# Several Classes of Negabent Functions over Finite Fields 

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#### Abstract

Negabent functions as a class of generalized bent functions have attracted a lot of attention recently due to their applications in cryptography and coding theory. In this paper, we consider the constructions of negabent functions over finite fields. First, by using the compositional inverses of certain binomial and trinomial permutations, we present several classes of negabent functions of the form $f(x)=\operatorname{Tr}_{1}^{n}\left(\lambda x^{2^{k}+1}\right)+\operatorname{Tr}_{1}^{n}(u x) \operatorname{Tr}_{1}^{n}(v x)$, where $\lambda \in \mathbb{F}_{2^{n}}, 2 \leq k \leq n-1,(u, v) \in \mathbb{F}_{2^{n}}^{*} \times \mathbb{F}_{2^{n}}^{*}$, and $\operatorname{Tr}_{1}^{n}(\cdot)$ is the trace function from $\mathbb{F}_{2^{n}}$ to $\mathbb{F}_{2}$. Second, by using Kloosterman sum, we prove that the condition for the cubic monomials given by Zhou and Qu (Cryptogr. Commun., to appear, DOI $10.1007 / \mathrm{s} 12095-015-0167-0$.) to be negabent is also necessary. In addition, a conjecture on negabent monomials whose exponents are of Niho type is given.


Index Terms Finite field, Negabent function, Nega-Hadamard transform, Kloosterman sum, Niho exponent.

## 1 Introduction

Bent functions are an important class of Boolean functions which were introduced by Rothaus [11. A Boolean function is called bent if and only if it has a flat spectrum with respect to the Walsh-Hadamard transform. Bent functions have attracted a lot of attention due to their applications in coding theory and cryptography. As a logical extension of bent functions, Kumar, Scholtz, and Welch [5] gave the definition of $p$-ary bent functions from $\mathbb{Z}_{p}^{n}$ to $\mathbb{Z}_{p}$, where $p$ is an integer. Schmidt [12] introduced the generalized Boolean bent functions from $\mathbb{Z}_{2}^{m}$ to $\mathbb{Z}_{p}$ from the viewpoint of cyclic codes over Galois ring.

Motivated by a choice of local unitary transforms that are central to the structural analysis of pure $n$-qubit stabilizer quantum states, Riera and Parker [10] introduced some generalized bent criteria for Boolean functions. They considered Boolean functions that have a flat spectrum with respect to one or

[^0]more matrix transforms from the $\{I, H, N\}^{n}$ set of matrices or subsets thereof, where $I=\left(\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right)$, $H=\frac{1}{\sqrt{2}}\left(\begin{array}{rr}1 & 1 \\ 1 & -1\end{array}\right)$, and $N=\frac{1}{\sqrt{2}}\left(\begin{array}{rr}1 & \sqrt{-1} \\ 1 & -\sqrt{-1}\end{array}\right)$. A $2^{n} \times 2^{n}$ transform matrix, $U$, is in the set $\{I, H, N\}^{n}$ if it can be written as $U=U_{0} \otimes U_{1} \otimes \ldots \otimes U_{n-1}=\bigotimes_{j=0}^{n-1} U_{j}$, where $U_{j} \in\{I, H, N\}$ and $\otimes$ is the tensor product. Thus $\{I, H, N\}^{n}$ is a set of $3^{n}$ transform matrices. A negabent function is a Boolean function which has flat spectrum with respect to the negaHadamard, $N^{\otimes n}$, transform. Bent-negabent functions are Boolean functions that are both bent and negabent. In 2007, Parker and Pott [8] gave an important connection between bent and negabent functions, and showed that if $n$ is even, then one can obtain negabent functions from any bent ones. By using this connection, Stănicǎ [14] gave a class of $n$-variable bent-negabent functions with algebraic degree $\frac{n}{4}+1$. Su, Pott, and Tang 17 considered the negaHadamard spectra of negabent functions, and constructed a class of bent-negabent functions with optimal algebraic degree by using complete permutation polynomials. Recently, Zhang, Wei, and Pasalic [18 used the indirect sum construction proposed by Carlet 2 to construct the first class of bent-negabent functions which are not in the completed Maiorana-McFarland class. On the other hand, it is also important to construct negabent functions over finite fields. Sarkar [15] considered negabent functions over finite fields, and characterized all the quadratic negabent monomials over finite fields. Recently, Zhou and Qu 19 gave a class of cubic monomial negabent functions and a class of cubic negabent polynomials over finite fields.

In this paper, we first give the necessary and sufficient conditions for the functions $\operatorname{Tr}_{1}^{k}\left(\lambda x^{2^{k}+1}\right)+$ $\operatorname{Tr}_{1}^{n}(u x) \operatorname{Tr}_{1}^{n}(v x)$ to be negabent, where $n=2 k, \lambda \in \mathbb{F}_{2^{k}}$, and $(u, v) \in \mathbb{F}_{2^{n}}^{*} \times \mathbb{F}_{2^{n}}^{*}$. Then by using some permutation trinomials over $\mathbb{F}_{2^{n}}$, we present some classes of negabent functions of the form $\operatorname{Tr}_{1}^{n}\left(\lambda x^{2^{k}+1}\right)+\operatorname{Tr}_{1}^{n}(u x) \operatorname{Tr}_{1}^{n}(v x)$, where $0<k<n$. Third, we show that the condition for the cubic monomials given by Zhou and $\mathrm{Qu}[19$ to be negabent is also necessary. Kloosterman sum plays an important role in the proof. In addition, we present a conjecture on negabent monomials whose exponents are of Niho type.

The remainder of this paper is organized as follows. In Section 2 some preliminaries including Kloosterman sum and permutation polynomials over finite fields are introduced. In Section 3 by using the compositional inverses of some binomial and trinomial permutations, several classes of negabent functions of the form $\operatorname{Tr}_{1}^{n}\left(\lambda x^{2^{k}+1}\right)+\operatorname{Tr}_{1}^{n}(u x) \operatorname{Tr}_{1}^{n}(v x)$ are given. A class of negabent monomials over finite fields is considered in Section 4, and some concluding remarks are given in Section 5.

## 2 Preliminaries

A Boolean function $f(x)$ is a mapping from $\mathbb{F}_{2}^{n}$ to $\mathbb{F}_{2}$. The Walsh-Hadamard transform of a function $f(x)$ at $a \in \mathbb{F}_{2}^{n}$ is defined by

$$
W_{f}(a)=\sum_{x \in \mathbb{F}_{2}^{n}}(-1)^{f(x)+a \cdot x}
$$

where $a \cdot x$ is the standard inner product. If for any $a \in \mathbb{F}_{2}^{n},\left|W_{f}(a)\right|=2^{\frac{n}{2}}$, then $f(x)$ is called a bent function. It is known that an $n$-variable Boolean function $f(x)$ is bent if and only if $f(x)+f(x+a)$ is balanced for all nonzero $a \in \mathbb{F}_{2}^{n}$. In [10], Riera and Parker introduced the notion of negabent function. The negaHadamard transform of $f(x)$ at $a \in \mathbb{F}_{2}^{n}$ is defined by

$$
N_{f}(a)=\sum_{x \in \mathbb{F}_{2}^{n}}(-1)^{f(x)+a \cdot x \sqrt{-1}^{w t(x)},, \text {, }, \text {. }}
$$

where $w t(x)$ is the weight of the vector $x=\left(x_{0}, x_{1}, \cdots, x_{n-1}\right)$, i.e., $w t(x)=\#\left\{i \mid x_{i}=1, i \in \mathbb{Z}_{n}\right\}$. A function $f(x)$ is called a negabent function if $\left|N_{f}(a)\right|=2^{\frac{n}{2}}$ for all $a \in \mathbb{F}_{2}^{n}$. Similarly, a function $f(x)$ is negabent if and only if $f(x)+f(x+a)+a \cdot x$ is balanced for all nonzero $a \in \mathbb{F}_{2}^{n}$.

In this paper, we focus on negabent functions over finite fields. It is well known that the vector space $\mathbb{F}_{2}^{n}$ is homomorphic to the finite field $\mathbb{F}_{2^{n}}$. Let $k$ be an integer such that $k \mid n$. The trace function from $\mathbb{F}_{2^{n}}$ onto $\mathbb{F}_{2^{k}}$ is defined by

$$
\operatorname{Tr}_{k}^{n}(x)=\sum_{i=0}^{n / k-1} x^{2^{i k}}, x \in \mathbb{F}_{2^{n}}
$$

If $k=1$, we call $\operatorname{Tr}_{1}^{n}(x)$ the absolute trace function from $\mathbb{F}_{2^{n}}$ to $\mathbb{F}_{2}$. Let $\left\{\alpha_{1}, \alpha_{2}, \cdots, \alpha_{n}\right\}$ be a self dual basis of $\mathbb{F}_{2^{n}}$ over $\mathbb{F}_{2}$. Let $x=\sum_{i=1}^{n} x_{i} \alpha_{i}$ and $a=\sum_{i=1}^{n} a_{i} \alpha_{i}$, then $\operatorname{Tr}_{1}^{n}(a x)=\sum_{i=1}^{n} a_{i} x_{i}=a \cdot x$. Thus we have the following equivalent definition of negabent functions over finite fields, which was first introduced by Sarkar in [15].

Theorem 1 [15] Let $f(x)$ be a Boolean function from $\mathbb{F}_{2^{n}}$ to $\mathbb{F}_{2}$. Then $f(x)$ is negabent if and only if

$$
\sum_{x \in \mathbb{F}_{2^{n}}}(-1)^{f(x)+f(x+a)+\operatorname{Tr}_{1}^{n}(a x)}=0
$$

for all nonzero a in $\mathbb{F}_{2^{n}}$.
In what follows we present some results on certain exponential sums and permutation polynomials over finite fields, which will play an important role in our proofs.

Let $a, b \in \mathbb{F}_{2^{n}}$, the Kloosterman sum over $\mathbb{F}_{2^{n}}$ is defined by

$$
K_{n}(a, b)=\sum_{x \in \mathbb{F}_{2}^{*} n}(-1)^{\operatorname{Tr}_{1}^{n}\left(a x+b x^{-1}\right)}
$$

Lemma 1 [6, Theorem 5.45] If $a, b \in \mathbb{F}_{2^{n}}$ are not both zero, then the Kloosterman sum satisfies

$$
\left|K_{n}(a, b)\right| \leq 2 \sqrt{2^{n}}
$$

Lemma 2 Let $k$ be a positive integer and $q=2^{k}$. For any $b \in \mathbb{F}_{q}^{*}$ and $c \in \mathbb{F}_{q}^{*}$, define $A=\#\{x \in$ $\left.\mathbb{F}_{q}^{*} \mid \operatorname{Tr}_{1}^{k}(b x)=0, \operatorname{Tr}_{1}^{k}\left(c x^{-1}\right)=1\right\}$. Then $A>0$ if $k>2$.

