Primal-Dual Incremental Gradient Method for Nonsmooth and Convex Optimization Problems

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Abstract

In this paper, we consider a nonsmooth convex finite-sum problem with a conic constraint. To overcome the challenge of projecting onto the constraint set and computing the full (sub)gradient, we introduce a primal-dual incremental gradient scheme where only a component function and two constraints are used to update each primal-dual sub-iteration in a cyclic order. We demonstrate an asymptotic sublinear rate of convergence in terms of suboptimality and infeasibility which is an improvement over the state-of-the-art incremental gradient schemes in this setting. Numerical results suggest that the proposed scheme compares well with competitive methods.

keywords: Incremental Gradient; Primal-Dual Method; Convex Optimization.

1 Introduction

Convex constrained optimization has a broad range of applications in many areas, such as machine learning (ML). As data gets more complex and the application of ML algorithms becomes more diversified, the goal of recent ML research is to improve the efficiency and scalability of algorithms. In this paper, we consider the following nonsmooth and convex constrained problem,

$$\min_{x \in X} f(x) \quad \text{s.t.} \quad Ax - b \in -\mathcal{K},\tag{1}$$

where $f(x) = \sum_{i=1}^{m} f_i(x), A = \begin{bmatrix} A_1^T & \dots & A_m^T \end{bmatrix}^T, b = \begin{bmatrix} b_1^T & \dots & b_m^T \end{bmatrix}^T, \mathcal{K} = \prod_{i=1}^{m} \mathcal{K}_i$. For each $i \in \{1, \dots, m\}$, the function $f_i : \mathbb{R}^n \to \mathbb{R}$ is convex (possibly nonsmooth), $A_i \in \mathbb{R}^{d_i \times n}, b_i \in \mathbb{R}^{d_i}$ and $\mathcal{K}_i \subseteq \mathbb{R}^{d_i}$ is a closed convex cone, and $X \subset \mathbb{R}^n$ is a compact and convex set. We assume that the projection onto \mathcal{K}_i can be computed efficiently while the projection onto the preimage set $\{x \mid A_i x - b_i \in -\mathcal{K}_i\}$ is assumed to be impractical for any $i \in \{1, \dots, m\}$. Letting $d \triangleq \sum_{i=1}^{m} d_i$, we introduce a dual multiplier $y = [y_i]_{i=1}^m \in \mathbb{R}^d$ for the constraint in (1)

Letting $d \triangleq \sum_{i=1}^{m} d_i$, we introduce a dual multiplier $y = [y_i]_{i=1}^{m} \in \mathbb{R}^d$ for the constraint in (1) and y^* denotes a dual optimal solution. Suppose a constant B > 0 exists such that $||y^*|| \leq B$. Such a bound B can be computed efficiently if a slater point of (1) is available, see Lemma 2. Let $Y = \prod_{i=1}^{m} Y_i, Y_i = \{y_i \in \mathbb{R}^{d_i} | \sqrt{m} ||y_i|| \leq B+1\}$ which implies that $||y|| \leq B+1$, for all $y \in Y$, then problem (1) can be equivalently written as the following saddle point (SP) problem:

$$\min_{x \in X} \max_{y = [y_i]_{i=1}^m \in Y \cap \mathcal{K}^*} \phi(x, y) \triangleq \sum_{i=1}^m f_i(x) + y_i^T (A_i x - b_i),$$
(2)

where \mathcal{K}^* denotes the dual cone of \mathcal{K} , i.e., $\mathcal{K}^* \triangleq \{u \in \mathbb{R}^d : \langle u, v \rangle \ge 0, \ \forall v \in \mathcal{K}\}.$

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Motivation. Problem (1) has a broad range of applications in ML, wireless sensor networks, signal processing, etc. Next, we illustrate examples written in the form of problem (1) in which projecting on the constraint is challenging.

Example 1. (Basis Pursuit Denoising (BPD) problem) Let x^* be a solution of a linear system of equations Ax = b, where A and b represent a transformation matrix and the observation vector, respectively. This problem arises in signal processing, image compression and compressed sensing [7] to recover a sparse solution x given A and b. In particular, one needs to solve $\min_x \{ ||x||_1 | A_i x = b_i, \forall i \in \{1, \ldots, m\} \}$. In real-world applications, the observations b might be noisy [9]. Therefore, the problem can be formulated as follows:

min
$$||x||_1$$
, s.t. $||A_ix - b_i|| \le \delta/\sqrt{m}, \forall i \in \{1, \dots, m\}$.

BDP problem is a special case of (1) and the constraint can be written as $(b_i - A_i x, -\delta/\sqrt{m}) \in -\mathcal{K}_i$ where $\mathcal{K}_i = \{(y,t) \in \mathbb{R}^{d_i} \times \mathbb{R} \mid ||y|| \leq t\}$ is a second-order cone. Projection onto the second-order cone can be computed as:

$$\mathbf{\Pi}_{\mathcal{K}}(y,t) = \begin{cases} (y,t) & \text{if } \|y\| \le t;\\ (0,0) & \text{if } \|y\| \le -t;\\ \frac{\|y\|+t}{2} \left(\frac{y}{\|y\|},1\right) & \text{otherwise,} \end{cases}$$

where $\Pi_{\mathcal{K}}(y)$ denotes the projection of y onto \mathcal{K} [2]; however, projection onto the preimage set $\{x \mid A_i x - b_i \in -\mathcal{K}_i\}$ can be impractical.

Example 2. (Constrained Lasso problem) Let $y \in \mathbb{R}^s$, $B \in \mathbb{R}^{s \times n}$, and $x \in \mathbb{R}^n$ denote the response vector, the design matrix of predictors, and the vector of unknown regression coefficients. Then the general Lasso problem can be written as follows [10]:

$$\min_{x} \frac{1}{2} \|y - Bx\|^{2} + \lambda_{1} \|x\|_{1} + \lambda_{2} \sum_{j=2}^{n} |x_{j} - x_{j-1}|, \text{ s.t. } Ax = b, \text{ and } Cx \le d_{2}$$

where $\lambda_1, \lambda_2 \geq 0$ are the tuning parameters. The constrained Lasso problem above is a special case of problem (1) by defining $f_i(x) = \frac{1}{2m} ||B_i x - y_i||^2 + \frac{\lambda_1}{m} ||x||_1 + \frac{\lambda_2}{m} \sum_{j=2}^n |x_j - x_{j-1}|$, where $B = [B_1^T, \ldots, B_m^T]^T$ and $y = [y_1^T, \ldots, y_m^T]^T$. The constraint can be written as $\begin{bmatrix} A_i \\ C_i \end{bmatrix} x - \begin{bmatrix} b_i \\ d_i \end{bmatrix} \in -\mathcal{K}$, where $A_i \in \mathbb{R}^{p_i \times n}, C_i \in \mathbb{R}^{q_i \times n}, p_i + q_i = d, \mathcal{K} = \{\mathbf{0}_{p_i}\} \times \mathbb{R}^{q_i}_+$ and $\{\mathbf{0}_{p_i}\} \in \mathbb{R}^{p_i}$.

