Proximal Operator and Optimality Conditions for Ramp Loss SVM

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Received: date / Accepted: date

Abstract Support vector machines with ramp loss (dubbed as L_r -SVM) have attracted wide attention due to the boundedness of ramp loss. However, the corresponding optimization problem is non-convex and the given Karush-Kuhn-Tucker (KKT) conditions are only the necessary conditions. To enrich the optimality theory of L_r -SVM and go deep into its statistical nature, we first introduce and analyze the proximal operator for ramp loss, and then establish a stronger optimality conditions: P-stationarity, which is proved to be the first-order necessary and sufficient conditions for local minimizer of L_r -SVM. Finally, we define the L_r support vectors based on the concept of P-stationary point, and show that all L_r support vectors fall into the support hyperplanes, which possesses the same feature as the one of hard margin SVM.

Keywords Ramp loss SVM \cdot proximal operator \cdot minimizer \cdot P-stationary point \cdot support vectors

1 Introduction

Support vector machines (SVM) were first introduced by Vapnik and Cortes [6] and have been widely applied into many fields, including text and image

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classification [12, 25], disease detection [8, 16], etc. The decision hyperplane of SVM classifier, $\langle \mathbf{w}, \mathbf{x} \rangle + b = 0$ with $\mathbf{w} \in \mathbb{R}^n$ and $b \in \mathbb{R}$, is trained from data set $\{(\mathbf{x}_i, y_i), i \in \mathbb{N}_m\}$ where $\mathbf{x}_i \in \mathbb{R}^n$, $y_i \in \{-1, 1\}$ and $\mathbb{N}_m := \{1, 2, \dots, m\}$ by optimizing the following problem

$$\min_{\mathbf{w}\in\mathbb{R}^n,b\in\mathbb{R}} \quad \frac{1}{2} \|\mathbf{w}\|^2 + C \sum_{i=1}^m \ell_h (1 - y_i(\langle \mathbf{w}, \mathbf{x}_i \rangle + b)), \tag{1}$$

where C > 0 is a penalty parameter and $\ell_h(t) = \max\{t, 0\}$ is the hinge loss, which has no cost for t < 0, but pays linear cost for $t \ge 0$. The cost given to the outliers by the hinge loss is quite huge since it is unbounded function [9–11, 14, 15, 20]. To remove the impact of outliers, one method for increasing the robustness of SVM is to use the ramp loss [18] (see Fig. 1 (a)). The ramp loss is defined as follows,

$$\ell_r(t) = \begin{cases} 1, & t \ge 1, \\ t, & 0 \le t < 1, \\ 0, & t < 0, \end{cases}$$
(2)

which is also called truncated hinge loss in [22]. It has no cost for t < 0, but it pays linear cost for $0 \le t < 1$ and a fixed cost at 1 for $t \ge 1$. Authors in [4, 5] also extended and studied ramp loss by replacing all constant 1 in (2) by adjustable parameter μ ($\mu > 0$). Since the theoretical results have no essential difference between the fixed 1 and parameter μ for ramp loss, for simplicity of the proof, we use formula (2) directly. Bartlett et al. [1] investigated some of the theoretical properties of the ramp loss, while SVM with ramp loss (dubbed as L_r -SVM) was first proposed by Shen et al. [18]. Replacing $\ell_h(\cdot)$ by $\ell_r(\cdot)$ in (1), the optimization problem of L_r -SVM is

$$\min_{\mathbf{w}\in\mathbb{R}^n,b\in\mathbb{R}} f_r(\mathbf{w};b) := \frac{1}{2} \|\mathbf{w}\|^2 + CL_r(\mathbf{1} - A\mathbf{w} - b\mathbf{y}),$$
(3)

where $A := [y_1 \mathbf{x}_1 \ y_2 \mathbf{x}_2 \ \cdots \ y_m \mathbf{x}_m]^\top \in \mathbb{R}^{m \times n}, \mathbf{y} := (y_1, y_2, \cdots, y_m)^\top \in \mathbb{R}^m, \mathbf{1} := (1, 1, \cdots, 1)^\top \in \mathbb{R}^m, \ L_r(\mathbf{u}) := \sum_{i=1}^m \ell_r(u_i) \text{ with } \mathbf{u} := (u_1, u_2, \cdots, u_m)^\top = \mathbf{1} - A\mathbf{w} - b\mathbf{y} \in \mathbb{R}^m$, which computes the sum of positive elements for all $u_i < 1$ and the number of elements for all $u_i \ge 1$ in \mathbf{u} . Due to the non-convexity of ramp loss, the L_r -SVM (3) is a non-differentiable non-convex optimization problem. Thus, to find a satisfying solution, scholars have paid many efforts in optimality conditions and algorithms for L_r -SVM.

