# Conditions for the existence, identification and calculus rules of the threshold of prox-boundedness 

C. Planiden*

September 12, 2019


#### Abstract

This work advances knowledge of the threshold of prox-boundedness of a function; an important concern in the use of proximal point optimization algorithms and in determining the existence of the Moreau envelope of the function. In finite dimensions, we study general prox-bounded functions and then focus on some useful classes such as piecewise functions and Lipschitz continuous functions. The thresholds are explicitly determined when possible and bounds are established otherwise. Some calculus rules are constructed; we consider functions with known thresholds and find the thresholds of their sum and composition.


## 2000 Mathematics Subject Classification:

Primary 49J53; Secondary 26A06, 90C30
Keywords: Fenchel conjugate, infimal convolution, Lipschitz continuous, Moreau envelope, MoreauYosida regularization, piecewise function, prox-bounded, proximal mapping, regularization, threshold.

## 1 Introduction

The Moreau envelope function, also known as Moreau-Yosida regularization, is a particular infimal convolution that first came about in the 1960s [32]. Given a function $f$ on a finite-dimensional space, the Moreau envelope of $f$ employs a nonnegative parameter $r$ and is denoted $e_{r} f$ :

$$
\begin{equation*}
e_{r} f(x)=\inf _{y \in \mathbb{R}^{n}}\left\{f(y)+\frac{r}{2}\|y-x\|^{2}\right\} \tag{1.1}
\end{equation*}
$$

It is a well-established, regularizing function that has many desirable properties when $f$ has reasonable structure [38, 40]:

[^0]- if $f$ is convex nonsmooth, $e_{r} f$ is convex smooth and there is an explicit formula for the gradient $\nabla e_{r} f$;
- the functions $f$ and $e_{r} f$ have the same minimum and minimizers when $f$ is convex;
- as $r \rightarrow \infty, e_{r} f \rightarrow f$.

The set of all solution points to (1.1) is known as the proximal mapping of $f$, denoted by $\operatorname{Prox}_{f}$. The proximal mapping is a key component of many optimization algorithms, such as the proximal point method and its variants [5, $8,13,16,20,21,37]$. Because of the above and other nice features, the Moreau envelope and proximal mapping have been thoroughly researched and applied to many situations in the convex [10, 17, 24, 29, 33, 35] and nonconvex [22, 25, 26, 28, 30, 34, 39] settings.

For a particular function $f$, its Moreau envelope may or may not exist, or may exist only for certain $x$ and/or certain $r$. If there does exist a point $x$ such that $e_{r} f(x) \in \mathbb{R}$ for some $r \geq 0$, we say that $f$ is prox-bounded. If a function is not prox-bounded, then its Moreau envelope does not exist anywhere, for any choice of $r$. It is the parameter $r$ that is of primary interest in this work. There are many theoretically proved-convergent proximal algorithms (see [1, 2, 7, 9, 11, 12] and the references therein), but in practice, it has been observed that the initial choice of $r$ and the manner of adjusting it as the algorithm runs are of critical importance, in order to obtain reliable performance [6, 14, 15, 23, 27, 36]. We explore the threshold of prox-boundedness of $f$ : the infimum of the set of $r \geq 0$ such that $e_{r} f$ exists at at least one point.

In [19], the class of piecewise linear-quadratic (PLQ) functions on $\mathbb{R}^{n}$ was examined in the context of prox-boundedness. The main result of [19] is a theorem that explicitly identifies the threshold and the domain of the Moreau envelope of any finite-dimensional PLQ function. In that setting, the threshold is max $r_{i}$, where $r_{i}$ is the threshold of $f_{i}$ for each $i$. One of the aims of the present work is to generalize that result in two aspects. We consider the cases where
(1) the functions $f_{i}$ are not necessarily linear nor quadratic and
(2) the domains dom $f_{i}$ are not necessarily polyhedral.

The main question on which we focus is this: what are the minimal conditions needed on $f_{i}$ in order to be sure that the threshold of the piecewise function is max $r_{i}$ ? We establish bounds and illustrate several counterexamples for functions with conditions that one might suspect sufficient to guarantee prox-boundedness, but are not. Under certain conditions, the threshold can be determined exactly.

The second focus of this work is the establishing of calculus rules for thresholds of proxbounded functions. We explore classes of functions with known thresholds and study the conditions needed to determine the threshold of their sum and composition, or to produce an upper bound when an exact result cannot be found. By making use of the Fenchel conjugate representation of the Moreau envelope and some other previously-established properties and characterizations of prox-bounded functions, we determine sufficient conditions for the existence of a sum rule and a composition rule for thresholds of prox-boundedness.

The remainder of this work is organized as follows. Section 2 presents the notation used throughout and several definitions and known facts regarding prox-bounded functions and their
thresholds. In Section 3, we examine the family of piecewise functions and determine the minimal conditions needed for finding the threshold. An example of what can go wrong when these conditions are not met is provided. Section 4 is dedicated to forming calculus rules for the threshold of the sum and the composition of prox-bounded functions. Section 5 offers concluding remarks and suggests interesting areas of further research in this vein.

## 2 Preliminaries

### 2.1 Notation

Throughout this paper, we work in finite-dimensional space $\mathbb{R}^{n}$, endowed with inner product $\langle\cdot, \cdot\rangle$ and induced norm $\|\cdot\|$. The set $\mathbb{R}^{n} \cup\{+\infty\}$ is denoted by $\overline{\mathbb{R}}$. We generally conform to the notation used in [38], including the terms proper and lower semicontinuous (lsc) defined therein. We denote the domain of a function $f$ by $\operatorname{dom} f$ and the gradient of $f$ by $\nabla f$. The distance from a point $x \in \mathbb{R}^{n}$ to a set $C$ is denoted $d_{C}(x)$, and the projection of $x$ onto $C$ is denoted $P_{C} x$.

