

ON REPRESENTATIONS OF THE FEASIBLE SET IN CONVEX OPTIMIZATION

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ABSTRACT. We consider the convex optimization problem $\min_{\mathbf{x}} \{f(\mathbf{x}) : g_j(\mathbf{x}) \leq 0, j = 1, \dots, m\}$ where f is convex, the feasible set \mathbf{K} is convex and Slater's condition holds, but the functions g_j 's are *not* necessarily convex. We show that for any representation of \mathbf{K} that satisfies a mild nondegeneracy assumption, every minimizer is a Karush-Kuhn-Tucker (KKT) point and conversely every KKT point is a minimizer. That is, the KKT optimality conditions are necessary and sufficient as in convex programming where one assumes that the g_j 's are convex. So in convex optimization, and as far as one is concerned with KKT points, what really matters is the geometry of \mathbf{K} and not so much its representation.

1. INTRODUCTION

Given differentiable functions $f, g_j : \mathbb{R}^n \rightarrow \mathbb{R}, j = 1, \dots, m$, consider the following convex optimization problem:

$$(1.1) \quad f^* := \inf_{\mathbf{x}} \{ f(\mathbf{x}) : \mathbf{x} \in \mathbf{K} \}$$

where f is convex and the feasible set $\mathbf{K} \subset \mathbb{R}^n$ is convex and represented in the form:

$$(1.2) \quad \mathbf{K} = \{ \mathbf{x} \in \mathbb{R}^n : g_j(\mathbf{x}) \leq 0, j = 1, \dots, m \}.$$

Convex optimization usually refers to minimizing a convex function over a convex set without precising its representation (see e.g. Ben-Tal and Nemirovsky [1, Definition 5.1.1] or Bertsekas et al. [3, Chapter 2]), and it is well-known that convexity of the function f and of the set \mathbf{K} imply that every local minimum is a global minimum. An elementary proof only uses the geometry of \mathbf{K} , not its representation by the defining functions g_j ; see e.g. Bertsekas et al. [3, Prop. 2.1.2].

The convex set \mathbf{K} may be represented by different choices of the (not necessarily convex) defining functions $g_j, j = 1, \dots, m$. For instance, the set

$$\mathbf{K} := \{ \mathbf{x} \in \mathbb{R}^2 : 1 - x_1x_2 \leq 0; \mathbf{x} \geq 0 \}$$

is convex but the function $\mathbf{x} \mapsto 1 - x_1x_2$ is not convex on \mathbb{R}_+^2 . Of course, depending on the choice of the defining functions (g_j), several properties may or may not hold. In particular, the celebrated Karush-Kuhn-Tucker (KKT) optimality conditions *depend* on the representation of \mathbf{K} . Recall that $\mathbf{x} \in \mathbf{K}$ is a KKT point if

$$(1.3) \quad \nabla f(\mathbf{x}) + \sum_{j=1}^m \lambda_j \nabla g_j(\mathbf{x}) = 0 \quad \text{and} \quad \lambda_j g_j(\mathbf{x}) = 0, j = 1, \dots, m,$$

for some nonnegative vector $\lambda \in \mathbb{R}^m$. (More precisely (\mathbf{x}, λ) is a KKT point.)

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Convex programming refers to the situation where f is convex and the defining functions g_j of \mathbf{K} are also convex. See for instance Ben-Tal and Nemirovsky [1, p. 335], Berkovitz [2, p. 179], Boyd and Vandenberghe [4, p. 7], Bertsekas et al. [3, §3.5.5], Nesterov and Nemirovskii [6, p. 217-218], and Hiriart-Urruty [5].

A crucial feature of convex programming is that when Slater's condition holds¹, the KKT optimality conditions (1.3) are necessary and sufficient, which shows that a representation of the convex set \mathbf{K} with convex functions (g_j) has some very attractive features.

The purpose of this note is to show that in fact, when \mathbf{K} is convex and as far as one is concerned with KKT points, what really matters is the geometry of \mathbf{K} and not so much its representation. Indeed, we show that if \mathbf{K} is convex and Slater's condition holds then the KKT optimality conditions (1.3) are also necessary and sufficient for *all* representations of \mathbf{K} that satisfy a mild nondegeneracy condition, no matter if the g_j 's are convex. So this attractive feature is not specific to representations of \mathbf{K} with convex functions.

That a KKT point is a local (hence global) minimizer follows easily from the convexity of \mathbf{K} . More delicate is the fact that any local (hence global) minimizer is a KKT point. Various *constraint qualifications* are usually required to hold at a minimizer, and when the g_j 's are convex the simple Slater's condition is enough. Here we show that Slater's condition is also sufficient for all representations of \mathbf{K} that satisfy a mild additional nondegeneracy assumption on the boundary of \mathbf{K} . Moreover under Slater's condition this mild nondegeneracy assumption is automatically satisfied if the g_j 's are convex.

