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Agent's Optimal Compensation Under Inflation Risk by Using Dynamic Contract Model*

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Abstract This paper studies the problem of principal-agent with moral hazard in continuous time. The firm's cash flow is described by geometric Brownian motion (hereafter GBM). The agent affects the drift of the firm's cash flow by her hidden effort. Meanwhile, the firm rewards the agent with corresponding compensation and equity which depend on the output. The model extends dynamic optimal contract theory to an inflation environment. Firstly, the authors obtain the dynamic equation of the firm's real cash flow under inflation by using the Itô formula. Then, the authors use the martingale representation theorem to obtain agent's continuation value process. Moreover, the authors derive the Hamilton-Jacobi-Bellman (HJB) equation of investor's value process, from which the authors derive the investors' scaled value function by solving the second-order ordinary differential equation. Comparing with He[1], the authors find that inflation risk affects the agent's optimal compensation depending on the firm's position in the market.

Keywords Equity incentive, inflation risk, Itô formula, principal-agent problem, the martingale representation theorem.

1 Introduction

The principal-agent problem with the moral hazard is investigated in this article. Traditionally, both parties have the same belief on the observable output process, but the principal can not measure agent's effort, where the drift of firm size process is controlled by agent's effort.

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Hence the principal pays the agent compensation only depending on the output level. This perspective with important economic implications was widely applied to reality. But the inflation risk affects the firm cash flow and the agent compensation. Generally speaking, the real value of the agent compensation is reducing and the incentive compatibility will be reconsidered in this case.

The optimal contract problem has always been a central project in corporate governance theory (see, e.g., Bloton, et al.[2]). A large number of literature focus on the optimal contract containing moral hazard in continuous time, for example, DeMarzo and Sannikov[3], Sannikov[4] and Williams[5], Biais, et al.[6] and DeMarzo and Fishman[7]. Zheng, et al.[8] reviewed a dynamic financial contract of optimal financing structure with moral hazard constraint. And one of the important factors in provision is the agent compensation based on output process, and Gertler and Gilchrist[9] found that when cash flow is tight, the agent of small-scale companies will be more constrained. Considering the agent cash reserves, debt and equity, Miao and Rivela[10] implemented the optimal contract. Moreover, He[1] investigated the agent optimal compensation when the firm cash flow is described by GBM, and finds that the agency problem is more severe for smaller firms. So our question is that: How is the output process affected by inflation risk from the market? And is the optimal contract implemented in this situation?

Since the principal and the agent hold the same belief in the output process, it is easy to think about how to establish the optimal contract when the output process is disturbed by uncertainty risk. Moreover, in view of the uncertainty of the statistics model, Fei, et al.[11], Fei, et al.[12] and Wang, et al.[13] discussed the continuous-time principal-agent problem under Knightian uncertainty, and Fei, et al.[14] explored the optimal contract design problem of unilateral limited commitment under Knightian uncertainty. Besides, considering the inflation uncertainty risk of the financial market, Fei, et al.[15] investigated the optimal dynamic contract between the executive and the stockholders.

As we all know, global inflation is not only a hot topic in society today, but also a real problem that investors, fund managers and investment institutions have to face with. For example, from the financial crisis to the COVID-19, countries around the world are rushing to adopt quantitative easing monetary and fiscal policies, leading to increasingly severe inflation. Hence, inflation is one of the most important factor for investors to consider. Based on the seminal work of Merton[16], many researchers focus on the optimal asset allocation problems considering the inflation risk. For example, Brennan and Xia[17] used the martingale approach to study the dynamic portfolio optimization problem with inflation. Kothari and Shanken[18] and Roll[19] found that the treasury inflation-protected securities improved diversified portfolio between stocks and nominal bonds. Cartea, et al.[20] used the treasury inflation-protected securities to solve investor's real wealth for maximizing returns of an investor. Chiarella, et al.[21] explored the intertemporal investment strategy with stochastic price index under inflation risk. Mkaouar, et al.[22] extended the long-term investment portfolio with inflation risk and considered the stochastic interest rate on the basis of Chiarella, et al.[21]. Munk, et al.[23] studied the dynamic asset allocation under mean-reverting returns, stochastic interest rates and inflation uncertainty, and discussed whether the popular recommendations are consistent with rational behavior. Besides, Bensoussan, et al.[23] investigated the optimal consumption and portfolio decisions with partially observed real prices.

Next, Siu[25] studied the long-term strategic asset allocation problem with inflation risk, and operates the martingale method to get the optimal allocation strategy. Based on the research method of Brennan and Xia[17], Yao, et al.[26] studied the continuous time mean-variance model under inflation risk when time-horizon is uncertain. Dai and Zhang[27] did the research on the relationship between asset price and inflation based on ARDL technical analysis. Fei and Li[28] analyzed the impact of inflation and Knightian uncertainty on the optimal investment strategy, and obtains an explicit solution of the investment strategy. Liang, et al.[29] investigated the impact of Knightian uncertainty on investor's investment strategies in an inflationary environment. Moreover, Fei, et al.[30] investigated the optimal investment strategy choice for an ambiguity aversive investor who is exposed to extreme events in an inflation environment. Fei, et al.[31] considered the optimal consumption and investment problems with inflation in the case of recursive utility, and obtains the optimal consumption and investment strategy by using the dynamic programming method. Fei, et al.[32] investigated the optimal investment strategies of hedge funds with incentives under the influence of inflation and other factors. Moreover, Wu and Dong[33] discussed the optimal investment strategies retirement with inflation risk. Fei and Fei[34] did the research on the optimal control of Markovian switching systems with applications to portfolio decisions under inflation. Fei[35] studied the optimal consumption and portfolio under inflation and Markovian switching. Fei, et al.[36] analyzed the entrepreneur's investment-consumption and hedging under inflation risk.

We try to extend the model of optimal dynamic contract in He[1] to that with inflation risk. Moreover, we are faced with three following problems.

First, the suitable incentive is necessary for an agent. The dependency between CEO's compensation pattern and her previous performance has been investigated by Ai and Li[37]. If the performance is bad, which is the growth rate of firm's cash flow does not meet investor's expectation, the agent will be punished. However, there are two problems in the compensation and punishment system in reality. On the one hand, the agent only bears the limited liability, so her compensation is positive, while the punishment will be triggered when the firm is liquidated, and investors fire the agent. On the other hand, in view of the incentive compatibility in an optimal dynamic contract, it is necessary to provide an effective incentive to motivate the agent for the better performance. Based on the real wealth of principal and agent, this paper focuses on the mechanism of the agent compensation under inflation risk.

Second, we consider two types of agents with different patience levels, which has been also discussed in He[1]. This paper also explores how the inflation risk affects the optimal contract designing for both the patience agents and the impatience agents. In a traditional view, agent's value contains cash and equity. In the cash payment component of optimal contract, Yang, et al.[38] showed that the time-inconsistent investors tend to payout lower compensations earlier and more frequently than time-consistent investors. Hence, the need for immediate consumption of impatient agent is stronger than patient one. However, early cash compensation leads to a corresponding reduction in agent's continuation value, which causes the potential risk of firm liquidation. And the inflation risk causes the distortion in firm's cash process, which is related to agent's performance. Therefore, the optimal contract sets an optimal cash payment threshold under inflation risk.

