
Connection between the clique number and the Lagrangian of 3-uniform hypergraphs

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Received: date / Accepted: date

Abstract There is a remarkable connection between the clique number and the Lagrangian of a 2-graph proved by Motzkin and Straus in 1965. It is useful in practice if similar results hold for hypergraphs. However the obvious generalization of Motzkin and Straus' result to hypergraphs is false. Frankl and Füredi conjectured that the r -uniform hypergraph with m edges formed by taking the first m sets in the colex ordering of $\mathbb{N}^{(r)}$ has the largest Lagrangian of all r -uniform hypergraphs with m edges. For $r = 2$, Motzkin and Straus' theorem confirms this conjecture. For $r = 3$, it is shown by Talbot that this conjecture is true when m is in certain ranges. In this paper, we explore the connection between the clique number and Lagrangians for 3-uniform hypergraphs. As an application of this connection, we confirm that Frankl and Füredi's conjecture holds for bigger ranges of m when $r=3$. We also obtain two weaker versions of Turán type theorem for left-compressed 3-uniform hypergraphs.

Keywords Cliques of hypergraphs · Colex ordering · Lagrangians of hypergraphs · Polynomial optimization.

Mathematics Subject Classification (2010) 05C35 · 05C65 · 05D99 · 90C27

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1 Introduction

For a set V and a positive integer r , let $V^{(r)}$ be the family of all r -subsets of V . An r -uniform graph or r -graph G consists of a set $V(G)$ of vertices and a set $E(G) \subseteq V(G)^{(r)}$ of edges. An edge $e = \{a_1, a_2, \dots, a_r\}$ will be simply denoted by $a_1 a_2 \dots a_r$. An r -graph H is a *subgraph* of an r -graph G , denoted by $H \subseteq G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Let $K_t^{(r)}$ denote the complete r -graph on t vertices, that is the r -graph on t vertices containing all possible edges. A complete r -graph on t vertices is also called a clique with order t . A clique is said to be maximal if there is no other clique containing it, while it is called maximum if it has maximum cardinality. The clique number of a r -graph G , denoted as $\omega(G)$, is defined as the cardinality of the maximum clique. Let \mathbb{N} be the set of all positive integers. For an integer $n \in \mathbb{N}$, let $[n]$ denote the set $\{1, 2, 3, \dots, n\}$. Let $[n]^{(r)}$ represent the complete r -graph on the vertex set $[n]$. When $r = 2$, an r -graph is a simple graph. When $r \geq 3$, an r -graph is often called a hypergraph.

For an r -graph $G := (V, E)$, denote the $(r - 1)$ -neighborhood of a vertex $i \in V$ by $E_i := \{A \in V^{(r-1)} : A \cup \{i\} \in E\}$. Similarly, denote the $(r - 2)$ -neighborhood of a pair of vertices $i, j \in V$ by $E_{ij} := \{B \in V^{(r-2)} : B \cup \{i, j\} \in E\}$. Denote the complement of E_i by $E_i^c := \{A \in V^{(r-1)} : A \cup \{i\} \in V^{(r)} \setminus E\}$. Also, denote the complement of E_{ij} by $E_{ij}^c := \{B \in V^{(r-2)} : B \cup \{i, j\} \in V^{(r)} \setminus E\}$. Denote $E_{i \setminus j} := E_i \cap E_j^c$. An r -graph $G = ([n], E)$ is *left-compressed* if $j_1 j_2 \dots j_r \in E$ implies $i_1 i_2 \dots i_r \in E$ provided $i_p \leq j_p$ for every $p, 1 \leq p \leq r$. Equivalently, an r -graph $G = ([n], E)$ is *left-compressed* if $E_{j \setminus i} = \emptyset$ for any $1 \leq i < j \leq n$.

Definition 1 For an r -uniform graph G with the vertex set $[n]$, edge set $E(G)$, and a vector $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, we associate a homogeneous polynomial in n variables, denoted by $\lambda(G, \mathbf{x})$ as follows:

$$\lambda(G, \mathbf{x}) = \sum_{i_1 i_2 \dots i_r \in E(G)} x_{i_1} x_{i_2} \dots x_{i_r}.$$

Let $S = \{\mathbf{x} = (x_1, x_2, \dots, x_n) : \sum_{i=1}^n x_i = 1, x_i \geq 0 \text{ for } i = 1, 2, \dots, n\}$. Let $\lambda(G)$ represent the maximum of the above homogeneous multilinear polynomial of degree r over the standard simplex S . Precisely

$$\lambda(G) = \max\{\lambda(G, \mathbf{x}) : \mathbf{x} \in S\}.$$

The value x_i is called the *weight* of the vertex i . A vector $\mathbf{x} := (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ is called a feasible weighting for G iff $\mathbf{x} \in S$. A vector $\mathbf{y} \in S$ is called an *optimal weighting* for G iff $\lambda(G, \mathbf{y}) = \lambda(G)$. We call $\lambda(G)$ the Graph-Lagrangian of hypergraph G , for abbreviation, the Lagrangian of G .