2. MAIN RESULT

Let $\mathbf{K} \subset \mathbb{R}^n$ be as in (1.2). We first start with the following non degeneracy assumption:

Assumption 2.1 (nondegeneracy). *For every $j = 1, \dots, m$,*

$$(2.1) \quad \nabla g_j(\mathbf{x}) \neq 0, \quad \text{whenever } \mathbf{x} \in \mathbf{K} \quad \text{and} \quad g_j(\mathbf{x}) = 0.$$

Observe that under Slater's condition, (2.1) is automatically satisfied if g_j is convex. Indeed if $g_j(\mathbf{x}) = 0$ and $\nabla g_j(\mathbf{x}) = 0$ then by convexity 0 is the global minimum of g_j on \mathbb{R}^n . Hence there is no $\mathbf{x}_0 \in \mathbf{K}$ with $g_j(\mathbf{x}_0) < 0$. We next state the following characterization of convexity.

Lemma 2.2. *With $\mathbf{K} \subset \mathbb{R}^n$ as in (1.2), let Assumption 2.1 and Slater's condition both hold for \mathbf{K} . Then \mathbf{K} is convex if and only if for every $j = 1, \dots, m$:*

$$(2.2) \quad \langle \nabla g_j(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle \leq 0, \quad \forall \mathbf{x}, \mathbf{y} \in \mathbf{K} \quad \text{with} \quad g_j(\mathbf{x}) = 0.$$

Proof. Only if part. Assume that \mathbf{K} is convex and $\langle \nabla g_j(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle > 0$ for some $j \in \{1, \dots, m\}$ and some $\mathbf{x}, \mathbf{y} \in \mathbf{K}$ with $g_j(\mathbf{x}) = 0$. Then $g_j(\mathbf{x} + t(\mathbf{y} - \mathbf{x})) > 0$ for all sufficiently small t , in contradiction with $\mathbf{x} + t(\mathbf{y} - \mathbf{x}) \in \mathbf{K}$ for all $0 \leq t \leq 1$ (by convexity of \mathbf{K}).

If part. By (2.2), at every point \mathbf{x} on the boundary of \mathbf{K} , there exists a supporting hyperplane for \mathbf{K} . As \mathbf{K} is closed with nonempty interior, by [8][Th. 1.3.3] the set \mathbf{K} is convex². \square

¹Slater's condition holds for \mathbf{K} if for some $\mathbf{x}_0 \in \mathbf{K}$, $g_j(\mathbf{x}_0) < 0$ for every $j = 1, \dots, m$.

²The author wishes to thank Prof. L. Tuncel for providing him with the reference [8].

Theorem 2.3. *Consider the nonlinear programming problem (1.1) and let Assumption 2.1 and Slater's condition both hold. If f is convex then every minimizer is a KKT point and conversely, every KKT point is a minimizer.*

Proof. Let $\mathbf{x}^* \in \mathbf{K}$ be a minimizer (hence a global minimizer) with $f^* = f(\mathbf{x}^*)$. We first prove that \mathbf{x}^* is a KKT point. The Fritz-John optimality conditions state that

$$\lambda_0 \nabla f(\mathbf{x}^*) + \sum_{j=1}^m \lambda_j \nabla g_j(\mathbf{x}^*) = 0; \quad \lambda_j g_j(\mathbf{x}^*) = 0, \quad j = 1, \dots, m,$$

for some non trivial nonnegative vector $0 \neq \lambda \in \mathbb{R}^{m+1}$. See e.g. Hiriart-Urruty [5, Th. page 77] or Polyak [7, Theor. 1, p. 271]. We next prove that $\lambda_0 \neq 0$. Suppose that $\lambda_0 = 0$ and let $J := \{j \in \{1, \dots, m\} : \lambda_j > 0\}$. As $\lambda \neq 0$ and $\lambda_0 = 0$, the set J is nonempty. Next, as $g_j(\mathbf{x}_0) < 0$ for every $j = 1, \dots, m$, there is some $\rho > 0$ such that $B(\mathbf{x}_0, \rho) := \{\mathbf{z} \in \mathbb{R}^n : \|\mathbf{z} - \mathbf{x}_0\| < \rho\} \subset \mathbf{K}$ and $g_j(\mathbf{z}) < 0$ for all $\mathbf{z} \in B(\mathbf{x}_0, \rho)$ and all $j \in J$. Therefore we obtain

$$\sum_{j \in J} \lambda_j \langle \nabla g_j(\mathbf{x}^*), \mathbf{z} - \mathbf{x}^* \rangle = 0 \quad \forall \mathbf{z} \in B(\mathbf{x}_0, \rho),$$

which, by Lemma 2.2, implies that $\langle \nabla g_j(\mathbf{x}^*), \mathbf{z} - \mathbf{x}^* \rangle = 0$ for every $j \in J$ and every $\mathbf{z} \in B(\mathbf{x}_0, \rho)$. But this clearly implies that $\nabla g_j(\mathbf{x}^*) = 0$ for every $j \in J$, in contradiction with Assumption 2.1. Hence $\lambda_0 > 0$ and we may and will set $\lambda_0 = 1$, so that the KKT conditions hold at \mathbf{x}^* .