Finally, similar to Frydman and Saks[39] and He[1], we discuss the equity incentive mechanism based on agent's performance in the corporation asset value. In fact, Albuquerque and Hopenhayn[40] indicated that when agent's stake becomes high enough in the contract, the agency problem will be completely solved. However, the premise of the contract is conforming the incentive compatibility condition. In He[1], the incentive points trace the proportion of agent's continuation payoff value based on firm's nominal cash flow. When agent's incentive points are accumulated to the previous level, the agent convert the incentive points into a certain proportion of the internal equity. We extend the model of He[1] with inflation risk, the optimal equity granting point based on firm's real flow process is characterized.

On the basis of the GBM framework of He[1], we describe the output process, which is related to agent's performance, essentially inescapably affected by inflation risk. Hence, this paper has some innovations as follows. We first get the firm real cash process under inflation risk by using Itô formula. Next, we derive the corresponding investors' scaled value function by solving the second-order ordinary differential equation, thus get the optimal contract under inflation risk. Finally, comparing with the results of He[1], we attempt to explain the reality economic meaning of optimal contract from inflation risk by numerical analysis.

The remaind parts of this paper are organized as follows. In Section 2, we establish basic dynamic contract model of the principal-agent problem in continuous time under inflation risk. In Section 3, we give the solving of the optimal contract. In Section 4, we analyse the properties of the optimal dynamic contract with inflation risk. In Section 5, we make a conclusion.

2 Basic Model

Similar to the principal-agent dynamic model of He[1] with moral hazard in continuous time, we suppose that there is a sustainable firm, and the risk-neutral investors hire a risk-neutral agent to manage the firm. The firm's nominal cash flow is described by the GBM dynamic model

$$d\delta_t = a_t\delta_t dt + \sigma\delta_t dZ_t, \tag{1}$$

where $Z = \{Z_t, \mathcal{F}_t; 0 \leq t < \infty\}$ is a standard Brownian motion on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$, and $a_t \in \{0, \mu\}$ is the agent's binary effort choice. Here, $a_t = \mu > 0$ stands for "working", while $a_t = 0$ stands for "shirking".

Since the firm's nominal cash flow is affected by inflation risk. The standard Brownian motion $B = \{B_t, \mathcal{F}_t; 0 \leq t < \infty\}$ is set in the same probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$, and the market inflation process driven by standard Brownian motion B as in Brennan and Xia[17]:

$$dI_t = iI_tdt + \varsigma I_t dB_t,$$

where i is the expected inflation rate, and ς is the expected inflation volatility. We use the Itô formula to derive

$$d\left(\frac{1}{I_t}\right) = \left(\frac{1}{I_t}\right)\left((-i+\varsigma^2)dt - \varsigma dB_t\right),$$

and set the firm's real cash flow process $Y_t \triangleq \frac{\delta_t}{I_t}$, and the correlation coefficient of two Brownian motions is ρ, i.e., $dB_t \cdot dZ_t = \rho dt$ ($\rho \in [-1,1]$). Derived from the Itô product rule:

$$\begin{aligned} dY_t = d\left(\frac{\delta_t}{I_t}\right) &= \delta_t d\left(\frac{1}{I_t}\right) + \left(\frac{1}{I_t}\right) d\delta_t + (d\delta_t) d\left(\frac{1}{I_t}\right) \\ &= Y_t\left((a_t + (-i+\varsigma^2) - \rho\sigma\varsigma)dt + \sigma dZ_t - \varsigma dB_t\right) \\ &= Y_t\left((a_t+\kappa)dt + \sigma dZ_t - \varsigma dB_t\right) \\ &= Y_t\left(\overline{a}_t dt + \sigma dZ_t - \varsigma dB_t\right). \end{aligned} \tag{2}$$

We set $\kappa = -i + \varsigma^2 - \rho\sigma\varsigma$. Here $\overline{a}_t = \{\kappa, \mu+\kappa\} = \{\kappa, \overline{\mu}\}$, which contains the agent effort choice and inflation risk. Even when agent shirks her work, the drift part never drops to zero, which is different from He[1]. Indeed, when agent works hard, the real "growth" of the output process is suffered by inflation risk. If the agent works all the time, then from the view of investors, from Formula (2), the firm's first-best real value at time t is

$$E_t\left[\int_t^{\infty} \mathrm{e}^{-r(s-t)} Y_s ds\right] = \frac{1}{r-\overline{\mu}} Y_t + E\left[\frac{1}{r-\overline{\mu}} Y_t(\sigma dZ_t - \varsigma dB_t)|\mathcal{F}_t\right] = \frac{1}{r-\overline{\mu}} Y_t, \tag{3}$$

which follows a GBM as well and the market interests rate r, and $r > \overline{\mu} > 0$.

For the agent, she has a right-continuous-left-limit nondecreasing cumulative wage process $\left\{\overline{U}_t : 0 \le t \le \tau\right\}$ within the contract period. Similar to He[1], we assume that the agent's reservation value is zero. We denote such a contract by $\overline{\pi} \equiv \left\{\left\{\overline{U}_t\right\}, \tau\right\}$, where elements are $Y-$measurable, and the stopping time τ could take the value ∞. We impose the usual square-integrable condition on $\overline{\pi}$ as follows:

$$E\left[\left(\int_0^{\tau} \mathrm{e}^{-\gamma t} d\overline{U}_t\right)^2\right] < \infty. \tag{4}$$

A contract $\overline{\pi}$ is incentive-compatible if it motivates the agent to work until liquidation; in other words, if $\{\overline{a}_t = \overline{\mu} : 0 \le t < \tau\}$ solves the following agents problem:

$$\max_{\overline{a}=\{\overline{a}_t \in \{\kappa,\overline{\mu}\}:0\le t<\tau\}} E^{\overline{a}}\left[\int_0^{\tau} \mathrm{e}^{-\gamma t}\left(d\overline{U}_t + \psi\kappa\left(1 - \frac{\overline{a}_t - \kappa}{\overline{\mu}-\kappa}\right) Y_t dt\right)\right].$$

The agent is risk neutral and discounts her consumption at rate γ.

Remark 2.1 Typically, the intertemporal marginal rate of substitution for a borrowing-constrained agent is greater than the market-interest rate r. To capture this detail in a risk-neutral setting, [3] assumes $\gamma > r$. That also means the agent is (weakly) less patient than investors. When the issue of relative consumption timing is absent, postponing cash payments has zero cost, that is the agent as patient as the investors, He[1] obtain a optimal payment boundary in the case $\gamma = r$.

The limited liability of the agent avoids the negative wages, that means the agent can get profit from shirking which is considered as moral hazard. If the agent shirks responsibility, the investors' profits loss $\psi\kappa Y_t dt$ until the agent is fired or replaced, where $\psi\kappa > 0$.