Proof: Let $B=\#\left\{x \in \mathbb{F}_{q}^{*} \mid \operatorname{Tr}_{1}^{k}(b x)=1, \operatorname{Tr}_{1}^{k}\left(c x^{-1}\right)=0\right\}, C=\#\left\{x \in \mathbb{F}_{q}^{*} \mid \operatorname{Tr}_{1}^{k}(b x)=0, \operatorname{Tr}_{1}^{k}\left(c x^{-1}\right)=\right.$ $0\}$, and $D=\#\left\{x \in \mathbb{F}_{q}^{*} \mid \operatorname{Tr}_{1}^{k}(b x)=1, \operatorname{Tr}_{1}^{k}\left(c x^{-1}\right)=1\right\}$. Then it is readily to verify that $A+C=$ $2^{k-1}-1, B+D=2^{k-1}$ and $A+D=2^{k-1}$. This together with Lemma 1 i.e., $|A+B-C-D| \leq 2 \sqrt{q}$, leads to $\left|4 A-2^{k}+1\right| \leq 2 \sqrt{q}$, which implies that $A>0$ if $k>2$. This completes the proof.

A polynomial $f \in \mathbb{F}_{q}[x]$ is called a permutation polynomial if the associated polynomial mapping $f: c \mapsto f(c)$ from $\mathbb{F}_{q}$ to itself is a permutation of $\mathbb{F}_{q}[6]$.

Lemma 3 [6, p.118] Let $q$ be a prime power and $f(x)=\sum_{i=0}^{m-1} a_{i} x^{q^{i}} \in \mathbb{F}_{q}[x]$. Then $f(x)$ is a permutation polynomial over $\mathbb{F}_{q^{m}}$ if and only if $\operatorname{gcd}\left(\sum_{i=0}^{m-1} a_{i} x^{i}, x^{m}-1\right)=1$. Moreover, if $g(x)$ is the compositional inverse of $f(x)$, i.e., $f(g(x)) \equiv x \bmod \left(x^{q^{m}}-x\right)$, then $g(x)$ is a $q$-polynomial over $\mathbb{F}_{q}$.

Lemma 4 Let $k$ be a positive integer and $f(x)=x+x^{2^{k}}+x^{2^{2 k}}$, then $f(x)$ is a permutation polynomial over $\mathbb{F}_{2^{n}}$ if and only if $\operatorname{gcd}(n, 3 k)=\operatorname{gcd}(n, k)$. Further, let $g(x)$ be the compositional inverse of $f(x)$. Then $g(x)$ is a 2-polynomial over $\mathbb{F}_{2}$ and $\operatorname{Tr}_{1}^{n}(g(x))=\operatorname{Tr}_{1}^{n}(x)$.

Proof: According to Lemma 3, $f(x)$ is a permutation polynomial over $\mathbb{F}_{2^{n}}$ if and only if $\operatorname{gcd}\left(\frac{x^{3 k}-1}{x^{k}-1}, x^{n}-\right.$ $1)=1$. Note that $\operatorname{gcd}\left(\frac{x^{3 k}-1}{x^{k}-1}, x^{k}-1\right)=\operatorname{gcd}\left(3, x^{k}-1\right)=1$. This implies that $\operatorname{gcd}\left(x^{3 k}-1, x^{n}-1\right)=$ $\operatorname{gcd}\left(\frac{x^{3 k}-1}{x^{k}-1}, x^{n}-1\right) \cdot \operatorname{gcd}\left(x^{k}-1, x^{n}-1\right)$ which leads to $\operatorname{gcd}\left(\frac{x^{3 k}-1}{x^{k}-1}, x^{n}-1\right)=\frac{x^{\operatorname{gcd}(n, 3 k)}-1}{x^{\operatorname{gcd}(n, k)}-1}$. Thus, $f(x)$ is a permutation polynomial over $\mathbb{F}_{2^{n}}$ if and only if $\operatorname{gcd}(n, 3 k)=\operatorname{gcd}(n, k)$.

If $g(x)$ is the compositional inverse of $f(x)$, then we have $g(x)$ is a 2-polynomial over $\mathbb{F}_{2}$ due to Lemma 3. Moreover, we have $g(1)=1$ since $f(1)=1$, i.e., $g(x)$ has odd number of terms. This leads to $\operatorname{Tr}_{1}^{n}(g(x))=\operatorname{Tr}_{1}^{n}(x)$ since $g(x)$ is a 2-polynomial over $\mathbb{F}_{2}$. This completes the proof.

Lemma 5 Let $n=r k$ and $f(x)=\lambda x+x^{2^{k}}+\lambda x^{2^{2 k}}$, where $r, k$ are positive integers and $\lambda \in \mathbb{F}_{2^{k}}^{*}$. Then $f(x)$ is a permutation polynomial over $\mathbb{F}_{2^{n}}$ if and only if $\operatorname{gcd}\left(\lambda+x+\lambda x^{2}, x^{r}-1\right)=1$. Further, let $g(x)$ be the compositional inverse of $f(x)$. Then $g(x)$ is a $2^{k}$-polynomial over $\mathbb{F}_{2^{k}}$ and $\operatorname{Tr}_{1}^{n}(g(x))=\operatorname{Tr}_{1}^{n}(x)$.

Proof: Note that $f(x)$ is a $2^{k}$-polynomial over $\mathbb{F}_{2^{k}}$. Thus the first assert follows directly from Lemma 3. Further, by Lemma 3 we have that $g(x)$ is also a $2^{k}$-polynomial over $\mathbb{F}_{2^{k}}$ if $g(x)$ is the compositional inverse of $f(x)$. Suppose that $g(x)=\sum_{i=0}^{r-1} c_{i} x^{2^{k i}}$, where $c_{i} \in \mathbb{F}_{2^{k}}$. Then, we have $\operatorname{Tr}_{1}^{n}(g(x))=$ $\operatorname{Tr}_{1}^{k}\left(\operatorname{Tr}_{k}^{r k}(g(x))\right)=\operatorname{Tr}_{1}^{k}\left(\operatorname{Tr}_{k}^{r k}\left(\sum_{i=0}^{r-1} c_{i} x^{2^{k i}}\right)\right)=\operatorname{Tr}_{1}^{k}\left(\sum_{i=0}^{r-1} c_{i} \operatorname{Tr}_{k}^{r k}\left(x^{2^{k i}}\right)\right)=\operatorname{Tr}_{1}^{k}\left(g(1) \operatorname{Tr}_{k}^{r k}(x)\right)$. Then the result follows from the fact that $g(1)=1$ since $f(1)=1$. This completes the proof.

## 3 Some classes of negabent polynomials

In this section, by using some permutation polynomials over $\mathbb{F}_{2^{n}}$, we present several classes of negabent functions of the form $\operatorname{Tr}_{1}^{n}\left(\lambda x^{2^{k}+1}\right)+\operatorname{Tr}_{1}^{n}(u x) \operatorname{Tr}_{1}^{n}(v x)$ over $\mathbb{F}_{2^{n}}$, where $2 \leq k \leq n-1, \lambda \in \mathbb{F}_{2^{n}}$, and $(u, v) \in \mathbb{F}_{2^{n}}^{*} \times \mathbb{F}_{2^{n}}^{*}$.

Theorem 2 Let $n=2 k, \lambda \in \mathbb{F}_{2^{k}}$ and $(u, v) \in \mathbb{F}_{2^{n}}^{*} \times \mathbb{F}_{2^{n}}^{*}$. Then $f(x)=\operatorname{Tr}_{1}^{k}\left(\lambda x^{2^{k}+1}\right)+\operatorname{Tr}_{1}^{n}(u x) \operatorname{Tr}_{1}^{n}(v x)$ is negabent on $\mathbb{F}_{2^{n}}$ if and only if one of the following conditions is satisfied:

1. $\lambda \neq 1,\left(\operatorname{Tr}_{1}^{n}\left(\frac{u}{1+\lambda}\right), \operatorname{Tr}_{1}^{n}\left(\frac{\left(\lambda u^{2^{k}}+u\right) v}{1+\lambda^{2}}\right), \operatorname{Tr}_{1}^{n}\left(\frac{v}{1+\lambda}\right)\right) \in\{(0,0,0),(0,0,1),(1,0,0),(1,1,1)\}$;
2. $\lambda=1, k=2, u, v, u+v \notin \mathbb{F}_{2^{k}}$;
3. $\lambda=1, k=1, u \neq v$.

Proof: According to Theorem to complete this proof, it is sufficient to prove that $f(x)+f(x+a)+$ $\operatorname{Tr}_{1}^{n}(a x)$ is balanced for all nonzero $a \in \mathbb{F}_{2^{n}}$ if and only if $\lambda, u, v$ satisfy one of the conditions given in Theorem 2 A direct calculation gives

$$
\begin{aligned}
f(x)+f(x+a)+\operatorname{Tr}_{1}^{n}(a x)= & \operatorname{Tr}_{1}^{k}\left(\lambda\left(a^{2^{k}} x+a x^{2^{k}}\right)\right)+\operatorname{Tr}_{1}^{n}(u a) \operatorname{Tr}_{1}^{n}(v x)+\operatorname{Tr}_{1}^{n}(v a) \operatorname{Tr}_{1}^{n}(u x)+\operatorname{Tr}_{1}^{n}(a x) \\
& +\operatorname{Tr}_{1}^{k}\left(\lambda a^{2^{k}+1}\right)+\operatorname{Tr}_{1}^{n}(u a) \operatorname{Tr}_{1}^{n}(v a) \\
= & \operatorname{Tr}_{1}^{n}\left(\left(\lambda a^{2^{k}}+a\right) x\right)+\operatorname{Tr}_{1}^{n}\left(v \operatorname{Tr}_{1}^{n}(u a) x\right)+\operatorname{Tr}_{1}^{n}\left(u \operatorname{Tr}_{1}^{n}(v a) x\right) \\
& +\operatorname{Tr}_{1}^{k}\left(\lambda a^{2^{k}+1}\right)+\operatorname{Tr}_{1}^{n}(u a) \operatorname{Tr}_{1}^{n}(v a) .
\end{aligned}
$$

This implies that $f(x)+f(x+a)+\operatorname{Tr}_{1}^{n}(a x)$ is balanced if and only if $\lambda a^{2^{k}}+a+v \operatorname{Tr}_{1}^{n}(u a)+u \operatorname{Tr}_{1}^{n}(v a) \neq 0$.
Notice that $\lambda a^{2^{k}}+a$ is a $2^{k}$-polynomial and $\operatorname{gcd}\left(\lambda a^{k}+1, a^{2 k}+1\right)=\operatorname{gcd}\left(\lambda a^{k}+1,\left(a^{k}+1\right)^{2}\right)=\operatorname{gcd}(\lambda+$ $\left.1, a^{k}+1\right)=1$ only if $\lambda \neq 1$. This together with Lemma 3 shows that $\lambda a^{2^{k}}+a$ is permutation polynomial if $\lambda \neq 1$. Moreover, for any $\lambda \neq 1$ and $b \in \mathbb{F}_{2^{n}}$, if $\lambda a^{2^{k}}+a=b$, then one gets $\lambda a+a^{2^{k}}=b^{2^{k}}$ since $n=2 k$ and $\lambda \in \mathbb{F}_{2^{k}}$. These two identities lead to

$$
\begin{equation*}
a=\frac{b+\lambda b^{2^{k}}}{\lambda^{2}+1} \tag{1}
\end{equation*}
$$

which is the unique solution to $\lambda a^{2^{k}}+a=b$.
For simplicity, define $h(a)=\lambda a^{2^{k}}+a+v \operatorname{Tr}_{1}^{n}(u a)+u \operatorname{Tr}_{1}^{n}(v a)$. Then by (1), for $\lambda \neq 1$ we have

