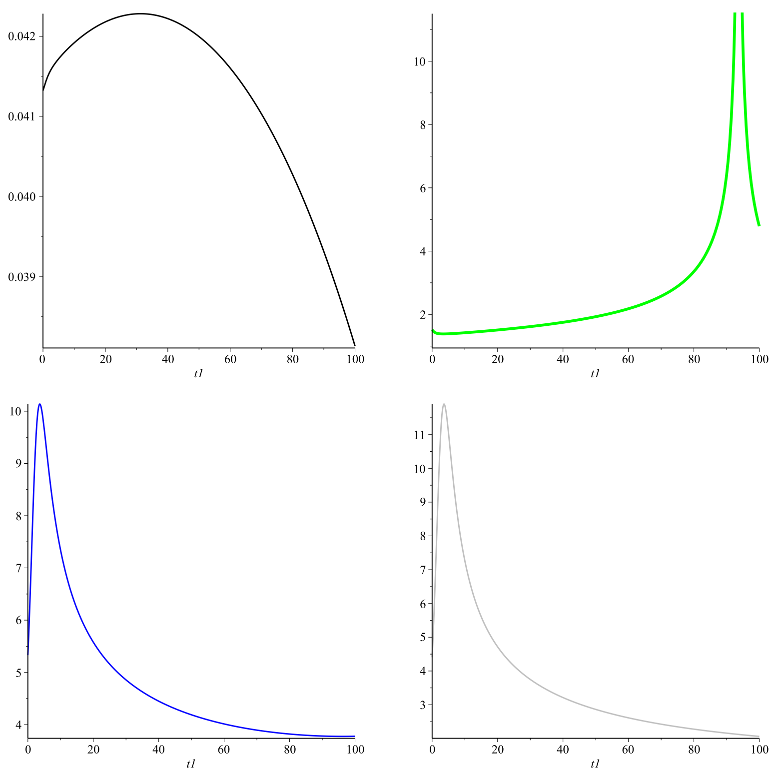

Figure 3: Functions $e_j(t_1, 5)$, $j = 1, \ldots, 4$ for $t_1 \in (0,\, 100)$

Now, we compute

$$\mathcal{D}_2(\,\bar{t}\,, s) = G(\mathcal{L}_2(a, b, c, s)) = B_4(a, b, c)s^4 + \cdots + B_0(a, b, c),$$

and the functions

$$e_1(a_0, b_0, c_0) := 4\left|\frac{B_0(a_0, b_0, c_0)}{B_1(a_0, b_0, c_0)}\right|, \qquad e_2(a_0, b_0, c_0) := 6\left|\frac{B_0(a_0, b_0, c_0)}{B_2(a_0, b_0, c_0)}\right|^{\frac{1}{2}},$$

$$e_3(a_0, b_0, c_0) := 4\left|\frac{B_0(a_0, b_0, c_0)}{B_3(a_0, b_0, c_0)}\right|^{\frac{1}{3}}, \qquad e_4(a_0, b_0, c_0) := \left|\frac{B_0(a_0, b_0, c_0)}{B_4(a_0, b_0, c_0)}\right|^{\frac{1}{4}},$$

where $F(a_0, b_0, c_0) = 0$. Using Theorem 3.3 (statement 2), and reasoning similarly as above, one may estimate $d((a_0, b_0, c_0), \mathcal{W}), (a_0, b_0, c_0) \in \mathcal{V}$. In this case, let us apply Corollary 3.4 and we maximize the function $e_1(a_0, b_0, c_0)$. For this purpose, we use Lagrange multipliers under the constrain $F(a_0, b_0, c_0) = 0$. Under these conditions, if we are far away of the points where $\nabla G(a_0, b_0, c_0)$ and $\mathcal{N}(a_0, b_0, c_0)$ are orthogonal, it holds that

$$d((a_0, b_0, c_0), \mathcal{W}) \le e_1(a_0, b_0, c_0) \le 0.00021.$$

3.2 Numerical Parametric Ruled Problem

In the following, we deal with the Numerical Parametric Ruled Problem. More precisely, given a surface $\mathcal{V}$ defined by a rational parametrization $\mathcal{M}(\bar{t}) = (m_1(\bar{t}), m_2(\bar{t}), m_3(\bar{t})) \in \mathbb{C}(\bar{t})^3$ with perturbed float coefficients, we present an algorithm that outputs a rational parametrization $\mathcal{P}(\bar{t}) = (p_1(r_1(t_1)) + t_2 q_1(r_2(t_1)), p_2(r_1(t_1)) + t_2 q_2(r_2(t_1)), t_2)$ parametrizing a ruled surface $\mathcal{W}$. In this case, we say that the surface $\mathcal{V}$ is an approximate rational ruled surface. In Theorem 3.7 and Corollary 3.8, we show how to compute the distance between the input surface $\mathcal{V}$ and the output surface $\mathcal{W}$.

