Convergence Rate Analysis of Proximal Iteratively Reweighted ℓ_1 Methods for ℓ_p Regularization Problems

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Abstract

In this paper, we focus on the local convergence rate analysis of the proximal iteratively reweighted ℓ_1 algorithms for solving ℓ_p regularization problems, which are widely applied for inducing sparse solutions. We show that if the Kurdyka-Lojasiewicz (KL) property is satisfied, the algorithm converges to a unique first-order stationary point; furthermore, the algorithm has local linear convergence or local sublinear convergence. The theoretical results we derived are much stronger than the existing results for iteratively reweighted ℓ_1 algorithms.

Keywords: Kurdyka-Lojasiewicz property, iteratively reweighted algorithm, ℓ_p regularization, convergence rate

1 Introduction

In recent years, sparse optimization problems arises in a wide range of fields including machine learning, image processing and compressed sensing (17; 10; 6; 9; 23; 18). A common technique to enforce sparsity is to add the ℓ_p (0 < p < 1) regularization term to the objective function, which is called the ℓ_p regularized problem

$$\min_{x \in \mathbb{R}^n} F(x) := f(x) + \lambda ||x||_p^p \quad \text{with } ||x||_p^p := \sum_{i=1}^n |x_i|^p$$
 (P)

where $f: \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function, $p \in (0,1)$ and $\lambda > 0$ is the regularization parameter. It is generally believed that ℓ_p can have superior ability to induce sparse solutions of a system compared with traditional convex regularization techniques. For example, when $p \to 0$, this problem approximates the ℓ_0 -norm optimization problem, that is usually useful for image processing; when p = 1, that is the well-known ℓ_1 -norm regularized problem.

However, it is full of challenges to seek the solution of ℓ_p -norm optimization problems due to the nonconvex and nonsmooth property of ℓ_p -norm. In fact, (11) proved that finding the global minimal value of the problem with ℓ_p -norm regularization term is strongly NP-Hard.

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Recently, effective methods have been proposed to construct smooth approximation models for the ℓ_p regularization problem. Some works (6; 16; 7) focus on constructing Lipshcitz continuous approximation to replace $|x_i|^p$. Other works (8) and (12) take the smoothing technique which adds perturbation to each $|x_i|$ to form the ϵ -approximation of the ℓ_p -norm. In the later case, the approximate objective function becomes

$$f(x) + \lambda \sum_{i=1}^{n} (|x_i| + \epsilon)^p, \qquad (1)$$

with $\epsilon > 0$. Iteratively reweighted ℓ_1 methods (16; 19; 22) were proposed for solving approximation (1). At each iteration, it replaces each component of the ϵ -approximation via linearizing $(\cdot)^p$ at x^k , i.e.,

$$p(|x_i^k| + \epsilon_i)^{p-1}|x_i|. (2)$$

There is a tradeoff in the choice of ϵ . Large ϵ smoothes out many local minimizers, while small values make the subproblems difficult to solve due to bad local minimizers. In order to approximate (P) effectively, (16) improved these weights by dynamically updating perturbation parameter ϵ_i at each iteration. Recently, it is shown in (21) that the general framework of iteratively reweighted ℓ_1 methods is equivalent to solving a weighted ℓ_1 regularization problem, based on which the global convergence and O(1/k) worst-case complexity of optimality residual were analyzed.

In this paper, we focus on the local convergence rate analysis of the proximal iteratively reweighted ℓ_1 methods for the ℓ_p regularization problem. This type of algorithms was first presented and investigated in (15) with fixed $\epsilon > 0$ and there was no convergence rate established. Our purpose is to show that local linear convergence or sublinear convergence can be obtained under mild assumptions. The Kurdyka-Łojasiewicz (KL) property (5; 4) is generally believed to capture a broad spectrum of the local geometries that a nonconvex function can have and has been shown to hold ubiquitously for most practical functions. It has been exploited extensively to analyze the convergence rate of various first-order algorithms for nonconvex optimization (1; 13; 4; 24). However, it has not been exploited to establish the convergence rate of iteratively reweighted methods. In this paper, we exploit the KL property of f to provide a comprehensive study of the convergence rate of iteratively reweighted ℓ_1 methods for ℓ_p regularization problems. We anticipate our study to substantially advance the existing understanding of the convergence of iteratively reweighted methods to a much broader range of nonconvex regularization problems.

1.1 Notation

We denote \mathbb{R} and \mathbb{Q} as the set of real numbers and rational numbers. In \mathbb{R}^n , denote $\|\cdot\|_p$ as the ℓ_p norm with $p \in (0, +\infty)$, i.e., $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$. Note that for $p \in (0, 1)$, this does not define a proper norm due to its lack of subadditivity. If function $f: \mathbb{R}^n \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ is convex, then the subdifferential of f at \overline{x} is given by

$$\partial f(\bar{x}) := \{ z \mid f(\bar{x}) + \langle z, x - \bar{x} \rangle \le f(x), \ \forall x \in \mathbb{R}^n \}.$$

In particular, for $x \in \mathbb{R}^n$, we use $\partial ||x||_1$ to denote the set $\{\xi \in \mathbb{R}^n \mid \xi_i \in \partial |x_i|, i = 1, \dots, n\}$. Given a lower semi-continuous function f, the limiting subdifferential at a is defined as

$$\bar{\partial}f(a) := \{z^* = \lim_{x^k \to a, f(x^k) \to f(a)} z^k, \ z^k \in \partial_F f(x^k)\}.$$

The Frechet subdifferential of f at a defined as

$$\partial_F f(a) := \{ z \in \mathbb{R}^n \mid \liminf_{x \to a} \frac{f(x) - f(a) - \langle z, x - a \rangle}{\|x - a\|_2} \ge 0 \}.$$

The Clarke subdifferential $\partial_c f$ is the convex hull of the limiting subdifferential. It holds true that $\partial f(a) \subset \bar{\partial} f(a) \subset \partial_c f(a)$. For convex functions, $\partial f(a) = \partial_F f(a) = \bar{\partial} f(a) = \partial_c f(a)$ and for differentiable f, $\partial f(a) = \partial_F f(a) = \bar{\partial} f(a) = \partial_c f(a) = \{\nabla f(a)\}$.

For $f: \mathbb{R}^n \to \mathbb{R}$ and index sets \mathcal{A} and \mathcal{I} satisfying $\mathcal{A} \cup \mathcal{I} = \{1, \dots, n\}$, let $f(x_{\mathcal{I}})$ be the function in the reduced space $\mathbb{R}^{|\mathcal{I}|}$ by fixing $x_i = 0, i \in \mathcal{A}$. For $a, b \in \mathbb{R}^n$, $a \leq b$ means the inequality holds for each component, i.e., $a_i \leq b_i$ for $i = 1, \dots, n$. For closed convex set $\chi \subset \mathbb{R}^n$, define the Euclidean distance of point $a \in \mathbb{R}^n$ to β as $\text{dist}(a, \chi) = \min_{b \in \chi} \|a - b\|_2$. Let $\{-1, 0, +1\}^n$ be the set of vectors in \mathbb{R}^n filled with elements in $\{-1, 0, +1\}$. The support of $x \in \mathbb{R}^n$ is defined as $\mathcal{I}(x) := \{i \mid x_i \neq 0\}$. For $a, b \in \mathbb{R}$, let $a \mod b$ denote the remainder of a divided by b.