1) $\left(\operatorname{Tr}_{1}^{n}(u a), \operatorname{Tr}_{1}^{n}(v a)\right)=(0,0)$ : For this case, $h(a)=0$ has the only solution $a=0$.
2) $\left(\operatorname{Tr}_{1}^{n}(u a), \operatorname{Tr}_{1}^{n}(v a)\right)=(0,1):$ By (11), $a=\frac{u+\lambda u^{2^{k}}}{\lambda^{2}+1}$ is the unique solution to $\lambda a^{2^{k}}+a+u=0$. Note that $\operatorname{Tr}_{1}^{n}(u a)=\operatorname{Tr}_{1}^{n}\left(u \cdot \frac{\lambda u^{2^{k}}+u}{1+\lambda^{2}}\right)=\operatorname{Tr}_{1}^{n}\left(\frac{\lambda u^{2^{k}+1}}{1+\lambda^{2}}\right)+\operatorname{Tr}_{1}^{n}\left(\frac{u^{2}}{1+\lambda^{2}}\right)=\operatorname{Tr}_{1}^{n}\left(\frac{u}{1+\lambda}\right)$ since $n=2 k$ and $\frac{\lambda u^{2^{k}+1}}{1+\lambda^{2}} \in \mathbb{F}_{2^{k}}$. Thus, in this case $h(a)=0$ has the only solution $a=\frac{u+\lambda u^{2}}{\lambda^{2}+1}$ if and only if $\operatorname{Tr}_{1}^{n}\left(\frac{u}{1+\lambda}\right)=0$ and $\operatorname{Tr}_{1}^{n}(v a)=\operatorname{Tr}_{1}^{n}\left(v \cdot \frac{\lambda u^{2^{k}}+u}{1+\lambda^{2}}\right)=1$.
3) $\left(\operatorname{Tr}_{1}^{n}(u a), \operatorname{Tr}_{1}^{n}(v a)\right)=(1,0)$ : Similar as above, for this case $h(a)=0$ has the only solution $a=$ $\frac{v+\lambda v^{2^{k}}}{\lambda^{2}+1}$ if and only if $\operatorname{Tr}_{1}^{n}\left(\frac{v}{1+\lambda}\right)=0$ and $\operatorname{Tr}_{1}^{n}(u a)=\operatorname{Tr}_{1}^{n}\left(u \cdot \frac{\lambda 2^{2^{k}}+v}{1+\lambda^{2}}\right)=1$.
4) $\left(\operatorname{Tr}_{1}^{n}(u a), \operatorname{Tr}_{1}^{n}(v a)\right)=(1,1)$ : In this case, $a=\frac{u+v+\lambda(u+v)^{2^{k}}}{\lambda^{2}+1}$ is the unique solution to $\lambda a^{2^{k}}+a+u+$ $v=0$ due to (11). By the same techniques used in Cases 2) and 3) one can conclude that $h(a)=0$ has the only solution if and only if $\operatorname{Tr}_{1}^{n}\left(\frac{u}{1+\lambda}+\frac{\left(\lambda v^{2^{k}}+v\right) u}{1+\lambda^{2}}\right)=1$ and $\operatorname{Tr}_{1}^{n}\left(\frac{v}{1+\lambda}+\frac{\left(\lambda u^{2^{k}}+u\right) v}{1+\lambda^{2}}\right)=1$.

Notice that $\operatorname{Tr}_{1}^{n}\left(\frac{\left(\lambda v^{2^{k}}+v\right) u}{1+\lambda^{2}}\right)=\operatorname{Tr}_{1}^{n}\left(\frac{\left(\lambda v u^{2^{k}} 2^{k}\right.}{\left(1+\lambda^{2}\right)^{2^{k}}}\right)+\operatorname{Tr}_{1}^{n}\left(\frac{v u}{1+\lambda^{2}}\right)=\operatorname{Tr}_{1}^{n}\left(\frac{\lambda v u^{2}}{1+\lambda^{2}}\right)+\operatorname{Tr}_{1}^{n}\left(\frac{v u}{1+\lambda^{2}}\right)=\operatorname{Tr}_{1}^{n}\left(\frac{\left(\lambda u^{2^{k}}+u\right) v}{1+\lambda^{2}}\right)$ due to $n=2 k$ and $\lambda \in \mathbb{F}_{2^{k}}$. Therefore, if $\lambda \neq 1$, by combining Cases 1)-4), one has that $h(a)=$ $\lambda a^{2^{k}}+a+v \operatorname{Tr}_{1}^{n}(u a)+u \operatorname{Tr}_{1}^{n}(v a) \neq 0$ for any nonzero $a \in \mathbb{F}_{2^{n}}$ if and only if the first condition in Theorem 2 is satisfied.

Now we consider the case of $\lambda=1$. First we discuss the number of solutions of $h(a)=\lambda a^{2^{k}}+a+$ $v \operatorname{Tr}_{1}^{n}(u a)+u \operatorname{Tr}_{1}^{n}(v a)$ under the condition $\left(\operatorname{Tr}_{1}^{n}(u a), \operatorname{Tr}_{1}^{n}(v a)\right)=(0,0)$. In this case, $h(a)=0$ is equivalent to $a \in \mathbb{F}_{2^{k}}$. Let $N(u, v)$ denote the number of nonzero $a \in \mathbb{F}_{2^{k}}$ such that $\left(\operatorname{Tr}_{1}^{n}(u a), \operatorname{Tr}_{1}^{n}(v a)\right)=(0,0)$, where $(u, v) \in \mathbb{F}_{2^{n}}^{*} \times \mathbb{F}_{2^{n}}^{*}$. Then, according to the balanced property of the trace function and the fact that $\left(\operatorname{Tr}_{1}^{n}(u a), \operatorname{Tr}_{1}^{n}(v a)\right)=\left(\operatorname{Tr}_{1}^{k}\left(a\left(u+u^{2^{k}}\right)\right), \operatorname{Tr}_{1}^{k}\left(a\left(v+v^{2^{k}}\right)\right)\right)$, it can be readily verified that $N(u, v)=2^{k}-1$ if $u, v \in \mathbb{F}_{2^{k}}, N(u, v)=2^{k-1}-1$ if exactly one of $u, v$ belongs to $\mathbb{F}_{2^{k}}, N(u, v)=2^{k-1}-1$ if $u, v \notin \mathbb{F}_{2^{k}}$ with $u+v \in \mathbb{F}_{2^{k}}$ and $N(u, v)=2^{k-2}-1$ if $u, v, u+v \notin \mathbb{F}_{2^{k}}$ respectively. This implies that $h(a)=0$ under the condition $\left(\operatorname{Tr}_{1}^{n}(u a), \operatorname{Tr}_{1}^{n}(v a)\right)=(0,0)$ has at least one nonzero solution for any given $u, v \in \mathbb{F}_{2^{n}}$ if $k>2$, i.e., $f(x)$ cannot be negabent if $\lambda=1$ and $k>2$. The conditions on $u, v \in \mathbb{F}_{2^{n}}$ such that $f(x)$ is negabent for $k=1,2$ can be easily verified based on a simple discussion. This completes the proof.

Remark 1 Let $u=v$ in Theorem 圆, then $f(x)$ is negabent on $\mathbb{F}_{2^{n}}$ if and only if $\lambda \neq 1$, which is Proposition 5 in [16].

Corollary 1 Let $f(x)$ with $u \neq v$ be given as in Theorem $⿴ 囗 ⿱ 一 𧰨 刂$ and $\mathbb{N}_{\lambda}$ denote the number of ordered pairs $(u, v)$ such that $f(x)$ is negabent．Then $\mathbb{N}_{\lambda}=\left(2^{n-1}-2\right)\left(2^{n}-1\right)$ for any fixed $\lambda \neq 1$ and $\mathbb{N}_{1}=6,96$ for $k=1,2$ respectively．

Proof：We only give the proof for $\lambda \neq 1$ since the proof for $\lambda=1$ is trivial due to Theorem2，For $\lambda \neq 1$ ， we first determine the number of ordered pairs $(u, v)$ such that $\left(\operatorname{Tr}_{1}^{n}\left(\frac{u}{1+\lambda}\right), \operatorname{Tr}_{1}^{n}\left(\frac{\left(\lambda u^{2^{k}}+u\right) v}{1+\lambda^{2}}\right), \operatorname{Tr}_{1}^{n}\left(\frac{v}{1+\lambda}\right)\right) \in$ $\{(0,0,0),(0,0,1)\}$ ．Note that $\left(\operatorname{Tr}_{1}^{n}\left(\frac{u}{1+\lambda}\right), \operatorname{Tr}_{1}^{n}\left(\frac{\left(\lambda u^{2^{k}}+u\right) v}{1+\lambda^{2}}\right), \operatorname{Tr}_{1}^{n}\left(\frac{v}{1+\lambda}\right)\right) \in\{(0,0,0),(0,0,1)\}$ is equivalent to $\left(\operatorname{Tr}_{1}^{n}\left(\frac{u}{1+\lambda}\right), \operatorname{Tr}_{1}^{n}\left(\frac{\left(\lambda u^{2^{k}}+u\right) v}{1+\lambda^{2}}\right)\right)=(0,0)$ ．Clearly，the number of $u \in \mathbb{F}_{2^{n}}^{*}$ satisfying $\operatorname{Tr}_{1}^{n}\left(\frac{u}{1+\lambda}\right)=0$ is $2^{n-1}-1$ ，and for each such $u$ ，there are $2^{n-1}-2 v$＇s in $\mathbb{F}_{2^{n}}^{*} \backslash\{u\}$ such that $\operatorname{Tr}_{1}^{n}\left(\frac{\left(\lambda u^{2^{k}}+u\right) v}{1+\lambda^{2}}\right)=0$ ．Thus， in this case we get $\left(2^{n-1}-1\right)\left(2^{n-1}-2\right)$ ordered pairs $(u, v)$ such that $f(x)$ is negabent．

Next we count the number of the pairs $(u, v)$ such that $\left(\operatorname{Tr}_{1}^{n}\left(\frac{u}{1+\lambda}\right), \operatorname{Tr}_{1}^{n}\left(\frac{\left(\lambda u^{2^{k}}+u\right) v}{1+\lambda^{2}}\right), \operatorname{Tr}_{1}^{n}\left(\frac{v}{1+\lambda}\right)\right) \in$ $\{(1,0,0),(1,1,1)\}$ ，which is equivalent to counting the number of the pairs $(u, v)$ satisfying $\operatorname{Tr}_{1}^{n}\left(\frac{u}{1+\lambda}\right)=1$ and $\operatorname{Tr}_{1}^{n}\left(\frac{\left(\lambda u^{2^{k}}+u\right) v}{1+\lambda^{2}}\right)+\operatorname{Tr}_{1}^{n}\left(\frac{v}{1+\lambda}\right)=\operatorname{Tr}_{1}^{n}\left(\frac{\left(\lambda u^{2^{k}}+u+1+\lambda\right) v}{1+\lambda^{2}}\right)=0$ ．Similar as above，for this case the number of $u \in \mathbb{F}_{2^{n}}^{*}$ satisfying $\operatorname{Tr}_{1}^{n}\left(\frac{u}{1+\lambda}\right)=1$ is $2^{n-1}$ ，and for each such $u$ ，there are $2^{n-1}-2 v$＇s in $\mathbb{F}_{2^{n}}^{*} \backslash\{u\}$ such that $\operatorname{Tr}_{1}^{n}\left(\frac{\left(\lambda u^{2^{k}}+u+1+\lambda\right) v}{1+\lambda^{2}}\right)=0$ ，i．e．，we have $2^{n-1}\left(2^{n-1}-2\right)$ ordered pairs $(u, v)$ such that $f(x)$ is negabent．This completes the proof．