Definition 2.1. Given $K>0$, the function $f: \mathbb{R}^{n} \rightarrow \mathbb{R}$ is locally $K$-Lipschitz continuous about $z \in \operatorname{dom} f$ with radius $\sigma>0$ if

$$
\|f(y)-f(x)\| \leq K\|y-x\| \text { for all } x, y \in B_{\sigma}(z)
$$

and $f$ is globally $K$-Lipschitz continuous if $\sigma$ can be taken to be $\infty$.
Definition 2.2. The indicator function of a set $S$ is denoted $\iota_{S}$ and defined by

$$
\iota_{s}(x)= \begin{cases}0, & x \in S \\ \infty, & x \notin S\end{cases}
$$

Definition 2.3. For proper functions $f_{i}: \mathbb{R}^{n} \rightarrow \overline{\mathbb{R}}, i \in\{1,2, \ldots, m\}$, the piecewise function $f: \mathbb{R}^{n} \rightarrow \overline{\mathbb{R}}$ is defined by

$$
f(x)= \begin{cases}f_{1}(x), & x \in S_{1} \\ & \vdots \\ f_{m}(x), & x \in S_{m}\end{cases}
$$

where $\bigcup_{i} S_{i}=\mathbb{R}^{n}$ and $S_{i} \cap \operatorname{int} S_{j}=\varnothing$ for $i \neq j$.
Notice that a piecewise function is not necessarily continuous, as we do not require $f_{i}(x)=f_{j}(x)$ on $S_{i} \cap S_{j}$.

Definition 2.4. Given a finite number of functions $f_{1}, f_{2}, \ldots, f_{m}: \mathbb{R}^{n} \rightarrow \overline{\mathbb{R}}$, the finite-max function $f$ is defined by

$$
f(x)=\max \left\{f_{i}(x)\right\} .
$$

The active set of indices for $f$ at a point $x$ is the set

$$
A_{x}=\left\{i: f_{i}(x)=f(x)\right\} .
$$

Note that a finite-max function is a piecewise function with $S_{i}=\left\{x: i \in A_{x}\right\}$.
Definition 2.5. A function $f: \mathbb{R}^{n} \rightarrow \overline{\mathbb{R}}$ is prox-bounded if there exists $r \geq 0$ such that $e_{r} f(x)>$ $-\infty$ for some $x \in \mathbb{R}^{n}$. The infimum of all such $r$ is called the threshold of prox-boundedness of $f$.

Our interest in this work is to identify thresholds of prox-boundedness of functions and the conditions for their existence. To that end, we list the following results from previous works.

Fact 2.6. [19, Lemma 2.4] Let $f: \mathbb{R}^{n} \rightarrow \overline{\mathbb{R}}$ be proper and lsc. Then $f$ is bounded below if and only if its threshold $\bar{r}=0$ and $\operatorname{dom} e_{\bar{r}} f=\mathbb{R}^{n}$.

Fact 2.7. [38, Theorem 1.25] Let $f: \mathbb{R}^{n} \rightarrow \mathbb{R}$ be proper, lsc and prox-bounded with threshold $\bar{r}$. Then for all $r>\bar{r}$, dom $e_{r} f=\mathbb{R}^{n}$.

Fact 2.8. [38, Example 3.28] Let $f: \mathbb{R}^{n} \rightarrow \mathbb{R}$ be such that

$$
\liminf _{\|x\| \rightarrow \infty} \frac{f(x)}{\|x\|}>-\infty
$$

Then $f$ is prox-bounded with threshold $\bar{r}=0$.
Fact 2.9. [38, Theorem 2.26] Let $f: \mathbb{R}^{n} \rightarrow \mathbb{R}$ be proper, lsc and convex. Then $f$ is prox-bounded with threshold $\bar{r}=0$.

Note that by definition of prox-regular, if $r<\bar{r}$, then $e_{r} f(x)=-\infty$ for all $x \in \mathbb{R}^{n}$. By Fact 2.7, if $r>\bar{r}$, then $e_{r} f(x)>-\infty$ for all $x \in \mathbb{R}^{n}$. At the threshold itself, however, there is no such universal behaviour of the Moreau envelope. Depending on the nature of $f$, $\operatorname{dom} e_{\bar{r}} f$ can be empty, full-domain or a proper subset of $\mathbb{R}^{n}$, even for very simple functions (see 19 , Examples $2.5-2.7]$ ). This is partly why the choice of initial prox-parameter $r$ and the manner in which it changes are so crucial in many minimization algorithms; vastly different proximal behaviour is possible with distinct values of $r$. In the next section, we explore existence conditions for the threshold of prox-boundedness of piecewise functions.

## 3 Existence of thresholds of prox-boundedness

The conditions for prox-boundedness in the case of PLQ functions was thoroughly examined in [19]. Now we move beyond that class of functions. We concentrate primarily on piecewise functions as defined in Definition 2.3, as that is a natural extension to what has been accomplished already. We begin by establishing the fact that Lipschitz continuous functions are prox-bounded.

Proposition 3.1. Let $f: \mathbb{R}^{n} \rightarrow \mathbb{R}$ be proper and globally $K$-Lipschitz. Then $f$ is prox-bounded with threshold $\bar{r}=0$.

Proof. Suppose that $f$ is not prox-bounded, i.e. $e_{r} f(x)=-\infty$ for all $r \geq 0$, for all $x \in \mathbb{R}^{n}$. Let $r>0$ be fixed and arbitrary. Then for any $\bar{x} \in \mathbb{R}^{n}$, there exists a sequence $\left\{x_{\alpha}\right\}_{\alpha=1}^{\infty} \subseteq \mathbb{R}^{n}$ such that

$$
\begin{equation*}
\lim _{\alpha \rightarrow \infty}\left\{f\left(x_{\alpha}\right)+\frac{r}{2}\left\|\bar{x}-x_{\alpha}\right\|^{2}\right\}=-\infty \tag{3.1}
\end{equation*}
$$

Then $\lim _{\alpha \rightarrow \infty} f\left(x_{\alpha}\right)=-\infty$, since the other term $\frac{r}{2}\left\|\bar{x}-x_{\alpha}\right\|^{2}$ is always nonnegative. Hence,

$$
\lim _{\alpha \rightarrow \infty}\left|f(\bar{x})-f\left(x_{\alpha}\right)\right|=\infty \quad(f \text { is proper, so } f(\bar{x}) \neq-\infty)
$$

and since $f$ is $K$-Lipschitz, we have

$$
\lim _{\alpha \rightarrow \infty} K\left\|\bar{x}-x_{\alpha}\right\|=\infty
$$