Conversely, let $\mathbf{x} \in \mathbf{K}$ be an arbitrary KKT point, i.e., $\mathbf{x} \in \mathbf{K}$ satisfies

$$\nabla f(\mathbf{x}) + \sum_{j=1}^m \lambda_j \nabla g_j(\mathbf{x}) = 0; \quad \lambda_j g_j(\mathbf{x}) = 0, \quad j = 1, \dots, m,$$

for some nonnegative vector $\lambda \in \mathbb{R}^m$. Suppose that there exists $\mathbf{y} \in \mathbf{K}$ with $f(\mathbf{y}) < f(\mathbf{x})$. Then we obtain the contradiction:

$$\begin{aligned} 0 &> f(\mathbf{y}) - f(\mathbf{x}) \\ &\geq \langle \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle \quad [\text{by convexity of } f] \\ &= - \sum_{j=1}^m \lambda_j \langle \nabla g_j(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle \geq 0 \end{aligned}$$

where the last inequality follows from $\lambda \geq 0$ and Lemma 2.2. Hence \mathbf{x} is a minimizer. \square

Hence if \mathbf{K} is convex and both Assumption 2.1 and Slater's condition hold, there is a one-to-one correspondence between KKT points and minimizers. That is, the KKT optimality conditions are necessary and sufficient for all representations of \mathbf{K} that satisfy Slater's condition and Assumption 2.1.

However there is an important additional property when all the defining functions g_j are convex. Dual methods of the type

$$\sup_{\lambda \in \mathbb{R}_+^m} \left\{ \inf_{\mathbf{x}} f(\mathbf{x}) + \sum_{j=1}^m \lambda_j g_j(\mathbf{x}) \right\},$$

are well defined because $\mathbf{x} \mapsto f(\mathbf{x}) + \sum_{j=1}^m \lambda_j g_j(\mathbf{x})$ is a convex function. In particular, the Lagrangian $\mathbf{x} \mapsto L_f(\mathbf{x}) := f(\mathbf{x}) - f^* + \sum_{j=1}^m \lambda_j g_j(\mathbf{x})$, defined from an

arbitrary KKT point $(\mathbf{x}^*, \lambda) \in \mathbf{K} \times \mathbb{R}_+^m$, is convex and nonnegative on \mathbb{R}^n , with \mathbf{x}^* being a global minimizer. If the g_j 's are not convex this is not true in general.

Example 1. Let $n = 2$ and consider the problem

$$\mathbf{P} : f^* = \min \{ f(\mathbf{x}) : a - x_1 x_2 \leq 0; \mathbf{Ax} \leq \mathbf{b}; \mathbf{x} \geq 0 \},$$

where $a > 0$, $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$, and f is convex and differentiable. The set

$$\mathbf{K} := \{ \mathbf{x} \in \mathbb{R}^2 : a - x_1 x_2 \leq 0; \mathbf{Ax} \leq \mathbf{b}; \mathbf{x} \geq 0 \}$$

is convex and it is straightforward to check that Assumption 2.1 holds. Therefore, by Theorem 2.3, if Slater's condition holds, every KKT point is a global minimizer. However, the Lagrangian

$$\mathbf{x} \mapsto f(\mathbf{x}) - f^* + \psi(a - x_1 x_2) + \langle \lambda, \mathbf{Ax} - \mathbf{b} \rangle - \langle \mu, \mathbf{x} \rangle,$$

with nonnegative $(\psi, \lambda, \mu) \in \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^n$, may not be convex whenever $\psi \neq 0$ (for instance if f is linear). On the other hand, notice that \mathbf{K} has the equivalent convex representation

$$\mathbf{K} := \left\{ \mathbf{x} \in \mathbb{R}^2 : \begin{bmatrix} x_1 & \sqrt{a} \\ \sqrt{a} & x_2 \end{bmatrix} \succeq 0; \mathbf{Ax} \leq \mathbf{b} \right\},$$

where for a real symmetric matrix \mathbf{B} , the notation $\mathbf{B} \succeq 0$ stands for \mathbf{B} is positive semidefinite.

A topic of further investigation is concerned with computational efficiency. Can *efficient* algorithms be devised for some class of convex problems (1.1) where the defining functions g_j of \mathbf{K} are not necessarily convex?

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