The function $f(x)$ in Theorem2 has been investigated recently by Mesnager［7］in order to construct new classes of bent functions．

Theorem 3 L7］Let $n=2 k, \lambda \in \mathbb{F}_{2^{k}}^{*}$ and $(u, v) \in \mathbb{F}_{2^{n}}^{*} \times \mathbb{F}_{2^{n}}^{*}$ ，then $f(x)=\operatorname{Tr}_{1}^{k}\left(\lambda x^{2^{k}+1}\right)+\operatorname{Tr}_{1}^{n}(u x) \operatorname{Tr}_{1}^{n}(v x)$ is bent if and only if $\operatorname{Tr}_{1}^{n}\left(\lambda^{-1} u^{2^{k}} v\right)=0$ ．

Combining Theorem 2 and Theorem 3 we have the following corollary．

Corollary 2 Let $n=2 k, \lambda \in \mathbb{F}_{2^{k}}^{*}$ and $(u, v) \in \mathbb{F}_{2^{n}}^{*} \times \mathbb{F}_{2^{n}}^{*}$ ．Then $f(x)=\operatorname{Tr}_{1}^{k}\left(\lambda x^{2^{k}+1}\right)+\operatorname{Tr}_{1}^{n}(u x) \operatorname{Tr}_{1}^{n}(v x)$ is bent－negabent on $\mathbb{F}_{2^{n}}$ if and only if one of the following conditions is satisfied：

1．$\lambda \neq 1, \quad\left(\operatorname{Tr}_{1}^{n}\left(\frac{u}{1+\lambda}\right), \operatorname{Tr}_{1}^{n}\left(\frac{\left(\lambda u^{2^{k}}+u\right) v}{1+\lambda^{2}}\right), \operatorname{Tr}_{1}^{n}\left(\lambda^{-1} u^{2^{k}} v\right)\right)=(0,0,0)$ or $\left(\operatorname{Tr}_{1}^{n}\left(\frac{u}{1+\lambda}\right), \operatorname{Tr}_{1}^{n}\left(\frac{\left(\lambda u^{2^{k}}+u+1+\lambda\right) v}{1+\lambda^{2}}\right)\right.$ ， $\left.\operatorname{Tr}_{1}^{n}\left(\lambda^{-1} u^{2^{k}} v\right)\right)=(1,0,0) ;$

2．$\lambda=1, k=2, u, v, u+v \notin \mathbb{F}_{2^{k}}$ and $\operatorname{Tr}_{1}^{n}\left(u^{2^{k}} v\right)=0$ ．
As a special case of Theorem 2 if $\lambda=0$ ，then it gives the necessary and sufficient conditions for $\operatorname{Tr}_{1}^{n}(u x) \operatorname{Tr}_{1}^{n}(v x)$ to be negabent on $\mathbb{F}_{2^{n}}$ for even $n$ ．In the following we consider the negabent property of $\operatorname{Tr}_{1}^{n}(u x) \operatorname{Tr}_{1}^{n}(v x)$ for both even and odd $n$ ．

Theorem 4 Let $f(x)=\operatorname{Tr}_{1}^{n}(u x) \operatorname{Tr}_{1}^{n}(v x)$, where $(u, v) \in \mathbb{F}_{2^{n}}^{*} \times \mathbb{F}_{2^{n}}^{*}$. Then $f(x)$ is negabent on $\mathbb{F}_{2^{n}}$ if and only if one of the following conditions is satisfied:

1. $\operatorname{Tr}_{1}^{n}(u)=0$ and $\operatorname{Tr}_{1}^{n}(u v)=0$;
2. $\operatorname{Tr}_{1}^{n}(u)=1$ and $\operatorname{Tr}_{1}^{n}((u+1) v)=0$.

Proof: According to Theorem it is sufficient to prove that

$$
f(x)+f(x+a)+\operatorname{Tr}_{1}^{n}(a x)=\operatorname{Tr}_{1}^{n}\left(\left(\operatorname{Tr}_{1}^{n}(v a) u+\operatorname{Tr}_{1}^{n}(u a) v+a\right) x\right)+\operatorname{Tr}_{1}^{n}(u a) \operatorname{Tr}_{1}^{n}(v a)
$$

is balanced for all nonzero $a \in \mathbb{F}_{2^{n}}$, which is equivalent to show that $\operatorname{Tr}_{1}^{n}(v a) u+\operatorname{Tr}_{1}^{n}(u a) v+a \neq 0$ for all nonzero $a$. Let $h(a)=\operatorname{Tr}_{1}^{n}(v a) u+\operatorname{Tr}_{1}^{n}(u a) v+a$, we have

1) $\left(\operatorname{Tr}_{1}^{n}(u a), \operatorname{Tr}_{1}^{n}(v a)\right)=(0,0)$ : For this case, $h(a)=0$ has the only solution $a=0$.
2) $\left(\operatorname{Tr}_{1}^{n}(u a), \operatorname{Tr}_{1}^{n}(v a)\right)=(0,1)$ : In this case, $h(a)=0$ has the only solution $a=u$ if and only if $\operatorname{Tr}_{1}^{n}(u)=0$ and $\operatorname{Tr}_{1}^{n}(u v)=1$.
3) $\left(\operatorname{Tr}_{1}^{n}(u a), \operatorname{Tr}_{1}^{n}(v a)\right)=(1,0)$ : Similar as above, for this case $h(a)=0$ has the only solution $a=v$ if and only if $\operatorname{Tr}_{1}^{n}(u v)=1$ and $\operatorname{Tr}_{1}^{n}(v)=0$.
4) $\left(\operatorname{Tr}_{1}^{n}(u a), \operatorname{Tr}_{1}^{n}(v a)\right)=(1,1)$ : In this case, $a=u+v$ is the only solution to $\operatorname{Tr}_{1}^{n}(v a) u+\operatorname{Tr}_{1}^{n}(u a) v+a=$ 0 if and only if $\operatorname{Tr}_{1}^{n}(u(u+v))=1$ and $\operatorname{Tr}_{1}^{n}(v(u+v))=1$.

Based on Cases 1)-4), it can be seen that $\operatorname{Tr}_{1}^{n}(v a) u+\operatorname{Tr}_{1}^{n}(u a) v+a \neq 0$ for all nonzero $a$ if and only if one of the two conditions in Theorem 4 is satisfied.

Remark 2 Theorem 4 shows that $\operatorname{Tr}_{1}^{n}(x) \operatorname{Tr}_{1}^{n}(v x)$ is negabent for any nonzero $v \in \mathbb{F}_{2^{n}}$ when $n$ is odd and $u=1$, which was given in Theorem 8 in [19]. Note that the negabent property is not preserved by linear transform, i.e., $f(x)$ is negabent on $\mathbb{F}_{2^{n}}$ does not imply that $f(a x)$ is negabent on $\mathbb{F}_{2^{n}}$ for all $a \in \mathbb{F}_{2^{n}}^{*}$ [13]. Thus, Theorem 4 is not a special case of Theorem 8 in [19].

Theorem 5 Let $n$ be an even integer and $k$ be a positive integer such that $\operatorname{gcd}(n, 3 k)=\operatorname{gcd}(n, k)$. Then $f(x)=\operatorname{Tr}_{1}^{n}\left(x^{2^{k}+1}\right)+\operatorname{Tr}_{1}^{n}(x) \operatorname{Tr}_{1}^{n}(v x)$ is negabent on $\mathbb{F}_{2^{n}}$ if $\operatorname{Tr}_{1}^{n}(v)=0$.

Proof: According to Theorem we only need to show that $f(x)+f(x+a)+\operatorname{Tr}_{1}^{n}(a x)$ is balanced for all nonzero $a \in \mathbb{F}_{2^{n}}$ if $\operatorname{Tr}_{1}^{n}(v)=0$. A direct calculation gives

$$
\begin{aligned}
f(x)+f(x+a)+\operatorname{Tr}_{1}^{n}(a x)= & \operatorname{Tr}_{1}^{n}\left(a^{2^{k}} x+a x^{2^{k}}\right)+\operatorname{Tr}_{1}^{n}(a) \operatorname{Tr}_{1}^{n}(v x)+\operatorname{Tr}_{1}^{n}(v a) \operatorname{Tr}_{1}^{n}(x)+\operatorname{Tr}_{1}^{n}(a x) \\
& +\operatorname{Tr}_{1}^{n}\left(a^{2^{k}+1}\right)+\operatorname{Tr}_{1}^{n}(a) \operatorname{Tr}_{1}^{n}(v a) \\
= & \operatorname{Tr}_{1}^{n}\left(\left(a^{2^{k}}+a^{2^{-k}}+a\right) x\right)+\operatorname{Tr}_{1}^{n}\left(\left(v \operatorname{Tr}_{1}^{n}(a)\right) x\right)+\operatorname{Tr}_{1}^{n}\left(\operatorname{Tr}_{1}^{n}(v a) x\right) \\
& +\operatorname{Tr}_{1}^{n}\left(a^{2^{k}+1}\right)+\operatorname{Tr}_{1}^{n}(a) \operatorname{Tr}_{1}^{n}(v a)
\end{aligned}
$$

This shows that $f(x)+f(x+a)+\operatorname{Tr}_{1}^{n}(a x)$ is balanced if and only if $a^{2^{k}}+a^{2^{-k}}+a+v \operatorname{Tr}_{1}^{n}(a)+\operatorname{Tr}_{1}^{n}(v a) \neq 0$, i.e., $a+a^{2^{k}}+a^{2^{2 k}}+v^{2^{k}} \operatorname{Tr}_{1}^{n}(a)+\operatorname{Tr}_{1}^{n}(v a) \neq 0$. Notice that $a+a^{2^{k}}+a^{2^{2 k}}$ is a permutation of $\mathbb{F}_{2^{n}}$ due to Lemma 4. Let $g(a)=a+a^{2^{k}}+a^{2^{2 k}}+v^{2^{k}} \operatorname{Tr}_{1}^{n}(a)+\operatorname{Tr}_{1}^{n}(v a)$ and $h(a)$ be the compositional inverse of $a+a^{2^{k}}+a^{2^{2 k}}$, then we have