The following fact is easily implied by Definition 1.

Fact 1 *Let G_1, G_2 be r -uniform graphs and $G_1 \subseteq G_2$. Then $\lambda(G_1) \leq \lambda(G_2)$.*

The maximum clique problem is a classical problem in combinatorial optimization which has important applications in various domains. In [6], Motzkin and Straus established a remarkable connection between the clique number and the Lagrangian of a graph.

Theorem 1 ([6]) *If G is a 2-graph with clique number t then $\lambda(G) = \lambda(K_t^{(2)}) = \frac{1}{2}(1 - \frac{1}{t})$.*

The obvious generalization of Motzkin and Straus' result to hypergraphs is false because there are many examples of hypergraphs that do not achieve their Lagrangian on any proper subhypergraph. Lagrangians of hypergraphs has been proved to be a useful tool, for example, it is useful to hypergraph extremal problems. Applications of Lagrangian method can be found in

[2-5, 10]. In most applications, an upper bound is needed. Frankl and Füredi [2] asked the following question. Given $r \geq 3$ and $m \in \mathbb{N}$ how large can the Lagrangian of an r -graph with m edges be? For distinct $A, B \in \mathbb{N}^{(r)}$ we say that A is less than B in the *colex ordering* if $\max(A \triangle B) \in B$, where $A \triangle B = (A \setminus B) \cup (B \setminus A)$. For example we have $246 < 156$ in $\mathbb{N}^{(3)}$ since $\max(\{2, 4, 6\} \triangle \{1, 5, 6\}) \in \{1, 5, 6\}$. In colex ordering, $123 < 124 < 134 < 234 < 125 < 135 < 235 < 145 < 245 < 345 < 126 < 136 < 236 < 146 < 246 < 346 < 156 < 256 < 356 < 456 < 127 < \dots$. Note that the first $\binom{t}{r}$ r -tuples in the colex ordering of $\mathbb{N}^{(r)}$ are the edges of $[t]^{(r)}$. The following conjecture of Frankl and Füredi (if it is true) proposes a solution to the question mentioned above.

Conjecture 1 ([2]) The r -graph with m edges formed by taking the first m sets in the colex ordering of $\mathbb{N}^{(r)}$ has the largest Lagrangian of all r -graphs with m edges. In particular, the r -graph with $\binom{t}{r}$ edges and the largest Lagrangian is $[t]^{(r)}$.

This conjecture is true when $r = 2$ by Theorem 1. For the case $r = 3$, Talbot in [12] proved the following.

Theorem 2 ([12]) *Let m and t be integers satisfying $\binom{t-1}{3} \leq m \leq \binom{t-1}{3} + \binom{t-2}{2} - (t-1)$. Then Conjecture 1 is true for $r = 3$ and this value of m . Conjecture 1 is also true for $r = 3$ and $m = \binom{t}{3} - 1$ or $m = \binom{t}{3} - 2$.*

Further evidence that supports Conjecture 1 can be found in [13, 14]. In particular, Conjecture 1 is true for $r = 3$ and $\binom{t}{3} - 6 \leq m \leq \binom{t}{3}$ (see [13, 14]).

Although the obvious generalization of Motzkin and Straus' result to hypergraphs is false, we attempt to explore the relationship between the Lagrangian of a hypergraph and its cliques number for hypergraphs when the number of edges is in certain ranges. In [7], it is conjectured that the following Motzkin and Straus type results are true for hypergraphs.

Conjecture 2 Let t, m , and $r \geq 3$ be positive integers satisfying $\binom{t-1}{r} \leq m \leq \binom{t-1}{r} + \binom{t-2}{r-1}$. Let G be an r -graph with m edges and G contain a clique of order $t-1$. Then $\lambda(G) = \lambda([t-1]^{(r)})$.

The upper bound $\binom{t-1}{r} + \binom{t-2}{r-1}$ in this conjecture is the best possible. When $m = \binom{t-1}{r} + \binom{t-2}{r-1} + 1$, let $C_{r,m}$ be the r -graph with the vertex set $[t]$ and the edge set $[t-1]^{(r)} \cup \{i_1 \cdots i_{r-1}t : i_1 \cdots i_{r-1} \in [t-2]^{(r-1)}\} \cup \{1 \cdots (r-2)(t-1)t\}$. Take a legal weighting $\mathbf{x} := (x_1, \dots, x_t)$, where $x_1 = x_2 = \dots = x_{t-2} = \frac{1}{t-1}$ and $x_{t-1} = x_t = \frac{1}{2(t-1)}$. Then $\lambda(C_{r,m}) \geq \lambda(C_{r,m}, \mathbf{x}) > \lambda([t-1]^{(r)})$.