Once the contract is terminated at the stopping time τ, the investors recover a value LY_τ from the firm's asset, if the liquidation value is less than the discounted value $(L < 1/(r-\overline{\mu}))$ (recall (3)), the liquidation is inefficient and the value of the firm is damaged. In this paper, unless otherwise stated, the expectation operator is under the measure induced by $\{\overline{a}_t = \overline{\mu} : 0 \le t < \tau\}$. We denote that $\overline{\pi}$ is incentive compatible if $\{\overline{a}_t = \overline{\mu} : 0 \le t < \tau\}$ maximizes the agent's total expected utility (see Sannikov[4]). The firm's problem is

$$\max_{\overline{\pi}} E^{\overline{a}_t}\left[\int_0^\tau \mathrm{e}^{-rt} Y_t dt + \mathrm{e}^{-r\tau} LY_\tau - \int_0^\tau \mathrm{e}^{-rt} d\overline{U}_t\right],$$

and the solution for this problem as $\overline{\pi}^* = \left\{\left\{\overline{U}_t{}^*\right\}, \tau^*\right\}$. The goal of this paper is to design an optimal contract with inflation risk.

3 Model Solution and Optimal Contract Under Inflation Risk

In this section, we establish the optimal contract with the incentive compatible condition under inflation risk, and compare with the results in He[1].

Remark 3.1 we discuss the $\rho = 1$, $\rho = 0$ and $\rho = -1$ below.

3.1 Continuation Payoff and Incentive Compatibility

We give a key proposition when the agent chooses to work hard, thus her behavior is $\overline{a}_t = \{\overline{\mu}; 0 \le t < \tau\}$ during the performance of contract. According to the definition of the continuation value of the agent in He[1], the agent's continuation payment at time t:

$$V_t(\overline{\pi}) \equiv E_t\left[\int_t^\tau \mathrm{e}^{-\gamma(s-t)} d\overline{U}_s\right].$$

In the following Proposition 3.2, we give the expression of the continuation payoff process and the incentive compatibility condition of the optimal contract.

Proposition 3.2 *For any contract $\overline{\pi} \equiv \left\{\left\{\overline{U}_t\right\}, \tau\right\}$, there exists a progressively measurable process $\left\{\beta_t^V : 0 \le t < \tau\right\}$ and $\overline{a} = \{\overline{a}_t = \overline{\mu} : 0 \le t < \tau\}$. Then the agent's continuation value process based on the firm's real cash flow process:*

$$dV_t = \gamma V_t dt - d\overline{U}_t + \beta_t^V Y_t(\sigma dZ_t - \varsigma dB_t). \tag{5}$$

The contract $\overline{\pi}$ is incentive-compatible, if and only if $\beta_t^V \ge \eta$ for $t \in [0, \tau)$, where $\eta = \psi\kappa/\overline{\mu}$ is the least incentive compatibility boundary condition.

(5) states that the agent's instantaneous compensation, which the wage dU_t plus the change of continuation payoff dV_t, has a predetermined drift part γV_t, and a diffusion part

$$\beta_t^V Y_t(\sigma dZ_t - \varsigma dB_t) = \beta_t^V (dY_t - \overline{\mu} Y_t dt),$$

which links to her effort choice and provides working incentive. If the agent chooses to shirk, the drift part becomes $\kappa Y_t dt$ due to the inflation effect, so she gains profit $\psi\kappa Y_t dt$ from the output process, but relatively loses $\beta_t^V \overline{\mu} Y_t dt$ in compensation (the drift of dY_t becomes $\kappa Y_t dt$ under shirking). For fixing incentive compatible, she chooses to work if and only if $\beta_t^V \overline{\mu} Y_t dt \geq \psi\kappa Y_t dt$, that is, $\beta_t^V \geq \frac{\psi\kappa}{\overline{\mu}} = \eta$.

Proof For any contract $\overline{\pi} \equiv \left\{\left\{\overline{U}_t\right\}, \tau\right\}$, $D_t = E^{\overline{a}_t}\left[\int_0^\tau \mathrm{e}^{-\gamma s} d\overline{U}_s\right]$ for $t \in [0, \tau)$ is defined as the agent's expected discounted accumulative wage. With Condition (4), $\{D_t : 0 \leq t < \tau\}$ is a square-integrable martingale until τ. By using the martingale representation theorem as in Øksendal[41], there exists a progressively measurable process $\left\{\beta_t^V : 0 \leq t < \tau\right\}$ such that:

$$D_t = D_0 + \int_0^t \mathrm{e}^{-\gamma s} Y_s \beta_s^V \left(\sigma dZ_t - \varsigma dB_t\right),$$

for $t \in [0, \tau)$. Under the assumption $\overline{a} = \{\overline{a}_t = \overline{\mu} : 0 \leq t < \tau\}$. Now since $V_t = E_t\left[\int_t^\tau \mathrm{e}^{-\gamma(s-t)} d\overline{U}_s\right]$, we have $D_t = \int_0^t \mathrm{e}^{-\gamma s} d\overline{U}_s + \mathrm{e}^{-\gamma t} V_t$. Thus, we get the evolution of V_t by taking derivative on both sides.

We now show that $\overline{\pi}$ is incentive-compatible if and only if $\beta_t^V \geq \eta$. Consider any effort $\overline{a} = \{\overline{a}_t \in \{\kappa, \kappa + \overline{\mu}\} : 0 \leq t < \tau\}$. For $t < \tau$, we construct a function for $\overline{a}$, and her associated value process is

$$\begin{aligned} D_t\left(\overline{a}\right) = & D_0 + \int_0^t \mathrm{e}^{-\gamma s} \beta_s^V \frac{\overline{\mu}}{\overline{\mu} - \kappa}\left(\overline{a}_s Y_s ds - \overline{\mu} Y_s ds\right) + \int_0^t \mathrm{e}^{-\gamma s} \beta_s^V Y_s \left(\sigma dZ_t - \varsigma dB_t\right) \\ & + \int_0^t \mathrm{e}^{-\gamma s} \eta \frac{\overline{\mu}}{\overline{\mu} - \kappa} Y_s \left(\overline{\mu} - \overline{a}_s\right) ds, \end{aligned}$$

thus we have

$$\begin{aligned} dD_t\left(\overline{a}\right) &= \mathrm{e}^{-\gamma t} Y_t \beta_t^V \frac{\overline{\mu}}{\overline{\mu} - \kappa}\left(\left(\overline{a}_t - \overline{\mu}\right) dt + \left(\sigma dZ_t - \varsigma dB_t\right)\right) + \mathrm{e}^{-\gamma t} \eta \frac{\overline{\mu}}{\overline{\mu} - \kappa} Y_t \left(\overline{\mu} - \overline{a}_t\right) dt \\ &= \mathrm{e}^{-\gamma t} Y_t \left(\beta_t^V - \eta\right) \frac{\overline{\mu}}{\overline{\mu} - \kappa}\left(\overline{a}_t - \overline{\mu}\right) dt + \mathrm{e}^{-\gamma t} Y_t \beta_t^V \left(\sigma dZ_t - \varsigma dB_t\right). \end{aligned}$$

If $\beta_t^V \geq \eta$, then D_t has a nonpositive drift, and is a martingale if $\{\overline{a}_t = \overline{\mu} : 0 \leq t < \tau\}$. If there is a positive probability event that $\beta_t^V < \eta$ for $t \in [0, \tau)$, the agent will deviate to $\overline{a}_t = \kappa$ and $\{\overline{a}_t = \overline{\mu} : 0 \leq t < \tau\}$ is suboptimal. Therefore, $\overline{\pi}$ is incentive compatible, if and only if $\beta_t^V \geq \eta$. ∎