Thus, $\lim _{\alpha \rightarrow \infty} \frac{r}{2}\left\|\bar{x}-x_{\alpha}\right\|^{2}=\infty$. This together with (3.1) says that as $\alpha \rightarrow \infty, f\left(x_{\alpha}\right) \rightarrow-\infty$ faster than $\frac{r}{2}\left\|\bar{x}-x_{\alpha}\right\|^{2} \rightarrow \infty$, i.e., $\frac{r}{2}\left\|\bar{x}-x_{\alpha}\right\|^{2}=o\left(f\left(x_{\alpha}\right)\right)$. Since $\frac{r}{2}\left\|\bar{x}-x_{\alpha}\right\|^{2}$ is not a Lipschitz continuous function and $f\left(x_{\alpha}\right)$ grows even faster, we have that $f\left(x_{\alpha}\right)$ is not Lipschitz either, a contradiction. Therefore, there must exist at least one $\bar{x} \in \mathbb{R}^{n}$ such that $e_{r} f(\bar{x})>-\infty$ and we have that $f$ is prox-bounded. Since this is true for any arbitrary $r>0$, it is true for all $r>0$. The threshold of prox-boundedness is the infimum of all such $r$, so $\bar{r}=0$.

Now we focus on the family of piecewise functions and say what we can about their thresholds. Henceforth, any mention of a piecewise function refers to a function defined as in Definition 2.3.

Proposition 3.2. Let $f: \mathbb{R}^{n} \rightarrow \overline{\mathbb{R}}$ be a proper, lsc, piecewise function. Then $f$ is prox-bounded if and only if $f_{i}+\iota_{S_{i}}$ is prox-bounded for each $i$.

Proof. $(\Rightarrow)$ Let $f$ be prox-bounded with threshold $\bar{r}$. Suppose that there exists $j$ such that $f_{j}+\iota_{S_{j}}$ is not prox-bounded. Fix $r>\bar{r}$, so that $e_{r} f(x) \in \mathbb{R}$ for all $x$ (Fact 2.6). By definition of $e_{r} f$, for $\bar{x}$ fixed we have

$$
\begin{align*}
e_{r} f(\bar{x}) & =\inf _{y}\left\{f(y)+\frac{r}{2}\|y-\bar{x}\|^{2}\right\} \\
& =\min \left[\inf _{y}\left\{f_{1}(y)+\iota_{S_{1}}(y)+\frac{r}{2}\|y-\bar{x}\|^{2}\right\}, \ldots, \inf _{y}\left\{f_{m}(y)+\iota_{S_{m}}(y)+\frac{r}{2}\|y-\bar{x}\|^{2}\right\}\right] . \tag{3.2}
\end{align*}
$$

Since $f_{j}+\iota_{s_{j}}$ is not prox-bounded, we have

$$
\inf _{y}\left\{f_{j}(y)+\iota_{S_{j}}(y)+\frac{r}{2}\|y-\bar{x}\|^{2}\right\}=-\infty,
$$

hence $e_{r} f(\bar{x})=-\infty$. This is a contradiction to the fact that $e_{r}(\bar{x}) \in \mathbb{R}$. Therefore, $f_{i}+\iota_{S_{i}}$ is prox-bounded for all $i \in\{1,2, \ldots, m\}$.
$(\Leftarrow)$ Let $f_{i}+\iota_{S_{i}}$ be prox-bounded with threshold $r_{i}$ for each $i \in\{1,2, \ldots, m\}$. Let $\bar{r}=\max r_{i}$, and choose any $r>\bar{r}$. Then $r>r_{i}$ for all $i$, so that $e_{r} f_{i}(x) \in \mathbb{R}$ for all $i$ and for all $x$. Then each infimum in (3.2) is a real number, hence the minimum of (3.2) exists. Therefore, $e_{r} f(\bar{x}) \in \mathbb{R}$, and we have that $f$ is prox-bounded.

Theorem 3.3. Let $f: \mathbb{R}^{n} \rightarrow \overline{\mathbb{R}}$ be a proper, lsc, piecewise function and let each $f_{i}+\iota_{S_{i}}$ be prox-bounded with threshold $r_{i}$. Then the threshold of $f$ is $\bar{r}=\max r_{i}$.
Proof. Choose any $r>\max r_{i}$. By Proposition 3.2, there exists $\bar{x}$ such that $e_{r} f(\bar{x}) \in \mathbb{R}$. Since $r$ is arbitrary, we have that $f$ is prox-bounded for all $r>\max r_{i}$. Hence, $\bar{r} \leq \max r_{i}$. Now choose any $r<\max r_{i}$. Then there exists $j \in\{1,2, \ldots, m\}$ such that $r<r_{j}$. By definition of proxboundedness, $e_{r}\left(f_{j}+\iota_{S_{j}}\right)(x)=-\infty$ for all $x$. The Moreau envelope of $f$ is the expression of (3.2), whose minimand contains at least one instance of $-\infty$ due to $f_{j}+\iota_{S_{j}}$. Hence, $e_{r} f(x)=-\infty$ for all $x$, and we have that $\bar{r} \geq \max r_{i}$. Therefore, $\bar{r}=\max r_{i}$.