Conjecture 3 Let G be an r -graph with m edges without containing a clique of size $t-1$, where $\binom{t-1}{r} \leq m \leq \binom{t-1}{r} + \binom{t-2}{r-1}$. Then $\lambda(G) < \lambda([t-1]^{(r)})$.

Let $C_{r,m}$ denote the r -graph with m edges formed by taking the first m sets in the colex ordering of $\mathbb{N}^{(r)}$. The following result was given in [12].

Lemma 1 [12] *For any integers m, t , and r satisfying $\binom{t-1}{r} \leq m \leq \binom{t-1}{r} + \binom{t-2}{r-1}$, we have $\lambda(C_{r,m}) = \lambda([t-1]^{(r)})$.*

Remark 1 Conjectures 2 and 3 refine Conjecture 1 when $\binom{t-1}{r} \leq m \leq \binom{t-1}{r} + \binom{t-2}{r-1}$. If Conjectures 2 and 3 are true, then Conjecture 1 is true for this range of m .

In [7], we showed that Conjecture 2 holds when $r = 3$ as in the following Theorem.

Theorem 3 ([7]) *Let m and t be positive integers satisfying $\binom{t-1}{3} \leq m \leq \binom{t-1}{3} + \binom{t-2}{2}$. Let G be a 3-graph with m edges and contain a clique of order $t-1$. Then $\lambda(G) = \lambda([t-1]^{(3)})$.*

In this paper, we will show the following.

Theorem 4 *Let m and t be integers satisfying $\binom{t-1}{3} \leq m \leq \binom{t-1}{3} + \binom{t-2}{2} - \frac{1}{2}(t-1)$. Let G be a 3-graph with m edges without containing a clique order of $t-1$, then $\lambda(G) < \lambda([t-1]^{(3)})$.*

Combing Theorems 3 and 4, we have the follow result on Conjecture 1.

Corollary 1 *Let m and t be integers satisfying $\binom{t-1}{3} \leq m \leq \binom{t-1}{3} + \binom{t-2}{2} - \frac{1}{2}(t-1)$. Then Conjecture 1 is true for $r=3$ and this value of m .*

Note that Theorem 4 supports Conjecture 3 and Corollary 1 improves Thoerem 2.

The rest of the paper is organized as follows. In section 3, we prove Theorem 4. In section 4, we explore the connection between the clique number and the Lagrangians of some left-compressed 3-graphs. As an application, we obtain two weaker versions of Tuán type result. First we give some useful results.

2 Useful Results

We will impose one additional condition on any optimal weighting $\mathbf{x} = (x_1, x_2, \dots, x_n)$ for an r -graph G :

$$\begin{aligned} |\{i : x_i > 0\}| \text{ is minimal, i.e. if } \mathbf{y} \text{ is a legal weighting for } G \text{ satisfying} \\ |\{i : y_i > 0\}| < |\{i : x_i > 0\}|, \text{ then } \lambda(G, \mathbf{y}) < \lambda(G). \end{aligned} \quad (1)$$

When the theory of Lagrange multipliers is applied to find the optimum of $\lambda(G, \mathbf{x})$, subject to $\sum_{i=1}^n x_i = 1$, notice that $\lambda(E_i, \mathbf{x})$ corresponds to the partial derivative of $\lambda(G, \mathbf{x})$ with respect to x_i . The following lemma gives some necessary conditions of an optimal weighting for G .

Lemma 2 ([3]) *Let $G := (V, E)$ be an r -graph on the vertex set $[n]$ and $\mathbf{x} = (x_1, x_2, \dots, x_n)$ be an optimal weighting for G with k ($\leq n$) non-zero weights x_1, x_2, \dots, x_k satisfying condition (1). Then for every $\{i, j\} \in [k]^{(2)}$, (a) $\lambda(E_i, \mathbf{x}) = \lambda(E_j, \mathbf{x}) = r\lambda(G)$, (b) there is an edge in E containing both i and j .*

Remark 2 (a) In Lemma 2, part(a) implies that

$$x_j \lambda(E_{ij}, \mathbf{x}) + \lambda(E_{i \setminus j}, \mathbf{x}) = x_i \lambda(E_{ij}, \mathbf{x}) + \lambda(E_{j \setminus i}, \mathbf{x}).$$

In particular, if G is left-compressed, then

$$(x_i - x_j) \lambda(E_{ij}, \mathbf{x}) = \lambda(E_{i \setminus j}, \mathbf{x})$$

for any i, j satisfying $1 \leq i < j \leq k$ since $E_{j \setminus i} = \emptyset$.