2 Proximal iteratively reweighted ℓ_1 method

In this section, we present the Proximal Iteratively Reweighted ℓ_1 (PIRL1) methods and examine their properties when applied to (P). The PIRL1 method is based on the smoothed approximation of F by adding perturbation ϵ_i to each component of |x|

$$F(x,\epsilon) := f(x) + \lambda \sum_{i=1}^{n} (|x_i| + \epsilon_i)^p,$$

where $\epsilon \in \mathbb{R}^n_{++}$ is the perturbation vector. At the kth iteration, PIRL1 solves the subproblem

$$\min_{x} \nabla f(x^{k})^{T} (x - x^{k}) + \frac{\beta}{2} ||x - x^{k}||^{2} + \lambda \sum_{i=1}^{n} w_{i}^{k} |x_{i}|$$

with $\beta > L_f/2$ and the weight w_i^k is defined as $w_i^k := p(|x_i^k| + \epsilon_i^k)^{p-1}$ with $\epsilon_i \to 0$. The framework of the PIRL1 is presented in Algorithm 1.

Algorithm 1 Proximal Iteratively Reweighted ℓ_1 Methods (PIRL1)

- 1: **Input:** $\mu \in (0,1), \beta > L_f/2, \epsilon^0 \in \mathbb{R}^n_{++} \text{ and } x^0. \text{ Set } k = 0$
- 2: repeat
- 3: Compute weights: $w_i^k = p(|x_i^k| + \epsilon_i^k)^{p-1}$.
- 4: Compute new iterate:

$$x^{k+1} \leftarrow \underset{x \in \mathbb{R}^n}{\operatorname{argmin}} \left\{ \nabla f(x^k)^T (x - x^k) + \frac{\beta}{2} ||x - x^k||^2 + \lambda \sum_{i=1}^n w_i^k |x_i| \right\}.$$
 (3)

- 5: Choose $\epsilon^{k+1} \leq \mu \epsilon^k$.
- 6: Set $k \leftarrow k + 1$.
- 7: until convergence

We make the following assumptions about the functions in (P). formulation

Assumption 1. f is Lipschitz differentiable with constant $L_f \geq 0$. The initial point (x^0, ϵ^0) is such that $\mathcal{L}(F^0) := \{x \mid F(x) \leq F^0 := F(x^0, \epsilon^0)\}$ is contained in a bounded ball $\mathbb{B}_R := \{x \mid ||x||_2 \leq R\}$.

2.1 Basic properties

The first-order necessary optimality condition of (P) is given (16),

$$\nabla_i f(x^*) + \lambda p |x_i^*|^{p-1} \operatorname{sign}(x_i^*) = 0 \quad \text{for} \quad i \in \mathcal{I}(x^*), \tag{4}$$

which is equivalent to

$$x_i \nabla_i f(x^*) + \lambda p |x_i^*|^p = 0 \quad \text{for} \quad i = 1, \dots, n.$$

$$(5)$$

We call any point satisfying (4) or (5) is stationary for F(x,0).

Proposition 2. Assume $\{x^k\}$ is generated by Algorithm 1 and Assumption 1 holds. Let Γ be the cluster point set of $\{x^k\}$. We have the following

(a)
$$F(x^{k+1}, \epsilon^{k+1}) \leq F(x^k, \epsilon^k) - \hat{\beta} \|x^{k+1} - x^k\|_2^2$$
 with $\hat{\beta} := \beta - \frac{L_f}{2}$ and $\{x^k\} \subset \mathcal{L}(F^0) \subset \mathbb{B}_R$.

(b) \exists constant ζ such that $F(x^*, 0) = \zeta$, $\forall x^* \in \Gamma$.

(c)
$$\sum_{k=0}^{\infty} ||x^{k+1} - x^k||_2^2 < +\infty$$
.

(d) All points in Γ are stationary for F(x,0).

Proof. (a). Lipschitz differentiability of f gives

$$f(x^{k+1}) \le f(x^k) + \nabla f(x^k)^T (x^{k+1} - x^k) + \frac{L_f}{2} ||x^{k+1} - x^k||_2^2$$
 (6)

The concavity of a^p on \mathbb{R}_{++} gives $a_1^p \leq a_2^p + pa_2^{p-1}(a_1 - a_2)$ for any $a_1, a_2 \in \mathbb{R}_{++}$. Hence we have

$$\begin{split} (|x_i^{k+1}| + \epsilon_i^k)^p &\leq (|x_i^k| + \epsilon_i^k)^p + p(|x_i^k| + \epsilon_i^k)^{p-1}(|x_i^{k+1}| - |x_i^k|) \\ &= (|x_i^k| + \epsilon_i^k)^p + w_i^k(|x_i^{k+1}| - |x_i^k|). \end{split}$$

Summing the above inequality over i yields

$$\sum_{i=1}^{n} (|x_i^{k+1}| + \epsilon_i^k)^p \le \sum_{i=1}^{n} (|x_i^k| + \epsilon_i^k)^p + \sum_{i=1}^{n} w_i^k (|x_i^{k+1}| - |x_i^k|). \tag{7}$$

The optimality condition of subproblems implies there exists $\xi^{k+1} \in \partial ||x^{k+1}||_1$ such that

$$\nabla f(x^k) + \beta^k (x^{k+1} - x^k) + \lambda w^k \circ \xi^{k+1} = 0.$$
 (8)

The definition of subgradient implies $|y_i| \leq |x_i| + \xi_i(y_i - x_i)$ with $\xi_i \in \partial |y_i|$. Thus, we have

$$F(x^{k+1}, \epsilon^{k+1}) - F(x^k, \epsilon^k)$$

$$= f(x^{k+1}) + \lambda \sum_{i=1}^n (|x_i^{k+1}| + \epsilon_i^k)^p - \left(f(x^k) + \lambda \sum_{i=1}^n (|x_i^k| + \epsilon_i^k)^p \right)$$

$$\leq \nabla f(x^k)^T (x^{k+1} - x^k) + \frac{L_f}{2} ||x^{k+1} - x^k||_2^2 + \lambda \sum_{i=1}^n w_i^k (|x_i^{k+1}| - |x_i^k|)$$

$$\leq \nabla f(x^k)^T (x^{k+1} - x^k) + \frac{L_f}{2} ||x^{k+1} - x^k||_2^2 + \lambda \sum_{i=1}^n w_i^k \xi_i^{k+1} (x_i^{k+1} - x_i^k)$$

$$= (\nabla f(x^k) + \beta (x^{k+1} - x^k) + \lambda w^k \circ \xi^{k+1})^T (x^{k+1} - x^k) - (\beta - \frac{L_f}{2}) ||x^{k+1} - x^k||_2^2$$

$$= -(\beta - \frac{L_f}{2}) ||x^{k+1} - x^k||_2^2,$$

$$(9)$$

where the first inequality follows from (6) and (7) and the last equality is due to (8). Therefore, (a)

holds true with $\hat{\beta} = \beta - L_f/2$. (b). Monotonicity of $\{F(x^k, \epsilon^k)\}$ gives $\zeta := \lim_{k \to \infty \atop k \in S} F(x^k, \epsilon^k) = F(x^*, 0)$ for any $x^* \in \Gamma$ with subsequence $\{x^k\}_{\mathcal{S}} \to x^*$.