3.2 Optimality Equations and Boundary Conditions

There are two state variables in the optimality contract equation: The firm's real cash flow Y_t and the agent's continuation value V_t. The investors' value function $b\,(Y, V)$, which is twice differentiable in both arguments, is the firm's highest expected future profit, given these two state variables. When the agent works all the time, the firm's real cash flow is

$$dY_t = \overline{\mu} Y_t dt + Y_t(\sigma dZ_t - \varsigma dB_t),$$

and the agent's continuation payoff V_t is

$$dV_t = \gamma V_t dt - d\overline{U}_t + \beta_t^V \left(dY_t - \overline{\mu} Y_t dt\right).$$

Similar to DeMarzo and Sannikov[3] and He[1], the concavity of the investor's value function implies that the optimal contract provides the incentive-compatible condition, i.e., $\beta_t^V = \eta$, and the optimal cash payment policy depends on $\frac{\partial b}{\partial V}$. If $\frac{\partial b}{\partial V} > -1$, then promising one dollar of continuation payoff from the agent costs the firm less than paying one dollar cash. Assuming that the agent's outside option values zero, thus the agent's continuation value process, the agent's shirking benefit, and the firm's liquidation value are linear in the firm's real cash flow. Hence, the investors' value function $b\,(Y, V)$ becomes the form $Y c\,(h)$ (see He[1]), where the agent's scaled continuation payoff $h = V/Y$ is the relevant state variable, and $c\,(\cdot) \in C^2$ is a univariate smooth function. We call $c\,(\cdot)$ the investors' scaled value function.

In Appendix A, we give the Hamilton-Jacobi-Bellman equation for $b\,(Y, V)$. Moreover, we find that $c\,(\cdot)$ solve the following second-order ordinary differential equation when there is no cash payment ($d\overline{U}_t = 0$):

$$(r - \overline{\mu})c\,(h) = 1 + (\gamma - \overline{\mu})\, hc'\,(h) + \frac{1}{2}(\eta - h)^2 \widehat{\sigma}^2 c''\,(h), \quad \text{where} \quad \widehat{\sigma}^2 = \sigma^2 - 2\rho\sigma\varsigma + \varsigma^2. \tag{6}$$

This equation plays a key role in the optimal contract. Lemma A.1 has proved that $c(\cdot)$ in Equation (6) is a concave function.

We consider two conditions in the optimal cash payment strategy: First, there is an optimal payment boundary $\overline{V}_t$; second, the optimal cash payment boundary is linear with the firm size, i.e., $\overline{V}_t \equiv \overline{h} Y_t$, where $\overline{h}$ is a positive constant to be solved in the optimal contract. For the impatient agent, the cash compensation boundary needs $V_t > \overline{V}_t$(see Figure 3). In order to return V_t to the optimal level $\overline{V}_t$, the investors pay the agent the equivalent cash $V_t - \overline{V}_t$. Similar to the conditions of the optimality equation in He[1], we have the smooth-pasting condition $\frac{\partial b}{\partial V}\left(Y_t, \overline{h} Y_t\right) = -1$, and the second-order differential condition $\frac{\partial^2 b}{\partial V^2}\left(Y_t, \overline{h} Y_t\right) = 0$. So for $c\,(\cdot)$, the constraint conditions on the optimal solution $\overline{h}$ is

$$c'\left(\overline{h}\right) = -1, \tag{7}$$

$$c''\left(\overline{h}\right) = 0. \tag{8}$$

We get another boundary at the firm's liquidation moment. Let τ be the first hitting time at $V_t = 0$, the agent is fired and investors liquidate the firm for LY_τ. Hence,

$$c\,(0) = L, \tag{9}$$

and $c\,(\cdot)$ solves Equation (6) with boundary conditions (7)–(9).

Applying (7) and (8) conditions to equation (6), we get the linear funnction $f\,(h)$ and find that at $\overline{h}$, where $c\,(\cdot)$ intersects the function $f\,(h) = \frac{1}{r - \overline{\mu}} - \frac{\gamma - \overline{\mu}}{r - \overline{\mu}} h$ with slope -1, then we extend $c\,(\cdot)$ linearly for $h > \overline{h}$ based on the optimal wage strategy.

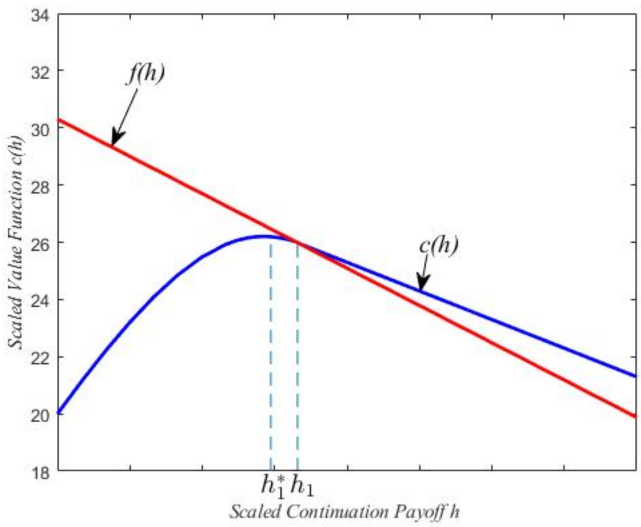

Figure 1 The investors' scaled value function under inflation($\rho = 1$) in the case of $\gamma > r$

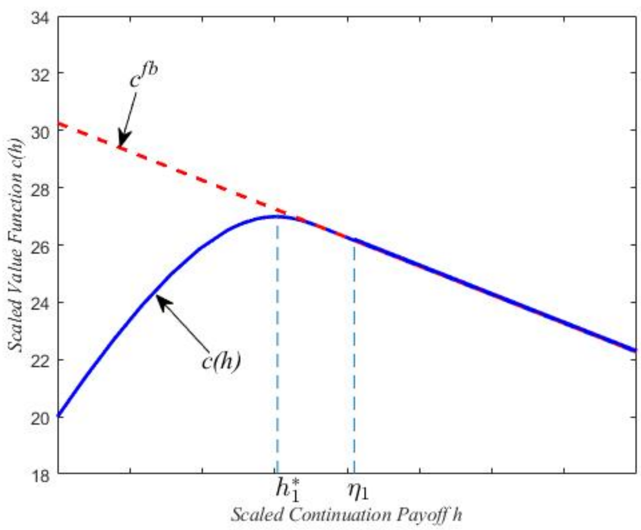

Figure 2 The investors' scaled value function under inflation ($\rho = 1$) in the case of $\gamma = r$

Remark 3.3 The common parameters are $r = 4\%$, $\mu = 1\%$, $\sigma = 31.6\%$, $L = 20$, and the different parameters are $\gamma = 5\%$ in Figure 1 and $\gamma = r = 4\%$ in Figure 2, as in He[1]. In addition, the inflation related parameters are $i = 0.3\%$, $\varsigma = 3\%$, $\overline{\mu} = \mu + \left(-i + \varsigma^2\right) - \sigma\varsigma = 0.72\%$, $\eta_1 = 4.1$.