In recent decades, on the one hand, some approaches such as convex relaxation and equivalent reformulation as mixed integer nonlinear programming (MINP) problem were proposed for dealing with L_r -SVM. Xu et al. [23] reformulated L_r -SVM as a semidefinite programming problem by convex relaxation which was solved by the MATLAB software package SDPT3. Brooks [2] transformed L_r -SVM into a MINP problem, and then proposed a branch and bound algorithm to solve it. Carrizos et al. [3] developed heuristic method to handle the MINP problem of L_r -SVM on large datasets. On the other hand, Collobert et al. [4] translated L_r -SVM (3) into an equivalent difference of convex functions (DC) programming and took the subdifferential of $f_r(\mathbf{w}; b)$ to establish the first-order necessary conditions: Karush-Kuhn-Tucker (KKT) conditions, and used the concave-convex procedure (CCCP) [24] for solving the DC programming. To improve the computational speed, Wang et al. [21] designed an efficient working set selection strategy based on KKT conditions and then used the CCCP with working set for solving the DC programming. Since the KKT conditions provided effective characterization of optimal solution to L_r -SVM (3), this class of approaches continued to be widely studied in theory as well as algorithms. For more details, see, e.g., [7, 13, 17, 19] and references therein.

A natural question arises. Is there stronger first-order optimality conditions of L_r -SVM (3) to provide more effective characterization of optimal solution? It is this question that motivates the work in our paper. The main results of this paper are summarized as follow: (i) We derive the explicit expression of proximal operator of ramp loss. (ii) We introduce a novel optimality conditions: P-stationarity, which is proved to be the necessary and sufficient conditions for local minimizer of L_r -SVM. (iii) We prove that the set of the support vectors defined by P-stationary point fall into the support hyperplanes.

This paper is organized as follows. In Section 2, we derive the explicit expression of proximal operator of ramp loss. In Section 3, we introduce a concept of P-stationary point, and reveal the relationship between P-stationary point and local/global minimizer of L_r -SVM. In Section 4, we introduce L_r support vectors based on the P-stationary point and discuss its properties. Conclusions are made in Section 5.

2 Proximal operator for ramp loss

In this section, we derive the explicit expression of proximal operator of ramp loss, which will be used to study the new first-order optimality conditions of L_r -SVM in the next section.

2.1 ℓ_r proximal operator

We first give the definition of ℓ_r proximal operator in one-dimensional case.

Definition 1 (ℓ_r proximal operator) For any given $\gamma, C > 0$ and $s \in \mathbb{R}$, the proximal operator of $\ell_r(v)$ (dubbed as ℓ_r proximal operator) is defined as

$$\operatorname{prox}_{\gamma C\ell_r}(s) = \arg\min_{v \in \mathbb{R}} C\ell_r(v) + \frac{1}{2\gamma}(v-s)^2.$$
(4)

The following two propositions state that the ℓ_r proximal operator admits a closed form solution for $0 < \gamma C < 2$ or $\gamma C \ge 2$. **Proposition 1** (Solution to ℓ_r proximal operator for $0 < \gamma C < 2$) For any given $\gamma, C > 0$ and $0 < \gamma C < 2$, the solution to ℓ_r proximal operator at $s \in \mathbb{R}$ is given as

$$\operatorname{prox}_{\gamma C\ell_r}(s) := \begin{cases} s, & s > 1 + \frac{\gamma C}{2}, \\ s \text{ or } s - \gamma C, & s = 1 + \frac{\gamma C}{2}, \\ s - \gamma C, & \gamma C \le s < 1 + \frac{\gamma C}{2}, \\ 0, & 0 < s < \gamma C, \\ s, & s \le 0. \end{cases}$$
(5)