We have our first results for piecewise functions. However, the result of Theorem 3.3 is weakened if the term $\iota_{S_{i}}$ is removed from the statement and we require $f_{i}$ itself to be prox-bounded, as Theorem 3.4 shows.
Theorem 3.4. For $i \in\{1,2, \ldots, m\}$, let $f_{i}: \mathbb{R}^{n} \rightarrow \overline{\mathbb{R}}$ be proper, lsc and prox-bounded with threshold $r_{i}$. With these $f_{i}$, define a piecewise function $f$ as per Definition 2.3 Then $f$ is proxbounded with threshold $\bar{r} \leq \max r_{i}$.
Proof. Since $f_{i}$ is prox-bounded with threshold $r_{i}$, there exists $\bar{x} \in \mathbb{R}^{n}$ such that $e_{r_{i}} f_{i}(\bar{x})>-\infty$. Since

$$
e_{r_{i}} f_{i}(\bar{x})=\inf _{y \in \mathbb{R}^{n}}\left\{f_{i}(y)+\frac{r_{i}}{2}\|y-\bar{x}\|^{2}\right\} \leq \inf _{y \in S_{i}}\left\{f_{i}(y)+\frac{r_{i}}{2}\|y-\bar{x}\|^{2}\right\}=e_{r_{i}}\left(f_{i}+\iota_{S_{i}}\right)(\bar{x}),
$$

we have that $e_{r_{i}}\left(f_{i}+\iota_{S_{i}}\right)(\bar{x})>-\infty$. Hence, $f_{i}+\iota_{S_{i}}$ is prox-bounded with threshold $\tilde{r}_{i} \leq r_{i}$. This is true for all $i \in\{1,2, \ldots, m\}$, so Theorem 3.3 applies and we have $\bar{r}=\max \tilde{r}_{i} \leq \max r_{i}$.

The best we can do is an upper bound in this case. One might hope to establish a lower bound for $\bar{r}$ as well, such as $\min r_{i}$. However, this cannot be done in the general setting of Theorem 3.4, Example 3.5 illustrates why not.
Example 3.5. Let $f_{1}, f_{2}: \mathbb{R} \rightarrow \mathbb{R}$,

$$
f_{1}(x)=\left\{\begin{array}{ll}
x^{2}, & x<0, \\
-x^{2}, & x \geq 0,
\end{array} \quad f_{2}(x)= \begin{cases}-x^{2}, & x<0 \\
x^{2}, & x \geq 0\end{cases}\right.
$$

## Define

$$
f(x)= \begin{cases}f_{1}(x), & x<0 \\ f_{2}(x), & x \geq 0\end{cases}
$$

Then $r_{1}=r_{2}=2$ and $\bar{r}=0$.
Proof. Considering $f_{1}$ first, we define

$$
\varphi_{r}(y)=f_{1}(y)+\frac{r}{2}|y-x|^{2}= \begin{cases}\left(1+\frac{r}{2}\right) y^{2}-r x y+\frac{r}{2} x^{2}, & y<0 \\ \left(-1+\frac{r}{2}\right) y^{2}-r x y+\frac{r}{2} x^{2}, & y \geq 0\end{cases}
$$

For any $r>2$, both pieces of $\varphi_{r}$ are strictly convex quadratic. Thus, $e_{r} f_{1}(x)=\inf \phi_{r}(y)>-\infty$ and $r_{1} \leq 2$. For any $r<2$, the second piece of $\varphi_{r}$ is concave quadratic, so $e_{r} f_{1}(x)=-\infty$ for all $x$ and $r_{1} \geq 2$. Therefore, $r_{1}=2$. Similarly, $r_{2}=2$. But $f(x)=x^{2}$ has $\bar{r}=0$ by Fact 2.6.


Figure 1: $f_{1}$ (red) and $f_{2}$ (green) have threshold 2, but $f=\max \left\{f_{1}, f_{2}\right\}$ (black) has threshold 0.
It is equally simple to construct an example where $\bar{r}=\max r_{i}$ for a piecewise function, for instance

$$
f(x)= \begin{cases}f_{1}(x), & x \geq 0 \\ f_{2}(x), & x<0\end{cases}
$$

where $f_{1}, f_{2}$ are defined in Example 3.5, In that case, $f(x)=-x^{2}$ and $\bar{r}=2=\max \left\{r_{1}, r_{2}\right\}$. So we cannot do better than bounding $\bar{r}$ from above in this most general setting. Furthermore, one can obtain a prox-bounded function from the sum of two functions that are not prox-bounded. For instance, $f_{1}(x)=x^{3}$ and $f_{2}(x)=-x^{3}$ are not prox-bounded, but their sum is the constant function zero, with threshold zero. The next section considers more specific cases of both the sum and the composition of functions, where we can make some tighter conclusions about exact thresholds.

## 4 Calculus of the threshold of prox-boundedness

In this section, we consider the thresholds of the sum and the composition of prox-bounded functions. The functions here are no longer (necessarily) piecewise functions, as they were in the previous section. The following definition and facts will be useful.

Definition 4.1 (Fenchel conjugate). For any function $f: \mathbb{R}^{n} \rightarrow \overline{\mathbb{R}}$, the Fenchel conjugate of $f$ is the function $f^{*}: \mathbb{R}^{n} \rightarrow \overline{\mathbb{R}}$ defined by

$$
f^{*}(y)=\sup _{x \in \mathbb{R}^{n}}\{\langle y, x\rangle-f(x)\} .
$$

The Fenchel conjugate and the Moreau envelope enjoy a beautiful equivalence, as the following fact states.

Fact 4.2. [34, Proposition 2.1] For any proper function $f: \mathbb{R}^{n} \rightarrow \overline{\mathbb{R}}$,

$$
e_{r} f(x)=\frac{r}{2}\|x\|^{2}-g^{*}(r x),
$$

where $g(x)=f(x)+\frac{r}{2}\|x\|^{2}$.

Fact 4.3. [38, Exercise 1.24] For a proper, Isc function $f: \mathbb{R}^{n} \rightarrow \overline{\mathbb{R}}$, the following are equivalent:
(i) $f$ is prox-bounded;
(ii) $f$ majorizes a quadratic function;
(iii) there exists $r \in \mathbb{R}$ such that $f+\frac{r}{2}\|\cdot\|^{2}$ is bounded below;
(iv) $\liminf _{\|x\| \rightarrow \infty} \frac{f(x)}{\|x\|^{2}}>-\infty$.

If $\hat{r}$ is the infimum of all $r$ for which (iii) holds, the limit in (iv) is $-\frac{\hat{r}}{2}$ and the threshold for $f$ is $\bar{r}=\max \{0, \hat{r}\}$.

First, we address the quadratic function mentioned in Fact 4.3(ii). Rockafellar and Wets state that it exists, but give no details as to its form. Lemma 4.4 below describes the curvature that such a quadratic function must have, in terms of the threshold.