1) $\left(\operatorname{Tr}_{1}^{n}(a), \operatorname{Tr}_{1}^{n}(v a)\right)=(0,0)$ : For this case, $g(a)=0$ has the only solution $a=0$.
2) $\left(\operatorname{Tr}_{1}^{n}(a), \operatorname{Tr}_{1}^{n}(v a)\right)=(0,1)$ : In this case, $g(a)=0$ means that $a+a^{2^{k}}+a^{2^{2 k}}=1$, i.e., $a=h(1)=1$. However, $\operatorname{Tr}_{1}^{n}(v a)=\operatorname{Tr}_{1}^{n}(v)=0$, which shows that $g(a)=0$ has no solution in this case.
3) $\left(\operatorname{Tr}_{1}^{n}(a), \operatorname{Tr}_{1}^{n}(v a)\right)=(1,0)$ : In this case, $g(a)=0$ is reduced to $a+a^{2^{k}}+a^{2^{2 k}}=v^{2^{k}}$, i.e., $a=h\left(v^{2^{k}}\right)$. However, by Lemma 4, $\operatorname{Tr}_{1}^{n}(a)=\operatorname{Tr}_{1}^{n}\left(h\left(v^{2^{k}}\right)\right)=\operatorname{Tr}_{1}^{n}\left(v^{2^{k}}\right)=\operatorname{Tr}_{1}^{n}(v)=0$. This shows that $g(a)=0$ has no solution in this case.
4) $\left(\operatorname{Tr}_{1}^{n}(a), \operatorname{Tr}_{1}^{n}(v a)\right)=(1,1)$ : Similar as above, $g(a)=0$ implies that $a+a^{2^{k}}+a^{2^{2 k}}=1+v^{2^{k}}$, i.e., $a=h\left(1+v^{2^{k}}\right)$. Note that $\operatorname{Tr}_{1}^{n}(1)=0$ since $n$ is even. From Lemma 4 . $\operatorname{Tr}_{1}^{n}(a)=\operatorname{Tr}_{1}^{n}\left(h\left(1+v^{2^{k}}\right)\right)=$ $\operatorname{Tr}_{1}^{n}\left(1+v^{2^{k}}\right)=\operatorname{Tr}_{1}^{n}(v)=0$, which shows that $g(a)=0$ has no solution in this case.

From the above Cases 1)-4), we can see that $a+a^{2^{k}}+a^{2^{2 k}}+v^{2^{k}} \operatorname{Tr}_{1}^{n}(a)+\operatorname{Tr}_{1}^{n}(v a) \neq 0$ for all nonzero $a \in \mathbb{F}_{2^{n}}$ if $\operatorname{Tr}_{1}^{n}(v)=0$. This completes the proof.

By the same techniques used in the proof of Theorem 5. we can derive the following result.
Theorem 6 Let $r$ and $k$ be two integers such that $r k$ is even. Let $n=r k, \lambda \in \mathbb{F}_{2^{k}}^{*}$ and $\operatorname{gcd}(\lambda+x+$ $\left.\lambda x^{2}, x^{r}-1\right)=1$. Then $f(x)=\operatorname{Tr}_{1}^{n}\left(\lambda x^{2^{k}+1}\right)+\operatorname{Tr}_{1}^{n}(x) \operatorname{Tr}_{1}^{n}(v x)$ is negabent on $\mathbb{F}_{2^{n}}$ if $\operatorname{Tr}_{1}^{n}(v)=0$.

Proof: According to Theorem 1 it is enough to prove that $f(x)+f(x+a)+\operatorname{Tr}_{1}^{n}(a x)$ is balanced for all nonzero $a \in \mathbb{F}_{2^{n}}$ for the $v \in \mathbb{F}_{2^{n}}$ satisfying $\operatorname{Tr}_{1}^{n}(v)=0$. Note that

$$
\begin{aligned}
f(x)+f(x+a)+\operatorname{Tr}_{1}^{n}(a x)= & \operatorname{Tr}_{1}^{n}\left(\lambda\left(a^{2^{k}} x+a x^{2^{k}}\right)\right)+\operatorname{Tr}_{1}^{n}(a) \operatorname{Tr}_{1}^{n}(v x)+\operatorname{Tr}_{1}^{n}(v a) \operatorname{Tr}_{1}^{n}(x)+\operatorname{Tr}_{1}^{n}(a x) \\
& +\operatorname{Tr}_{1}^{n}\left(\lambda a^{2^{k}+1}\right)+\operatorname{Tr}_{1}^{n}(a) \operatorname{Tr}_{1}^{n}(v a) \\
= & \operatorname{Tr}_{1}^{n}\left(\left(\lambda a^{2^{k}}+(\lambda a)^{2-k}+a\right) x\right)+\operatorname{Tr}_{1}^{n}\left(v \operatorname{Tr}_{1}^{n}(a) x\right)+\operatorname{Tr}_{1}^{n}\left(\operatorname{Tr}_{1}^{n}(v a) x\right) \\
& +\operatorname{Tr}_{1}^{n}\left(\lambda a^{2^{k}+1}\right)+\operatorname{Tr}_{1}^{n}(a) \operatorname{Tr}_{1}^{n}(v a) .
\end{aligned}
$$

Thus, $f(x)+f(x+a)+\operatorname{Tr}_{1}^{n}(a x)$ is balanced if and only if

$$
\begin{equation*}
\lambda a^{2^{k}}+(\lambda a)^{2^{-k}}+a+v \operatorname{Tr}_{1}^{n}(a)+\operatorname{Tr}_{1}^{n}(v a) \neq 0 \tag{2}
\end{equation*}
$$

Raising both sides of (21) to the $2^{k}$-th power, we get $\lambda a^{2^{2 k}}+\lambda a+a^{2^{k}}+v^{2^{k}} \operatorname{Tr}_{1}^{n}(a)+\operatorname{Tr}_{1}^{n}(v a) \neq 0$ due to $\lambda \in \mathbb{F}_{2^{k}}$. Let $g(a)=\lambda a+a^{2^{k}}+\lambda a^{2^{2 k}}+v^{2^{k}} \operatorname{Tr}_{1}^{n}(a)+\operatorname{Tr}_{1}^{n}(v a)$. According to Lemma 5, $\lambda a+a^{2^{k}}+\lambda a^{2^{2 k}}$ is a permutation of $\mathbb{F}_{2^{n}}$ since $\operatorname{gcd}\left(\lambda+x+\lambda x^{2}, x^{r}-1\right)=1$. Let $h(a)$ be the compositional inverse of $\lambda a+a^{2^{k}}+\lambda a^{2^{2 k}}$. Similar as in the proof of Theorem 5, we have

1) $\left(\operatorname{Tr}_{1}^{n}(a), \operatorname{Tr}_{1}^{n}(v a)\right)=(0,0)$ : For this case, $g(a)=0$ has the only solution $a=0$.
2) $\left(\operatorname{Tr}_{1}^{n}(a), \operatorname{Tr}_{1}^{n}(v a)\right)=(0,1)$ : In this case, $g(a)=0$ means that $\lambda a+a^{2^{k}}+\lambda a^{2^{2 k}}=1$, i.e., $a=h(1)=1$ since $\lambda \cdot 1+1^{2^{k}}+\lambda \cdot 1^{2^{2 k}}=1$. However, $\operatorname{Tr}_{1}^{n}(v a)=\operatorname{Tr}_{1}^{n}(v)=0$, which shows that $g(a)=0$ has no solution in this case.
3) $\left(\operatorname{Tr}_{1}^{n}(a), \operatorname{Tr}_{1}^{n}(v a)\right)=(1,0)$ : In this case, $g(a)=0$ means that $\lambda a+a^{2^{k}}+\lambda a^{2^{2 k}}=v^{2^{k}}$, i.e., $a=h\left(v^{2^{k}}\right)$. From Lemma [5. $\operatorname{Tr}_{1}^{n}(a)=\operatorname{Tr}_{1}^{n}\left(h\left(v^{2^{k}}\right)\right)=\operatorname{Tr}_{1}^{n}\left(v^{2^{k}}\right)=\operatorname{Tr}_{1}^{n}(v)=0$, which shows that $g(a)=0$ has no solution in this case.
4) $\left(\operatorname{Tr}_{1}^{n}(a), \operatorname{Tr}_{1}^{n}(v a)\right)=(1,1)$ : Similar as above, $g(a)=0$ implies that $\lambda a+a^{2^{k}}+\lambda a^{2^{2 k}}=1+v^{2^{k}}$, i.e., $a=h\left(1+v^{2^{k}}\right)$. Note that $\operatorname{Tr}_{1}^{n}(1)=0$ due to $n$ is even. Again by Lemma $5 \operatorname{Tr}_{1}^{n}(a)=$ $\operatorname{Tr}_{1}^{n}\left(h\left(1+v^{2^{k}}\right)\right)=\operatorname{Tr}_{1}^{n}\left(1+v^{2^{k}}\right)=\operatorname{Tr}_{1}^{n}(v)=0$. This implies that $g(a)=0$ has no solution in this case.

From the above Cases 1)-4), we can see that if $\operatorname{Tr}_{1}^{n}(v)=0$, then $\lambda a+a^{2^{k}}+\lambda a^{2^{2 k}}+v^{2^{k}} \operatorname{Tr}_{1}^{n}(a)+\operatorname{Tr}_{1}^{n}(v a) \neq 0$ for all nonzero $a \in \mathbb{F}_{2^{n}}$. This completes the proof.

Remark 3 Notice that if one takes $n=r k$ in Theorem 5 then Theorem 5 is a special case of Theorem (6) due to the fact that $\operatorname{gcd}\left(1+x+x^{2}, x^{r}-1\right)=1$ if and only if $\operatorname{gcd}(r k, 3 k)=\operatorname{gcd}(r k, k)$. For the values of $n, k$ with $\operatorname{gcd}(n, k) \neq k$, the results in Theorem 5 are not covered by Theorem 6 .

By Theorem 6 we can obtain the following results if we take $r=3,4,5$ respectively.
Corollary 3 Let $k$ be an even integer and $n=3 k$. Let $\lambda \in \mathbb{F}_{2^{k}} \backslash\{0,1\}$. Then $f(x)=\operatorname{Tr}_{1}^{n}\left(\lambda x^{2^{k}+1}\right)+$ $\operatorname{Tr}_{1}^{n}(x) \operatorname{Tr}_{1}^{n}(v x)$ is negabent on $\mathbb{F}_{2^{n}}$ if $\operatorname{Tr}_{1}^{n}(v)=0$.

Proof: According to Theorem 6, it is sufficient to show that $\operatorname{gcd}\left(\lambda+x+\lambda x^{2}, x^{3}-1\right)=1$ if $\lambda \neq 1$. Then result follows from the fact that $\operatorname{gcd}\left(\lambda+x+\lambda x^{2}, x^{3}-1\right)=\operatorname{gcd}\left(\lambda+x+\lambda x^{2}, x^{2}+x+1\right)=$ $\operatorname{gcd}\left(\lambda\left(x^{2}+x+1\right)+(\lambda+1) x, x^{2}+x+1\right)=\operatorname{gcd}\left((\lambda+1) x, x^{2}+x+1\right)$.