(c). From (a), we have

$$\hat{\beta} \sum_{k=0}^{t} \|x^{k+1} - x^k\|_2^2 \le F(x^0, \epsilon^0) - F(x^{t+1}, \epsilon^{t+1}).$$

Then, taking the limit as $t \to \infty$,

$$\hat{\beta} \sum_{k=0}^{\infty} \|x^{k+1} - x^k\|_2^2 \le F(x^0, \epsilon^0) - \lim_{t \to \infty} F(x^{t+1}, \epsilon^{t+1}) < \infty.$$

(d). Let x^* be a limit point with $\{x^k\}_{\mathcal{S}} \to x^*$. The optimal condition of the kth subproblem implies

$$\nabla_i f(x^{k-1}) + \beta(x_i^k - x_i^{k-1}) + \lambda p(|x_i^{k-1}| + \epsilon_i^{k-1})^{p-1} \operatorname{sign}(x_i^k) = 0, \quad \forall i \in \mathcal{I}(x^k).$$

Taking the limit on S, we have for each $i \in \mathcal{I}(x^*)$,

$$\begin{split} 0 &= \lim_{k \to \infty \atop k \in S} \nabla_i f(x^{k-1}) + \beta(x_i^k - x_i^{k-1}) + \lambda p(|x_i^{k-1}| + \epsilon_i^{k-1})^{p-1} \mathrm{sign}(x_i^k) \\ &= \lim_{k \to \infty \atop k \in S} \nabla_i f(x^k) + \beta(x_i^{k+1} - x_i^k) + \lambda p|x_i^*|^{p-1} \mathrm{sign}(x_i^*) \\ &= \lim_{k \to \infty \atop k \in S} \nabla_i f(x^k) + \lambda p|x_i^*|^{p-1} \mathrm{sign}(x_i^*) \\ &= \nabla_i f(x^*) + \lambda p|x_i^*|^{p-1} \mathrm{sign}(x_i^*). \end{split}$$

Here the second equality is from $\epsilon_i^k \to 0$ for all $i \in \mathcal{I}(x^*)$. Therefore, x^* is a stationary point of F(x,0).

Algorithm 1 belongs to the framework of iteratively reweighted ℓ_1 methods proposed in (21). From (21), the following properties hold true.

Theorem 3. (21, Theorem 1) Assume Assumption 1 holds and let $\{(x^k, \epsilon^k)\}$ be a sequence generated by Algorithm 1. Define constant $C = \sup_{x \in \mathbb{B}_R} \|\nabla f(x)\|_2 + 2R\beta$. Then we have the following

- (i) If $w(x_i^{\tilde{k}}, \epsilon_i^{\tilde{k}}) > C/\lambda$ for some $\tilde{k} \in \mathbb{N}$, then $x_i^k \equiv 0$ for all $k > \tilde{k}$. Conversely, if there exists $\hat{k} > \tilde{k}$ for any $\tilde{k} \in \mathbb{N}$ such that $x_i^{\hat{k}} \neq 0$, then $w_i^k \leq C/\lambda$ for all $k \in \mathbb{N}$.
- (ii) There exist index sets $\mathcal{I}^* \cup \mathcal{A}^* = \{1, \dots, n\}$ and $\bar{k} > 0$, such that $\forall k > \bar{k}$, $\mathcal{I}(x^k) \equiv \mathcal{I}^*$ and $\mathcal{A}(x^k) \equiv \mathcal{A}^*$.
- (iii) For any $i \in \mathcal{I}^*$, there holds that

$$|x_i^k| > \left(\frac{C}{p\lambda}\right)^{\frac{1}{p-1}} - \epsilon_i^k > 0, \quad i \in \mathcal{I}^*.$$

$$\tag{10}$$

Therefore, $\{|x_i^k|, i \in \mathcal{I}^*, k \in \mathbb{N}\}\$ are bounded away from 0 after some $\hat{k} \in \mathbb{N}$.

(iv) For any cluster point x^* of $\{x^k\}$, it holds that $\mathcal{I}(x^*) = \mathcal{I}^*$, $\mathcal{A}(x^*) = \mathcal{A}^*$ and

$$|x_i^*| \ge \left(\frac{C}{p\lambda}\right)^{\frac{1}{p-1}}, \quad i \in \mathcal{I}^*. \tag{11}$$

The above theorem shows locally the support of the iterates remains unchanged and the nonzeros are bounded away from 0. The next theorem shows that the signs of iterates stay stable locally.

Theorem 4. (21, Theorem 2) Let $\{x^k\}$ be a sequence generated by Algorithm 1 and Assumption 1 is satisfied. There exists $\bar{k} \in \mathbb{N}$, such that the sign of $\{x^k\}$ are fixed for all $k > \bar{k}$, i.e., $sign(x^k) \equiv s$ for some $s \in \{-1,0,+1\}^n$.

2.2 Kurdyka-Łojasiewicz property

(2) have proved a series of convergence results of descent methods for semi-algebraic problems under the assumption that the objective satisfies the Kurdyka-Łojasiewicz (KL) property. In fact, this assumption covers a wide range of problems such as nonsmooth semi-algebraic minimization problem (4). The definition of KL property is given below. **Definition 5** (Kurdyka-Łojasiewicz property). The function $f: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said to have the Kurdyka-Łojasiewicz property at $x^* \in dom\bar{\partial}f$ if there exists $\eta \in (0, +\infty]$, a neighborhood U of x^* and a continuous concave function $\phi: [0, \eta) \to \mathbb{R}_+$ such that:

- (i) $\phi(0) = 0$,
- (ii) ϕ is C^1 on $(0, \eta)$,
- (iii) for all $s \in (0, \eta), \phi'(s) > 0$
- (iv) for all x in $U \cap [f(x^*) < f < f(x^*) + \eta]$, the Kurdyka-Lojasiewicz inequality holds

$$\phi'(f(x) - f(x^*)) \operatorname{dist}(0, \bar{\partial}f(x)) \ge 1.$$

If f is smooth, then condition (iv) reverts to (2)

$$\|\nabla(\phi \circ f)(x)\| \ge 1.$$

Since for sufficiently large k, the iterates $\{x_{\mathcal{I}^*}^k\}$ remains in the same orthant of $\mathbb{R}^{|\mathcal{I}^*|}$ and are bounded away from the axis, or equivalently,

$$\{x_{\mathcal{I}^*}^k\} \in \Omega \subset \mathbb{R}_s^{|\mathcal{I}^*|}$$

where Ω is in the interior of an orthant and is bounded away from the axis. To further analyze the property of iterates $\{(x^k, \epsilon^k)\}$, denote $\delta_i = \sqrt{\epsilon_i}$. Therefore, we can write $F(x, \delta)$ as a function of (x, δ) for simplicity. We can assume the reduced function $F(x_{\mathcal{I}^*}, \delta_{\mathcal{A}^*})$ has the KL property at $(x_{\mathcal{I}^*}^*, 0_{\mathcal{I}^*})$. In fact, we only need to make assumption on f. To see this, we introduce the concept of semi-algebraic functions, which is a weak condition and can cover most common functions.

Definition 6 (Semi-algebraic functions). A subset of \mathbb{R}^n is called semi-algebraic if it can be written as a finite union of sets of the form

$$\{x \in \mathbb{R}^n : h_i(x) = 0, \ q_i(x) < 0, \ i = 1, \dots, p\},\$$

where h_i, q_i are real polynomial functions. A function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is semi-algebraic if its graph is a semi-algebraic subset of \mathbb{R}^{n+1} .