Related work. One of the main approaches to solve problem (1), when the projection is cheap, is using the Projected Incremental Gradient (PIG) scheme [16] where the (sub)gradient of the function is approximated in a deterministic manner and cyclic order. Let $C \triangleq \{x \in X \mid Ax - b \in -\mathcal{K}\}$ denote the constraint set in (1), then each iteration of PIG has the following main steps: for $i = 1, \ldots, m$

- 1. Set $x_{k,1} = x_k$ and pick stepsize γ_k ;
- 2. $x_{k,i+1} = \mathbf{\Pi}_C(x_{k,i} \gamma_k g_{k,i});$
- 3. Set $x_{k+1} = x_{k,m+1}$,

end

where $g_{k,i} \in \partial f_i(x_{k,i})$, and $\partial f_i(x)$ denotes subdifferential of function $f_i(x)$, for all $i \in \{1, \ldots, m\}$. When the problem is nonsmooth and convex, the convergence rate of $\mathcal{O}(1/\sqrt{k})$ has been shown for PIG. The accelerated variant of Incremental Gradient (IG) scheme is studied in [5, 8, 11, 15]. These methods require storing a variable of size $\mathcal{O}(mn)$ at each iteration, hence, are impractical for large-scale problems and/or when the projection is hard to compute. One avenue to handle the constraints is by leveraging iterative regularization schemes [1, 19]. Recently in [14], authors introduced averaged iteratively regularized IG method that does not involve any hard-to-project computation to solve and require storing a variable of size $\mathcal{O}(n)$. However, their suboptimality and infeasibility rates are $\mathcal{O}(1/k^{0.5-b})$ and $\mathcal{O}(1/k^b)$, respectively, for some $b \in (0, 0.5)$. In contrast to the existing methods, in this paper, we address the challenge of projection by introducing a primal-dual scheme requiring memory of $\mathcal{O}(n + d/m)$. Moreover, our new primal-dual IG scheme improves the rate results to $\mathcal{O}(1/\sqrt{k})$ in terms of suboptimality and infeasibility.

Convex constrained optimization problems can be viewed as a special case of saddle point problems using Lagrangian duality. Different primal-dual methods have been introduced to solve such problems. Consider a saddle point problem of the form $\min_{x \in X} \max_{y \in Y} f(x) + \phi(x, y) - g_i(y_i)$, where $\phi(x, y) = \sum_{i=1}^{m} \langle A_i x - b_i, y_i \rangle$. When the objective function is strongly-convex stronglyconcave and smooth, a linear convergence rate has been shown in [20, 21, 22] using stochastic methods by randomly selecting the dual and/or primal coordinates. Assuming a merely convexconcave setting, the convergence rate of $\mathcal{O}(1/k)$ has been shown in [6, 17]. Moreover, Xu [18] considered problem (1) with nonlinear constraint $h_i \leq 0$ where h_i is convex, and bounded function and ∂h_i is bounded. They proposed a stochastic augmented Lagrangian scheme with convergence rate of $\mathcal{O}(1/\sqrt{k})$. In this paper, we aim to recover the rate of $\mathcal{O}(1/\sqrt{k})$ by approximating the subgradient in a deterministic manner and considering weaker assumptions. Finally, in our recent work [13], we considered $\min_x \max_y \sum_{i=1}^m f_i(x_i) + \sum_{j=1}^p \phi_j(x, y) - \sum_{\ell=1}^n h_\ell(y_\ell)$ where f_i, h_ℓ are convex and nonsmooth with efficiently computable proximal map and $\phi(x, y)$ is a smooth convex-concave function. The convergence rate of $\mathcal{O}(\log(k)/k)$ is obtained for merely convex setting by sampling the component functions using an increasing sample size. However, in this paper, we introduce a deterministic method to solve a nonsmooth optimization problem with a conic constraint.

Contribution. In this paper, we consider a nonsmooth minimization with a conic constraint. Considering the equivalent saddle point formulation, we propose a novel primal-dual incremental gradient (PDIG) scheme. In particular, the proposed method comprises a deterministic cycle in which only two constraints, and one objective function component, f_i , are utilized to update the iterates. This new approach significantly improves the previous state-of-the-art incremental gradient method for constrained minimization problems [14] from $\mathcal{O}(1/k^{\frac{1}{4}})$ to $\mathcal{O}(1/\sqrt{k})$ in terms of suboptimality/infeasibility. Moreover, the proposed scheme guarantees a convergence rate in a deterministic manner, in contrast to randomized methods [18] where the convergence rate is in the expectation sense.

In Section 2, we provide the main assumptions and definitions, required for the convergence analysis. Next, in Section 3, we introduce PDIG method and show the convergence rate of $\mathcal{O}(1/\sqrt{k})$ for both suboptimality and infeasibility. Finally, in Section 4 we implemented the proposed algorithm to solve the constrained Lasso problem and compare it with other competitive methods.

2 Assumptions and definitions

In this section, we outline some important notations, definitions and the required assumptions that we consider for the analysis of the method.

Notation. Throughout the paper, $\|.\|$ denotes the Euclidean norm and $\operatorname{relint}(X)$ denotes the relative interior of the set X. We define $\operatorname{dist}_{\mathcal{K}}(u) \triangleq \|\mathbf{\Pi}_{\mathcal{K}}(u) - u\| = \||\mathbf{\Pi}_{-\mathcal{K}^*}(u)\|$. Also, \mathbf{I}_d denotes $d \times d$ identity matrix.

Definition 1. Define $U_i \in \mathbb{R}^{d \times d_i}$ for $i \in \{1, \ldots, m\}$ such that $\mathbf{I}_d = [U_1, \ldots, U_m]$.

We impose the following requirements on problem (1).

Assumption 1. For all $i \in \{1, ..., m\}$, the following hold:

(a) A primal-dual solution, (x^*, y^*) , of problem (1) exists.