If $r=4$, then $\operatorname{gcd}\left(\lambda+x+\lambda x^{2}, x^{4}-1\right)=\operatorname{gcd}\left(\lambda+x+\lambda x^{2}, x-1\right)=1$ for any $\lambda \in \mathbb{F}_{2^{k}}^{*}$. Thus, we have
Corollary 4 Let $n=4 k$ and $\lambda \in \mathbb{F}_{2^{k}}^{*}$. Then $f(x)=\operatorname{Tr}_{1}^{n}\left(\lambda x^{2^{k}+1}\right)+\operatorname{Tr}_{1}^{n}(x) \operatorname{Tr}_{1}^{n}(v x)$ is negabent on $\mathbb{F}_{2^{n}}$ if $\operatorname{Tr}_{1}^{n}(v)=0$.

Corollary 5 Let $k$ be an even integer and $n=5 k$. Let $\lambda \in \mathbb{F}_{2^{k}} \backslash\left\{0, \omega, \omega^{2}\right\}$, where $\omega$ is a primitive element of $\mathbb{F}_{2^{2}}$. Then $f(x)=\operatorname{Tr}_{1}^{n}\left(\lambda x^{2^{k}+1}\right)+\operatorname{Tr}_{1}^{n}(x) \operatorname{Tr}_{1}^{n}(v x)$ is negabent on $\mathbb{F}_{2^{n}}$ if $\operatorname{Tr}_{1}^{n}(v)=0$.

Proof: According to Theorem 6. we need to determine the condition on $\lambda$ such that $\operatorname{gcd}(\lambda+x+$ $\left.\lambda x^{2}, x^{5}-1\right)=1$. Notice that $\operatorname{gcd}\left(\lambda+x+\lambda x^{2}, x^{5}-1\right)=\operatorname{gcd}\left(\lambda+x+\lambda x^{2}, x^{4}+x^{3}+x^{2}+x+1\right)$. By a simple calculation, we have $x^{4}+x^{3}+x^{2}+x+1=\left(1+\mu x+x^{2}\right)\left(\mu^{2}+\mu+(\mu+1) x+x^{2}\right)+\left(\mu^{2}+\mu+1\right)(\mu x+1)$, where $\mu=\lambda^{-1}$. This leads to $\operatorname{gcd}\left(\lambda+x+\lambda x^{2}, x^{4}+x^{3}+x^{2}+x+1\right)=\operatorname{gcd}\left(1+\mu x+x^{2}, x^{4}+x^{3}+x^{2}+x+1\right)=$ $\operatorname{gcd}\left(1+\mu x+x^{2},\left(\mu^{2}+\mu+1\right)(\mu x+1)\right)=1$ if and only of $\mu^{2}+\mu+1 \neq 0$. This completes the proof.

## 4 On a class of monomial negabent functions

In [19], Zhou and Qu showed that $\operatorname{Tr}_{1}^{2 k}\left(\lambda x^{d}\right)$ is negabent on $\mathbb{F}_{2^{2 k}}$ if $\lambda \in \mathbb{F}_{2}$, where $d=2^{k}+3$ and $k \geq 3$ is odd. In this section, we will show that $\lambda \in \mathbb{F}_{2}$ is also necessary for $\operatorname{Tr}_{1}^{2 k}\left(\lambda x^{d}\right)$ to be negabent.

Theorem 7 Let $n=2 k, q=2^{k}$ and $d=q+3$, where $k \geq 3$ is odd. Then $\operatorname{Tr}_{1}^{n}\left(\lambda x^{d}\right)$ is negabent on $\mathbb{F}_{2^{n}}$ if and only if $\lambda \in \mathbb{F}_{2}$.

Proof: Since $k$ is odd, then $f(x)=x^{2}+x+1$ is irreducible over $\mathbb{F}_{2^{k}}$ as it is irreducible over $\mathbb{F}_{2}$. Let $\omega$ be a root of $f(x)$. Then $\mathbb{F}_{2^{n}}=\mathbb{F}_{2^{k}}[\omega]$, i.e., each $x \in \mathbb{F}_{2^{n}}$ can be uniquely represented as $x_{0}+x_{1} \omega$, where $x_{i} \in \mathbb{F}_{2^{k}}$. Then

$$
\begin{equation*}
x^{d}=\left(x_{0}+x_{1} \omega\right)^{d}=x_{0}^{4}+x_{1}^{4}+x_{1} x_{0}^{3}+x_{0} x_{1}^{3}+\left(x_{0}^{2} x_{1}^{2}+x_{0} x_{1}^{3}+x_{1}^{4}\right) \omega \tag{3}
\end{equation*}
$$

and

$$
\begin{align*}
(x+a)^{d}= & \left(x_{0}+a_{0}\right)^{4}+\left(x_{1}+a_{1}\right)^{4}+\left(x_{1}+a_{1}\right)\left(x_{0}+a_{0}\right)^{3}+\left(x_{0}+a_{0}\right)\left(x_{1}+a_{1}\right)^{3} \\
& +\left(\left(x_{0}+a_{0}\right)^{2}\left(x_{1}+a_{1}\right)^{2}+\left(x_{0}+a_{0}\right)\left(x_{1}+a_{1}\right)^{3}+\left(x_{1}+a_{1}\right)^{4}\right) \omega \tag{4}
\end{align*}
$$

where $a=a_{0}+a_{1} \omega$.
Note that $\operatorname{Tr}_{k}^{2 k}(1)=0$ and $\operatorname{Tr}_{k}^{2 k}(\omega)=\omega+\omega^{2^{k}}=1$ since $k$ is odd and $\omega$ is a root of $x^{2}+x+1$. Let $\lambda=\lambda_{0}+\lambda_{1} \omega$. Then from (3), (4) and $\operatorname{Tr}_{k}^{2 k}(a x)=a_{0} x_{1}+a_{1} x_{0}+a_{1} x_{1}$, we have

$$
\begin{align*}
& \operatorname{Tr}_{k}^{2 k}\left(\lambda x^{d}+\lambda(x+a)^{d}+a x\right) \\
= & \lambda_{1} x_{0}^{2} a_{1}^{2}+\lambda_{0} x_{0} a_{1}^{3}+\lambda_{0} x_{0}^{2} a_{1}^{2}+\lambda_{1} a_{1} x_{0}^{3}+\lambda_{0} a_{0} x_{1}^{3}+a_{1} x_{0}+\lambda_{1} x_{1} x_{0} a_{0}^{2}+\lambda_{1} x_{1} x_{0}^{2} a_{0}+\lambda_{1} a_{1} x_{0} a_{0}^{2} \\
& +\lambda_{1} a_{1} x_{0}^{2} a_{0}+x_{1} a_{1}+\lambda_{0} a_{0} x_{1}^{2} a_{1}+\lambda_{0} x_{0} x_{1}^{2} a_{1}+\lambda_{0} x_{0} x_{1} a_{1}^{2}+\lambda_{0} a_{0} x_{1} a_{1}^{2}+\lambda_{0} a_{0} a_{1}^{3}+\lambda_{0} a_{1}^{4} \\
& +\lambda_{1} a_{0}^{4}+\lambda_{1} x_{1} a_{0}^{3}+\lambda_{1} a_{1} a_{0}^{3}+\lambda_{1} a_{0}^{2} x_{1}^{2}+\lambda_{1} a_{0}^{2} a_{1}^{2}+\lambda_{0} a_{0}^{2} x_{1}^{2}+\lambda_{0} a_{0}^{2} a_{1}^{2}+a_{0} x_{1}=G\left(x_{0}, x_{1}\right) . \tag{5}
\end{align*}
$$

Suppose that $\lambda_{1} \neq 0$. We will show that for each $\lambda=\lambda_{0}+\lambda_{1} \omega$ with $\lambda_{1} \neq 0$, there exists at least one nonzero $a=a_{0}+a_{1} \omega \in \mathbb{F}_{2^{n}}$ such that $\operatorname{Tr}_{1}^{n}\left(\lambda x^{d}+\lambda(x+a)^{d}+a x\right)=\operatorname{Tr}_{1}^{k}\left(G\left(x_{0}, x_{1}\right)\right)$ is not balanced. We consider this in three cases.

Case (i) $\lambda_{1} \neq 0, \lambda_{0}^{2}+\lambda_{1}^{2}+\lambda_{0} \lambda_{1}+1 \neq 0$.
In this case, let $a_{1}=0$ and $a_{0} \neq 0$. Then

$$
\begin{align*}
& \sum_{x_{0}, x_{1} \in \mathbb{F}_{q}}(-1)^{\operatorname{Tr}_{1}^{k}\left(G\left(x_{0}, x_{1}\right)\right)} \\
= & \sum_{x_{0}, x_{1} \in \mathbb{F}_{q}}(-1)^{\operatorname{Tr}_{1}^{k}\left(\lambda_{1} x_{1} x_{0}^{2} a_{0}+\lambda_{1} x_{1} x_{0} a_{0}^{2}+\lambda_{1} a_{0}^{2} x_{1}^{2}+\lambda_{0} a_{0} x_{1}^{3}+\lambda_{0} a_{0}^{2} x_{1}^{2}+\lambda_{1} a_{0}^{4}+\lambda_{1} x_{1} a_{0}^{3}+a_{0} x_{1}\right)} \\
= & \sum_{x_{1} \in \mathbb{F}_{q}}(-1)^{\operatorname{Tr}_{1}^{k}\left(\lambda_{1} a_{0}^{2} x_{1}^{2}+\lambda_{0} a_{0} x_{1}^{3}+\lambda_{0} a_{0}^{2} x_{1}^{2}+\lambda_{1} a_{0}^{4}+\lambda_{1} x_{1} a_{0}^{3}+a_{0} x_{1}\right)} \sum_{x_{0} \in \mathbb{F}_{q}}(-1)^{\operatorname{Tr}_{1}^{k}\left(\left(\lambda_{1} x_{1} a_{0}+\lambda_{1}^{2} x_{1}^{2} a_{0}^{4}\right) x_{0}^{2}\right)} \\
= & 2^{k} \sum_{x_{1}=0 \text { or } x_{1}=\left(\lambda_{1} a_{0}^{3}\right)^{-1}}(-1)^{\operatorname{Tr}_{1}^{k}\left(\lambda_{1} a_{0}^{2} x_{1}^{2}+\lambda_{0} a_{0} x_{1}^{3}+\lambda_{0} a_{0}^{2} x_{1}^{2}+\lambda_{1} a_{0}^{4}+\lambda_{1} x_{1} a_{0}^{3}+a_{0} x_{1}\right)} \\
= & 2^{k}\left((-1)^{\operatorname{Tr}_{1}^{k}\left(\lambda_{1} a_{0}^{4}\right)}+(-1)^{\operatorname{Tr}_{1}^{k}\left(\lambda_{1} a_{0}^{2} t^{2}+\lambda_{0} a_{0} t^{3}+\lambda_{0} a_{0}^{2} t^{2}+\lambda_{1} a_{0}^{4}+\lambda_{1} t a_{0}^{3}+a_{0} t\right)}\right), \tag{6}
\end{align*}
$$