Semi-algebraic functions satisfy KL property with $\phi(x) = cs^{1-\theta}$, for some $\theta \in [0,1) \cap \mathbb{Q}$ and some c > 0 (5; 3). This non-smooth result generalizes the famous Łojasiewicz inequality for real-analytic function (14). Finite sums of semi-algebraic functions are semi-algebraic; for $p \in \mathbb{Q}$, $\sum_{i \in \mathcal{I}^*} (|x_i| + \epsilon_i)^p$ is semi-algebraic around $(x_{\mathcal{I}^*}^*, 0_{\mathcal{A}^*})$ by (20). Therefore, we only need to assume $f(x_{\mathcal{I}^*})$ is semi-algebraic in a neighborhood around x^* .

We state this assumption formally below.

Assumption 7. Suppose $p \in \mathbb{Q}$ and $f(x_{\mathcal{I}^*})$ is semi-algebraic in $\mathbb{R}_s^{|\mathcal{I}^*|}$, where x^* is a limit point of $\{x^k\}$ generated by the PIRL1 methods.

For simplicity of the following analysis and without loss of generality, we assume $\mathcal{I}^* = \{1, ..., n\}$ and $\mathcal{A}^* = \emptyset$, so that for sufficiently large k, the iterates $\{x_{\mathcal{I}^*}^k\}$ remains in the same orthant are bounded away from the axis.

3 The uniqueness of limit points

We investigate the uniqueness of limit points under KL property of F.

Lemma 8. Let $\{x^k\}$ be a sequence generated by Algorithm 1. The following statements hold.

(i) There exists $D_1 > 0$ such that for all k

$$\|\nabla F(x^k, \delta^k)\|_2 \le D_1(\|x^k - x^{k-1}\|_2 + \|\delta^{k-1}\|_1 - \|\delta^k\|_1),$$
and $\lim_{k \to \infty} \|\nabla F(x^k, \delta^k)\|_2 = 0.$

(ii) $\{F(x^k, \delta^k)\}\$ is monotonically decreasing, and there exists $\hat{\beta} > 0$ such that

$$F(x^{k+1}, \delta^{k+1}) - F(x^k, \delta^k) \ge \hat{\beta} \|x^{k+1} - x^k\|_2^2$$

 $(iii) \ F(x^*,0) = \zeta = \lim_{k \to \infty} F(x^k,\delta^k) \ for \ all \ x^* \in \Gamma, \ where \ \Gamma \ is \ the \ set \ of \ the \ cluster \ points \ of \ \{x^k\}.$

Proof. (i) The gradient of F at (x^k, δ^k) is

$$\nabla_x F(x^k, \delta^k) = \nabla f(x^k) + \lambda w^k \circ \operatorname{sign}(x^k),$$

$$\nabla_\delta F(x^k, \delta^k) = 2\lambda w^k \circ \delta^k.$$
 (12)

We first derive an upper bound for $\|\nabla_x F(x^k, \delta^k)\|_2$. The first-order optimality condition of the (k-1)th subproblem at x^k is

$$\nabla f(x^{k-1}) + \beta^k (x^k - x^{k-1}) + \lambda w^{k-1} \circ \text{sign}(x^k) = 0.$$

Hence, we have

$$\nabla_x F(x^k, \delta^k) = \nabla f(x^k) - \nabla f(x^{k-1}) - \beta^k (x^k - x^{k-1}) + \lambda (w^k - w^{k-1}) \circ \operatorname{sign}(x^k). \tag{13}$$

By the Lipschitz property of f, the first two terms in (13) is bounded by

$$\|\nabla f(x^k) - \nabla f(x^{k-1}) - \beta^k (x^k - x^{k-1})\|_2 \le (L_f + \beta) \|x^k - x^{k-1}\|_2.$$

Now we give an upper bound for the third term. It follows from Lagrange's mean value theorem that $\exists z_i^k$ between $|x_i^k| + (\delta_i^k)^2$ and $|x_i^{k-1}| + (\delta_i^{k-1})^2$, such that

$$\begin{split} \left| (w_i^k - w_i^{k-1}) \cdot \operatorname{sign}(x_i^k) \right| &= \left| w_i^k - w_i^{k-1} \right| \\ &= \left| p(|x_i^k| + (\delta_i^k)^2)^{p-1} - p(|x_i^{k-1}| + (\delta_i^{k-1})^2)^{p-1} \right| \\ &= \left| p(1-p)(z_i^k)^{p-2} (|x_i^k| - |x_i^{k-1}| + (\delta_i^k)^2 - (\delta_i^{k-1})^2) \right| \\ &\leq p(1-p)(z_i^k)^{p-2} (|x_i^k - x_i^{k-1}| + (\delta_i^{k-1})^2 - (\delta_i^k)^2) \\ &\leq p(1-p)(z_i^k)^{p-2} (|x_i^k - x_i^{k-1}| + 2\delta_i^0 (\delta_i^{k-1} - \delta_i^k)) \\ &\leq p(1-p) \left(\frac{p\lambda}{C} \right)^{\frac{p-2}{1-p}} (|x_i^k - x_i^{k-1}| + 2\delta_i^0 (\delta_i^{k-1} - \delta_i^k)), \end{split}$$

where the first equality is by the fact that $x_i^k \neq 0$ and the last inequality by observing the following. From Theorem 3(i), we know

$$|x_i^k| + (\delta_i^k)^2 = \left(\frac{w_i^k}{p}\right)^{\frac{1}{p-1}} \ge \left(\frac{C}{p\lambda}\right)^{\frac{1}{p-1}} = \left(\frac{p\lambda}{C}\right)^{\frac{1}{1-p}}$$

$$|x_i^{k-1}| + (\delta_i^{k-1})^2 = \left(\frac{w_i^{k-1}}{p}\right)^{\frac{1}{p-1}} \ge \left(\frac{C}{p\lambda}\right)^{\frac{1}{p-1}} = \left(\frac{p\lambda}{C}\right)^{\frac{1}{1-p}},$$

$$(14)$$

hence

$$(z_i^k)^{p-2} \le \left(\frac{p\lambda}{C}\right)^{\frac{p-2}{1-p}}.$$

Now we can obtain an upper bound for the third term in (13),

$$\begin{aligned} \|(w^{k} - w^{k-1}) \circ \operatorname{sign}(x^{k})\|_{2} &\leq \|(w^{k} - w^{k-1}) \circ \operatorname{sign}(x^{k})\|_{1} \\ &= \sum_{i=1}^{n} \left| (w_{i}^{k} - w_{i}^{k-1}) \cdot \operatorname{sign}(x_{i}^{k}) \right| \\ &\leq \sum_{i=1}^{n} p(1-p) \left(\frac{p\lambda}{C} \right)^{\frac{p-2}{1-p}} (|x_{i}^{k} - x_{i}^{k-1}| + 2\delta_{i}^{0}(\delta_{i}^{k-1} - \delta_{i}^{k})) \\ &\leq \bar{D} \left(\|x^{k} - x^{k-1}\|_{1} + 2\|\delta^{0}\|_{\infty} (\|\delta^{k-1}\|_{1} - \|\delta^{k}\|_{1}) \right) \\ &\leq \bar{D} \left(\sqrt{n} \|x^{k} - x^{k-1}\|_{2} + 2\|\delta^{0}\|_{\infty} (\|\delta^{k-1}\|_{1} - \|\delta^{k}\|_{1}) \right), \end{aligned}$$

$$(15)$$

where $\bar{D} := p(1-p) \left(\frac{p\lambda}{C}\right)^{\frac{p-2}{1-p}}$. Putting together the bounds for all three terms in (13), we have

$$\|\nabla_x F(x^k, \delta^k)\|_2 \le (L_f + \beta) \|x^k - x^{k-1}\|_2 + 2\bar{D} \|\delta^0\|_{\infty} (\|\delta^{k-1}\|_1 - \|\delta^k\|_1). \tag{16}$$