(b) Function f_i is convex and nonsmooth.

(c) f_i is Lipschitz continuous with constant L.

(d) X is a compact and convex set, i.e., $\exists D > 0 \text{ s.t. } \|x\| \leq D, \forall x \in X.$

(e) There exists a constant B > 0 such that $||y^*|| \le B$.

Assumption 1(c) is a common assumption for nonsmooth problems and it implies that f at every point x admits a subgradient g(x) such that $||g(x)|| \leq L$. We assume that this small norm subgradient g(x) is exactly the one reported by the first-order oracle as called with input x and this is not a severe restriction, since at least in the interior of the domain all subgradients of f are "small" in the outlined sense (see section 5.3 in [3] for more details). The following lemma states an important relation required for our convergence results.

Lemma 1. Suppose a convex function $f : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz continuous with constant L. Then $f(x) \leq f(y) + g(y)^T (x-y) + 2L ||x-y||$ holds for any $x, y \in \mathbb{R}^n$, where g is a subgradient of function f.

Proof. Using convexity of function f, Cauchy-Schwarz inequality, and boundedness of the subgradient, we can show the desired result as follows:

$$\langle g(y), y - x \rangle = \langle g(y) - g(x), y - x \rangle + \langle g(x), y - x \rangle \\ \leq \|g(y) - g(x)\| \|x - y\| + f(y) - f(x) \leq 2L \|x - y\| + f(y) - f(x).$$

In addition, note that the dual bound B in Assumption 1(e) can be computed efficiently if a slater point of (1) is available using the following lemma.

Lemma 2. [12] Let \hat{x} be a slater point of (1), i.e. $\hat{x} \in \text{relint}(\text{dom}(f))$ such that $Ax - b \in int(-\mathcal{K})$, and $h : \mathbb{R}^d \to \mathbb{R} \cup \{-\infty\}$ denote the dual function, i.e.,

$$h(y) = \begin{cases} \inf_x f(x) + \langle Ax - b, y \rangle, & y \in \mathcal{K}^* \\ -\infty, & o.w. \end{cases}$$

For any $\hat{y} \in \mathbf{dom}(h)$, let $Q_{\hat{y}} = \{y \in \mathbf{dom}(h) : h(y) \ge h(\hat{y})\} \subset \mathcal{K}^*$ denotes the corresponding superlevel set. Then for all $\hat{y} \in \mathbf{dom}(h)$, $Q_{\hat{y}}$ can be bounded as $\|y\| \le \frac{f(\hat{x}) - h(\hat{y})}{r^*}$, $\forall y \in Q_{\hat{y}}$ where $0 < r^* \triangleq \min_u \{-\langle A\hat{x} - y, u \rangle : \|u\| = 1, \ u \in \mathcal{K}^*\}.$

3 Convergence analysis

In this section, we propose the Primal-Dual Incremental Gradient (PDIG) method, displayed in Algorithm 1 to solve problem (2).

Algorithm 1 Primal-Dual Incremental Gradient (PDIG) method

input: $x_1 \in \mathbb{R}^n$, $y_1 \in \mathbb{R}^d$, positive sequences $\{\gamma_k\}_k$ and $\{\eta_k\}_k$, and let $x_{1,0} \leftarrow x_1$ for $k = 1 \dots K$ do $(x_{k,1}, y_{k,1}) \leftarrow (x_k, y_k)$; for $i = 1, \dots, m$ do $A_0 \leftarrow A_m$ and $U_0 \leftarrow U_m$; $y_{k,i+1} \leftarrow \Pi_{Y \cap \mathcal{K}^*} (y_{k,i} + \eta_k U_i(A_i x_{k,i} - b_i) + \eta_k U_{i-1} A_{i-1}(x_{k,i} - x_{k,i-1}))$; $x_{k,i+1} \leftarrow \Pi_X (x_{k,i} - \gamma_k (g_i(x_{k,i}) + A_i^T U_i^T y_{k,i+1}))$, where $g_i(x) \in \partial f_i(x)$; end for $x_{k+1,0} \leftarrow x_{k,m}, (x_{k+1}, y_{k+1}) \leftarrow (x_{k,m+1}, y_{k,m+1})$; end for

In the following theorem, we state our main result which is the convergence rate of PDIG in terms of suboptimality and infeasibility.

Theorem 1. Suppose Assumption 1 holds. Let $\{x_k, y_k\}_{k\geq 1}$ be the iterates generated by Algorithm 1, with the step-sizes chosen as $\eta_k = \frac{1}{a_{\max}\sqrt{k}}$ and $\gamma_k = \frac{1}{a_{\max}+\sqrt{k}}$ for all $k \geq 1$, where $a_{\max} = \max_{1\leq i\leq m}\{\|A_i\|\}$. Then the following result holds

$$\max\left\{|f(\bar{x}_K) - f(x^*)|, \operatorname{dist}_{-\mathcal{K}}(A\bar{x}_K - b)\right\} \le \phi(\bar{x}_K, \tilde{y}) - \phi(x^*, \bar{y}_K) \le \mathcal{O}(1/\sqrt{K}),$$

where $\tilde{y} \triangleq (\|y^*\| + 1) \Pi_{\mathcal{K}^*} (A\bar{x}_K - b) \|\Pi_{\mathcal{K}^*} (A\bar{x}_K - b)\|^{-1}$ and $(\bar{x}_K, \bar{y}_K) \triangleq \frac{1}{K} \sum_{k=1}^{K} (x_k, y_k).$

Before proving Theorem 1, we state a technical lemma for projection mappings and then provide a one-step analysis of the algorithm in Lemma 4.

Lemma 3. [4] Let $X \subseteq \mathbb{R}^n$ be a nonempty closed and convex set. Then the following hold: (a) $\|\Pi_X[u] - \Pi_X[v]\| \le \|u - v\|$ for all $u, v \in \mathbb{R}^n$; (b) $(\Pi_X[u] - u)^T(x - \Pi_X[u]) \ge 0$ for all $u \in \mathbb{R}^n$ and $x \in X$.