where $t=\left(\lambda_{1} a_{0}^{3}\right)^{-1}$. By (6), if there exists $a_{0} \in \mathbb{F}_{q}^{*}$ such that $\operatorname{Tr}_{1}^{k}\left(\lambda_{1} a_{0}^{2} t^{2}+\lambda_{0} a_{0} t^{3}+\lambda_{0} a_{0}^{2} t^{2}+\lambda_{1} t a_{0}^{3}+a_{0} t\right)=$ 0 , then $\sum_{x_{0}, x_{1} \in \mathbb{F}_{q}}(-1)^{\operatorname{Tr}_{1}^{k}\left(G\left(x_{0}, x_{1}\right)\right)}=(-1)^{\operatorname{Tr}_{1}^{k}\left(\lambda_{1} a_{0}^{4}\right)} \cdot 2^{k+1} \neq 0$, i.e., $\operatorname{Tr}_{1}^{k}\left(G\left(x_{0}, x_{1}\right)\right)$ is not balanced for such $a_{0} \in \mathbb{F}_{q}^{*}$. Since $t=\left(\lambda_{1} a_{0}^{3}\right)^{-1}$, we have $\operatorname{Tr}_{1}^{k}\left(\lambda_{1} a_{0}^{2} t^{2}+\lambda_{0} a_{0} t^{3}+\lambda_{0} a_{0}^{2} t^{2}+\lambda_{1} t a_{0}^{3}+a_{0} t\right)=$ $\operatorname{Tr}_{1}^{k}\left(\frac{\lambda_{1}^{2}+\lambda_{0}^{2}+1+\lambda_{0} \lambda_{1}}{\lambda_{1}^{4}}\left(a_{0}^{8}\right)^{-1}+1\right)$, which implies that there exists $a_{0} \neq 0$ such that $\operatorname{Tr}_{1}^{k}\left(\frac{\lambda_{1}^{2}+\lambda_{0}^{2}+1+\lambda_{0} \lambda_{1}}{\lambda_{1}^{4}}\left(a_{0}^{8}\right)^{-1}\right)+$ $1=0$ if $\lambda \in \mathbb{F}_{2^{n}}$ satisfying $\lambda_{0}^{2}+\lambda_{1}^{2}+\lambda_{0} \lambda_{1}+1 \neq 0$ and $\lambda_{1} \neq 0$.

Case (ii) $\lambda_{1} \neq 0, \lambda_{0}^{2}+\lambda_{1}^{2}+\lambda_{0} \lambda_{1}+1=0$ and $\lambda_{0} \neq 0$.
In this case, let $a_{0}=0$ and $a_{1} \neq 0$. Then

$$
\begin{align*}
& \sum_{x_{0}, x_{1} \in \mathbb{F}_{q}}(-1)^{\operatorname{Tr}_{1}^{k}\left(G\left(x_{0}, x_{1}\right)\right)} \\
= & \sum_{x_{0}, x_{1} \in \mathbb{F}_{q}}(-1)^{\operatorname{Tr}_{1}^{k}\left(\lambda_{0} x_{0} x_{1}^{2} a_{1}+\left(\lambda_{0} a_{1}^{2} x_{0}+a_{1}\right) x_{1}+\lambda_{1} a_{1} x_{0}^{3}+\left(\lambda_{1} a_{1}^{2}+\lambda_{0} a_{1}^{2}\right) x_{0}^{2}+\left(\lambda_{0} a_{1}^{3}+a_{1}\right) x_{0}+\lambda_{0} a_{1}^{4}\right)} \\
= & \sum_{x_{0} \in \mathbb{F}_{q}}(-1)^{\operatorname{Tr}_{1}^{k}\left(\lambda_{1} a_{1} x_{0}^{3}+\left(\lambda_{1} a_{1}^{2}+\lambda_{0} a_{1}^{2}\right) x_{0}^{2}+\left(\lambda_{0} a_{1}^{3}+a_{1}\right) x_{0}+\lambda_{0} a_{1}^{4}\right)} \sum_{x_{1} \in \mathbb{F}_{q}}(-1)^{\operatorname{Tr}_{1}^{k}\left(\left(\lambda_{0} x_{0} a_{1}+\lambda_{0}^{2} a_{1}^{4} x_{0}^{2}+a_{1}^{2}\right) x_{1}^{2}\right)} \\
= & 2^{k} \sum_{x_{0}=y_{1} \text { or } x_{0}=y_{2}}(-1)^{\operatorname{Tr}_{1}^{k}\left(\lambda_{1} a_{1} x_{0}^{3}+\left(\lambda_{1} a_{1}^{2}+\lambda_{0} a_{1}^{2}\right) x_{0}^{2}+\left(\lambda_{0} a_{1}^{3}+a_{1}\right) x_{0}+\lambda_{0} a_{1}^{4}\right)}, \tag{7}
\end{align*}
$$

where $y_{1}$ and $y_{2}$ are the two roots of $\lambda_{0} x_{0} a_{1}+\lambda_{0}^{2} a_{1}^{4} x_{0}^{2}+a_{1}^{2}=0$ ( $x_{0}$ as the indeterminate variable) under the condition $\operatorname{Tr}_{1}^{k}\left(a_{1}\right)=0$. Thus, $y_{1}+y_{2}=\frac{1}{\lambda_{0} a_{1}^{3}}$ and $y_{1} y_{2}=\frac{1}{\lambda_{0}^{2} a_{1}^{2}}$. By (7), if there exists $a_{1} \in \mathbb{F}_{q}^{*}$
such that $\operatorname{Tr}_{1}^{k}\left(a_{1}\right)=0$ and $\operatorname{Tr}_{1}^{k}\left(\lambda_{1} a_{1}\left(y_{1}^{3}+y_{2}^{3}\right)+\left(\lambda_{1} a_{1}^{2}+\lambda_{0} a_{1}^{2}\right)\left(y_{1}+y_{2}\right)^{2}+\left(\lambda_{0} a_{1}^{3}+a_{1}\right)\left(y_{1}+y_{2}\right)\right)=0$, then $\sum_{x_{0}, x_{1} \in \mathbb{F}_{q}}(-1)^{\operatorname{Tr}_{1}^{k}\left(G\left(x_{0}, x_{1}\right)\right)}= \pm 2^{k+1} \neq 0$, i.e., $\operatorname{Tr}_{1}^{k}\left(G\left(x_{0}, x_{1}\right)\right)$ is not balanced for such $a_{1} \in \mathbb{F}_{q}^{*}$. By $y_{1}^{3}+y_{2}^{3}=\left(y_{1}+y_{2}\right)^{3}+y_{1} y_{2}\left(y_{1}+y_{2}\right)=\frac{1}{\lambda_{0}^{3}}\left(\frac{1}{a_{1}^{9}}+\frac{1}{a_{1}^{s}}\right)$, one obtains that

$$
\begin{aligned}
& \operatorname{Tr}_{1}^{k}\left(\lambda_{1} a_{1}\left(y_{1}^{3}+y_{2}^{3}\right)+\left(\lambda_{1} a_{1}^{2}+\lambda_{0} a_{1}^{2}\right)\left(y_{1}+y_{2}\right)^{2}+\left(\lambda_{0} a_{1}^{3}+a_{1}\right)\left(y_{1}+y_{2}\right)\right) \\
= & \operatorname{Tr}_{1}^{k}\left(\left(\frac{\lambda_{1}^{2}}{\lambda_{0}^{6}}+\frac{\lambda_{0}^{2}+\lambda_{1}^{2}+\lambda_{0} \lambda_{1}+1}{\lambda_{0}^{4}}\right) \frac{1}{a_{1}^{8}}+1\right)=\operatorname{Tr}_{1}^{k}\left(\frac{\lambda_{1}^{2}}{\lambda_{0}^{6}} \cdot \frac{1}{a_{1}^{8}}+1\right) .
\end{aligned}
$$

According to Lemma 2 for odd $k>2$, there exists $a_{1} \in \mathbb{F}_{q}^{*}$ such that $\operatorname{Tr}_{1}^{k}\left(\frac{\lambda_{1}^{2}}{\lambda_{0}^{\circ}} \cdot \frac{1}{a_{1}^{\mathrm{s}}}+1\right)=\operatorname{Tr}_{1}^{k}\left(\left(\frac{\lambda_{1}^{2}}{\lambda_{0}^{\circ}}\right)^{-8} \cdot \frac{1}{a_{1}}\right)+$ $1=0$ and $\operatorname{Tr}_{1}^{k}\left(a_{1}\right)=0$. Thus, for any $\lambda \in \mathbb{F}_{2^{n}}$ such that $\lambda_{0}^{2}+\lambda_{1}^{2}+\lambda_{0} \lambda_{1}+1=0$ and $\lambda_{0} \lambda_{1} \neq 0$, there exists $a_{1} \neq 0$ such that $\operatorname{Tr}_{1}^{k}\left(a_{1}\right)=0$ and $\operatorname{Tr}_{1}^{k}\left(\lambda_{1} a_{1}\left(y_{1}^{3}+y_{2}^{3}\right)+\left(\lambda_{1} a_{1}^{2}+\lambda_{0} a_{1}^{2}\right)\left(y_{1}+y_{2}\right)^{2}+\left(\lambda_{0} a_{1}^{3}+a_{1}\right)\left(y_{1}+y_{2}\right)\right)=0$. That is, $\operatorname{Tr}_{1}^{k}\left(G\left(x_{0}, x_{1}\right)\right)$ is not balanced for such $a_{1} \in \mathbb{F}_{q}^{*}$.

Case (iii) $\lambda_{1} \neq 0, \lambda_{0}^{2}+\lambda_{1}^{2}+\lambda_{0} \lambda_{1}+1=0$ and $\lambda_{0}=0$.
For this case, $\lambda_{1}=1$ and $\lambda_{0}=0$. Let $a_{0}=a_{1} \neq 0$. Then

$$
\begin{align*}
& \sum_{x_{0}, x_{1} \in \mathbb{F}_{q}}(-1)^{\operatorname{Tr}_{1}^{k}\left(G\left(x_{0}, x_{1}\right)\right)} \\
= & \sum_{x_{0}, x_{1} \in \mathbb{F}_{q}}(-1)^{\operatorname{Tr}_{1}^{k}\left(a_{0}^{2} x_{1}^{2}+\left(x_{0}^{2} a_{0}+x_{0} a_{0}^{2}+a_{0}^{3}\right) x_{1}+x_{0}^{3} a_{0}+\left(a_{0}^{3}+a_{0}\right) x_{0}+a_{0}^{4}\right)} \\
= & \sum_{x_{0} \in \mathbb{F}_{q}}(-1)^{\operatorname{Tr}_{1}^{k}\left(x_{0}^{3} a_{0}+\left(a_{0}^{3}+a_{0}\right) x_{0}+a_{0}^{4}\right)} \sum_{x_{1} \in \mathbb{F}_{q}}(-1)^{\operatorname{Tr}_{1}^{k}\left(\left(a_{0}+x_{0}^{2} a_{0}+x_{0} a_{0}^{2}+a_{0}^{3}\right) x_{1}\right)} \\
= & 2^{k} \sum_{x_{0}=y_{1} \text { or } x_{0}=y_{2}}(-1)^{\operatorname{Tr}_{1}^{k}\left(x_{0}^{3} a_{0}+\left(a_{0}^{3}+a_{0}\right) x_{0}+a_{0}^{4}\right)}, \tag{8}
\end{align*}
$$