On the other hand,

$$\|\nabla_{\delta}F(x^{k},\delta^{k})\|_{2} \leq \|\nabla_{\delta}F(x^{k},\delta^{k})\|_{1}$$

$$= \sum_{i=1}^{n} 2\lambda w_{i}^{k} \delta_{i}^{k}$$

$$\leq \sum_{i=1}^{n} 2\lambda \frac{C}{\lambda} \frac{\sqrt{\mu}}{1 - \sqrt{\mu}} (\delta_{i}^{k-1} - \delta_{i}^{k})$$

$$\leq \frac{2C\sqrt{\mu}}{1 - \sqrt{\mu}} (\|\delta^{k-1}\|_{1} - \|\delta^{k}\|_{1}),$$
(17)

where the second inequality is by Theorem 3(i) and $\delta^k \leq \sqrt{\mu} \delta^{k-1}$. Overall, we obtain from (16) and (17) that Part (i) holds true by setting

$$D_1 = \max \left(\beta + L_f, 2\bar{C} \|\delta^0\|_{\infty} + \frac{2C\sqrt{\mu}}{1-\sqrt{\mu}} \right).$$

Part (ii) and (iii) follows directly from Proposition 2(a) and 2(b), respectively. \Box

Now we are ready to prove the global convergence under KL property.

Theorem 9. Let $\{x^k\}$ be a sequence generated by Algorithm 1 and F is a KL function at $(x^*, 0)$ with $x^* \in \Gamma$. Then $\{x^k\}$ converges to a stationary point of F(x, 0); moreover,

$$\sum_{k=0}^{\infty} \|x^{k+1} - x^k\|_2 < \infty.$$

Proof. By Proposition 2, every cluster point is stationary for F(x,0), it is sufficient to show that $\{x^k\}$ has a unique cluster point.

By Lemma 8, $F(x^k, \delta^k)$ is monotonically decreasing and converging to ζ . If $F(x^k, \delta^k) = \zeta$ after some k_0 , then from Lemma 8(ii), we know $x^{k+1} = x^k$ for all $k > k_0$, meaning $x^k \equiv x^{k_0} \in \Gamma$, so that the proof is done.

We next consider the case that $F(x^k, \delta^k) > \zeta$ for all k. Since F has the KL property at every $(x^*, 0) \in \bar{\Gamma}$, there exists a continuous concave function ϕ with $\eta > 0$ and neighborhood $U = \{(x, \delta) \in \mathbb{R}^n \times \mathbb{R}^n : \operatorname{dist}((x, \delta), \bar{\Gamma}) < \tau\}$ such that

$$\phi'(F(x,\delta) - \zeta)\operatorname{dist}((0,0), \nabla F(x,\delta)) \ge 1 \tag{18}$$

for all $(x, \delta) \in U \cap \{(x, \delta) \in \mathbb{R}^n \times \mathbb{R}^n : \zeta < F(x, \delta) < \zeta + \eta\}.$

Let $\bar{\Gamma} \subset \mathbb{R}^{2n}$ be the set of limit points of $\{(x^k, \delta^k)\}$, i.e., $\bar{\Gamma} := \{(x^*, 0) \mid x^* \in \Gamma\}$, by Proposition 2(ii), we have

$$\lim_{k \to \infty} \operatorname{dist}((x^k, \delta^k), \bar{\Gamma}) = 0.$$

Hence, there exist $k_1 \in \mathbb{N}$ such that $\operatorname{dist}((x^k, \delta^k), \bar{\Gamma}) < \tau$ for any $k > k_1$. On the other hand, since $\{F(x^k, \delta^k)\}$ is monotonically decreasing and converges to ζ , there exists $k_2 \in \mathbb{N}$ such that $\zeta < F(x^k, \delta^k) < \zeta + \eta$ for all $k > k_2$. Letting $\bar{k} = \max\{k_1, k_2\}$ and noticing that F is smooth at (x^k, δ^k) for all $k > \bar{k}$, we know from (18) that

$$\phi'(F(x^k, \delta^k) - \zeta) \|\nabla F(x^k, \delta^k)\|_2 \ge 1, \quad \text{for all } k \ge \bar{k}.$$

$$\tag{19}$$

It follows that for any $k > \bar{k}$.

$$\begin{split} & \left[\phi(F(x^k, \delta^k) - \zeta) - \phi(F(x^{k+1}, \delta^{k+1}) - \zeta) \right] \cdot D_1(\|x^k - x^{k-1}\|_2 + \|\delta^{k-1}\|_1 - \|\delta^k\|_1) \\ & \geq \left[\phi(F(x^k, \delta^k) - \zeta) - \phi(F(x^{k+1}, \delta^{k+1}) - \zeta) \right] \cdot \|\nabla F(x^k, \delta^k)\|_2 \\ & \geq \phi'(F(x^k, \delta^k) - \zeta) \cdot \|\nabla F(x^k, \delta^k)\|_2 \cdot \left[F(x^k, \delta^k) - F(x^{k+1}, \delta^{k+1}) \right] \\ & \geq F(x^k, \delta^k) - F(x^{k+1}, \delta^{k+1}) \\ & \geq \hat{\beta} \|x^{k+1} - x^k\|_2^2, \end{split}$$

where the first inequality is by Lemma 8(i), the second inequality is by the concavity of ϕ , and the third inequality is by (19) and the last inequality is by Lemma 8(ii). Rearranging and taking the square root of both sides, and using the inequality of arithmetic and geometric means inequality, we have

$$||x^{k} - x^{k+1}||_{2} \leq \sqrt{\frac{2D_{1}}{\hat{\beta}} [\phi(F(x^{k}, \delta^{k}) - \zeta) - \phi(F(x^{k+1}, \delta^{k+1}) - \zeta)]} \times \sqrt{\frac{||x^{k} - x^{k-1}||_{2} + (||\delta^{k-1}||_{1} - ||\delta^{k}||_{1})}{2}}$$

$$\leq \frac{D_{1}}{\hat{\beta}} [\phi(F(x^{k}, \delta^{k}) - \zeta) - \phi(F(x^{k+1}, \delta^{k+1}) - \zeta)]$$

$$+ \frac{1}{4} [||x^{k} - x^{k-1}||_{2} + (||\delta^{k-1}||_{1} - ||\delta^{k}||_{1})].$$

Subtracting $\frac{1}{4}||x^k - x^{k+1}||_2$ from both sides, we have

$$\begin{split} \frac{3}{4}\|x^{k+1} - x^k\|_2 &\leq \frac{D_1}{\hat{\beta}} \Big[\phi(F(x^k, \delta^k) - \zeta) - \phi(F(x^{k+1}, \delta^{k+1}) - \zeta) \Big] \\ &+ \frac{1}{4} (\|x^k - x^{k-1}\|_2 - \|x^{k+1} - x^k\|_2 + \|\delta^{k-1}\|_1 - \|\delta^k\|_1). \end{split}$$

Summing up both sides from \bar{k} to t, we have

$$\frac{3}{4} \sum_{k=\bar{k}}^{t} \|x^{k+1} - x^{k}\|_{2} \leq \frac{D_{1}}{\hat{\beta}} \left[\phi(F(x^{\bar{k}}, \delta^{\bar{k}}) - \zeta) - \phi(F(x^{t+1}, \delta^{t+1}) - \zeta) \right] \\
+ \frac{1}{4} (\|x^{\bar{k}} - x^{\bar{k}-1}\|_{2} - \|x^{t+1} - x^{t}\|_{2} + \|\delta^{\bar{k}-1}\|_{1} - \|\delta^{t}\|_{1}).$$