Lemma 4. Suppose Assumption 1 holds. Let $\{x_k, y_k\}_{k\geq 1}$ be the iterates generated by Algorithm 1, with the step-sizes chosen as $\eta_k = \frac{1}{a_{\max}\sqrt{k}}$ and $\gamma_k = \frac{1}{a_{\max}+\sqrt{k}}$ for all $k \geq 1$, where $a_{\max} = \max_{1\leq i\leq m}\{\|A_i\|\}$. Then the following holds for any $y \in Y \cap \mathcal{K}^*$.

$$\begin{split} \phi(\bar{x}_k, y) - \phi(x^*, \bar{y}_k) &\leq \frac{1}{K} \left(\frac{1}{a_{\max} + 1} + 2\sqrt{K} \right) (\tilde{C}_3 + \tilde{C}_1) + \frac{1}{K} \left(\frac{1}{a_{\max}} + \frac{\sqrt{K}}{a_{\max}} \right) \tilde{C}_2 \\ &+ \frac{2mL^2}{K\sqrt{K}} + \frac{2(B+1)^2 a_{\max}\sqrt{K}}{K} + \frac{2D^2(a_{\max} + \sqrt{K})}{K} \\ &+ 4D^2 \left(\frac{a_{\max} + \sqrt{K}}{2K} - \frac{\sqrt{K}}{K} \right) \leq \mathcal{O}(1/\sqrt{K}), \end{split}$$

for some constants $\tilde{C}_1, \tilde{C}_2, \tilde{C}_3 \ge 0$ where $(\bar{x}_K, \bar{y}_K) \triangleq \frac{1}{K} \sum_{k=1}^K (x_k, y_k)$.

Proof. For any $k \ge 1$, we have the following for any $y \in \mathbb{R}^d$,

$$\begin{aligned} \|y_{k,i+1} - y\|^2 &= \|y_{k,i+1} - y_{k,i}\|^2 + \|y_{k,i} - y\|^2 + 2\langle y_{k,i+1} - y_{k,i}, y_{k,i} - y \pm y_{k,i+1} \rangle \\ &= \|y_{k,i+1} - y_{k,i}\|^2 + \|y_{k,i} - y\|^2 - 2\|y_{k,i+1} - y_{k,i}\|^2 \\ &+ 2\langle y_{k,i+1} - y_{k,i}, y_{k,i+1} - y \rangle \\ &= \|y_{k,i} - y\|^2 - \|y_{k,i+1} - y_{k,i}\|^2 + \underbrace{2\langle y_{k,i+1} - y_{k,i}, y_{k,i+1} - y \rangle}_{\text{Term (a)}} \end{aligned}$$

From the definition of $y_{k,i+1}$ and Lemma 3(b) the following holds:

$$0 \leq (y_{k,i+1} - (y_{k,i} + \eta_k U_i (A_i x_{k,i} - b_i) + \eta_k U_{i-1} A_{i-1} (x_{k,i} - x_{k,i-1}))^T (y - y_{k,i+1}) = (y_{k,i+1} - y_{k,i})^T (y - y_{k,i+1}) + (\eta_k U_i (A_i x_{k,i} - b_i) + \eta_k U_{i-1} A_{i-1} (x_{k,i} - x_{k,i-1}))^T (y_{k,i+1} - y).$$
(3)

Therefore, term (a) can be written as

$$2\langle y_{k,i+1} - y_{k,i}, y_{k,i+1} - y \rangle \le 2(\eta_k U_i (A_i x_{k,i} - b_i) + \eta_k U_{i-1} A_{i-1} (x_{k,i} - x_{k,i-1}))^T (y_{k,i+1} - y)$$

Hence, we have the following:

$$\begin{split} \|y_{k,i+1} - y\|^2 \\ &\leq \|y_{k,i} - y\|^2 - \|y_{k,i+1} - y_{k,i}\|^2 + 2(\eta_k U_i(A_i x_{k,i} - b_i) \\ &+ \eta_k U_{i-1} A_{i-1}(x_{k,i} - x_{k,i-1}) \pm \eta_k U_i A_i x_{k,i+1}))^T (y_{k,i+1} - y) \\ &= \|y_{k,i} - y\|^2 - \|y_{k,i+1} - y_{k,i}\|^2 + 2\eta_k (y_{k,i+1} - y)^T U_i(A_i x_{k,i+1} - b_i) \\ &+ 2\eta_k (y_{k,i+1} - y \pm y_{k,i})^T U_{i-1} (A_{i-1}(x_{k,i} - x_{k,i-1})) \\ &- 2\eta_k (y_{k,i+1} - y)^T U_i (A_i (x_{k,i+1} - x_{k,i})) \\ &= \|y_{k,i} - y\|^2 - \|y_{k,i+1} - y_{k,i}\|^2 + 2\eta_k (y_{k,i+1} - y)^T U_i (A_i x_{k,i+1} - b_i) \\ &+ 2\eta_k (y_{k,i} - y)^T U_{i-1} (A_{i-1}(x_{k,i} - x_{k,i-1})) \\ &+ 2\eta_k (y_{k,i+1} - y_{k,i})^T U_{i-1} (A_{i-1} (x_{k,i} - x_{k,i-1})) \\ &- 2\eta_k (y_{k,i+1} - y)^T U_i (A_i (x_{k,i+1} - x_{k,i})), \end{split}$$

Now using Young's inequality, i.e., $a^T b \leq \frac{1}{2\alpha_k} \|a\|^2 + \frac{\alpha_k}{2} \|b\|^2$, for any $a, b \in \mathbb{R}^d$ and $\alpha_k > 0$, we conclude that

$$\begin{aligned} \|y_{k,i+1} - y\|^2 &\leq \|y_{k,i} - y\|^2 + \left(\frac{\eta_k}{\alpha_k} - 1\right) \|y_{k,i+1} - y_{k,i}\|^2 \\ &+ 2\eta_k (y_{k,i+1} - y)^T U_i (A_i x_{k,i+1} - b_i) \\ &+ 2\eta_k (y_{k,i} - y)^T U_{i-1} (A_{i-1} (x_{k,i} - x_{k,i-1})) \\ &+ \eta_k \alpha_k \|U_{i-1} A_{i-1} (x_{k,i} - x_{k,i-1})\|^2 - 2\eta_k (y_{k,i+1} - y)^T U_i (A_i (x_{k,i+1} - x_{k,i})). \end{aligned}$$
(4)

Similar to (3), from the update of $x_{k,i+1}$ and Lemma 3(b) the following holds:

$$(x_{k,i+1} - x_{k,i})^T (x_{k,i+1} - x^*) \le \left(\gamma_k (g_i(x_{k,i}) + A_i^T U_i^T y_{k,i+1})\right)^T (x^* - x_{k,i+1}).$$