Lemma 4.4. Let $f: \mathbb{R}^{n} \rightarrow \overline{\mathbb{R}}$ be proper, lsc and prox-bounded with threshold $\bar{r}>0$. Then $f$ is bounded below by $-\frac{\bar{r}}{2}\|\cdot\|^{2}+m$ for some $m \in \mathbb{R}$. Furthermore, for any choice of $r<\bar{r}$ there does not exist $m \in \mathbb{R}$ such that $f$ is bounded below by $-\frac{r}{2}\|\cdot\|^{2}+m$. Therefore, $\frac{\bar{r}}{2}$ is the smallest possible curvature of a quadratic function that is a minorant of $f$.
Proof. By Fact 4.3 (iii), we have that $f+\frac{\bar{r}}{2}\|\cdot\|^{2}$ is bounded below, i.e., there exists $m \in \mathbb{R}$ such that for all $x \in \operatorname{dom} f, f(x)+\frac{\bar{r}}{2}\|x\|^{2} \geq m$. Rearranging, we have

$$
f(x) \geq-\frac{\bar{r}}{2}\|x\|^{2}+m \quad \forall x \in \operatorname{dom} f
$$

Suppose that for some $r<\bar{r}$, there exists $m$ such that the above inequality holds, replacing $\bar{r}$ with $r$. Then we have that $f+\frac{r}{2}\|\cdot\|^{2}$ is bounded below, which by Fact4.3(iii) and the postamble contradicts the fact that $\bar{r}$ is the threshold of $f$. Therefore, $\frac{\bar{r}}{2}$ is the minimum curvature of a quadratic minorant of $f$.

Remark 4.5. In Lemma 4.4 the condition $\bar{r}>0$ is necessary. We cannot make such a determination of the quadratic curvature in the case of $\bar{r}=0$, as there exist functions with threshold zero (such as affine functions) that are bounded below by a quadratic of curvature zero (i.e. affine function) and others that are not. For instance, the function $f: \mathbb{R} \rightarrow \mathbb{R}, f(x)=-|x|$ has threshold zero and is not bounded below by any affine function, but is bounded below by a concave quadratic function of any curvature greater than zero. We see this by noting that the inequality

$$
-|x| \geq-\frac{r}{2} x^{2}+m
$$

can be made true for all $x$ by making $m=m_{r}$ and shifting $m_{r}$ downwards as $r$ decreases.
Now we focus on the threshold of the sum of two prox-bounded functions. As in the case of the piecewise function with prox-bounded pieces, we will find that an exact threshold cannot be obtained and we settle for an upper bound in the general case. If certain restrictions are put on one or both of the functions, an exact threshold can be determined.

Lemma 4.6. Let $f_{1}, f_{2}: \mathbb{R}^{n} \rightarrow \overline{\mathbb{R}}$ be proper, lsc and prox-bounded with respective thresholds $r_{1}, r_{2}$. If $f_{1} \leq f_{2}$, then for any $r>\max \left\{r_{1}, r_{2}\right\}$, we have

$$
e_{r} f_{1} \leq e_{r} f_{2}
$$

Proof. Setting $g_{1}(x)=f_{1}(x)+\frac{r}{2}\|x\|^{2}$ and $g_{2}(x)=f_{2}(x)+\frac{r}{2}\|x\|^{2}$, we have $g_{1} \leq g_{2}$. By [4, Proposition 13.14(ii)], $g_{1}^{*}(r x) \geq g_{2}^{*}(r x)$. By Fact 4.2, we have

$$
\begin{aligned}
e_{r} f_{1}(x) & =\frac{r}{2}\|x\|^{2}-g_{1}^{*}(r x), \\
& \leq \frac{r}{2}\|x\|^{2}-g_{2}^{*}(r x), \\
& =e_{r} f_{2}(x) .
\end{aligned}
$$

Corollary 4.7. Let $f_{1}, f_{2}: \mathbb{R}^{n} \rightarrow \overline{\mathbb{R}}$ be proper, lsc and prox-bounded with respective thresholds $r_{1}, r_{2}$. If $f_{1} \leq f_{2}$, then $r_{1} \geq r_{2}$.

Proposition 4.8. Let $f_{1}, f_{2}: \mathbb{R}^{n} \rightarrow \overline{\mathbb{R}}$ be proper, lsc and prox-bounded with respective thresholds $r_{1}, r_{2}$. Define $f(x)=\left(f_{1}+f_{2}\right)(x)$. Then $f$ is prox-bounded with threshold $\bar{r} \leq r_{1}+r_{2}$. Moreover, if $f_{2}$ is bounded, then $\bar{r}=r_{1}$.

Proof. The first part of this proposition appears as part of [18, Lemma 2.4], but we provide a full proof for the sake of completeness. For any $\varepsilon>0$, we have

$$
\begin{aligned}
e_{r_{1}+r_{2}+2 \varepsilon} f(x) & =\inf _{y}\left\{f_{1}(y)+f_{2}(y)+\frac{r_{1}+r_{2}+2 \varepsilon}{2}\|y-x\|^{2}\right\} \\
& =\inf _{y}\left\{\left[f_{1}(y)+\frac{r_{1}+\varepsilon}{2}\|y-x\|^{2}\right]+\left[f_{2}(y)+\frac{r_{2}+\varepsilon}{2}\|y-x\|^{2}\right]\right\} \\
& \geq \inf _{y}\left\{f_{1}(y)+\frac{r_{1}+\varepsilon}{2}\|y-x\|^{2}\right\}+\inf _{y}\left\{f_{2}(y)+\frac{r_{2}+\varepsilon}{2}\|y-x\|^{2}\right\} \\
& =e_{r_{1}+\varepsilon} f_{1}(x)+e_{r_{2}+\varepsilon} f_{2}(x)>-\infty \forall x \in \mathbb{R}^{n} .
\end{aligned}
$$