Now letting $t\to\infty$, we know $\|\delta^t\|_1\to 0$ and $\|x^{t+1}-x^t\|_2\to 0$ by Proposition 2(c), and that $\phi(F(x^{t+1},\delta^{t+1})-\zeta)\to\phi(\zeta-\zeta)=\phi(0)=0$. Therefore, we have

$$\sum_{k=\bar{k}}^{\infty} \|x^{k+1} - x^k\|_2 \le \frac{4D_1}{3\hat{\beta}} \phi(F(x^{\bar{k}}, \delta^{\bar{k}}) - \zeta) + \frac{1}{3} (\|x^{\bar{k}} - x^{\bar{k}-1}\|_2 + \|\delta^{\bar{k}-1}\|_1) < \infty.$$
 (20)

Hence $\{x^k\}$ is a Cauchy sequence, and consequently it is a convergent sequence.

4 Local convergence rate

We have shown that there is only one unique limit point of $\{x^k\}$ under KL property. Now we investigate the local convergence rate of Algorithm 1 by assuming that ϕ in the KL definition taking the form $\phi(s) = cs^{1-\theta}$ for some $\theta \in [0,1)$ and c>0. By the discussion in §2.2, this additional requirement is satisfied by the semialgebraic functions, which is also commonly satisfied by a wide range of functions.

Theorem 10. Suppose $\{x^k\}$ is generated by Algorithm 1 and converges to x^* . Assume that F is a KL function with ϕ in the KL definition taking the form $\phi(s) = cs^{1-\theta}$ for some $\theta \in [0,1)$ and c > 0. Then the following statements hold.

- (i) If $\theta = 0$, then there exists $k_0 \in \mathbb{N}$ so that $x^k \equiv x^*$ for any $k > k_0$;
- (ii) If $\theta \in (0, \frac{1}{2}]$, then there exist $\gamma \in (0, 1), c_1 > 0$ such that

$$||x^k - x^*||_2 < c_1 \gamma^k \tag{21}$$

 $for \ sufficiently \ large \ k;$

(iii) If $\theta \in (\frac{1}{2}, 1)$, then there exist $c_2 > 0$ such that

$$||x^k - x^*||_2 < c_2 k^{-\frac{1-\theta}{2\theta - 1}} \tag{22}$$

for sufficiently large k.

Proof. (i) If $\theta = 0$, then $\phi(s) = cs$ and $\phi'(s) \equiv c$. We claim that there must exist $k_0 > 0$ such that $F(x^{k_0}, \delta^{k_0}) = \zeta$. Suppose by contradiction this is not true so that $F(z^k) > \zeta$ for all k. Since $\lim_{k \to \infty} x^k = x^*$ and the sequence $\{F(x^k, \delta^k)\}$ is monotonically decreasing to ζ by Lemma8. The KL inequality implies that all sufficiently large k,

$$c\|\nabla F(x^k, \delta^k)\|_2 \ge 1,$$

contradicting $\|\nabla F(x^k, \delta^k)\|_2 \to 0$ by Lemma 8(i). Thus, there exists $k_0 \in \mathbb{N}$ such that $F(x^k, \delta^k) = F(x^{k_0}, \delta^{k_0}) = \zeta$ for all $k > k_0$. Hence, we conclude from Lemma8(ii) that $x^{k+1} = x^k$ for all $k > k_0$, meaning $x^k \equiv x^* = x^{k_0}$ for all $k \geq k_0$. This proves (i).

(ii)-(iii) Now consider $\theta \in (0,1)$. First of all, if there exists $k_0 \in \mathbb{N}$ such that $F(x^{k_0}, \delta^{k_0}) = \zeta$, then using the same argument of the proof for (ii), we can see that $\{x^k\}$ converges finitely. Thus, we only need to consider the case that $F(x^k, \delta^k) > \zeta$ for all k.

Define $S^{k} = \sum_{l=k}^{\infty} ||x^{l+1} - x^{l}||_{2}$. It holds that

$$||x^k - x^*||_2 = ||x^k - \lim_{t \to \infty} x^t||_2 = ||\lim_{t \to \infty} \sum_{l=k}^t (x^{l+1} - x^l)||_2 \le \sum_{l=k}^\infty ||x^{l+1} - x^l||_2 = S^k.$$

Therefore, we only have to prove S^k also has the same upper bound as in (21) and (22).

To derive the upper bound for S^k , by KL inequality with $\phi'(s) = c(1-\theta)s^{-\theta}$, for $k > \bar{k}$,

$$c(1-\theta)(F(x^k, \delta^k) - \zeta)^{-\theta} \|\nabla F(x^k, \delta^k)\|_2 \ge 1.$$
(23)

On the other hand, using (8)(i) and the definition of S^k , we see that for all sufficiently large k,

$$\|\nabla F(x^k, \delta^k)\|_2 \le D_1(S^{k-1} - S^k + \|\delta^{k-1}\|_1 - \|\delta^k\|_1)$$
(24)

Combining (23) with (24), we have

$$(F(x^k, \delta^k) - \zeta)^{\theta} \le D_1 c (1 - \theta) (S^{k-1} - S^k + \|\delta^{k-1}\|_1 - \|\delta^k\|_1).$$

Taking a power of $(1-\theta)/\theta$ to both sides of the above inequality and scaling both sides by c, we obtain that for all $k > \bar{k}$

$$\phi(F(x^{k}, \delta^{k}) - \zeta) = c \left[F(x^{k}, \delta^{k}) - \zeta \right]^{1-\theta}$$

$$\leq c \left[D_{1}c(1-\theta)(S^{k-1} - S^{k} + \|\delta^{k-1}\|_{1} - \|\delta^{k}\|_{1}) \right]^{\frac{1-\theta}{\theta}}$$

$$\leq c \left[D_{1}c(1-\theta)(S^{k-1} - S^{k} + \|\delta^{k-1}\|_{1}) \right]^{\frac{1-\theta}{\theta}},$$
(25)

From (20), we have

$$S^{k} \le \frac{4D_{1}}{3\hat{\beta}}\phi(F(x^{k},\delta^{k}) - \zeta) + \frac{1}{3}(\|x^{k} - x^{k-1}\|_{2} + \|\delta^{k-1}\|_{1}). \tag{26}$$