In Figures 1 and 2, we get the optimal contract about investors' scaled value function under inflation, respectively in the cases of $\gamma > r$ and $\gamma = r$. In Figure 1, $\overline{h}_1 < \eta_1$ is a reflecting barrier. $c(\cdot)$ attaches $f(h) = \frac{1}{r-\overline{\mu}} - \frac{\gamma-\overline{\mu}}{r-\overline{\mu}}h$ with a slope -1, and is extended for $h > \overline{h}_1$ with a slope -1. For the agent who is as patient as the investors ($\gamma = r$), as showing in Figure 2, we have the first-best value function $c^{fb}(h) = 1/(r - \overline{\mu}) - h$ for later references. The scaled value function $c(\cdot)$ attaches $c^{fb}(h) = 1/(r - \overline{\mu}) - h$ smoothly, and $h = \eta_1$ is an optimal state.

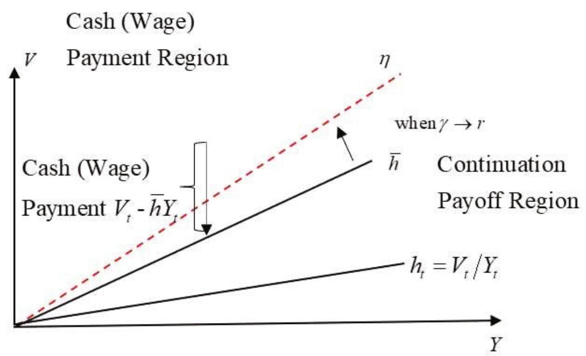

Figure 3 The optimal cash payment and incentive strategy

In Figure 3, we show that the optimal cash payment strategy and incentive strategy in a dynamic proportion of the value h_t. When $V_t > \overline{h}Y_t$, the firm immediately pays the cash to the agent $V_t - \overline{h}Y_t$, and the agent's continuation value ratio back to $\overline{h}$. If the agent is as patient as investor ($\gamma = r$), it is not necessary to consider cash payment in advance. The agent's scaled continuation payment rate h_t grows in the interval $[0, \eta]$ and eventually reaches the optimal incentive point $h_t = \eta$, where the agent becomes shareholder and holds enough incentives to work for the firm forever.

In Figures 4 and 5, we show the optimal contract under inflation in different ρ. For case of $\gamma > r$, the endogenous optimal payment boundary in different inflation coefficient $\rho = -1$, $\rho = 0$ and $\rho = 1$ is $\overline{h}_3$, $\overline{h}_2$ and $\overline{h}_1$ respectively, and $\overline{h}_3 > \overline{h}_1 > \overline{h}_2$.

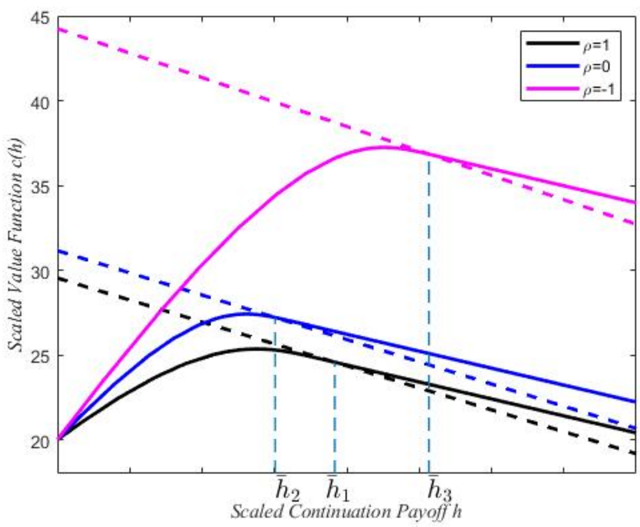

Figure 4 The investors' scaled value function in different ρ when $\gamma > r$

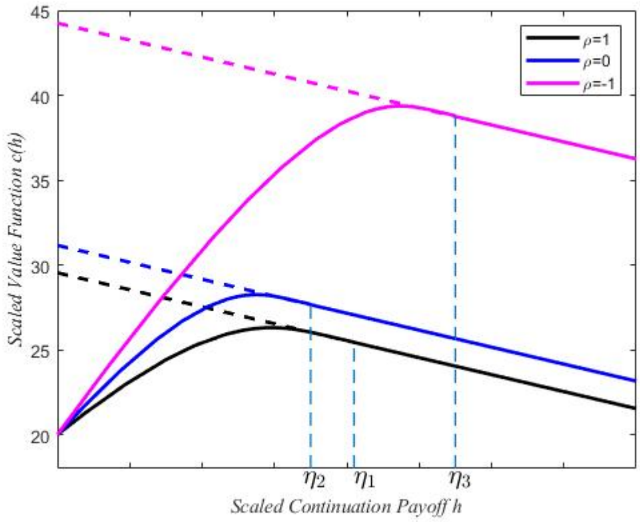

Figure 5 The investors' scaled value function in different ρ when $\gamma = r$

3.3 Comparing the Results in Different ρ with He[1]

He[1] uses the GBM model to investigate the optimal executive compensation, which provides a reference for this paper. Considering the agent's behaviour model is $a_t = \mu$, the firm's nominal cash flow process in He[1] is

$$d\delta_t = \mu\delta_t dt + \sigma\delta_t dZ_t,$$

the agent's continuation value in He[1] is

$$dW_t = \gamma W_t dt - dU_t + \lambda\delta_t dZ_t.$$

Taking the ratio of the agent's continuation value to the firm's nominal cash flow as the only state variable, i.e., $k = W/\delta$. The scaled value equation of the investors based on firm's nominal cash flow in He[1] is $e(k)$, and the corresponding optimal equation is

$$(r-\mu)e(k) = 1 + (\gamma-\mu)\, ke'(k) + \frac{1}{2}(\lambda-k)^2\sigma^2 e''(k), \tag{10}$$

and the cash payment threshold point is $\overline{k}$.

Now, we compare the results in different ρ with He[1] in case of $\gamma > r$ respectively, and the case of $\gamma = r$ has a similar analysis.

In Figure 6, the parameters of dotted line is similar to He[1] in case $\gamma > r$, and the dotted line indicates that the investors' scaled function $e(k)$ attaches $g(k) = 1/(r-\mu) - (\gamma-\mu)\,h/(r-\mu)$ with slope -1, and is extended for $k > \overline{k}$ with a slope -1. The solid line in Figure 4 is the same as Figure 1.