This tells us that $f$ is prox-bounded and $\bar{r} \leq r_{1}+r_{2}$. Now suppose that $f_{2}$ is bounded. Since $f_{2}$ is bounded below, we have that $r_{2}=0$ by Fact 2.6. Hence, $\bar{r} \leq r_{1}+0=r_{1}$. Since $f_{2}$ is bounded above, there exists $M \in \mathbb{R}$ such that $M \geq f_{2}(x) \forall x$. Suppose that $r_{1}>0$. (Otherwise, trivially $\bar{r}=0=r_{1}$.) Then for any $r \in\left(0, r_{1}\right)$, we have

$$
\begin{aligned}
e_{r} f(x) & =\inf _{y}\left\{f_{1}(y)+f_{2}(y)+\frac{r}{2}\|y-x\|^{2}\right\}, \\
& \leq \inf _{y}\left\{f_{1}(y)+M+\frac{r}{2}\|y-x\|^{2}\right\}, \\
& =M+\inf _{y}\left\{f_{1}(y)+\frac{r}{2}\|y-x\|^{2}\right\}, \\
& =M+e_{r} f_{1}(x)=-\infty
\end{aligned}
$$

Hence, $\bar{r} \geq r_{1}$. Therefore, $\bar{r}=r_{1}$.

Corollary 4.9. Let $f_{1}, f_{2}: \mathbb{R}^{n} \rightarrow \overline{\mathbb{R}}$ be proper, lsc and prox-bounded with threshold 0 . Then $\left(f_{1}+f_{2}\right)(x)$ is prox-bounded with threshold 0 .

The very strong condition of $f_{2}$ being bounded above and below in Proposition 4.8 can be relaxed slightly, as the corollary below indicates, with the same proof as the proposition.

Corollary 4.10. Let $f_{1}, f_{2}: \mathbb{R}^{n} \rightarrow \overline{\mathbb{R}}$ be proper, lsc and prox-bounded with respective thresholds $r_{1}, r_{2}$. Define $f(x)=\left(f_{1}+f_{2}\right)(x)$. If $r_{2}=0$ and $f_{2}$ is bounded above, then $\bar{r}=r_{1}$.

Proposition 4.11. Let $f_{1}: \mathbb{R}^{n} \rightarrow \overline{\mathbb{R}}$ be proper and lsc. Let $f_{2}: \mathbb{R}^{n} \rightarrow \overline{\mathbb{R}}$ be an affine function. Then $f_{1}+f_{2}$ is prox-bounded with threshold $r_{1}$ if and only if $f_{1}$ is prox-bounded with threshold $r_{1}$.

Proof. We see in [3, Lemma 3.6] that the Moreau envelope of $f$ can be expressed as the sum of a quadratic function and a Moreau envelope of $f_{1}$ only, with $x$ plus a constant as the argument. Therefore, $e_{r} f$ exists if and only if $e_{r} f_{1}$ exists and we have that $f$ has the same threshold as $f_{1}$.

Proposition 4.11 invites another slight relaxation of the condition on $f_{2}$ in Proposition 4.8,
Corollary 4.12. Let $f_{1}, f_{2}: \mathbb{R}^{n} \rightarrow \overline{\mathbb{R}}$ be prox-bounded with respective thresholds $r_{1}, r_{2}$. Define $f=f_{1}+f_{2}$. If $r_{2}=0$ and $f_{2}$ is majorized by an affine function, then $f$ is prox-bounded with threshold $\bar{r}=r_{1}$.

Now we move on to sufficient conditions for a composition rule. This is a difficult issue; one can construct examples of composition where the resulting threshold is any nonnegative number one desires, or even nonexistent. The following simple examples demonstrate.

Example 4.13. For $a, b \geq 0$, define $f_{1}, f_{2}: \mathbb{R} \rightarrow \mathbb{R}$,

$$
f_{1}(x)=-b x, \quad f_{2}(x)=-\frac{a}{2} x^{2}
$$

Then the threshold of $f_{1} \circ f_{2}$ is $\bar{r}_{12}=0$, while the threshold of $f_{2} \circ f_{1}$ is $\bar{r}_{21}=a b^{2}$.
Proof. Since $\left(f_{1} \circ f_{2}\right)(x)=\frac{a b}{2} x^{2}$ is bounded below, by Fact 2.6 we have $\bar{r}_{12}=0$. Since $\left(f_{2} \circ\right.$ $\left.f_{1}\right)(x)=-\frac{a b^{2}}{2} x^{2}$, by the same method as the proof of Example 3.5 we find $\bar{r}_{21}=a b^{2}$.

Example 4.13 shows that with two basic prox-bounded functions it is possible to obtain a threshold for the composition that is any particular nonnegative number, by making appropriate choices of $a$ and $b$. The next example shows that we can just as easily use two prox-bounded functions to construct a function that is not prox-bounded.

Example 4.14. Define $f_{1}(x)=x^{2}, f_{2}(x)=-x^{2}$. Then $f_{1} \circ f_{2}$ has threshold $\bar{r}_{12}=0$, while $f_{2} \circ f_{1}$ is not prox-bounded.

Proof. Since $\left(f_{1} \circ f_{2}\right)(x)=x^{4}$ is bounded below, by Fact 2.6 we have that $\bar{r}_{12}=0$. Since $\left(f_{2} \circ\right.$ $\left.f_{1}\right)(x)=-x^{4}$, by Fact4.3(ii), $f_{2} \circ f_{1}$ is not prox-bounded.

Furthermore, one can compose two functions that are not prox-bounded to form a function that is prox-bounded. For instance, $f_{1}(x)=-x^{3}$ and $f_{2}(x)=\ln x$ are not prox-bounded, yet they yield the composition $\left(f_{1} \circ f_{2}\right)(x)=-\ln ^{3} x$, which is minorized by $-x^{2}$ and thus prox-bounded by Fact 4.3 (ii). So what can we say about the thresholds of the composition of prox-bounded functions? As in the case of the sum of prox-bounded functions, if we restrict ourselves to certain classes of functions, we can make some conclusions. We start by listing a known fact that is used in the proof of the subsequent proposition.

Fact 4.15. [18, Lemma 2.4] Let $f: \mathbb{R}^{n} \rightarrow \overline{\mathbb{R}}$ be proper, lsc and prox-bounded with threshold $\bar{r}$. Then for any $\lambda \geq 0, \lambda f$ is prox-bounded with threshold $\lambda \bar{r}$.