Therefore, one can conclude that

$$\begin{aligned} \|x_{k,i+1} - x^*\|^2 &= \|x_{k,i+1} - x_{k,i}\|^2 + \|x_{k,i} - x^*\|^2 + 2\langle x_{k,i+1} - x_{k,i}, x_{k,i} - x^* \pm x_{k,i+1} \rangle \\ &= \|x_{k,i} - x^*\|^2 - \|x_{k,i+1} - x_{k,i}\|^2 + 2\langle x_{k,i+1} - x_{k,i}, x_{k,i+1} - x^* \rangle \\ &\leq \|x_{k,i} - x^*\|^2 - \|x_{k,i+1} - x_{k,i}\|^2 + 2\left(\gamma_k(g_i(x_{k,i}) + A_i^T U_i^T y_{k,i+1})\right)^T \underbrace{(x^* - x_{k,i+1})}_{\text{Term (b)}} \end{aligned}$$

Indeed, adding and subtracting $x_{k,i}$ to term (b) leads to

$$\|x_{k,i+1} - x^*\|^2 \le \|x_{k,i} - x^*\|^2 - \|x_{k,i} - x_{k,i+1}\|^2 - 2\gamma_k (x_{k,i+1} - x_{k,i})^T (g_i(x_{k,i}) + A_i^T U_i^T y_{k,i+1}) - 2\gamma_k (x_{k,i} - x^*) (g_i(x_{k,i}) + A_i^T U_i^T y_{k,i+1}).$$
(5)

From Assumption 1(b), one can easily show that $-2\gamma_k(x_{k,i}-x^*)^T g_i(x_{k,i}) \leq -2\gamma_k(x_{k,i}-x^*)^T (f_i(x_{k,i})-f_i(x^*))$ and from Assumption 1(c) and Lemma 1, we have that $-2\gamma_k(x_{k,i+1}-x_{k,i}^T)g_i(x_{k,i}) \leq 2\gamma_k(f_i(x_{k,i})-f_i(x_{k,i+1}))+4\gamma_k L ||x_{k,i+1}-x_{k,i}||$. Moreover, for some $\beta_k > 0$ we know that $4\gamma_k L ||x_{k,i+1}-x_{k,i}|| \leq \frac{2\gamma_k L^2}{\beta_k} + 2\gamma_k \beta_k ||x_{k,i+1}-x_{k,i}||^2$, hence, (5) can be written as follows.

$$\begin{aligned} \|x_{k,i+1} - x^*\|^2 &\leq \|x_{k,i} - x^*\|^2 - \|x_{k,i} - x_{k,i+1}\|^2 \\ &+ 2\gamma_k (f(x_{k,i}) - f_i(x_{k,i+1})) + \frac{2\gamma_k L^2}{\beta_k} + 2\gamma_k \beta_k \|x_{k,i+1} - x_{k,i}\|^2 \\ &- 2\gamma_k (x_{k,i+1} - x^*)^T A_i^T U_i^T y_{k,i+1} - 2\gamma_k (f_i(x_{k,i}) - f_i(x^*)) \\ &= \|x_{k,i} - x^*\|^2 - (2\gamma_k \beta_k - 1) \|x_{k,i} - x_{k,i+1}\|^2 + 2\gamma_k (f_i(x^*) - f_i(x_{k,i+1})) \\ &+ \frac{2\gamma_k L^2}{\beta_k} - 2\gamma_k (x_{k,i+1} - x^*)^T A_i^T U_i^T y_{k,i+1}. \end{aligned}$$
(6)

Multiplying (4) by $\frac{1}{2\eta_k}$, (6) by $\frac{1}{2\gamma_k}$ and summing up the result leads to

$$\begin{aligned} f_{i}(x_{k,i+1}) - f_{i}(x^{*}) &- (y_{k,i+1} - y)^{T} U_{i}(A_{i}x_{k,i+1} - b_{i}) - (x^{*} - x_{k,i+1})^{T} A_{i}^{T} U_{i}^{T} y_{k,i+1} \\ &\leq \frac{1}{2\eta_{k}} \|y_{k,i} - y\|^{2} - \frac{1}{2\eta_{k}} \|y_{k,i+1} - y\|^{2} \\ &+ \left(\frac{1}{2\alpha_{k}} - \frac{1}{2\eta_{k}}\right) \|y_{k,i+1} - y_{k,i}\|^{2} + \frac{1}{2\gamma_{k}} \|x_{k,i} - x^{*}\|^{2} - \frac{1}{2\gamma_{k}} \|x_{k,i+1} - x^{*}\|^{2} \\ &+ \left(\beta_{k} - \frac{1}{2\gamma_{k}}\right) \|x_{k,i} - x_{k,i+1}\|^{2} + (y_{k,i} - y)^{T} U_{i-1} \left[A_{i-1}(x_{k,i} - x_{k,i-1})\right] \\ &+ \frac{\alpha_{k}}{2} \|U_{i-1}A_{i-1}(x_{k,i} - x_{k,i-1})\|^{2} - (y_{k,i+1} - y)^{T} \left[U_{i}A_{i}(x_{k,i+1} - x_{k,i})\right] + \frac{L^{2}}{\beta_{k}}. \end{aligned}$$

Next, by selecting η_k and γ_k such that $\frac{1}{2\alpha_k} - \frac{1}{2\eta_k} \leq 0$ and $\frac{\alpha_k}{2} \|U_{i-1}A_{i-1}\|^2 \leq \frac{1}{2\gamma_k} - \beta_k$, one can drop $\left(\frac{1}{2\alpha_k} - \frac{1}{2\eta_k}\right) \|y_{k,i+1} - y_{k,i}\|^2$ in the above inequality to obtain the following result

$$\begin{aligned} f_i(x_{k,i+1}) - f_i(x^*) &- (y_{k,i+1} - y)^T U_i(A_i x_{k,i+1} - b_i) - (x^* - x_{k,i+1})^T A_i^T U_i^T y_{k,i+1} \\ &\leq \frac{1}{2\eta_k} \left(\|y_{k,i} - y\|^2 - \|y_{k,i+1} - y\|^2 \right) + \frac{1}{2\gamma_k} \left(\|x_{k,i} - x^*\|^2 - \|x_{k,i+1} - x^*\|^2 \right) \\ &+ (y_{k,i} - y)^T \left[U_{i-1} A_{i-1}(x_{k,i} - x_{k,i-1}) \right] - (y_{k,i+1} - y)^T \left[U_i A_i(x_{k,i+1} - x_{k,i}) \right] \\ &+ \left(\beta_k - \frac{1}{2\gamma_k} \right) \left(\|x_{k,i} - x_{k,i+1}\|^2 - \|x_{k,i-1} - x_{k,i}\|^2 \right) + \frac{L^2}{\beta_k}. \end{aligned}$$