Proposition 2 (Solution to ℓ_r proximal operator for $\gamma C \ge 2$) For any given $\gamma, C > 0$ and $\gamma C \ge 2$, the solution to ℓ_r proximal operator at $s \in \mathbb{R}$ is given as

$$\operatorname{prox}_{\gamma C\ell_r}(s) := \begin{cases} s, & s > \sqrt{2\gamma C}, \\ s \text{ or } 0, & s = \sqrt{2\gamma C}, \\ 0, & 0 < s < \sqrt{2\gamma C}, \\ s, & s \le 0. \end{cases}$$
(6)

2.2 L_r proximal operator

Based on separate property of $L_r(\cdot)$, we extend (5) and (6) to multidimensional case.

Definition 2 (L_r proximal operator) For any given $\gamma, C > 0$, the proximal operator of $L_r(\mathbf{v})$ (dubbed as L_r proximal operator) at $\mathbf{s} = (s_1, s_2, \cdots, s_m)^\top \in \mathbb{R}^m$ is defined as

$$\operatorname{prox}_{\gamma CL_r}(\mathbf{s}) = \arg\min_{\mathbf{v}\in\mathbb{R}^m} CL_r(\mathbf{v}) + \frac{1}{2\gamma} \|\mathbf{v} - \mathbf{s}\|^2.$$
(7)

The following proposition states that the L_r proximal operator admits a closed form solution for two cases: $0 < \gamma C < 2$ or $\gamma C \ge 2$.

Proposition 3 (Solution to L_r proximal operator) For a given $\gamma, C > 0$, the solution to L_r proximal operator at $\mathbf{s} = (s_1, s_2, \cdots, s_m)^\top \in \mathbb{R}^m$ is given as

$$\operatorname{prox}_{\gamma CL_r}(\mathbf{s}) := \begin{bmatrix} \operatorname{prox}_{\gamma C\ell_r}(s_1) \\ \vdots \\ \operatorname{prox}_{\gamma C\ell_r}(s_m) \end{bmatrix},$$
(8)

where $\operatorname{prox}_{\gamma C\ell_r}(s_i)$ takes formula (5) as $0 < \gamma C < 2$ or (6) as $\gamma C \ge 2$.

Proof It follows from (7) that $[\operatorname{prox}_{\gamma CL_r}(\mathbf{s})]_i = \operatorname{prox}_{\gamma C\ell_r}(s_i)$, where

$$\operatorname{prox}_{\gamma C\ell_r}(s_i) = \arg\min_{v \in \mathbb{R}} C\ell_r(v) + \frac{1}{2\gamma}(v-s_i)^2, i \in \mathbb{N}_m.$$

Using (5) or (6) completes the proof.

3 First-order optimality conditions

In this section, we develop a first-order necessary and sufficient optimality conditions for (3). To proceed this, we introduce a variable $\mathbf{u} \in \mathbb{R}^m$ to equivalently reformulate (3) as

$$\min_{\mathbf{w}\in\mathbb{R}^{n},b\in\mathbb{R},\mathbf{u}\in\mathbb{R}^{m}}\frac{1}{2}\|\mathbf{w}\|^{2} + CL_{r}(\mathbf{u})$$
s.t. $\mathbf{u} + A\mathbf{w} + b\mathbf{y} = \mathbf{1}.$
(9)

Now let us define some notation

$$B := [A \mathbf{y}] \in \mathbb{R}^{m \times (n+1)}, \quad H := \begin{bmatrix} I_{n \times n} \mathbf{0} \\ \mathbf{0} \end{bmatrix} B^{\dagger}, \tag{10}$$

where $B^{\dagger} \in \mathbb{R}^{(n+1)\times m}$ is the generalized inverse of B and $I_{n\times n}$ is the identity matrix with order n. Denote $\lambda_H := \lambda_{\max}(H^{\top}H)$ where $\lambda_{\max}(H^{\top}H)$ is the maximum eigenvalue of $H^{\top}H$.