(b) If G is left-compressed, then for any i, j satisfying $1 \leq i < j \leq k$,

$$x_i - x_j = \frac{\lambda(E_{i \setminus j}, \mathbf{x})}{\lambda(E_{ij}, \mathbf{x})} \quad (2)$$

holds. If G is left-compressed and $E_{i \setminus j} = \emptyset$ for i, j satisfying $1 \leq i < j \leq k$, then $x_i = x_j$.

(c) By (2), if G is left-compressed, then an optimal legal weighting $\mathbf{x} = (x_1, x_2, \dots, x_n)$ for G must satisfy

$$x_1 \geq x_2 \geq \dots \geq x_n \geq 0. \quad (3)$$

The following lemma implies that we only need to consider left-compressed r -graphs when Conjecture 1 is explored.

Lemma 3 ([12]) *Let m, t be positive integers satisfying $m \leq \binom{t}{r} - 1$, then there exists a left-compressed r -graph G with m edges such that $\lambda(G) = \lambda_m^r$.*

3 Proof of Theorem 4

The following lemma showed in [9] implies that we only need to consider left-compressed 3-graphs G on t vertices to verify Conjecture 3 for $r = 3$. Denote

$$\lambda_{(m,t)}^- := \max\{\lambda(G) : G \text{ is an } r\text{-graph with } m \text{ edges and does not contain a clique of size } t\}.$$

Lemma 4 [9] *Let m and t be positive integers satisfying $\binom{t-1}{3} \leq m \leq \binom{t-1}{3} + \binom{t-2}{2}$. Then there exists a left-compressed 3-graph G on the vertex set $[t]$ with m edges and not containing a clique of order $t-1$ such that $\lambda(G) = \lambda_{(m,t-1)}^{3-}$.*

Proof of Theorem 4. Let $\binom{t-1}{3} \leq m \leq \binom{t-1}{3} + \binom{t-2}{2} - \frac{1}{2}(t-1)$. Let G be a 3-graph with m edges without containing $[t-1]^{(3)}$ such that $\lambda(G) = \lambda_{(m,t-1)}^{3-}$. To prove Theorem 4, we only need to prove $\lambda_{(m,t-1)}^{3-} = \lambda(G) < \lambda([t-1]^{(3)})$.

By Lemma 4, we can assume that G is left-compressed. Let $\mathbf{x} = (x_1, x_2, \dots, x_n)$ be an optimal weighting for G . By Remark 2(a), $x_1 \geq x_2 \geq \dots \geq x_k > x_{k+1} = \dots = x_n = 0$. If $k \leq t-1$, $\lambda(G) < \lambda([t-1]^{(3)})$ since G does not contain a clique order of $[t-1]$. So we assume $k \geq t$. First we show that $k = t$. We need the following lemma.

Lemma 5 [12] *Let $G := (V, E)$ be a left-compressed 3-graph with m edges such that $\lambda(G) = \lambda_m^3$. Let $b := |E_{(k-1)k}|$. Let $\mathbf{x} := (x_1, x_2, \dots, x_k)$ be an optimal weighting for G satisfying $x_1 \geq x_2 \geq \dots \geq x_k > x_{k+1} = \dots = x_n = 0$. Then*

$$|[k-1]^{(3)} \setminus E| \leq \lceil b(1 + \frac{k-(b+2)}{k-3}) \rceil.$$

Since G is left-compressed and $1(k-1)k \in E$, then $|[k-2]^{(2)} \cap E_k| \geq 1$. If $k \geq t+1$, then applying Lemma 5, we have $|[k-1]^{(3)} \setminus E| \leq k-2$. Hence

$$\begin{aligned} m = |E| &= |E \cap [k-1]^{(3)}| + |[k-2]^{(2)} \cap E_k| + |E_{(k-1)k}| \\ &\geq \binom{t}{3} - (t-1) + 2 \\ &\geq \binom{t-1}{3} + \binom{t-2}{2} + 1, \end{aligned} \tag{4}$$

which contradicts to the assumption that $m \leq \binom{t-1}{3} + \binom{t-2}{2}$. Recall that $k \geq t$, so we have

$$k = t.$$

Hence we can assume G is on vertex set $[t]$.

Next we prove an inequality.