Proposition 4.16. Let $f_{1}: \mathbb{R}^{m} \rightarrow \overline{\mathbb{R}}$ and $f_{2}: \mathbb{R}^{n} \rightarrow \mathbb{R}^{m}$ be prox-bounded with respective thresholds $r_{1}, r_{2}$. Let ran $f_{2} \subseteq \operatorname{dom} f_{1}$, define $f=f_{1} \circ f_{2}$ and denote the prox-threshold of $f$ as $\bar{r}$ when it exists. Then the following hold.
(i) If $f_{1}, f_{2}$ are Lipschitz continuous, then $\bar{r}=0$.
(ii) If $f_{1}$ is affine: $f_{1}(x)=a x+b$ with $a \geq 0$, then $\bar{r}=a r_{2}$.
(iii) If $f_{2}$ is affine: $f_{2}(x)=a x+b$, then $\bar{r}=a^{2} r_{1}$.

Proof. (i) Let $f_{1}$ be $K_{1}$-Lipschitz and $f_{2}$ be $K_{2}$-Lipschitz. Then

$$
\left\|f_{1}\left(f_{2}(y)\right)-f_{1}\left(f_{2}(x)\right)\right\| \leq K_{1}\left\|f_{2}(y)-f_{2}(x)\right\| \leq K_{1} K_{2}\|y-x\|
$$

which says that $f_{1} \circ f_{2}$ is $K_{1} K_{2}$-Lipschitz. By Proposition 3.1, $\bar{r}=0$.
(ii) We have $\left(f_{1} \circ f_{2}\right)(x)=a f_{2}(x)+b$, which yields $\bar{r}=a r_{2}$ by Fact 4.15 and the fact that the vertical shift by $b$ has no impact on the threshold.
(iii) Let $r>r_{1}$. Then by Fact 4.3(iii), $f_{1}+\frac{r}{2}\|\cdot\|^{2}$ is bounded below. We have, for some $m \in \mathbb{R}$ and for all $x \in \operatorname{dom} f_{1}$,

$$
\begin{align*}
f_{1}(x)+\frac{r}{2}\|x\|^{2} & \geq m \\
f_{1}(a x+b)+\frac{r}{2}\|a x+b\|^{2} & \geq m \\
f_{1}(a x+b)+a r\langle x, b\rangle+\frac{a^{2} r}{2}\|x\|^{2} & \geq m-\frac{r}{2}\|b\|^{2} . \tag{4.1}
\end{align*}
$$

Hence, $f_{1}(a x+b)+a r\langle x, b\rangle+\frac{a^{2} r}{2}\|x\|^{2}$ is bounded below. Since (4.1) is true for any arbitrary $r>r_{1}$, it is true for all $r>r_{1}$. By an identical argument, for any $r<r_{1}$ we have that $f_{1}(a x+b)+$ $\operatorname{ar}\langle x, b\rangle+\frac{a^{2} r}{2}\|x\|^{2}$ is not bounded below. Thus, $r_{1}$ is the infimum of all $r$ such that (4.1) is true. By Fact 4.3, the threshold of $f_{1}(a \cdot+b)+\operatorname{ar}\langle\cdot, b\rangle$ is $a^{2} r_{1}$. By Proposition4.11, we conclude that the threshold of $f_{1} \circ f_{2}$ is $a^{2} r_{1}$.

So we have that if both $f_{1}, f_{2}$ are Lipschitz continuous functions, or if one of $f_{1}, f_{2}$ is affine, then the threshold of the composition can be determined exactly. It is clear from Examples 4.13 and 4.14 that if one of $f_{1}, f_{2}$ is quadratic, chaos ensues. So far, it does not seem that other standard properties such as convexity and boundedness are any more promising in forming composition rules, not even in providing an upper bound for the threshold. We leave the further development of properties of the threshold of prox-boundedness to future consideration.

## 5 Conclusion and future work

The threshold of prox-boundedness of the objective function of a minimization problem is an important value to take into consideration when implementing optimization algorithms. In this work, we have determined the threshold of Lipschitz functions and bounds on the threshold of piecewise functions. We established properties of thresholds of the sum and the composition of functions under certain conditions and shown that when we do not have these conditions, functions can be constructed so that the threshold of the sum or composition is any nonnegative number.

This paper is the first step in determining thresholds for larger classes of functions, with the long-term goal of improving the efficiency of optimization routines that are based in the proximal point algorithm. At the moment, the conditions imposed are quite heavy; the search continues for other well-behaved functions whose thresholds can be identified or at least bounded. The work done here regarding piecewise functions, together with the results of [19] on PLQ functions, should open the way for exploration of thresholds of composition classes such as the fully subamenable functions of [31]. Such functions are an extension of fully amenable functions [38], respect a chain rule and are likely suitable for use in constrained composite modelling and optimization applications.