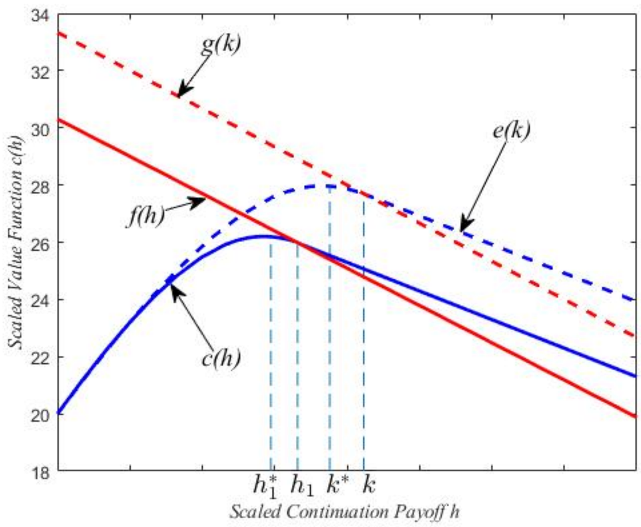

Figure 6 Comparison of the optimal contract when $\rho = 1$ in $\gamma > r$ with the one in He[1]

In Figure 6, we shall compare (6) with (10) in case $\gamma > r$. The main differences as follows: due to the inflation risk, $\overline{\mu}$ and $\widehat{\sigma}$ are numerically smaller in (6) than the drift term and the volatility term of the firm's nominal cash flow in He[1]; the parameter associated with the lowest incentive compatibility condition in (6) is smaller than (10), i.e., $\eta < \lambda$; the scaled value function $c(h)$ reaches the maximum value ratio at h^*, and in the formula (10) of He[1] without considering the inflation, the scaled value function $e(k^*)$ reaches the maximum value ratio at k^*, and $k^* > h_1^*$. Longitudinally, when the agent's continuation payoff ratio is the same, the investor's scaled value function under inflation risk is relatively small, because inflation risk interferes with the firm's real value, in other words, the real performance that an agent can bring to investors is relatively smaller. Under inflation, the agent reaches the optimal cash payment ratio earlier, which means that according to the firm's real cash flow, the cash payment threshold of the optimal contract needs to be adjusted accordingly. For the agent in case of $\gamma = r$, the consistent results $\overline{h} = \eta_1$, $\overline{k} = \lambda$ can be obtained by an analogous analysis, and $\eta_1 < \lambda$.

In Figures 7 and 8, we show that effect of the inflation coefficient induce the optimally scaled payoff $\overline{h}_2 < \overline{k} < \overline{h}_3$, $h_2^* < k^* < h_3^*$ in different ρ. In Figure 7, compared to $e(h)$, the scaled value $c(h)$ seemly translates a little to the left. But at same scaled value, the scaled payoff is different and $h_2^* < k^*$, inflation incurs the loss of the investors' value. On the contrary, there is a huge gap numerically in Figure 8 and seemly the investors can obtain more value in $\rho = -1$. But profits of the firm are essential influenced by inflation risk, and the agent has a higher payment threshold, that is, $\overline{h}_3 > \overline{k}$.

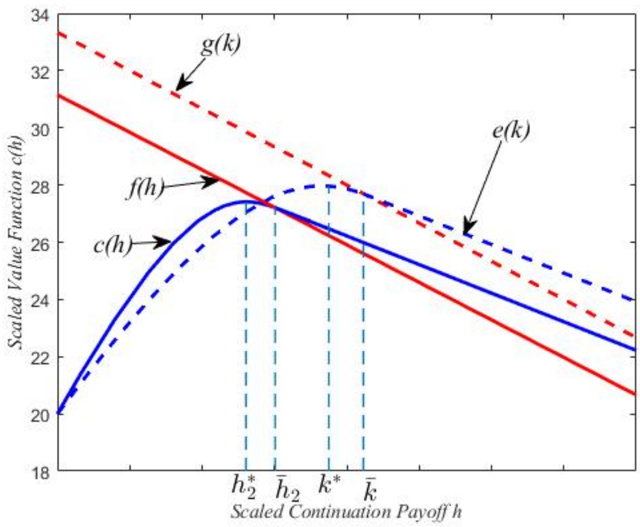

Figure 7 Comparison of the optimal contract when $\rho = 0$ in $\gamma > r$ with the one in He[1]

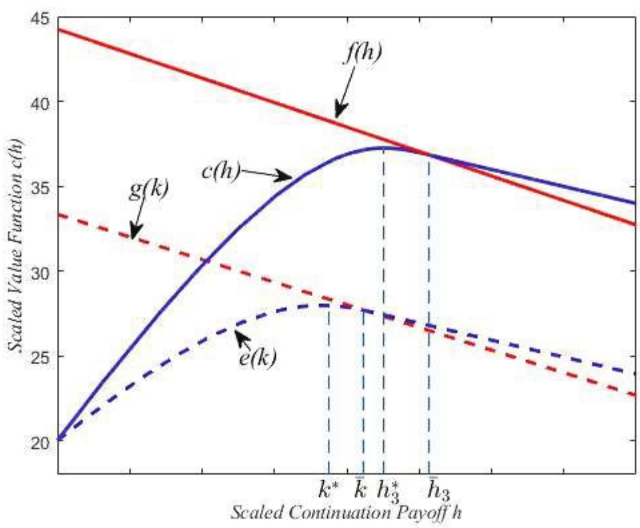

Figure 8 Comparison of the optimal contract when $\rho = -1$ in $\gamma > r$ with the one in He[1]

4 The Optimal Contract Under Inflation Risk

Consistent with the classification of the agent by Biais, et al.[6] and He[1], this paper studies that the optimal contract differs for the two cases $\gamma > r$ and $\gamma = r$. Due to inflation risk, on the one hand, in order to achieve the expected firm's real cash flow growth, the agent needs to work harder; on the other hand, we design the reasonable provision for different agents based on the real performance.

4.1 For an Impatience Agent, a Smaller Cash Payment Threshold is Set in the Optimal Contract

For brevity of notation, we use the h, $\overline{h}$ and η for different ρ case in the following.

Proposition 4.1 *When $\gamma > r$, we have $\overline{h} < \eta$. There exists a unique solution $c(\cdot)$ to Equation* (6) *with boundary conditions* (7)–(9). *And the solution $c(\cdot)$ is strictly concave on $\left[0, \overline{h}\right]$.*

Proof We show that $\overline{h} \neq \eta$. For $\overline{h} = \eta$, $c(\eta) = \frac{1}{r-\overline{\mu}} - \frac{\gamma-\overline{\mu}}{r-\overline{\mu}}\eta$. By the Taylor expansion we have

$$c(\eta - \varepsilon) = c(\eta) + \varepsilon + \frac{1}{2}c''(\theta_1)\varepsilon^2,$$

where $\theta_1 \in (\eta - \varepsilon, \eta)$, and (Taylor expansion for $c'(\eta - \varepsilon)$),

$$(r - \overline{\mu})\, c(\eta - \varepsilon) = 1 + (\gamma - \overline{\mu})(\eta - \varepsilon)(-1 - c''(\theta_2)\varepsilon) + \frac{\varepsilon^2 \widehat{\sigma}^2}{2}c''(\eta - \varepsilon),$$

where $\theta_2 \in (\eta - \varepsilon, \eta)$. It implies that

$$r - \gamma = \frac{r - \overline{\mu}}{2}c''(\theta_1)\varepsilon - c''(\theta_2)(\gamma - \overline{\mu})(\eta - \varepsilon) + \frac{\varepsilon\widehat{\sigma}^2}{2}c''(\eta - \varepsilon).$$