Lemma 6 *Let G be a 3-graph on the vertex set $[t]$. Let $\mathbf{x} := (x_1, x_2, \dots, x_t)$ be an optimal weighting for G satisfying $x_1 \geq x_2 \geq \dots \geq x_t \geq 0$. Then*

$$x_1 < x_{t-3} + x_{t-2} \text{ or } \lambda(G) \leq \frac{1}{6} \frac{(t-3)^2}{(t-2)(t-1)} < \lambda([t-1]^{(3)}).$$

Proof If $x_1 \geq x_{t-3} + x_{t-2}$, then

$$3x_1 + x_2 + \dots + x_{t-4} > x_1 + x_2 + \dots + x_{t-4} + x_{t-3} + x_{t-2} + x_{t-1} + x_t = 1.$$

Recall that $x_1 \geq x_2 \geq \dots \geq x_{t-4}$, we have $x_1 > \frac{1}{t-2}$. Using Lemma 2, we have

$$\begin{aligned} \lambda(G) &= \frac{1}{3} \lambda(E_1, \mathbf{x}) \leq \frac{1}{3} \binom{t-1}{2} \left(\frac{1 - \frac{1}{t-2}}{t-1} \right)^2 \\ &= \frac{1}{6} \frac{(t-3)^2}{(t-2)(t-1)} < \frac{1}{6} \frac{(t-3)(t-2)}{(t-1)^2} = \lambda([t-1]^{(3)}). \end{aligned}$$

The first inequality follows from Theorem 1. Hence $\lambda(G) < \lambda([t-1]^{(3)})$, which contradicts to $\lambda(G) \geq \lambda([t-1]^{(3)})$. This completes the proof. \square

The following lemma is proved in [15].

Lemma 7 ([15], Lemma 5.3) *Let G be a left-compressed 3-graph on the vertex set $[t]$. Let $\mathbf{x} := (x_1, x_2, \dots, x_t)$ be an optimal weighting for G . Then $|[t-1]^{(3)} \setminus E| \leq t-3$, or $\lambda(G) \leq \lambda([t-1]^{(3)})$.*

Remark 3 We can prove that $|[t-1]^{(3)} \setminus E| \leq t-3$, or $\lambda(G) < \lambda([t-1]^{(3)})$ under the condition of Lemma 7 through the method in [15].

Now we continue the proof of Theorem 4. Let $D = [t-1]^{(3)} \setminus E$. Let $b = |E_{(t-1)t}|$. By Lemma 5, we have $|D| \leq 2b$. So $\lfloor \frac{|D|}{2} \rfloor \leq b$ and the triples $1(t-1)t, \dots, \lfloor \frac{|D|}{2} \rfloor (t-1)t$ are in G . Let $G' = G \cup D \setminus \{1(t-1)t, \dots, \lfloor \frac{|D|}{2} \rfloor (t-1)t\}$. If $\lambda(G) < \lambda([t-1]^{(3)})$, we are done. Otherwise by Remark 3 we have $|D| \leq t-3$. So

$$\begin{aligned} |G'| &= |G| + |D| - \lfloor \frac{|D|}{2} \rfloor \leq \binom{t-1}{3} + \binom{t-2}{2} - \frac{1}{2}(t-1) + t-3 - \frac{t-3}{2} + 1 \\ &= \binom{t-1}{3} + \binom{t-2}{2}. \end{aligned}$$

Note that G' contains $[t-1]^{(3)}$. By Theorem 3, we have $\lambda(G', \mathbf{x}) \leq \lambda(G') = \lambda([t-1]^{(3)})$.

Next we show that $\lambda(G, \mathbf{x}) < \lambda(G', \mathbf{x})$. By Remark 2(b), $x_1 = x_2 = \dots = x_{\lfloor \frac{|D|}{2} \rfloor}$. Hence

$$\begin{aligned} \lambda(G', \mathbf{x}) - \lambda(G, \mathbf{x}) &= \lambda(D, \mathbf{x}) - \lfloor \frac{|D|}{2} \rfloor x_1 x_{t-1} x_t \\ &\geq |D| x_{t-3} x_{t-2} x_{t-1} - \lfloor \frac{|D|}{2} \rfloor x_1 x_{t-1} x_t \\ &> |D| x_{t-3} x_{t-2} x_{t-1} - \lfloor \frac{|D|}{2} \rfloor (x_{t-3} + x_{t-2}) x_{t-1} x_t. \end{aligned}$$

In the last step, we used Lemma 6. Recall that $x_1 \geq x_2 \geq \dots \geq x_t > 0$, we have

$$|D| x_{t-3} x_{t-2} x_{t-1} - \lfloor \frac{|D|}{2} \rfloor (x_{t-3} + x_{t-2}) x_{t-1} x_t \geq |D| x_{t-3} x_{t-2} x_{t-1} - |D| x_{t-3} x_{t-1} x_t \geq 0.$$

Hence $\lambda(G, \mathbf{x}) < \lambda(G', \mathbf{x}) \leq \lambda([t-1]^{(3)}) = \lambda(C_{3,m})$. This completes the proof of Theorem 4. \square

4 Connection between the clique number and the Lagrangians of some left-compressed 3-graphs

In this section, we will confirm Conjecture 1 and Conjecture 3 for some left-compressed 3-graphs with specified structures. As an application, we also obtain two weaker versions of Turán type result for left-compressed 3-graphs.