Summing the result over i from 1 to m we conclude that

$$\sum_{i=1}^{m} \left(f_i(x_{k,i+1}) - f_i(x^*) - (y_{k,i+1} - y)^T U_i(A_i x_{k,i+1} - b_i) \right) - (x^* - x_{k,i+1})^T A_i^T U_i^T y_{k,i+1}$$

$$\leq \frac{1}{2\eta_k} \left(\|y_{k,1} - y\|^2 - \|y_{k,m+1} - y\|^2 \right) + \frac{1}{2\gamma_k} \left(\|x_{k,1} - x^*\|^2 - \|x_{k,m+1} - x^*\|^2 \right)$$

$$+ (y_{k,1} - y)^T \left[U_0 A_0(x_{k,1} - x_{k,0}) \right] - (y_{k,m+1} - y)^T \left[U_m A_m(x_{k,m+1} - x_{k,m}) \right]$$

$$+ \left(\beta_k - \frac{1}{2\gamma_k} \right) \left(\|x_{k,m} - x_{k,m+1}\|^2 - \|x_{k,0} - x_{k,1}\|^2 \right) + \frac{mL^2}{\beta_k}.$$
(7)

Now we proceed by providing a lower bound for the left hand side of (7). In particular, using Lemma 3(a) the following holds:

$$\begin{aligned} \|x_{k,2} - x_k\| &= \|\mathbf{\Pi}_X(x_{k,1} - \gamma_k(g_1(x_{k,1}) + A_1^T U_1^T y_{k,3})) - \mathbf{\Pi}_X(x_k)\| \\ &\leq \gamma_k(\|g_1(x_{k,1})\| + \|A_1\| \|U_1^T y_{k,3}\|) \leq \gamma_k(L + (B+1)\|A_1\|). \\ \Rightarrow \|x_{k,3} - x_k\| &= \|\mathbf{\Pi}_X(x_{k,2} - \gamma_k(g_2(x_{k,2}) + A_2^T U_2^T y_{k,4})) - \mathbf{\Pi}_X(x_k)\| \\ &\leq \|x_{k,2} - x_k\| + \gamma_k(\|g_2(x_{k,2})\| + \|A_2\| \|U_2^T y_{k,4}\|) \\ &\leq \gamma_k(2L + (B+1)(\|A_1\| + \|A_2\|)). \end{aligned}$$

Therefore, continuing this procedure for any $i \ge 1$, we conclude that

$$\|x_{k,i} - x_k\| \le \gamma_k \left(iL + (B+1)\sum_{\ell=1}^i \|A_\ell\| \right) \le \gamma_k i \left(L + (B+1)a_{\max} \right), \tag{8}$$

where $a_{\max} = \max_{1 \le i \le m} \{ \|A_i\| \}$. Let $b_{\max} = \max_{1 \le i \le m} \{ \|b_i\| \}$, then similar to (8) one can also obtain the following for the dual iterates

$$||y_{k,i} - y_k|| \le \eta_k i (3Da_{\max} + b_{\max}).$$
 (9)

Now, we can obtain a lower bound for the left hand side of (7).

$$\sum_{i=1}^{m} \left(f_i(x_{k,i+1}) - f_i(x^*) - (y_{k,i+1} - y)^T U_i(A_i x_{k,i+1} - b_i) \right) - (x^* - x_{k,i+1})^T A_i^T U_i^T y_{k,i+1}$$

$$= f(x_k) - f(x^*) + \sum_{i=1}^{m} \left(f_i(x_{k,i+1}) - f_i(x_k) - (y_{k,i+1} - y)^T U_i(A_i x_{k,i+1} - b_i) \right)$$

$$- (x^* - x_{k,i+1})^T A_i^T U_i^T y_{k,i+1}$$

$$\ge f(x_k) - f(x^*) + \sum_{i=1}^{m} L \| x_{k,i+1} - x_k\| + \sum_{i=1}^{m} \left[(y - y_{k,i+1})^T U_i(A_i x_{k,i+1} - b_i) - (x^* - x_{k,i+1})^T A_i^T U_i^T y_{k,i+1} \right] \pm \left[(y - y_k)^T (A x_k - b) - y_k^T A(x^* - x_k) \right],$$
(10)

where in the last inequality we used Assumption 1(c). Also, using (8) and (9), we can obtain the following bound for any $y \in Y \cap \mathcal{K}^*$,

$$\Big|\sum_{i=1}^{m} \left[(y - y_{k,i+1})^T U_i (A_i x_{k,i+1} - b_i) \right] - (y - y_k)^T (A x_k - b)$$

$$+ \sum_{i=1}^{m} \left[y_{k,i+1}^{T} U_{i} A_{i}(x_{k,i+1} - x^{*}) \right] - y_{k}^{T} A(x_{k} - x^{*}) \Big|$$

$$= \Big| \sum_{i=1}^{m} \left[y^{T} U_{i}(A_{i} x_{k,i+1} - b_{i}) + y_{k,i+1}^{T} U_{i} b_{i} \right] - y^{T} (A x_{k} - b) - y_{k}^{T} b$$

$$- \sum_{i=1}^{m} \left[y_{k,i+1}^{T} U_{i} A_{i} x^{*} \right] + y_{k}^{T} A x^{*} \Big|$$

$$= \Big| \sum_{i=1}^{m} \left[y^{T} U_{i}(A_{i} x_{k,i+1} - b_{i}) \right] - y^{T} (A x_{k} - b)$$

$$- \sum_{i=1}^{m} \left[y_{k,i+1}^{T} U_{i}^{T} (A_{i} x^{*} - b_{i}) \right] + y_{k}^{T} (A x^{*} - b) \Big|$$

$$\le \Big| y^{T} \left(\sum_{i=1}^{m} U_{i} A_{i}(x_{k,i+1} - x_{k}) \right) - \left(\sum_{i=1}^{m} (y_{k,i+1} - y_{k})^{T} U_{i}(A_{i} x^{*} - b_{i}) \right) \Big|$$

$$\le \sum_{i=1}^{m} (B + 1) ||A_{i}|| ||x_{k,i+1} - x_{k}|| + \sum_{i=1}^{m} ||y_{k,i+1} - y_{k}|| ||U_{i}(A_{i} x^{*} - b_{i})||$$