The algorithm presented is obtained from Algorithm 2. We illustrate this algorithm with an example.

Algorithm 4: Computation of a rational ruled surface from an approximate parametric surface
[Step 1] Compute the polynomials $F(x_1, x_2, 0)$ and $\overline{F}(0, x_1, x_2, 1)$ by applying Theorem 2.7 and Remark 2.8. Check whether there exist two approximate rational plane curves $\mathcal{C}_1$ and $\mathcal{C}_2$ defined by an approximate factor of the above polynomials, respectively. In the affirmative case, let $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$ be these polynomials and go to Step 2. Otherwise, RETURN "$\mathcal{V}$ `is not an approximate rational ruled surface`".
[Step 2] Compute $\mathcal{P}_1 = (p_1, p_2) \in \mathbb{C}(t_1)^2$ and $\mathcal{P}_2 = (q_1, q_2) \in \mathbb{C}(t_1)^2$ approximate proper rational parametrizations of the curves $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively (see [19]).
[Step 3] Check whether there exists an approximate rational plane curve $\mathcal{D}$ defined by an approximate factor of the polynomial $R(x_1, x_2, t_1) = \mathrm{Res}_{t_2}(e_1, e_2)$, where $e_i(x_1, x_2, t_1, t_2) = \mathrm{numer}(p_i(x_1) + m_3 q_i(x_2) - m_i(\bar{t}))$, $i = 1, 2$. In the affirmative case, compute, $R(t_1) := (r_1(t_1), r_2(t_1)) \in (\mathbb{C}(t_1) \setminus \mathbb{C})^2$, an approximate proper rational parametrization of $\mathcal{D}$ (see [19]), and RETURN "$\mathcal{W}$ `is a ruled surface parametrized by`

$$\mathcal{P}(\bar{t}) = (p_1(r_1(t_1)) + t_2 q_1(r_2(t_1)), p_2(r_1(t_1)) + t_2 q_2(r_2(t_1)), t_2)."$$

Otherwise, RETURN "$\mathcal{V}$ `is not an approximate rational ruled surface`".

Example 3.6 Let $\mathcal{V}$ be the surface defined by the parametrization $\mathcal{M}(\bar{t}) = (m_1(\bar{t}), m_2(\bar{t}), m_3(\bar{t})) \in \mathbb{C}(\bar{t})^3$, where

$$m_1(\bar{t}) = \frac{0.9999t_1^2 + 1.9999t_2t_1 + t_2^2 - 2t_2 - 2.0003t_1}{(t_1 + t_2 + 2)t_1},$$

$$m_2(\bar{t}) = \frac{t_1^2 + 2t_2t_1 + t_2^2 - 4.9999t_2 - 0.00001t_1}{(t_1 + t_2 + 2)t_1}, \qquad m_3(\bar{t}) = \frac{1.0001t_2 - 0.0001t_1}{t_1}.$$

By applying Algorithm 2, one gets that $\mathcal{V}$ is not a ruled surface. Let us apply Algorithm 4 to check whether $\mathcal{V}$ is an approximate rational ruled surface and, in the affirmative case, we compute a parametrization of a ruled surface $\mathcal{W}$. Afterwards, we will measure the distance between $\mathcal{V}$ and $\mathcal{W}$ (see Theorem 3.7 and Corollary 3.8).

Figure 4: Input surface $\mathcal{V}$ (left), Output surface $\mathcal{W}$ (center), and both surfaces (right)

In Step 1 of Algorithm 4, we compute the polynomials

$$F(x_1, x_2, 0) = 0.4999850244 + 0.5001149706x_1 - x_2,$$

$$\overline{F}(0, x_1, x_2, 1) = 7x_1 - 4.000057144x_2 - 2.999642892$$

by applying Theorem 2.7 and Remark 2.8. We check whether there exist two approximate rational plane curves $\mathcal{C}_1$ and $\mathcal{C}_2$ defined by an approximate factor of the above polynomials, respectively. In Step 2, we compute

$$\mathcal{P}_1(t_1) = (p_1(t_1), p_2(t_1)) = (t_1, 0.4999850244 + 0.5001149706t_1) \in \mathbb{C}(t_1)^2,$$

$$\mathcal{P}_2(t_1) = (q_1(t_1), q_2(t_1)) = \left(t_1, \frac{7t_1 - 2.999642892}{44.000057144}\right) \in \mathbb{C}(t_1)^2$$

approximate proper approximate parametrizations of $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively.