Theorem 5 *Let $G := (V, E)$ be a left-compressed 3-graph on vertex set $[t]$ and G does not contain a clique order of $\lfloor \frac{t-2}{2} \rfloor$. Then*

$$\lambda(G) \leq \frac{1}{6} \frac{(t-3)^2}{(t-2)(t-1)} < \lambda([t-1]^{(3)}).$$

Proof The idea to prove Theorem 5 is similar to that in the proof of Lemma 6. Let $G := (V, E)$ be a left-compressed 3-graph with m edges and $\omega(G) \leq \lfloor \frac{t-2}{2} \rfloor$. Recall $\omega(G)$ is the clique number of G . If $t \leq 5$, Theorem 5 clearly holds. Next we assume $t \geq 6$. Let $\mathbf{x} = (x_1, x_2, \dots, x_t)$ be an optimal weighting for G satisfying, $x_1 \geq x_2 \geq \dots \geq x_t$. The clique number of E_{t-3} must be smaller than $\frac{t-2}{2}$, otherwise $\omega(G) > \lfloor \frac{t-2}{2} \rfloor$ since G is left-compressed. By Lemma 6, if $\lambda(G) > \frac{1}{6} \frac{(t-3)^2}{(t-2)(t-1)}$, we have $x_{t-3} > \frac{1}{2t}$. Using Lemma 2 and Theorem 1, we have

$$\begin{aligned} \lambda(G) &= \frac{1}{3} \lambda(E_t, \mathbf{x}) < \frac{1}{3} \binom{\lfloor \frac{t-2}{2} \rfloor}{2} \left(\frac{1 - \frac{1}{2t}}{\lfloor \frac{t-2}{2} \rfloor} \right)^2 \\ &\leq \frac{1}{6} \frac{t-4}{t-2} \frac{(2t-1)^2}{4t^2} \\ &< \frac{1}{6} \frac{(t-3)^2}{(t-2)(t-1)}. \end{aligned}$$

which is a contradiction. This completes the proof. \square

Corollary 2 *Let $G := (V, E)$ be a left-compressed 3-graph with t vertices and m edges. If $m \geq \frac{(t-3)^2 t^3}{6(t-2)(t-1)}$, then G contains a clique order of $\lfloor \frac{t-2}{2} \rfloor$.*

Proof Let $G := (V, E)$ be a 3-graph with t vertices and m edges. Assume that $m \geq \frac{(t-3)^2}{6(t-2)(t-1)} t^3$. Clearly, $x_1 = x_2 = \dots = x_t = \frac{1}{t}$ is a legal weighting for G . Hence $\lambda(G) \geq \frac{(t-3)^2}{6(t-2)(t-1)} t^3 \frac{1}{t^3} = \frac{(t-3)^2}{6(t-2)(t-1)}$. However by Theorem 5 we know that $\lambda(G) < \frac{(t-3)^2}{6(t-2)(t-1)}$ if G does not contain a clique order of $\lfloor \frac{t-2}{2} \rfloor$. This completes the proof. \square

For the case of forbidding a clique of order 4, we have the following result.

Proposition 1 *Let G be a left-compressed 3-uniform graph on $[t]$ with m edges. If G does not contain a clique of order 4, then $m \leq \frac{2}{27} t^3$.*

Proof Let $\mathbf{x} := (x_1, x_2, \dots, x_t)$ be an optimal vector of G . We claim that all edges in G must contain vertex 1. Otherwise, 234 is an edge of G and G contains the clique $[4]^{(3)}$ since G is left-compressed. So

$$\lambda(G) \leq x_1 \cdot \frac{1}{2} (x_2 + x_3 + \dots + x_k)^2 = \frac{1}{2} x_1 (1 - x_1)^2 \leq \frac{1}{2} \times \frac{4}{27} \left(x_1 + \frac{1-x_1}{2} + \frac{1-x_1}{2} \right)^3 = \frac{2}{27}.$$

Let $\mathbf{y} = (y_1, y_2, \dots, y_t)$ given by $y_i = \frac{1}{t}$ for each $i, 1 \leq i \leq t$. Then $\frac{2}{27} \geq \lambda(G) \geq \lambda(G, \mathbf{y}) = \frac{m}{t^3}$. Therefore, $m \leq \frac{2}{27}t^3$. \square

In [1], Buló and Pelillo proved the following theorem.