Definition 3 (P-stationary point of (9)) For a given C > 0, we call $(\mathbf{w}^*; b^*; \mathbf{u}^*)$ is a proximal stationary (P-stationary) point of (9) if there exists a Lagrangian multiplier vector $\lambda^* \in \mathbb{R}^m$ and a constant $\gamma > 0$ such that

$$\begin{cases} \mathbf{w}^* + A^\top \boldsymbol{\lambda}^* = \mathbf{0}, \\ \langle \mathbf{y}, \boldsymbol{\lambda}^* \rangle = 0, \\ \mathbf{u}^* + A \mathbf{w}^* + b^* \mathbf{y} = \mathbf{1}, \\ \operatorname{prox}_{\gamma CL_r} (\mathbf{u}^* - \gamma \boldsymbol{\lambda}^*) \ni \mathbf{u}^*. \end{cases}$$
(11)

Based on the above definition, we obtain the desired result in this section.

Theorem 1 (First-order necessary optimality conditions). Let *B* be full column rank. For a given C > 0, if $(\mathbf{w}^*; b^*; \mathbf{u}^*)$ is a global minimizer of (9), then it is a P-stationary point with $0 < \gamma < 1/\lambda_H$.

Theorem 2 (First-order sufficient optimality conditions). For a given C > 0, if $(\mathbf{w}^*; b^*; \mathbf{u}^*)$ with $\lambda^* \in \mathbb{R}^m$ and $\gamma > 0$ is a P-stationary point, then it is a local minimizer of (9).

At the end of this section, we analyze the following relationship between the P-stationary point and the KKT point given by [4]. Based on the problem of (9) and the necessary conditions results in [4], if $(\overline{\mathbf{w}}; \overline{b}; \overline{\mathbf{u}})$ is a local minimizer of (9), then we have the following KKT conditions,

$$\begin{cases} \overline{\mathbf{w}} + A^{\top} \overline{\lambda} = \mathbf{0}, \\ \langle \mathbf{y}, \overline{\lambda} \rangle = 0, \\ \overline{\mathbf{u}} + A \overline{\mathbf{w}} + \overline{b} \mathbf{y} = \mathbf{1}, \\ C \partial L_r(\overline{\mathbf{u}}) + \overline{\lambda} \ni \mathbf{0}, \end{cases}$$
(12)

where $\overline{\boldsymbol{\lambda}} \in \mathbb{R}^m$ is a multiplier vector, $(\overline{\mathbf{w}}; \overline{b}; \overline{\mathbf{u}})$ is called the KKT point of (9), and $\partial L_r(\overline{\mathbf{u}}) = (\partial \ell_r(\overline{u}_1), \partial \ell_r(\overline{u}_2), \cdots, \partial \ell_r(\overline{u}_m))^\top \in \mathbb{R}^m$ is the subdifferential of $L_r(\overline{\mathbf{u}})$ with

$$\partial \ell_r(\overline{u}_i) \begin{cases} \in [0,1], & \overline{u}_i \in \{0,1\}, \\ = 1, & \overline{u}_i \in (0,1), \\ = 0, & \overline{u}_i < 0 \text{ or } \overline{u}_i > 1, \end{cases} \quad i \in \mathbb{N}_m.$$

Furthermore, from the above formula and last formula of (12), we have

$$\mathbf{0} \in C\partial L_r(\overline{\mathbf{u}}) + \overline{\boldsymbol{\lambda}} \Leftrightarrow \begin{cases} \overline{\boldsymbol{\lambda}} \in \mathbb{R}^m : \overline{\lambda}_i \begin{cases} \in [-C,0], & \overline{u}_i \in \{0,1\}, \\ = -C, & \overline{u}_i \in (0,1), \\ = 0, & \overline{u}_i < 0 \text{ or } \overline{u}_i > 1. \end{cases}$$
(13)

Theorem 3 For a given C > 0, if $(\mathbf{w}^*; b^*; \mathbf{u}^*)$ with $\lambda^* \in \mathbb{R}^m$ and $\gamma > 0$ is a P-stationary point of (9), then it is also a KKT point of (9), but the converse does not hold.