In Step 3, we check whether there exists an approximate rational plane curve $\mathcal{D}$ defined by an approximate factor of the polynomial $R(x_1, x_2, t_1) = \mathrm{Res}_{t_2}(e_1, e_2)$, where $e_i(x_1, x_2, t_1, t_2) = \mathrm{numer}(p_i(x_1) + m_3 q_i(x_2) - m_i(\bar{t}\,))$, $i = 1, 2$. We get the curve $\mathcal{D}$ defined by the polynomial $h(x_1, x_2) = -x_2 + 0.000124971881 + 0.9997750481x_1$, and we compute an approximate proper parametrization of $\mathcal{D}$. We get

$$R(t_1) := (r_1(t_1), r_2(t_1)) = (t_1, 0.000124971881 + 0.9997750481t_1) \in (\mathbb{C}(t_1) \setminus \mathbb{C})^2.$$

Finally, we return a new rational ruled surface, $\mathcal{W}$, defined parametrically by $\mathcal{P}(\bar{t}\,) = (p_1(r_1(t_1)) + t_2 q_1(r_2(t_1)), p_2(r_1(t_1)) + t_2 q_2(r_2(t_1)), t_2)$, where

$$p_1(r_1(t_1)) + t_2 q_1(r_2(t_1)) = 0.000124971881t_2 + 0.9997750481t_2 t_1 + t_1,$$

$$p_2(r_1(t_1)) + t_2 q_2(r_2(t_1)) = -0.7496813123t_2 + 1.749581340t_2 t_1 + 0.4999850244 + 0.5001149706t_1.$$

In Figure 4, we plot the input surface and the output surface. One may check that both surfaces are very "close".

Analysis of the Error

Let $\mathcal{V}$ be the input parametric surface defined by $\mathcal{M}(\bar{t}) = (m_1(\bar{t}), m_2(\bar{t}), m_3(\bar{t})) \in \mathbb{C}(\bar{t})^3$. Let $\mathcal{W}$ be the output surface obtained by Algorithm 4. In addition, let $F(\overline{x})$ and $G(\overline{x})$ be the defining polynomials of $\mathcal{V}$ and $\mathcal{W}$, respectively, and let $\mathcal{P}(\bar{t}) \in \mathbb{C}(\bar{t})^3$ be the parametrization of $\mathcal{W}$ outputs by our algorithm. Similarly as in Subsection 3.1, we study the distance between both surfaces. For this purpose, we consider the normal line,

$$\mathcal{L}_1(\bar{t}, s) = \mathcal{P}(\bar{t}) + s\,\mathcal{T}_{\mathcal{P}}(\bar{t}), \qquad \mathcal{T}_{\mathcal{P}}(\bar{t}) = \frac{\frac{\partial \mathcal{P}}{\partial t_1} \times \frac{\partial \mathcal{P}}{\partial t_2}}{\|\frac{\partial \mathcal{P}}{\partial t_1} \times \frac{\partial \mathcal{P}}{\partial t_2}\|_2}$$

to $\mathcal{W}$ at the generic point $\mathcal{P}(\bar{t})$, and the normal line,

$$\mathcal{L}_2(\bar{t}, s) = \mathcal{M}(\bar{t}) + s\,\mathcal{T}_{\mathcal{M}}(\bar{t}), \qquad \mathcal{T}_{\mathcal{M}}(\bar{t}) = \frac{\frac{\partial \mathcal{M}}{\partial t_1} \times \frac{\partial \mathcal{M}}{\partial t_2}}{\|\frac{\partial \mathcal{M}}{\partial t_1} \times \frac{\partial \mathcal{M}}{\partial t_2}\|_2}$$

to $\mathcal{V}$ at the generic point $\mathcal{M}(\bar{t})$. Moreover, we introduce the polynomials

$$\mathcal{D}_1(\bar{t}, s) = F(\mathcal{L}_1(\bar{t}, s)) \in \overline{\mathbb{R}(\bar{t})}[s], \quad \mathcal{D}_2(\bar{t}, s) = G(\mathcal{L}_2(\bar{t}, s)) \in \overline{\mathbb{R}(\bar{t})}[s].$$