Combining (25) and (26), we have

$$S^{k} \leq C_{1}[S^{k-1} - S^{k} + \|\delta^{k-1}\|_{1}]^{\frac{1-\theta}{\theta}} + \frac{1}{3}(S^{k-1} - S^{k} + \|\delta^{k-1}\|_{1})$$

$$\leq C_{1}[S^{k-2} - S^{k} + \|\delta^{k-1}\|_{1}]^{\frac{1-\theta}{\theta}} + \frac{1}{3}[S^{k-2} - S^{k} + \|\delta^{k-1}\|_{1}]$$
(27)

where $C_1 = \frac{4D_1c}{3\hat{\beta}} \left(D_1 \cdot c(1-\theta)\right)^{\frac{1-\theta}{\theta}}$. It follows that

$$S^{k} + \frac{\sqrt{\mu}}{1-\mu} \|\delta^{k}\|_{1}$$

$$\leq C_{1} [S^{k-2} - S^{k} + \|\delta^{k-1}\|_{1}]^{\frac{1-\theta}{\theta}} + \frac{1}{3} [S^{k-2} - S^{k} + \|\delta^{k-1}\|_{1}] + \frac{\sqrt{\mu}}{1-\mu} \|\delta^{k}\|_{1}$$

$$\leq C_{1} [S^{k-2} - S^{k} + \|\delta^{k-1}\|_{1}]^{\frac{1-\theta}{\theta}} + \frac{1}{3} [S^{k-2} - S^{k} + \|\delta^{k-1}\|_{1}] + \frac{\mu}{1-\mu} \|\delta^{k-1}\|_{1}$$

$$\leq C_{1} [S^{k-2} - S^{k} + \|\delta^{k-1}\|_{1}]^{\frac{1-\theta}{\theta}} + C_{2} [S^{k-2} - S^{k} + \|\delta^{k-1}\|_{1}],$$
(28)

with $C_2 := \frac{1}{3} + \frac{\mu}{1-\mu}$ and the second inequality is by the update $\delta^k \leq \sqrt{\mu} \delta^{k-1}$. For part (ii), $\theta \in (0, \frac{1}{2}]$. Notice that

$$\frac{1-\theta}{\theta} \ge 1$$
 and $S^{k-2} - S^k + \|\delta^{k-1}\|_1 \to 0$.

Hence, there exists sufficient large k such that

$$\left[S^{k-2} - S^k + \|\delta^{k-1}\|_1\right]^{\frac{1-\theta}{\theta}} \leq S^{k-2} - S^k + \|\delta^{k-1}\|_1,$$

we assume the above inequality holds for all $k \geq \bar{k}$. This, combined with (28), yields

$$S^{k} + \frac{\sqrt{\mu}}{1 - \mu} \|\delta^{k}\|_{1} \le (C_{1} + C_{2}) \left[S^{k-2} - S^{k} + \|\delta^{k-1}\|_{1} \right]$$
(29)

for any $k \geq \bar{k}$. Using $\delta_k \leq \mu \delta_{k-1}$, we can show that

$$\delta^{k-1} \le \frac{\sqrt{\mu}}{1-\mu} (\delta^{k-2} - \delta^k). \tag{30}$$

Combining (29) and (30) gives

$$S^{k} + \frac{\sqrt{\mu}}{1-\mu} \|\delta^{k}\|_{1} \le (C_{1} + C_{2}) \left[\left(S^{k-2} + \frac{\sqrt{\mu}}{1-\mu} \|\delta^{k-2}\|_{1} \right) - \left(S^{k} + \frac{\sqrt{\mu}}{1-\mu} \|\delta^{k}\|_{1} \right) \right].$$

Rearranging this inequality gives

$$\begin{split} S^k + \frac{\sqrt{\mu}}{1-\mu} \|\delta^k\|_1 &\leq \frac{C_1 + C_2}{C_1 + C_2 + 1} \bigg[S^{k-2} + \frac{\sqrt{\mu}}{1-\mu} \|\delta^{k-2}\|_1 \bigg] \\ &\leq \left(\frac{C_1 + C_2}{C_1 + C_2 + 1} \right)^{\lfloor \frac{k}{2} \rfloor} \bigg[S^{k \bmod 2} + \frac{\sqrt{\mu}}{1-\mu} \|\delta^{k \bmod 2}\|_1 \bigg] \\ &\leq \left(\frac{C_1 + C_2}{C_1 + C_2 + 1} \right)^{\frac{k-1}{2}} \bigg[S^0 + \frac{\sqrt{\mu}}{1-\mu} \|\delta^0\|_1 \bigg]. \end{split}$$

Therefore, for any $k \geq \bar{k}$,

$$||x^k - x^*||_2 \le S^k + \frac{\sqrt{\mu}}{1 - \mu} ||\delta^k||_1 \le c_1 \gamma^k$$

with

$$c_1 = (S^0 + \frac{\sqrt{\mu}}{1-\mu} \|\delta^0\|) \left(\frac{C_1 + C_2}{C_1 + C_2 + 1}\right)^{-\frac{1}{2}}$$
 and $\gamma = \sqrt{\frac{C_1 + C_2}{C_1 + C_2 + 1}}$,

which complets the proof of (ii).

For part (iii), $\theta \in (\frac{1}{2}, 1)$. Notice that

$$\frac{1-\theta}{\theta} < 1$$
 and $S^{k-2} - S^k + \|\delta^{k-1}\|_1 \to 0$.

Hence, there exists sufficient large k such that

$$S^{k-2} - S^k + \|\delta^{k-1}\|_1 \le \left[S^{k-2} - S^k + \|\delta^{k-1}\|_1 \right]^{\frac{1-\theta}{\theta}},$$

we assume the above inequality holds for $k \geq \bar{k}$. This, combined with (28), yields

$$S^{k} + \frac{\sqrt{\mu}}{1-\mu} \|\delta^{k}\|_{1} \le (C_{1} + C_{2}) \left[S^{k-2} - S^{k} + \|\delta^{k-1}\|_{1} \right]^{\frac{1-\theta}{\theta}}.$$

This, combined with (30), yields

$$S^{k} + \frac{\sqrt{\mu}}{1 - \mu} \|\delta^{k}\|_{1} \le (C_{1} + C_{2}) \left[S^{k-2} + \frac{\sqrt{\mu}}{1 - \mu} \|\delta^{k-2}\|_{1} - (S^{k} + \frac{\sqrt{\mu}}{1 - \mu} \|\delta^{k}\|_{1}) \right]^{\frac{1 - \theta}{\theta}}.$$
 (31)

Raising to a power of $\frac{\theta}{1-\theta}$ of both sides of the above inequality, we see

$$\left[S^k + \frac{\sqrt{\mu}}{1-\mu} \|\delta^k\|_1 \right]^{\frac{\theta}{1-\theta}} \le C_3 \left[S^{k-2} + \frac{\sqrt{\mu}}{1-\mu} \|\delta^{k-2}\|_1 - (S^k + \frac{\sqrt{\mu}}{1-\mu} \|\delta^k\|_1) \right]$$
(32)

with $C_3 := (C_1 + C_2)^{\frac{\theta}{1-\theta}}$.

Consider the "even" subsequence of $\{\bar{k}, \bar{k}+1,\ldots\}$ and define $\{\Delta_t\}_{t\geq N_1}$ with $N_1:=\lceil \bar{k}/2\rceil$, and $\Delta_t:=S^{2t}+\frac{\sqrt{\mu}}{1-\mu}\|\delta^{2t}\|_1$. Then for all $t\geq N_1$, we have

$$\Delta_t^{\frac{\theta}{1-\theta}} \le C_3(\Delta_{t-1} - \Delta_t) \tag{33}$$

The remaining part of our proof is similar to (1, Theorem 2) (starting from (1, Equation (13))). Define $h:(0,+\infty)\to\mathbb{R}$ by $h(s)=s^{-\frac{\theta}{1-\theta}}$ and let $T\in(1,+\infty)$. Take $k\geq N_1$ and consider the case that $h(\Delta_k)\leq Th(\Delta_{k-1})$ holds. By rewriting (33) as

$$1 \le C_3(\Delta_{k-1} - \Delta_k) \Delta_k^{-\frac{\theta}{1-\theta}}.$$

we obtain that

$$\begin{split} &1 \leq C_3(\Delta_{k-1} - \Delta_k)h(\Delta_k) \\ &\leq TC_3(\Delta_{k-1} - \Delta_k)h(\Delta_{k-1}) \\ &\leq TC_3\int_{\Delta_k}^{\Delta_{k-1}} h(s)ds \\ &\leq TC_3\frac{1-\theta}{1-2\theta} [\Delta_{k-1}^{\frac{1-2\theta}{1-\theta}} - \Delta_k^{\frac{1-2\theta}{1-\theta}}]. \end{split}$$