Proof The former conclusion is obvious since the P-stationary point is the local minimizer of (9). The latter conclusion is from the following counterexample. Consider the training set with the positive vectors $\mathbf{x}_1 = (3,3)^{\top}$, $\mathbf{x}_2 = (6,-2)^{\top}$ and negative vector $\mathbf{x}_3 = (1,1)^{\top}$. Namely,

$$A := [y_1 \mathbf{x}_1 \ y_2 \mathbf{x}_2 \ y_3 \mathbf{x}_3]^\top = \begin{bmatrix} 3 & 6 & -1 \\ 3 & -2 & -1 \end{bmatrix}^\top, \quad \mathbf{y} = \begin{bmatrix} 1, \ 1, \ -1 \end{bmatrix}^\top.$$

For a given C = 0.25, we can verify that

$$\overline{\mathbf{w}} = (0.5, 0.5)^{\top}, \qquad \overline{b} = -2, \qquad \overline{\mathbf{u}} = (0, 1, 0)^{\top}, \overline{\boldsymbol{\lambda}} = (-0.25, 0, -0.25)^{\top}, \qquad \partial L_r(\overline{\mathbf{u}}) = (1, 0, 1)^{\top}$$

satisfy (12). This means $(\overline{\mathbf{w}}; \overline{b}; \overline{\mathbf{u}})$ with $\overline{\lambda}$ is a KKT point of (9). Particularly, for i = 2, we have $\overline{u}_2 = 1$ and $\overline{\lambda}_2 = 0$. However, for $0 < \gamma < 8$ and C = 0.25, i.e., $0 < \gamma C < 2$, from (5) and $\overline{u}_2 - \gamma \overline{\lambda}_2 \in (0, 1 + \frac{\gamma C}{2})$, we obtain $\operatorname{prox}_{\gamma C \ell_r}(\overline{u}_2 - \gamma \overline{\lambda}_2) = 0$ or $\overline{u}_2 - \gamma C$ but the both does not equal \overline{u}_2 . For $\gamma \geq 8$ and C = 0.25, i.e., $\gamma C \geq 2$, from (6) and $\overline{u}_2 - \gamma \overline{\lambda}_2 \in (0, \sqrt{2\gamma C})$, we get $\operatorname{prox}_{\gamma C \ell_r}(\overline{u}_2 - \gamma \overline{\lambda}_2) = 0 \neq \overline{u}_2$. To sum up, we have $\overline{u}_2 \notin \operatorname{prox}_{\gamma C \ell_r}(\overline{u}_2 - \gamma \overline{\lambda}_2)$, which means $(\overline{\mathbf{w}}; \overline{b}; \overline{\mathbf{u}})$ is not a P-stationary point of (9). \Box

$4 L_r$ support vectors

In this section, we define the support vectors for L_r -SVM by P-stationary point of (9), which are named as L_r support vectors since they are selected by the L_r proximal operator. Before proceeding, we first review the definitions of support vectors of hard margin SVM, hinge loss SVM and ramp loss SVM, respectively. i) Support vectors of hard margin SVM[6]: Let $(\widetilde{\mathbf{w}}; \widetilde{b})$ be a global minimizer of hard margin SVM and $\widetilde{\boldsymbol{\alpha}} = (\widetilde{\alpha}_1, \widetilde{\alpha}_2, \cdots, \widetilde{\alpha}_m)^{\top}$ with $\widetilde{\alpha}_i \geq 0$ be a solution of its dual problem. Then the global minimizer $\widetilde{\mathbf{w}}$ satisfies

$$\widetilde{\mathbf{w}} = \sum_{i \in \widetilde{J}^*} \widetilde{\alpha}_i y_i \mathbf{x}_i,$$

where $\widetilde{J}^* := \{i \in \mathbb{N}_m : \widetilde{\alpha}_i > 0\}$. The training vectors $\{\mathbf{x}_i, i \in \widetilde{J}^*\}$ are called support vectors. For any $i \in \widetilde{J}^*$, we have

$$y_i(\langle \widetilde{\mathbf{w}}, \mathbf{x}_i \rangle + \widetilde{b}) = 1.$$