We write $\mathcal{D}_1$ and $\mathcal{D}_2$ as

$$\mathcal{D}_1(\bar{t}, s) = A_n(\bar{t})s^n + \cdots + A_0(\bar{t}), \quad \mathcal{D}_2(\bar{t}, s) = B_n(\bar{t})s^n + \cdots + B_0(\bar{t}).$$

(note that $\deg_s(\mathcal{D}_1) = \deg_s(\mathcal{D}_2) = \deg(\mathcal{V}) = \deg(\mathcal{W}) = n$). We may reason similarly as above and we get the following results:

Theorem 3.7 *Let $\bar{t}_0 \in \mathbb{C}$ be such that $\mathcal{D}_1(\bar{t}_0, s)$ and $\mathcal{D}_2(\bar{t}_0, s)$ are well defined. Then,*

1. $d(\mathcal{P}(\bar{t}_0), \mathcal{V}) \leq \min\left\{ \binom{n}{i} \left|\frac{A_0(\bar{t}_0)}{A_i(\bar{t}_0)}\right|^{\frac{1}{i}} \text{ where } A_i(\bar{t}_0) \neq 0 \text{ and } 1 \leq i \leq n \right\}.$
2. $d(\mathcal{M}(\bar{t}_0), \mathcal{W}) \leq \min\left\{ \binom{n}{i} \left|\frac{B_0(\bar{t}_0)}{B_i(\bar{t}_0)}\right|^{\frac{1}{i}} \text{ where } B_i(\bar{t}_0) \neq 0 \text{ and } 1 \leq i \leq n \right\}.$

Using the expression of the coefficients given by the Taylor expansion, the next corollary also holds.

Corollary 3.8 *Let $\bar{t}_0 \in \mathbb{C}$ such that $\mathcal{D}_1(\bar{t}_0, s)$ and $\mathcal{D}_2(\bar{t}_0, s)$ are well defined. Then,*

1. *if $\nabla F(\mathcal{P}(\bar{t}_0))$ and $\mathcal{T}_{\mathcal{P}}(\bar{t}_0)$ are not orthogonal, then*

$$d(\mathcal{P}(\bar{t}_0), \mathcal{V}) \leq n \left|\frac{A_0(\bar{t}_0)}{A_1(\bar{t}_0)}\right| = n \left|\frac{F(\mathcal{P}(t_0))}{\nabla F(\mathcal{P}(\bar{t}_0)) \cdot \mathcal{T}_{\mathcal{P}}(\bar{t}_0)}\right|,$$

2. *if $\nabla G(\mathcal{M}(\bar{t}_0))$ and $\mathcal{T}_{\mathcal{M}}(\bar{t}_0)$ are not orthogonal, then*

$$d(\mathcal{M}(\bar{t}_0), \mathcal{W}) \leq n \left|\frac{B_0(\bar{t}_0)}{B_1(\bar{t}_0)}\right| = n \left|\frac{G(\mathcal{M}(t_0))}{\nabla G(\mathcal{M}(\bar{t}_0)) \cdot \mathcal{T}_{\mathcal{M}}(\bar{t}_0)}\right|.$$

In the example below we apply the above results, and we look for empirical evidences indicating that the input surface and the output surface are very “close”.

Example 3.9 Let $\mathcal{V}$ be the surface considered in Example 3.6, and the output parametrization $\mathcal{P}(\bar{t})$ obtained by applying Algorithm 4. Under these conditions, from Corollary 3.4 and reasoning similarly as in Example 3.5, if we are far away of the points where $\nabla F(\mathcal{P}(\bar{t}_0))$ and $\mathcal{T}_{\mathcal{P}}(\bar{t}_0)$ are orthogonal, then

$$d(\mathcal{P}(\bar{t}_0), \mathcal{V}) \le n \left| \frac{F(\mathcal{P}(t_0))}{\nabla F(\mathcal{P}(\bar{t}_0)) \cdot \mathcal{T}_{\mathcal{P}}(\bar{t}_0)} \right| \le 1.915937999 \cdot 10^{-13}.$$

Similarly, if we are far away of the points where $\nabla G(\mathcal{M}(\bar{t}_0))$ and $\mathcal{T}_{\mathcal{M}}(\bar{t}_0)$ are orthogonal, then

$$d(\mathcal{M}(\bar{t}_0), \mathcal{W}) \le n \left| \frac{G(\mathcal{M}(t_0))}{\nabla G(\mathcal{M}(\bar{t}_0)) \cdot \mathcal{T}_{\mathcal{M}}(\bar{t}_0)} \right| \le 3.186985570 \cdot 10^{-10}.$$

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