As $\varepsilon \to 0$, $c''(\theta_i) \to 0$ for both θ_i's and $c''(\eta - \varepsilon) \to 0$ due to $c \in C^2$, hence the right hand side goes to 0, but $r - \gamma < 0$, a contradiction. Notice that we argue this case but does not involve the information about $c'''(\lambda)$, because it might not exist due to the singularity of second-order term in Equation (6). Then we show that $c''(h) < 0$ for all $h \in [0, \overline{h})$. If it is not true, for $h = \overline{h} \neq \eta$, $\frac{1}{2}\left(\eta - \overline{h}\right)^2 \widehat{\sigma}^2 c'''\left(\overline{h}\right) = \gamma - r > 0$ implies that $c''\left(\overline{h} - \omega\right) < 0$ for some small $\omega > 0$. Define $x := \sup\left\{h \in \left[0, \overline{h}\right) : c''(h) \geq 0\right\}$. The continuity implies $c''(x) = 0$ and $c''(h) < 0$ for $h \in \left(x, \overline{h}\right)$. We have $c(x) = \frac{1}{r-\overline{\mu}} + \frac{\gamma-\overline{\mu}}{r-\overline{\mu}}xc'(x)$. Since $c(x) < \frac{1}{r-\overline{\mu}}$, $\frac{1}{2}\left(\eta - \overline{h}\right)^2 \widehat{\sigma}^2 c'''\left(\overline{h}\right) = (r - \gamma)\, c'(x) > 0$, which implies that $c''(x + \omega) > 0$, a contradiction. Therefore, $c(h)$ is strictly concave on $[0, h]$. Now the strict concavity implies that

$$c(\eta) < c\left(\overline{h}\right) - \left(\eta - \overline{h}\right) = \frac{1}{r - \overline{\mu}} - \frac{\gamma - \overline{\mu}}{r - \overline{\mu}}\overline{h} - \eta < \frac{1}{r - \overline{\mu}} - \frac{\gamma - \overline{\mu}}{r - \overline{\mu}}\overline{h}.$$

But we know that $c(\eta) \geq \frac{1}{r-\overline{\mu}} - \frac{\gamma-\overline{\mu}}{r-\overline{\mu}}\overline{h}$, because it can be achieved by granting $\alpha^* = (r - \overline{\mu})\,\eta$ shares of stock and the agent is working forever. Therefore, we have $h < \eta$.

Now we show uniqueness. Take $\overline{h} \in [0,\ \eta)$; we use initial conditions $c\left(\overline{h}\right) = \frac{1}{r-\overline{\mu}} - \frac{\gamma-\overline{\mu}}{r-\overline{\mu}}\overline{h}$ and $c'\left(\overline{h}\right) = -1$, and $c(\cdot)$ is unique on $\left[0, \overline{h}\right]$, and the solution $c\left(\cdot; \overline{h}\right)$ is continuous in $\overline{h}$. We want to show that $c\left(0; \overline{h}\right)$ is strictly increasing in $\overline{h}$. Suppose that $c(\cdot; h_1)$ and $c(\cdot; h_2)$ solve Equation (6) while taking $\overline{h}_1 < \overline{h}_2$ as upper boundaries respectively, and define $f(h) = c\left(h; \overline{h}_2\right) - c\left(h; \overline{h}_1\right)$ on $\left[0, \overline{h}_1\right]$. We have $f\left(\overline{h}_1\right) < 0$ and $f'\left(\overline{h}_1\right) > 0$. According to the proof process of concaveness of the solution to (6), $f(h) < 0$ for $h \in \left[0, \overline{h}_1\right]$, which implies that $f(0) < 0$. Therefore, $c\left(0; \overline{h}\right)$ is increasing in $\overline{h}$, and thus there is a unique $\overline{h}$ such that, $c\left(0; \overline{h}\right) = L$. ∎

As shown in Figure 1, when $t = 0$, the investors start to hire an agent and promise a continuation return $V_0 = h^* Y_0$ in the future. The agent's continuation payoff value is $dV_t = \gamma V_t dt + \eta\left(dY_t - \overline{\mu} Y_t dt\right)$. When V_t exceeds $\overline{h} Y_t$, the investors begin to pay the agent's cash to maintain continuation value payment level $\overline{h}Y_t$. When V_t hits zero at time τ, investors fire the agent and liquidate the firm. In addition, it is possible that h drops to zero if the impatient agent's future performance is poor. In order to motivate the agent to improve her performance and continue to work hard, the investors should pay the agent cash payment as early as possible, or reduce the agent's cash payment threshold.

In Figure 6, we compare the optimal contract with He[1] in case of $\gamma > r$. Inflation risk induces short-sighted behavior of agent, and the impatience agent has a more urgent need for cash compensation.

4.2 When the Agent is as Patient as the Investors

When the agent and the investors have the same patience level, there is no need to consider the relevant cash payment, see He[1], and the marginal cost of delaying the cash wage payment is zero. As shown in Figure 2, $\overline{h} = \eta$ is the optimal payment boundary, which is higher than the one obtained when $\gamma > r$ (see Figure 1), and there will be no further chance of liquidation once h attains η.

Proposition 4.2 *When $\gamma = r$, without loss of generality, we have $\overline{h} = \eta$. There exists a unique solution $c(\cdot)$ to Equation (6) with boundary conditions (7)–(9), and the solution is strictly concave on $[0, \eta]$.*

Proof Suppose $\overline{h} > \eta$. We have $c'\left(\overline{h}\right) = -1$, and $c''\left(\overline{h}\right) = 0$, the only solution to equation (6) on $(\eta\ , \overline{h}]$ is $f(h) = \frac{1}{r-\overline{\mu}} - h$. If $\overline{h} < \eta$, then $c'\left(\overline{h}\right) = -1$ and $c''\left(\overline{h}\right) = 0$ imply that, on $\left[0, \overline{h}\right]$ the solution is uniquely determined as $c(h) = f(h) = \frac{1}{r-\overline{\mu}} - h$, then $c(0) = \frac{1}{r-\overline{\mu}}$, contradicting with (9). Therefore $\overline{h} = \eta$. If $c''(\cdot) \geq 0$ for some point on $[0, \eta)$, then we can choose the closest one to η (call it $x < \eta$), with $c''(x) = 0$ and $c'(x) > -1$. But it directly means $c(h) > f(h)$, a contradiction. The existence and uniqueness proof is similar to the one of Proposition 4.1, and will be omitted. ∎

If V_t falls to zero, then investors liquidate the firm and fire the agent. However, once good fortune V_t drives to attain ηY_t, the agent receives cash payment $d\overline{U}_s = \eta(r - \overline{\mu}) Y_s ds$, and as an optimal state, her continuation payoff V_t stays at ηY_t forever. In other words, the agent's optimal scaled continuation payment is $h_t = \eta$, and lowest incentives to motivate the agent to work hard forever is $\eta(r - \overline{\mu})$ shares.

4.3 Economic Explanation of Optimal Contract under Inflation

This subsection provides an intuitive economic explanation for the optimal contract. We discuss the incentive strategy of optimal contract. In order to simplify the analysis, we will discuss the situation of patient agent in the inflation environment. The situation of the impatient agent can be discussed similarly.

Due to the time-varying firm's real cash flow in the framework of this paper, the agent's continuation value generates a part of the incentive. Suppose that at time t, investors decide to reward the agent with shares according to her continuation payoff and the agent values α fraction of the firm as $V_t = \alpha Y_t/(r - \overline{\mu})$ (given that she is working all the time). Therefore, the agent is qualified to own $\alpha = (r - \overline{\mu})\, h$ shares in this firm.

According to Proposition 4.2, when the agent's continuation payoff $V_t = \eta Y_t$, it holds that the value of η is equivalent to $\eta\,(r-\overline{\mu})$ internal shares at time t. Once shares are $\alpha = (r-\overline{\mu})\,h$, and $\alpha \geq \alpha^* \equiv \eta\,(r-\overline{\mu})$, the incentive is high enough to motivate the agent to work forever and there is no possibility of firm liquidation.