where $y_{1}$ and $y_{2}$ are the two roots of $a_{0}+x_{0}^{2} a_{0}+x_{0} a_{0}^{2}+a_{0}^{3}=0$ ( $x_{0}$ as the indeterminate variable) under the condition $\operatorname{Tr}_{1}^{k}\left(a_{0}^{-1}\right)=1$. Thus, $y_{1}+y_{2}=a_{0}$ and $y_{1} y_{2}=1+a_{0}^{2}$. By ( $\mathbb{8}$ ), if there exists $a_{0} \in \mathbb{F}_{q}^{*}$ such that $\operatorname{Tr}_{1}^{k}\left(a_{0}^{-1}\right)=1$ and $\operatorname{Tr}_{1}^{k}\left(\left(y_{1}^{3}+y_{2}^{3}\right) a_{0}+\left(a_{0}^{3}+a_{0}\right)\left(y_{1}+y_{2}\right)\right)=0$, then $\sum_{x_{0}, x_{1} \in \mathbb{F}_{q}}(-1)^{\operatorname{Tr}_{1}^{k}\left(G\left(x_{0}, x_{1}\right)\right)}= \pm 2^{k+1} \neq 0$. That is, $\operatorname{Tr}_{1}^{k}\left(G\left(x_{0}, x_{1}\right)\right)$ is not balanced for such $a_{0} \in \mathbb{F}_{q}^{*}$. Note that $y_{1}^{3}+y_{2}^{3}=\left(y_{1}+y_{2}\right)^{3}+y_{1} y_{2}\left(y_{1}+y_{2}\right)=$ $a_{0}^{3}+\left(1+a_{0}^{2}\right) a_{0}=a_{0}$, then $\operatorname{Tr}_{1}^{k}\left(\left(y_{1}^{3}+y_{2}^{3}\right) a_{0}+\left(a_{0}^{3}+a_{0}\right)\left(y_{1}+y_{2}\right)\right)=\operatorname{Tr}_{1}^{k}\left(a_{0}^{2}+\left(a_{0}^{3}+a_{0}\right) a_{0}\right)=\operatorname{Tr}_{1}^{k}\left(a_{0}\right)=0$. Again by Lemma 园, for odd $k>2$, there exists $a_{0} \in \mathbb{F}_{q}^{*}$ such that $\operatorname{Tr}_{1}^{k}\left(a_{0}\right)=0$ and $\operatorname{Tr}_{1}^{k}\left(a_{0}^{-1}\right)=1$. Thus, for $\lambda=\lambda_{0}+\lambda_{1} \omega=\omega$, there exists $a_{0} \neq 0$ such that $\operatorname{Tr}_{1}^{k}\left(a_{0}^{-1}\right)=1$ and $\operatorname{Tr}_{1}^{k}\left(\left(y_{1}^{3}+y_{2}^{3}\right) a_{0}+\left(a_{0}^{3}+a_{0}\right)\left(y_{1}+\right.\right.$ $\left.\left.y_{2}\right)\right)=0$, which implies that $\operatorname{Tr}_{1}^{k}\left(G\left(x_{0}, x_{1}\right)\right)$ is not balanced.

From the above Cases (i)-(iii), for each $\lambda=\lambda_{0}+\lambda_{1} \omega$ with $\lambda_{1} \neq 0$, there exists at least one nonzero $a=a_{0}+a_{1} \omega \in \mathbb{F}_{2^{n}}$ such that $\operatorname{Tr}_{1}^{n}\left(\lambda x^{d}+\lambda(x+a)^{d}+a x\right)=\operatorname{Tr}_{1}^{k}\left(G\left(x_{0}, x_{1}\right)\right)$ is not balanced.

In the following we assume that $\lambda_{1}=0$ and $\lambda=\lambda_{0}+\lambda_{1} \omega=\lambda_{0} \neq 0$. Let $a_{1}=0$. Then

$$
\begin{align*}
\sum_{x_{0}, x_{1} \in \mathbb{F}_{q}}(-1)^{\operatorname{Tr}_{1}^{k}\left(G\left(x_{0}, x_{1}\right)\right)} & =\sum_{x_{0}, x_{1} \in \mathbb{F}_{q}}(-1)^{\operatorname{Tr}_{1}^{k}\left(\lambda_{0} a_{0} x_{1}^{3}+\lambda_{0} a_{0}^{2} x_{1}^{2}+a_{0} x_{1}\right)} \\
& =2^{k} \sum_{x_{1} \in \mathbb{F}_{q}}(-1)^{\operatorname{Tr}_{1}^{k}\left(\lambda_{0} a_{0} x_{1}^{3}+\lambda_{0} a_{0}^{2} x_{1}^{2}+a_{0} x_{1}\right)} \\
& =2^{k} \sum_{x_{1} \in \mathbb{F}_{q}}(-1)^{\operatorname{Tr}_{1}^{k}\left(\lambda_{0} a_{0} x_{1}^{3}+\left(\lambda_{0}^{2^{k-1}} a_{0}+a_{0}\right) x_{1}\right)} \tag{9}
\end{align*}
$$

Since $k$ is odd, then $\operatorname{gcd}\left(3,2^{k}-1\right)=1$. Let $\lambda_{0}=r^{3}, a_{0}=t^{3}$, then from (9), one gets

$$
\begin{equation*}
\sum_{x_{0}, x_{1} \in \mathbb{F}_{q}}(-1)^{\operatorname{Tr}_{1}^{k}\left(G\left(x_{0}, x_{1}\right)\right)}=2^{k} \sum_{x_{1} \in \mathbb{F}_{q}}(-1)^{\operatorname{Tr}_{1}^{k}\left(x_{1}^{3}+\left(r^{3 \cdot 2^{k-1}}+1\right) r^{-1} t^{2} x_{1}\right)} \tag{10}
\end{equation*}
$$

Thus, if $\lambda_{0}=r^{3} \neq 1$, then $r^{3 \cdot 2^{k-1}}+1 \neq 0$. We claim that for any $r \in \mathbb{F}_{q}^{*}$ and $r \neq 1$, there must exist some $a_{0} \in \mathbb{F}_{q}^{*}$ such that $\sum_{x_{0}, x_{1} \in \mathbb{F}_{q}}(-1)^{\operatorname{Tr}_{1}^{k}\left(G\left(x_{0}, x_{1}\right)\right)}=2^{k} \sum_{x_{1} \in \mathbb{F}_{q}}(-1)^{\operatorname{Tr}_{1}^{k}\left(x_{1}^{3}+\left(r^{3 \cdot 2^{k-1}}+1\right) r^{-1} t^{2} x_{1}\right)} \neq 0$, i.e., $\operatorname{Tr}_{1}^{k}\left(G\left(x_{0}, x_{1}\right)\right)$ is not balanced. Otherwise, the Walsh-Hadamard transform of $\operatorname{Tr}_{1}^{k}\left(x^{3}\right)$ at any point $t \in \mathbb{F}_{q}$ is zero, which contradicts with Parseval's theorem 1 .

Therefore, if $\operatorname{Tr}_{1}^{n}\left(\lambda x^{d}\right)$ is negabent on $\mathbb{F}_{2^{n}}$, then $\lambda$ has to be in $\mathbb{F}_{2}$. Zhou and Qu [19, Theorem 6] proved that if $\lambda \in \mathbb{F}_{2}$, then $\operatorname{Tr}_{1}^{n}\left(\lambda x^{d}\right)$ is indeed negabent on $\mathbb{F}_{2^{n}}$. This completes the proof.

To end this section, we present a conjecture on negabent monomials whose exponents are of Niho type, namely the exponents of the form $d=r\left(2^{m}-1\right)+1$, where $m=n / 2$ and $1 \leq r \leq 2^{m}$. Notice that $d_{1}=r_{1}\left(2^{m}-1\right)+1$ and $d_{2}=r_{2}\left(2^{m}-1\right)+1$ lie in the same cyclotomic coset modulo $2^{n}-1$ if and only if $r_{1} \equiv r_{2}\left(\bmod 2^{m}+1\right)$ or $r_{1}+r_{2} \equiv 1\left(\bmod 2^{m}+1\right)$.

Sarkar [15] gave a class of negabent monomials whose exponents are of Niho type, as follows:

Theorem 8 [15] Let $n=2 m$ and $d=\left(2^{m-1}+1\right)\left(2^{m}-1\right)+1$. Then $\operatorname{Tr}_{1}^{n}\left(\alpha x^{d}\right)$ is negabent if and only if $\alpha+\alpha^{2^{m}} \neq 1$.

Based on our computer experiments, we have the following conjecture:
Conjecture 1 Let $n=2 m$ and $d=r\left(2^{m}-1\right)+1$, where $2 \leq r \leq 2^{m-1}+1$. Then $\operatorname{Tr}_{1}^{n}\left(\alpha x^{d}\right)$ is a negabent function if and only if one of the following two conditions holds:

1. $m$ is odd, $r=2^{m-2}+1 \equiv \frac{3}{4}\left(\bmod 2^{m}+1\right)$ and $\alpha \in \mathbb{F}_{2}$. (Cubic functions, Theorem 7 )

[^1]2. $r=2^{m-1}+1 \equiv \frac{1}{2}\left(\bmod 2^{m}+1\right)$ and $\alpha+\alpha^{2^{m}} \neq 1$. (Quadratic functions, Theorem (8))

This conjecture has been verified by Magma for $n \leq 14$.

## 5 Conclusion

Negabent functions as a generalization of bent functions are very useful in cryptography and coding theory. In this paper, several classes of negabent functions of the form $f(x)=\operatorname{Tr}_{1}^{n}\left(\lambda x^{2^{k}+1}\right)+$ $\operatorname{Tr}_{1}^{n}(u x) \operatorname{Tr}_{1}^{n}(v x)$ were given, where $0<k<n$ and $(u, v) \in \mathbb{F}_{2^{n}}^{*} \times \mathbb{F}_{2^{n}}^{*}$. In particular, we gave the necessary and sufficient conditions for $\operatorname{Tr}_{1}^{k}\left(\lambda x^{2^{k}+1}\right)+\operatorname{Tr}_{1}^{2 k}(u x) \operatorname{Tr}_{1}^{2 k}(v x)$ to be negabent on $\mathbb{F}_{2^{2 k}}$, where $\lambda \in \mathbb{F}_{2^{k}}$. We also showed that the condition $\lambda \in \mathbb{F}_{2}$ for $\operatorname{Tr}_{1}^{2 k}\left(\lambda x^{2^{k}+3}\right)$ to be negabent is necessary, where $k \geq 3$ is odd. Finally, based on our Magma results, we presented a conjecture on monomial negabent functions whose exponents are of Niho type.

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[^1]:    ${ }^{1}$ Parseval's theorem shows that for any Boolean function $f(x)$ from $\mathbb{F}_{2^{k}}$ to $\mathbb{F}_{2}$, its Walsh-Hadamard transform $W_{f}(u)$ satisfies $\sum_{u \in \mathbb{F}_{2^{k}}}\left(W_{f}(u)\right)^{2}=2^{2 k}$.