ii) Support vectors of hinge loss SVM[6]: Let $(\widehat{\mathbf{w}}; \widehat{b})$ be a global minimizer of hinge loss SVM and $\widehat{\alpha} = (\widehat{\alpha}_1, \widehat{\alpha}_2, \cdots, \widehat{\alpha}_m)^\top$ with $\widehat{\alpha}_i \in [0, C]$ be a solution of its dual problem. Then the global minimizer $\widehat{\mathbf{w}}$ satisfies

$$\widehat{\mathbf{w}} = \sum_{i \in \widehat{J}^*} \widehat{\alpha}_i y_i \mathbf{x}_i,$$

where $\widehat{J}^* := \{i \in \mathbb{N}_m : \widehat{\alpha}_i \in (0, C]\}$. The training vectors $\{\mathbf{x}_i, i \in \widehat{J}^*\}$ are called support vectors. For any $i \in \widehat{J}^*$, we have

$$\begin{cases} y_i(\langle \widehat{\mathbf{w}}, \mathbf{x}_i \rangle + \widehat{b}) = 1, \ i \in \{i \in \mathbb{N}_m : \widehat{\alpha}_i \in (0, C)\} \\ y_i(\langle \widehat{\mathbf{w}}, \mathbf{x}_i \rangle + \widehat{b}) \le 1, \ i \in \{i \in \mathbb{N}_m : \widehat{\alpha}_i = C\}. \end{cases}$$

iii) Support vectors of ramp loss SVM[4]: Let $(\overline{\mathbf{w}}; \overline{b}; \overline{\mathbf{u}})$ and $\overline{\lambda} \in \mathbb{R}^m$ with $\overline{\lambda}_i \in [-C, 0]$ satisfy KKT conditions. From (13) and the first equation of (12), the KKT point $\overline{\mathbf{w}}$ satisfies

$$\overline{\mathbf{w}} = -\sum_{i \in \overline{J}^*} \overline{\lambda}_i y_i \mathbf{x}_i,$$

where $\overline{J}^* := \{i \in \mathbb{N}_m : \overline{\lambda}_i \in [-C, 0]\}$. The training vectors $\{\mathbf{x}_i, i \in \overline{J}^*\}$ are called support vectors. For any $i \in \overline{J}^*$, from (13) and the third equation of (12), we have

$$\begin{cases} y_i(\langle \overline{\mathbf{w}}, \mathbf{x}_i \rangle + \overline{b}) \in \{0, 1\}, \ i \in \{i \in \mathbb{N}_m : \overline{\lambda}_i \in (-C, 0)\}, \\ y_i(\langle \overline{\mathbf{w}}, \mathbf{x}_i \rangle + \overline{b}) \in [0, 1], \ i \in \{i \in \mathbb{N}_m : \overline{\lambda}_i = -C\}. \end{cases}$$

The above results show that the hard margin SVM and hinge loss SVM define the support vectors at their global minimizer. However, authors in [4] define the support vectors at the KKT point of L_r -SVM. In the following, we define the L_r support vectors for L_r -SVM at the P-stationary point, which is also the local minimizer of L_r -SVM.

Summarizing the above analysis, we obtain the following interesting result.

Theorem 4 Let $(\mathbf{w}^*; b^*; \mathbf{u}^*)$ with $\lambda^* \in \mathbb{R}^m$ and $\gamma > 0$ be a P-stationary point of (9) for $\gamma C \geq 2$. Then all L_r support vectors must fall into the support hyperplanes $\langle \mathbf{w}^*, \mathbf{x} \rangle + b^* = \pm 1$, which possesses the same feature as the one of hard margin SVM.

5 Conclusions

In this paper, with the help of explicit expression of proximal operator for ramp loss, we have introduced and characterized a novel first-order necessary and sufficient optimality conditions of L_r -SVM. We have defined the L_r support vectors based on the concept of P-stationary point and showed that all L_r support vectors fall into the support hyperplanes for $\gamma C \geq 2$. Based on the advance of P-stationarity, could we design an efficient algorithm for L_r -SVM? We leave this topic to be investigated in the future.

Acknowledgements

This work is supported by the National Natural Science Foundation of China (11926348-9, 61866010, 11871183), and the Natural Science Foundation of Hainan Province (118QN181).

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