For an agent who is as patient as an investor, we consider the scaled continuation payoff h_t as a performance-based incentive point in the optimal contract. At the beginning of the contract, the investors promise the agent with provision incentive shares $\alpha^* = \eta\,(r-\overline{\mu})$ based on the time-varying firm's real cash flow, once the agent's continuation value ratio reaches the optimal point at $h_t = \eta$. When $h_t < \eta$, as required by Proposition 3.2, the optimal contract $\overline{\pi}$ imposes additional incentives $(\eta - h_t)\,\widehat{\sigma} Y_t$ to motivate the agent.

5 Conclusion

In reality, firms and individual inevitably consider the impact of inflation on their investment portfolios. This paper investigates the optimal dynamic contract problem under the inflation risk. The agent's hidden action affects the drift component of the firm's real cash flow. But the agent's performance is influenced by inflation risk. Thus we first derive the firm's real cash process under inflation risk by using Itô formula. Next, we get the corresponding investors' scaled value function by solving the second-order ordinary differential equation, and derive the optimal contract under inflation risk. Finally, comparing with the results of He[1], we attempt to explain the reality economic implications of optimal contract from inflation risk by the numerical analysis.

To make the discussion on the model more formal and comprehensive, we obtain the optimal contract under inflation and discuss three situations by choosing different correlation coefficients $\rho = -1, 0, 1$. The common ground is that inflation risk, which usually reduces the real value of both parties, changes the scaled optimal payoff and the investor scaled value by comparing the results of He[1]. When $\rho = 1$ and $\rho = 0$, our results confirms that inflation risk reduces the real value of firm's flow. When $\rho = -1$, it seemly improve the value of investors and agents.

In the study of agency problems in continuous time, this paper formulates a strategy that considers the dynamic contract with inflation risk. Starting from perspective of the uncertainty of the asset model, we can investigate the principal-agent dynamic contract problem with Knightian uncertainty in continuous time in the future.

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Appendix A

The following Lemma A.1 gives the proof of the concaveness of the solution to (6).

Lemma A.1 *Suppose $f(\cdot) \in C^2\left[0, \overline{h}\right]$ where $\overline{h} \leq \eta$, and it satisfies*

$$(r-\overline{\mu}) f(h) = 1 + (\gamma - \overline{\mu}) h f'(h) + \frac{1}{2}(\eta - h)^2 \widehat{\sigma}^2 f''(h).$$

Thus we have the following results:

(i) *For $h_1 \in (0, \eta)$, if $f(h_1) < 0$ and $f'(h_1) \geq 0$, then $f(h) < 0$, $f'(h) > 0$ and $f''(h) < 0$, for $h \in [0, h_1)$.*

(ii) *If $0 \leq h_1 \leq h_2 \leq \eta$, and $f(h_1) = f(h_2) = 0$, then $f(h) = 0$ for all $h \in [0, \eta]$.*

(iii) *If $0 \leq h_1 \leq h_2 \leq \eta$, and $f(h_1) < 0$ but $f(h_2) = 0$, then $f(h) < 0$, $f'(h) > 0$ and $f''(h) < 0$ for $h \in [0, h_2)$.*

Proof (i) First let us show $f'(h) > 0$ for $h \in (0, h_1)$. Note that $f'(h_1 - \varepsilon) > 0$ for some small $\varepsilon > 0$ (because even if $f'(h_1) = 0$, $f''(h_1) = \frac{2(r-\overline{\mu})}{(\eta - h_1)^2 \widehat{\sigma}^2} f(h_1) < 0$). Suppose that $f' < 0$ for some points on $[0, h_1]$; then $x \equiv \sup\{h \in [0, h_1) : f'(h) < 0\} < h_1$ is well defined, and $f'(x) = 0$, $f(x) < 0$ and $f'(x + \varepsilon) > 0$ for some small $\varepsilon > 0$. Thus, x is the local minimum points closest (from left) to h_1. But then $\frac{1}{2}(\eta - x)^2 \widehat{\sigma}^2 f''(x) = (r - \overline{\mu}) f(x) < 0$, contradicting with $f'(x + \varepsilon) > 0$. Therefore, f is increasing on $[0, h_1)$, which implies that $f(h) < 0$ for $h \in [0, h_1]$. Finally, suppose that $f'' \geq 0$ for some h, then define $y \equiv \sup\{h \in [0, h_1) : f''(h) \geq 0\}$, and $f''(y) = 0$. If $y = 0$, then $f(0) = 0$, a contradiction; if $y > 0$, then $f'(y) = \frac{(r-\overline{\mu}) f(y)}{(\gamma - \overline{\mu}) y} < 0$, a contradiction.

(ii) It is sufficient to consider the case $0 < h_1 < h_2 < \eta$. Without loss of generality, suppose there exists $x \in (h_1, h_2)$ such that $f(x) < 0$, and let $y \equiv \inf\{h \in [x, h_2] : f(h) \geq 0\}$. According to the intermediate value theorem, there exists $z \in (x, y)$ such that $f(z) < 0$ and $f'(z) > 0$. Result (i) implies that $f(h_1) < 0$, a contradiction. Therefore, we have $f(h) = 0$ for $h \in [h_1, h_2]$. Furthermore, on $[0, h_1]$ given the initial condition $f(h_1) = 0$ and $f'(h_1) = 0$, the solution $f = 0$ is unique. Similarly, for $h \in \left[h_2, \eta - \frac{1}{n}\right]$, we have $f = 0$ for $n = 1, 2, \cdots$. Invoking continuity, we have $f(\eta) = 0$.

(iii) Similar to arguments in (ii) and recalling the result in (i), we show that $f(h) < 0$ for all $h \in (h_1, h_2)$. Again, the intermediate value theorem shows that there exists $x \in (0, \eta)$ such that $f(x) < 0$ and $f'(x) > 0$, delivering our claim by the result in (i). ∎

The Hamilton-Jacobi-Bellman equation for $b(Y, V)$

$b(Y_t, V_t)$ satisfies the following Hamilton-Jacobi-Bellman equation:

$$rbdt = \sup_{dU_t \geq 0} \left\{ Y dt - d\overline{U}_t + b_1 \overline{\mu} Y dt + b_2 \left(\gamma V dt - d\overline{U}_t\right) \right.$$
$$\left. + \frac{1}{2}\left(\widehat{\sigma}^2 Y^2 b_{11} + 2\eta \widehat{\sigma}^2 Y^2 b_{12} + \eta^2 \widehat{\sigma}^2 Y^2 b_{22}\right) dt \right\},$$

where b_i and b_{ij} denote the first-order and second-order partial derivatives, respectively, and $\widehat{\sigma}^2 = \sigma^2 - 2\rho\sigma\varsigma + \varsigma^2$. Immediately we see that the optimal wage policy satisfies $d\overline{U}_t = 0$ when $b_2 > -1$. The optimality equation is derived by utilizing $b(Y, V) = Yc(h)$, where $h = V/Y$, hence $b_2 = c'(h)$, $b_1 = c(h) - hc'(h)$, and $Yb_{11} = -Yhb_{12} = Yh^2 b_{22} = h^2 c''(h)$. ∎