Theorem 6 ([1]) *An r -graph $G = (V, E)$ with m edges and t vertices, which contains no p -clique with $p \geq r$, then*

$$m \leq \binom{t}{r} - \frac{t}{(r-1)r} \left[\left(\frac{t}{p-1} \right)^{r-1} - 1 \right].$$

Remark 4 (1) We note that Theorem 5 and Corollary 2 establish a connection between Lagrangian and clique number for 3-graphs. They also provide evidence for Conjecture 3.

(2) For the case $r = 3$ and $p = \lfloor \frac{t-2}{2} \rfloor$, the upper bound in Theorem 6 is bigger than $\frac{(t^4 - 11t^3 + 39t^2 - 72t + 48)t}{6(t-4)^2}$. Since

$$\frac{(t^4 - 11t^3 + 39t^2 - 72t + 48)t}{6(t-4)^2} > \frac{(t-3)^2 t^3}{6(t-2)(t-1)}$$

when $t \geq 38$, the result in Corollary 2 is better than the result in Theorem 6 under the left-compressed condition.

(3) Again, for the case $r = 3$ and $p = 4$, the upper bound in Theorem 6 is bigger than the bound in Propostion 1 under the left-compressed condition.

Next we give the following partial result to Conjecture 1.

Theorem 7 *Let m, t , and a be positive integers satisfying $m = \binom{t-1}{3} + \binom{t-2}{2} + a$ where $1 \leq a \leq t-2$. Let $G = (V, E)$ be a left-compressed 3-graph on the vertex set $[t]$ with m edges satisfying $|E_{(t-1)t}| \leq \frac{2t+3a-4}{5}$. If G contains a clique of order $t-1$, then $\lambda(G) \leq \lambda(C_{3,m})$.*

Proof Let G be a 3-graph with m edges and containing a clique of order $t-1$. Assume $\mathbf{x} := (x_1, x_2, \dots, x_t)$ is an optimal weighting for G satisfying $x_1 \geq x_2 \geq \dots \geq x_t \geq 0$. We will prove that $\lambda(C_{3,m}, \mathbf{x}) - \lambda(G, \mathbf{x}) \geq 0$. Therefore $\lambda(C_{3,m}) \geq \lambda(C_{3,m}, \mathbf{x}) \geq \lambda(G, \mathbf{x}) = \lambda(G)$.

By Remark 2(b) and noting that G contains $[t-1]^{(3)}$, we have

$$x_1 = x_{t-3} + \frac{\lambda(E_{1 \setminus (t-3)}, \mathbf{x})}{\lambda(E_{1(t-3)}, \mathbf{x})} = x_{t-3} + \frac{\lambda(E_{t-3}^c, \mathbf{x})}{\lambda(E_{1(t-3)}, \mathbf{x})} = x_{t-3} + \frac{x_t \lambda(E_{(t-3)t}^c, \mathbf{x})}{\lambda(E_{1(t-3)}, \mathbf{x})}, \quad (5)$$

and

$$x_{t-2} = x_{t-1} + \frac{\lambda(E_{(t-2) \setminus (t-1)}, \mathbf{x})}{\lambda(E_{(t-2)(t-1)}, \mathbf{x})} = x_{t-1} + \frac{x_t \lambda(E_{(t-2)t} \cap E_{(t-1)t}^c, \mathbf{x})}{\lambda(E_{(t-2)(t-1)}, \mathbf{x})}. \quad (6)$$

Let $b := |E_{(t-1)t}|$. Since G contains the clique $[t-1]^{(3)}$, we have $|[t-2]^{(2)} \setminus E_t| = b - a$. Note that $|E_{(t-3)t}^c| \leq |E_{(t-2)t}^c|$ since G is left-compressed, we have $|E_{(t-3)t}^c| \leq \frac{b-a}{2} + 1$ (Note that $t-1 \in E_{(t-3)t}^c$). On the other hand, $|E_{(t-2)t}| = t-2 - |E_{(t-2)t}^c| \geq t-2 - (b-a) - 1$ (Note that $t-1 \in E_{(t-2)t}^c$) and $|E_{(t-2)t} \cap E_{(t-1)t}^c| \geq (t-2) - (b-a) - 1 - b = t-2b+a-1$. Recalling that $|E_{(t-1)t}| \leq \frac{2t+3a-4}{5}$, we have $|E_{(t-3)t}^c| \leq |E_{(t-2)t} \cap E_{(t-1)t}^c|$. Let i be the minimum integer in $E_{(t-3)t}^c$ and j be the minimum integer in $E_{(t-2)t} \cap E_{(t-1)t}^c$. Because G is left-compressed, we have $i \geq j$. Hence

$$\lambda(E_{(t-3)t}^c, \mathbf{x}) \leq \lambda(E_{(t-2)t} \cap E_{(t-1)t}^c, \mathbf{x}). \quad (7)$$