Thus if we set $u = \frac{2\theta - 1}{(1 - \theta)TC_3} > 0$ and $\nu = \frac{1 - 2\theta}{1 - \theta} < 0$ one obtains that

$$0 < u \le \Delta_k^{\nu} - \Delta_{k-1}^{\nu}. \tag{34}$$

Assume now that $h(\Delta_k) > Th(\Delta_k)$ and set $q = (\frac{1}{T})^{\frac{1-\theta}{\theta}} \in (0,1)$. It follows immediately that $\Delta_k \leq q\Delta_{k-1}$ and furthermore - recalling that ν is negative - we have

$$\Delta_k^{\nu} \ge q^{\nu} \Delta_{k-1}^{\nu}$$
 and $\Delta_k^{\nu} - \Delta_{k-1}^{\nu} \ge (q^{\nu} - 1) \Delta_{k-1}^{\nu}$.

Since $q^{\nu}-1>0$ and $\Delta_t\to 0^+$ as $t\to +\infty$, there exists $\bar{u}>0$ such that $(q^{\nu}-1)\Delta_{t-1}^{\nu}>\bar{u}$ for all $t \geq N_1$. Therefore we obtain that

$$\Delta_k^{\nu} - \Delta_{k-1}^{\nu} > \bar{u}. \tag{35}$$

If we set $\hat{u} = \min\{u, \bar{u}\} > 0$, one can combine (34) and (35) to obtain that

$$\Delta_k^{\nu} - \Delta_{k-1}^{\nu} \ge \hat{u} > 0$$

for all $k \geq N_1$. By summing those inequalities from N_1 to some t greater than N_1 we obtain that $\Delta_t^{\nu} - \Delta_{N_1}^{\nu} \ge \hat{u}(t - N_1)$, implying

$$\Delta_t \le \left[\Delta_{N_1}^{\nu} + \hat{u}(t - N_1)\right]^{1/\nu} \le C_4 t^{-\frac{1-\theta}{2\theta - 1}},\tag{36}$$

for some $C_4 > 0$.

As for the "odd" subsequence of $\{\bar{k}, \bar{k}+1,\ldots\}$, we can define $\{\Delta_t\}_{t\geq \lceil \bar{k}/2\rceil}$ with $\Delta_t:=S^{2t+1}+$ $\frac{\sqrt{\mu}}{1-\mu}\|\delta^{2t+1}\|_1$ and then can still show that (36) holds true. Therefore, for all sufficiently large and even number k,

$$||x^k - x^*||_2 \le \Delta_{\frac{k}{2}} \le 2^{\frac{1-\theta}{2\theta-1}} C_4 k^{-\frac{1-\theta}{2\theta-1}}.$$

For all sufficiently large and odd number k, there exists $C_5 > 0$ such that

$$||x^k - x^*||_2 \le \Delta_{\frac{k-1}{2}} \le 2^{\frac{1-\theta}{2\theta-1}} C_4 (k-1)^{-\frac{1-\theta}{2\theta-1}} \le 2^{\frac{1-\theta}{2\theta-1}} C_5 k^{-\frac{1-\theta}{2\theta-1}}.$$

Overall, we have

$$||x^k - x^*||_2 \le c_2 k^{-\frac{1-\theta}{2\theta-1}}$$

where

$$c_2 := 2^{\frac{1-\theta}{2\theta-1}} \max(C_4, C_5).$$

This completes the proof of (iii).

5 Numerical results

In this section, we demonstrate the local convergence rate of PIRL1 in practice. We test PIRL1 on sparse signal recovery experiments and observe its performance. The test problems can be formulated as

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$$\min_{x \in \mathbb{R}^n} \quad \frac{1}{2} \|y - Ax\|_2^2 + \lambda \|x\|_p^p, \tag{37}$$

where matrix $A \in \mathbb{R}^{m \times n}$ is generated uniformly at random from i.i.d. standard Gaussian entries. We set $L_f = 1$ by orthonormalizing the rows of A. The observation $y \in \mathbb{R}^m$ is generated by the sparse signal x_{true} and Gaussian noise $e \sim \mathbb{N}(0, \sigma^2)$, that is $y = Ax_{true} + e$. The objective of the experiments is to reconstruct a length n sparse signal x from m observations. All experiments start from a randomized initialized x_0 and use the same termination criterion

$$\max_{i=1,\ldots,n} |x_i \nabla_i f(x) + \lambda p |x_i|^p| \le \text{opttol},$$

In the experiments, we set $\sigma = 10^{-3}$, m = 1024, n = 2048, and the x_{true} contains 128 randomly placed ± 1 spikes.

Our purpose is to demonstrate the evolution of $||x^k - x^*||_2$ and verify whether the bounds (21) and (22) in Theorem 10 can be witnessed. Therefore, we first run the algorithm with sufficiently small tolerance opttol = 10^{-12} and the final iterate x^* used as the surrogate of the real solution x_{opt} , since in this case $||x_{opt} - x^*||_2$ is deemed sufficiently small. In this way, we can use

$$||x^k - x^*||_2 \le ||x^k - x_{opt}||_2 + ||x_{opt} - x^*||_2 \approx ||x^k - x_{opt}||_2$$

to examine the local behavior of PIRL1. The algorithm is rerun with opttol = 10^{-8} meaning x^k is sufficiently close to the optimal solution. The last 100 iterations of $\log_{10}(\|x^k - x^*\|_2)$ is plotted in Figure 1, where T represents the last iteration for each run. To see how performance is affected by different p and λ , we repeat this procedure for cases with p = 0.2, 0.5, 0.8 and $\lambda = 0.0001, 0.001, 0.01$. In each case, we randomly generate 5 problems.

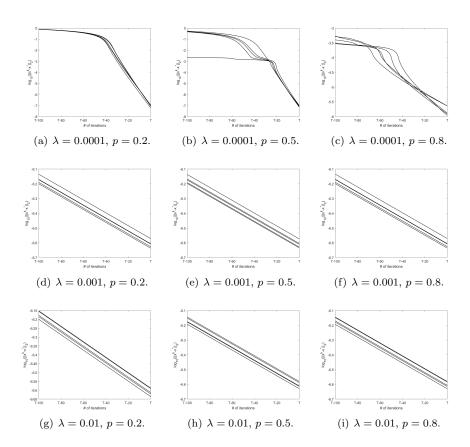


Figure 1: The last 100 iterations of $\log_{10}(\|x^k - x^*\|_2)$.

From Figure 1, we can see that PIRL1 exhibits linear convergence for all problems in all cases, meaning the bound (21) is always witnessed. This may indicate that the KL property can be satisfied for a wide range of test problems, and PIRL1 can then achieve local linear convergence in many cases.

6 Conclusion

In this paper, we have analyzed the global convergence and local convergence rate of the proximal iteratively reweighted ℓ_1 method for solving ℓ_p regularization problems under the KL property. We have shown that the iterates generated by this method have a unique limit point. It has a locally linear convergence or sublinear convergence under KL property. It should be noticed that our analysis can be easily extended to other types of nonconvex regularization problems under the assumption of the KL property for the loss function.

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