## References

[1] M. Annergren, A. Hansson, and B. Wahlberg. An ADMM algorithm for solving $l_{1}-$ regularized MPC. In 2012 IEEE 51st Conference on Decision and Control, pages 4486-4491. IEEE, 2012.
[2] H. Attouch and J. Peypouquet. Convergence of inertial dynamics and proximal algorithms governed by maximally monotone operators. Math. Program., 174(1-2, Ser. B):391-432, 2019.
[3] M. Bačak, J. Borwein, A. Eberhard, and B. Mordukhovich. Infimal convolutions and Lipschitzian properties of subdifferentials for prox-regular functions in Hilbert spaces. J. Convex Anal., 17(3-4):737-763, 2010.
[4] H. Bauschke and P. Combettes. Convex analysis and monotone operator theory in Hilbert spaces. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC. Springer, New York, 2011.
[5] H. Bauschke, E. Matoušková, and S. Reich. Projection and proximal point methods: convergence results and counterexamples. Nonlinear Anal., 56(5):715-738, 2004.
[6] S. Boyd, N. Parikh, E. Chu, B. Peleato, and J. Eckstein. Distributed optimization and statistical learning via the alternating direction method of multipliers. Found. Trend. Mach. Learn., $3(1): 1-122,2011$.
[7] L. Briceno-Arias, P. Combettes, J.-C. Pesquet, and N. Pustelnik. Proximal algorithms for multicomponent image recovery problems. J. Math. Imag. Vision, 41(1-2):3-22, 2011.
[8] R. Burachik and A. Iusem. A generalized proximal point algorithm for the nonlinear complementarity problem. RAIRO Oper. Res., 33(4):447-479, 1999.
[9] P. Calamai and J. Moré. Projected gradient methods for linearly constrained problems. Math. Program., 39(1):93-116, 1987.
[10] Y. Chen, C. Kan, and W. Song. The Moreau envelope function and proximal mapping with respect to the Bregman distances in Banach spaces. Vietnam J. Math., 40(2-3):181-199, 2012.
[11] M. Dao and M. Tam. Union averaged operators with applications to proximal algorithms for min-convex functions. J. Optim. Theory Appl., 181(1):61-94, 2019.
[12] J. Douglas and H. Rachford. On the numerical solution of heat conduction problems in two and three space variables. Trans. Amer. Math. Soc., 82(2):421-439, 1956.
[13] J. Eckstein and D. Bertsekas. On the Douglas-Rachford splitting method and the proximal point algorithm for maximal monotone operators. Math. Program., 55(3, Ser. A):293-318, 1992.
[14] C. Fougner and S. Boyd. Parameter selection and preconditioning for a graph form solver. In Emerging Applications of Control and Systems Theory, pages 41-61. Springer, 2018.
[15] A. Fuduli and M. Gaudioso. Tuning strategy for the proximity parameter in convex minimization. J. Optim. Theory Appl., 130(1):95-112, 2006.
[16] M. Fukushima and H. Mine. A generalized proximal point algorithm for certain nonconvex minimization problems. Internat. J. Systems Sci., 12(8):989-1000, 1981.
[17] O. Güler. On the convergence of the proximal point algorithm for convex minimization. SIAM J. Control Optim., 29(2):403-419, 1991.
[18] W. Hare. A proximal average for nonconvex functions: a proximal stability perspective. SIAM J. Optim., 20(2):650-666, 2009.
[19] W. Hare and C. Planiden. Thresholds of prox-boundedness of PLQ functions. J. Convex Anal., 23(3), 2014.
[20] W. Hare and C. Planiden. Computing proximal points of convex functions with inexact subgradients. Set-Valued and Variational Analysis, 26(3):469-492, 2018.
[21] W. Hare, C. Planiden, and C. Sagastizábal. A derivative-free VU-algorithm for convex finitemax problems. arXiv preprint arXiv:1903.11184, 2019.
[22] W. Hare and C. Sagastizábal. Computing proximal points of nonconvex functions. Math. Program., 116(1-2, Ser. B):221-258, 2009.
[23] B. He, H. Yang, and S. Wang. Alternating direction method with self-adaptive penalty parameters for monotone variational inequalities. J. Optim. Theory Appl., 106(2):337-356, 2000.
[24] J.-B. Hiriart-Urruty and H. Le. From Eckart and Young approximation to Moreau envelopes and vice versa. RAIRO Oper. Res., 47(3):299-310, 2013.
[25] A. Kaplan and R. Tichatschke. Proximal point methods and nonconvex optimization. J. Global Optim., 13(4):389-406, 1998. Workshop on Global Optimization (Trier, 1997).
[26] I. Kecis and L. Thibault. Moreau envelopes of $s$-lower regular functions. Nonlinear Anal., 127:157-181, 2015.
[27] Y. Li, H. Zhang, Z. Li, and H. Gao. Proximal gradient method with automatic selection of the parameter by automatic differentiation. Optim. Methods Softw., 33(4-6):708-717, 2018.
[28] Y. Lucet. Fast Moreau envelope computation. I. Numerical algorithms. Numer. Algorithms, 43(3):235-249 (2007), 2006.
[29] B. Martinet. Régularisation d'inéquations variationnelles par approximations successives. Rev. Française Informat. Recherche Opérationnelle, 4(Ser. R-3):154-158, 1970.
[30] R. Mifflin and C. Sagastizábal. $V U$-smoothness and proximal point results for some nonconvex functions. Optim. Methods Softw., 19(5):463-478, 2004.
[31] A. Mohammadi, B. Mordukhovich, and M. Sarabi. Variational analysis of composite models with applications to continuous optimization. arXiv preprint arXiv:1905.08837, 2019.
[32] J.-J. Moreau. Proximité et dualité dans un espace Hilbertien. Bull. Soc. Math. France, 93:273-299, 1965.
[33] C. Planiden and X. Wang. Strongly convex functions, Moreau envelopes, and the generic nature of convex functions with strong minimizers. SIAM J. Optim., 26(2):1341-1364, 2016.
[34] C. Planiden and X. Wang. Epi-convergence: the Moreau envelope and generalized linearquadratic functions. J. Optim. Theory Appl., 177(1):21-63, 2018.
[35] C. Planiden and X. Wang. Proximal mappings and Moreau envelopes of single-variable convex piecewise cubic functions and multivariable gauge functions. In Nonsmooth optimization and its applications, volume 170 of Internat. Ser. Numer. Math., pages 89-130. Birkhäuser/Springer, Cham, 2019.
[36] P. Rey and C. Sagastizábal. Dynamical adjustment of the prox-parameter in bundle methods. Optimization, 51(2):423-447, 2002.
[37] R. Rockafellar. Monotone operators and the proximal point algorithm. SIAM J. Control Optim., 14(5):877-898, 1976.
[38] R. Rockafellar and R. Wets. Variational analysis. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1998.
[39] W. Sun, R. Sampaio, and M. Candido. Proximal point algorithm for minimization of DC function. J. Comput. Math., 21(4):451-462, 2003.
[40] K. Yosida. Functional analysis. Die Grundlehren der Mathematischen Wissenschaften, Band 123. Academic Press, Inc., New York; Springer-Verlag, Berlin, 1965.


[^0]:    *School of Mathematics and Applied Statistics, University of Wollongong, Wollongong, NSW, 2522, Australia. chayne@uow.edu.au