Since $x_1 \geq x_2 \geq \dots \geq x_t$. Next we prove that

$$\lambda(E_{1(t-3)}, \mathbf{x}) - \lambda(E_{(t-2)(t-1)}, \mathbf{x}) = x_{t-2} + x_{t-1} + x_t - x_1 - x_{t-3} \geq 0. \quad (8)$$

To verify (8), by Remark 2(b), we have

$$x_1 = x_{t-1} + \frac{\lambda(E_{1 \setminus (t-1)}, \mathbf{x})}{\lambda(E_{1(t-1)}, \mathbf{x})} \leq x_{t-1} + \frac{(x_2 + \dots + x_{t-2})x_t}{x_2 + \dots + x_{t-2} + x_t} \leq x_{t-1} + x_t; \quad (9)$$

$$\begin{aligned} x_1 &= x_{t-2} + \frac{\lambda(E_{1 \setminus (t-2)}, \mathbf{x})}{\lambda(E_{1(t-2)}, \mathbf{x})} \\ &= x_{t-2} + \frac{\lambda(E_{(t-2)t}^c, \mathbf{x})}{1 - x_1 - x_{t-2}} x_t \\ &\leq x_{t-2} + \frac{\lambda(E_{(t-2)t}^c, \mathbf{x})}{1 - x_{t-3} - x_{t-1} - x_t} x_t; \end{aligned} \quad (10)$$

and

$$\begin{aligned} x_{t-3} &= x_{t-1} + \frac{\lambda(E_{(t-3) \setminus (t-1)}, \mathbf{x})}{\lambda(E_{(t-3)(t-1)}, \mathbf{x})} \\ &= x_{t-1} + \frac{\lambda(E_{(t-3)t} \cap E_{(t-1)t}^c, \mathbf{x})}{1 - x_{t-3} - x_{t-1} - x_t} x_t. \end{aligned} \quad (11)$$

Adding (10) and (11), we obtain that

$$x_1 + x_{t-3} \leq x_{t-2} + x_{t-1} + \frac{\lambda(E_{(t-2)t}^c, \mathbf{x}) + \lambda(E_{(t-3)t} \cap E_{(t-1)t}^c, \mathbf{x})}{1 - x_{t-3} - x_{t-1} - x_t} x_t.$$

Clearly $t-3 \notin E_{(t-3)t}$. Since G is left-compressed and $G \neq C_{3,m}$, we have $t-2 \notin E_{(t-3)t}$. On the other hand both $t-3$ and $t-2$ are in $E_{(t-1)t}^c$. Hence $|E_{(t-2)t}^c| + |E_{(t-3)t} \cap E_{(t-1)t}^c| \leq |E_{(t-2)t}^c| + |E_{(t-1)t}^c| - 2$. Recalling that $|E^c| \leq t-3$, we have $|E_{(t-2)t}^c| + |E_{(t-1)t}^c| \leq |E^c| \leq t-2$ (Note that $t-1 \in E_{(t-2)t}^c$ and $t-2 \in E_{(t-1)t}^c$) and $|E_{(t-2)t}^c| + |E_{(t-3)t} \cap E_{(t-1)t}^c| \leq t-4$. Clearly $b \geq 2$. Hence 2 is not in $E_{(t-1)t}^c$ and $E_{(t-2)t}^c$. Recalling that $x_1 \geq x_2 \geq \dots \geq x_t$, we have $\frac{\lambda(E_{(t-2)t}^c, \mathbf{x}) + \lambda(E_{(t-3)t} \cap E_{(t-1)t}^c, \mathbf{x})}{1 - x_{t-3} - x_{t-1} - x_t} \leq 1$. So, (8) is true. This implies that $\lambda(E_{(t-2)(t-1)}, \mathbf{x}) \leq \lambda(E_{1(t-3)}, \mathbf{x})$. Combining (5), (6) and (7), we obtain that $x_1 - x_{t-3} \leq x_{t-2} - x_{t-1}$ and $x_{t-3}x_{t-2}x_t - x_1x_{t-1}x_t \geq 0$. Hence

$$\begin{aligned} \lambda(C_{3,m}, \mathbf{x}) - \lambda(G, \mathbf{x}) &= \lambda([t-2]^{(2)} \setminus E_t, \mathbf{x}) - |[t-2]^{(2)} \setminus E_t| x_1 x_{t-1} x_t \\ &\geq |[t-2]^{(2)} \setminus E_t| (x_{t-3} x_{t-2} x_t - x_1 x_{t-1} x_t) \\ &\geq 0. \end{aligned} \quad (12)$$

This completes the proof. \square

Acknowledgements This research is partially supported by National Natural Science Foundation of China (No. 11271116).

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