$$\le \gamma_{k} \frac{m(m+3)}{2} ||y|| a_{\max}(L + a_{\max}(B + 1))$$

$$+ \eta_{k} \frac{m(m+3)}{2} (a_{\max} ||x^{*}|| + b_{\max}) (3Da_{\max} + b_{\max}) = \gamma_{k} \tilde{C}_{1} + \eta_{k} \tilde{C}_{2},$$

$$(11)$$

where we used the fact that $y \in Y$, i.e. $||y|| \leq B+1$, and we let $\tilde{C}_1 \triangleq \frac{m(m+3)}{2}(B+1)a_{\max}(L+a_{\max}(B+1))$ and $\tilde{C}_2 \triangleq \frac{m(m+3)}{2}(a_{\max}||x^*|| + b_{\max})(3Da_{\max} + b_{\max})$. Using (11) within (10) one can conclude the following for any $y \in Y \cap \mathcal{K}^*$,

$$\sum_{i=1}^{m} \left(f_i(x_{k,i+1}) - f_i(x^*) - (y_{k,i+1} - y)^T U_i(A_i x_{k,i+1} - b_i) \right) - (x^* - x_{k,i+1})^T A_i^T U_i^T y_{k,i+1}$$

$$\geq \phi(x_k, y) - \phi(x^*, y_k) - \frac{m(m+1)}{2} \gamma_k L(L + a_{\max}(B+1)) - \gamma_k \tilde{C}_1 - \eta_k \tilde{C}_2.$$

Therefore, the inequality (7) can be rewritten as follows for any $y \in Y \cap \mathcal{K}^*$,

$$\begin{aligned} \phi(x_{k}, y) &- \phi(x^{*}, y_{k}) \\ &\leq \gamma_{k} \tilde{C}_{3} + \gamma_{k} \tilde{C}_{1} + \eta_{k} \tilde{C}_{2} + \frac{1}{2\eta_{k}} \left(\|y_{k,1} - y\|^{2} - \|y_{k+1,1} - y\|^{2} \right) \\ &+ \frac{1}{2\gamma_{k}} \left(\|x_{k,1} - x^{*}\|^{2} - \|x_{k+1,1} - x^{*}\|^{2} \right) + (y_{k,1} - y)^{T} \left[U_{m} A_{m}(x_{k,1} - x_{k,0}) \right] \\ &- (y_{k+1,1} - y)^{T} \left[U_{m} A_{m}(x_{k+1,1} - x_{k+1,0}) \right] \\ &+ \left(\beta_{k} - \frac{1}{2\gamma_{k}} \right) \left(\|x_{k+1,0} - x_{k+1,1}\|^{2} - \|x_{k,0} - x_{k,1}\|^{2} \right) + \frac{mL^{2}}{\beta_{k}}, \end{aligned} \tag{12}$$

where $\tilde{C}_3 \triangleq \frac{m(m+1)}{2}L(L+a_{\max}(B+1))$ and we used $y_{k,m+1} = y_{k+1,1}, x_{k,m+1} = x_{k+1,1}, x_{k,m} = x_{k+1,0}, A_0 = A_m$, and $U_0 = U_m$. Before summing (12) over k, we state some helpful inequalities on the consecutive terms involved in (12).

$$\sum_{k=1}^{K} \frac{1}{2\gamma_k} (\|x_{k,1} - x^*\|^2 - \|x_{k+1,1} - x^*\|^2)$$

$$= \frac{1}{2\gamma_1} \|x_{1,1} - x^*\|^2 + \left[\sum_{k=2}^K (\frac{1}{2\gamma_k} - \frac{1}{2\gamma_{k-1}}) 4D^2\right] - \frac{1}{2\gamma_K} \|x_{K+1,1} - x^*\|^2$$

$$\leq \frac{1}{2\gamma_K} (4D^2 - \|x_{K+1,1} - x^*\|^2), \tag{13}$$

where we used Assumption 1(d) and $\{\gamma_k\}_k$ is a decreasing sequence. Similarly,

$$\sum_{k=1}^{K} \frac{1}{2\eta_k} (\|y_{k,1} - y\|^2 - \|y_{k+1,1} - y\|^2) \le \frac{1}{2\eta_K} (4(B+1)^2 - \|y_{K+1,1} - y\|^2).$$
(14)

Summing both sides of (12) over k from 1 to K and using (13) and (14) we conclude that for any $y \in Y \cap \mathcal{K}^*$,

$$\begin{split} &\sum_{k=1}^{K} \phi(x_{k}, y) - \phi(x^{*}, y_{k}) \\ &\leq \sum_{k=1}^{K} \left(\gamma_{k}(\tilde{C}_{3} + \tilde{C}_{1}) + \eta_{k}\tilde{C}_{2} + \frac{mL^{2}}{\beta_{k}} \right) + \frac{1}{2\eta_{K}} (4(B+1)^{2} - \|y_{K+1,1} - y\|^{2}) \\ &+ \frac{1}{2\gamma_{K}} (4D^{2} - \|x_{K+1,1} - x^{*}\|^{2}) - (y_{K,m+1} - y)^{T} (U_{m}A_{m}(x_{K,m+1} - x_{K,m})) \\ &+ \left(\frac{1}{2\gamma_{K}} - \beta_{K} \right) \left(4D^{2} - \|x_{K+1,0} - x_{K+1,1}\|^{2} \right) \\ &\leq \sum_{k=1}^{K} \left(\gamma_{k}(\tilde{C}_{3} + \tilde{C}_{1}) + \eta_{k}\tilde{C}_{2} + \frac{mL^{2}}{\beta_{k}} \right) + \frac{1}{2\eta_{K}} (4(B+1)^{2} - \|y_{K+1,1} - y\|^{2}) \\ &+ \frac{1}{2\gamma_{K}} (4D^{2} - \|x_{K+1,1} - x^{*}\|^{2}) + \frac{1}{2\alpha_{K}} \|y_{K,m+1} - y\|^{2} \\ &+ \frac{\alpha_{K}}{2} \|U_{m}A_{m}(x_{K,m+1} - x_{K,m})\|^{2} + \left(\frac{1}{2\gamma_{K}} - \beta_{K} \right) \left(4D^{2} - \|x_{K+1,0} - x_{K+1,1}\|^{2} \right). \end{split}$$

Now using the fact that $\frac{\alpha_K}{2} \|U_m A_m\|^2 \leq \frac{1}{2\gamma_K} - \beta_K$, $x_{k,m+1} = x_{k+1,1}$, $x_{k,m} = x_{k+1,0}$ and choosing $\eta_k = \alpha_k$, one can simplify the above inequality as follows

$$\sum_{k=1}^{K} \phi(x_k, y) - \phi(x^*, y_k) \le \sum_{k=1}^{K} \left(\gamma_k(\tilde{C}_3 + \tilde{C}_1) + \eta_k \tilde{C}_2 + \frac{mL^2}{\beta_k} \right) + \frac{2(B+1)^2}{\eta_K} + \frac{2D^2}{\gamma_K} + 4D^2 \left(\frac{1}{2\gamma_K} - \beta_K \right).$$

Choosing $\eta_k = \alpha_k = \frac{1}{a_{\max}\sqrt{k}}, \ \beta_k = \frac{1}{2}\sqrt{k}, \ \gamma_k = \frac{1}{a_{\max}+\sqrt{k}}$ and using the fact that $\sum_{k=1}^{K} \frac{1}{\sqrt{k}} \leq 1 + \int_{x=1}^{K} \frac{1}{\sqrt{x}} dx \leq 1 + \sqrt{K}$ and similarly $\sum_{k=1}^{K} \frac{1}{a_{\max}+\sqrt{k}} \leq \frac{1}{a_{\max}+1} + 2\sqrt{K}$, the desired result can be obtained.

Now we are ready to prove the main result of the paper.

Proof of Theorem 1. Let (x^*, y^*) be an arbitrary saddle point of (2). Using the fact that for any $u \in \mathbb{R}^d$, $u = \Pi_{-\mathcal{K}}(u) + \Pi_{\mathcal{K}^*}(u)$ and $\langle \Pi_{-\mathcal{K}}(u), \Pi_{\mathcal{K}^*}(u) \rangle = 0$, one can show that $\langle A\bar{x}_K - b, \tilde{y} \rangle =$ $(||y^*|| + 1)\mathbf{dist}_{-\mathcal{K}}(A\bar{x}_K - b)$. Therefore, $\phi(\bar{x}_K, \tilde{y}) = f(\bar{x}_K) + (||y^*|| + 1)\mathbf{dist}_{-\mathcal{K}}(A\bar{x}_K - b)$. Moreover, since (x^*, y^*) is a saddle point of (2) one can conclude that $f(x^*) = \phi(x^*, y^*) \ge \phi(x^*, \bar{y}_K)$ and by Lemma 4 at $y = \tilde{y} \in Y \cap \mathcal{K}^*$,

$$f(\bar{x}_K) - f(x^*) + (\|y^*\| + 1) \mathbf{dist}_{-\mathcal{K}}(A\bar{x}_K - b) \le \phi(\bar{x}_K, \tilde{y}) - \phi(x^*, \bar{y}_K) \le \mathcal{O}(1/\sqrt{K}).$$
(15)

In addition, using the fact that for any $y \in \mathbb{R}^d$, $\langle y^*, y \rangle \leq \langle y^*, \Pi_{\mathcal{K}^*}(y) \rangle \leq ||y^*|| \mathbf{dist}_{-\mathcal{K}}(y)$, the following can be obtain:

$$0 \le \phi(\bar{x}_K, y^*) - \phi(x^*, y^*) = f(\bar{x}_K) - f(x^*) + \langle A\bar{x}_K - b, y^* \rangle$$

$$\le f(\bar{x}_K) - f(x^*) + \|y^*\| \mathbf{dist}_{-\mathcal{K}}(A\bar{x}_K - b).$$
(16)

Combining (15) and (16) gives the desired result.

4 Numerical results

In this section, we compare the performance of PDIG with aIR-IG [14] and PDSG [18] to solve the following constrained Lasso problem.

$$\min_{x \in [-10,10]} \frac{1}{2} \sum_{i=1}^{m} \|C_i x - d_i\|^2 + \frac{\lambda}{m} \sum_{i=1}^{m} \|x\|_1, \text{ s.t. } Bx \le 0,$$
(17)

where matrix $C = [C_i]_{i=1}^m \in \mathbb{R}^{pm \times n}$, $d = [d_i]_{i=1}^m \in \mathbb{R}^{pm}$, and $B \in \mathbb{R}^{n-1 \times n}$. We set m = 1000, n = 40, p = n+5, and $\lambda = 0.1$. (17) is a special case of (1), if we set $f_i(x) = \frac{1}{2} ||C_i x - d_i||^2 + \frac{\lambda}{m} ||x||_1$, $A_i = B_i$ for $1 \le i \le n-1$ and $A_i = 0$ for $n \le i \le m$, $b_i = 0$, and $\mathcal{K}_i = \mathbb{R}_+$ for all $i \in \{1, \ldots, m\}$. The problem data is generated as follows. First, we generate a vector $\bar{x} \in \mathbb{R}^n$ whose first 10 and last 10 components are sampled from [-10, 0] and [0, 10] uniformly at random in ascending order, respectively, and the other 20 middle components are set to zero. Next, we set $d = C\bar{x} + \eta$, where $\eta \in \mathbb{R}^{pm}$ is a random vector with i.i.d. components with Gaussian distribution with mean zero and standard deviation 10^{-1} . We choose the stepsizes of PDIG as suggested in Theorem 1. For aIR-IG, according to [14], the stepsize is set to $1/(1 + \sqrt{k})$ and the regularizer is $10/(1 + k)^{0.25}$ and for PDSG, as suggested in [18], the primal and dual step sizes are set to $1/(\log(k+1)\sqrt{k+1})$.



Figure 1: Comparing suboptimality (left) and infeasibility (right) of PDIG, aIR-IG and PDSG.

In Figure 1, we compared the suboptimality and infeasibility of three methods. We observe that PDIG outperforms aIR-IG which matches with the faster convergence rate of PDIG. Also, the rate of $\mathcal{O}(1/\sqrt{k})$ for our proposed method is deterministic and our step-sizes diminish periodically, in contrast with PDSG where the step-sizes diminish with iteration counter.

5 Concluding remarks

Motivated by the finite sum constrained problems arising in machine learning and wireless sensor networks, we introduced a novel primal-dual incremental gradient scheme to solve nonsmooth and convex problems with linear conic constraints. We improved the existing rate results of the incremental gradient approach for this setting to $O(1/\sqrt{k})$ in terms of suboptimality and infeasibility in a deterministic manner.

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