Robust Feasibility of Systems of Quadratic Equations Using Topological Degree Theory

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Abstract

We consider the problem of measuring the *margin of robust feasibility* of solutions to a system of nonlinear equations. We study the special case of a system of quadratic equations, which shows up in many practical applications such as the power grid and other infrastructure networks. This problem is a generalization of quadratically constrained quadratic programming (QCQP), which is NP-Hard in the general setting. We develop approaches based on topological degree theory to estimate bounds on the robustness margin of such systems. Our methods use tools from convex analysis and optimization theory to cast the problems of checking the conditions for robust feasibility as a nonlinear optimization problem. We then develop *inner bound* and *outer bound* procedures for this optimization problem, which could be solved efficiently to derive lower and upper bounds, respectively, for the margin of robust feasibility. We evaluate our approach numerically on standard instances taken from the MATPOWER and NESTA databases of AC power flow equations that describe the steady state of the power grid. The results demonstrate that our approach can produce tight lower and upper bounds on the margin of robust feasibility for such instances.

1 Introduction

Solving systems of equations is ubiquitous in computational mathematics. In many applications, these problems are made challenging due to the functions in the equations being nonlinear and/or nonconvex. Another aspect adding to the problem complexity is the uncertainty in the problem parameters. Our work is motivated by two central computations performed as part of power systems operations are power flow (PF) studies and optimal power flow (OPF). PF studies ensure the power grid state (i.e., voltages and flows across the network) will remain within acceptable limits in spite of contingencies (e.g., loss of a generator or transmission line) and other uncertainties (e.g., shift-ing demand or renewable sources of power). OPF seeks further to choose values for controllable

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assets in the system (e.g., generators whose rate of power production could be controlled) so as to meet demand at minimum cost. These problems have inherent nonlinearities and nonconvexities, making them hard to solve in their general form.

To further complicate the problem, the rapid adoption of renewable energy sources such as wind and solar energy is adding unprecedented uncertainties to modern power systems. Since these sources depend on the weather, their energy output is not perfectly controllable. In fact, this output can be forecasted with only limited accuracy. While demand-side flexibility can be used to balance fluctuations in solar and wind generation, its amount can in turn be difficult to predict [31, 37]. Due to all these uncertainties, it is increasingly difficult to ensure there is sufficient power generation to meet demand while accounting for losses and network limits.

We study quadratic systems of equations with parameters, and take a *robust viewpoint* of uncertainty. Specifically, we aim to quantify the worst-case impact of uncertainty in parameters on feasibility. To this end, we study the *robust feasibility problem*, which includes the robust version of the standard PF problem as a special case. The power system can be described by a system of nonlinear equations in a set of variables that capture the state of the power grid, i.e., voltages at every point in the power network, and include the controllable inputs as well as uncertain inputs. In the main PF problem, we are given a fixed value of the controllable inputs and an uncertainty set for the uncertain inputs. The goal of the robust feasibility problem is to characterize whether the system has a solution within specified bounds (capturing engineering limits on voltages, flows, etc.) for *each* choice of the uncertain inputs in the uncertainty set.

More concretely, we study a system of quadratic equations $F(\mathbf{x}) = \mathbf{u}$ where $F : \mathbb{R}^n \mapsto \mathbb{R}^n$ is quadratic in \mathbf{x} for $\mathbf{x}, \mathbf{u} \in \mathbb{R}^n$. We consider situations where the parameters \mathbf{u} are uncertain, and we are interested in guaranteeing the existence of a solution to $F(\mathbf{x}) = \mathbf{u}$ within limits on \mathbf{x} and \mathbf{u} . We draw on results from topological degree theory and Borsuk's theorem from algebraic topology and nonlinear analysis to develop tests for existence of solutions. Using ideas from optimization such as convex relaxations of quadratic constraints, we develop rigorous and efficient algorithms based on these tests for robust feasibility. We develop efficient implementations of these algorithms capable of scalably solving large instances of PF problems. While we use power systems as the main application area, the methods we develop are fairly general, and could be applied to problems in other domains as well, e.g., stochastic processes and gas distribution networks.

1.1 Our Contributions

We study systems of quadratic equations, and define a *robustness margin* as a measure of the system's robust feasibility (see Definition 2.1). We develop approaches based on topological degree theory to estimate bounds on the robustness margin of such systems (see Section 3). We use tools from convex analysis and optimization theory to cast the problem of checking the conditions for robust feasibility as a nonlinear optimization problem. We then develop *inner bound* (Section 4) and *outer bound* (Section 5) formulations for this optimization problem, which could be solved efficiently to derive lower and upper bounds, respectively, for the margin of robust feasibility. We evaluate our approach numerically on standard instances taken from the MatPower database of AC power flow equations that describe the steady state of the power grid (Section 7). The results demonstrate that our approach can produce tight lower and upper bounds on the robustness margin for such instances.

1.2 Related Work

Robust feasibility and optimization have been well-studied by both the optimization and topology communities. What is lacking is an approach that can guarantee and quantify robust feasibility on large scale systems in an efficient manner. In this article we address this deficiency by developing theory that utilizes results from topological degree theory and convex optimization. We provide a theoretical foundation for determining robust feasibility of systems of quadratic equations and computational methods for producing lower and upper bounds on the maximum error bound for which one can guarantee robust solvability (the radius of robust solvability). To highlight the efficacy of our approach we derive procedures, which we test numerically on several quadratic systems constructed from the AC power flow equations that describe the steady state of the power grid with added uncertainty. The results show that our approach can be applied to large scale systems to produce tight lower and upper bounds on the radius of robust solvability, which we shall define as the robustness margin of the system.

In optimization, the focus has been on robust *convex* optimization where uncertainty sets are specified for the parameters of a convex optimization problem (typically an LP or conic program) [3], while the robust versions of generic polynomial programming problem are related by a hierarchy of SDP relaxations [27, 28]. Robust *nonconvex* optimization has received only limited attention (a notable exception is the work of Bertsimas et al. [6]). These approaches do not provide rigorous guarantees for robust feasibility with nonconvex constraints.

In algebraic topology, there have been a number of studies on these problems based on several approaches, including ones based on robustness of level sets and persistent homology [5, 16], well groups and diagrams [12, 19, 20], topological degree and robust satisfiability [18, 21], and on Borsuk's theorem and interval arithmetic [22, 23, 24]. While the theory developed by these approaches is fairly complete, the associated algorithms typically rely on explicit simplicial or cellular decompositions of the problem space. But the size of such decompositions typically grows exponentially in the problem dimension, and hence these algorithms are typically impractical for large-scale applications.

Looking specifically at applications such as the power systems, there has been significant interest in solving the non-robust version of the OPF problem to global optimality. The driver has been the development of strong convex relaxations of the nonconvex optimization problems combined with ideas from global optimization such as spatial branch-and-cut, bound tightening, etc. [10, 14]. Uncertainty has been handled in a chance-constrained framework [9, 43]. However, this approach has typically been applied only to linear approximations or convex relaxations of the AC power flow equations, and does not guarantee feasibility with respect to the true nonlinear power flow equations [9, 26, 36, 38].

There is significant empirical work on solving the PF equations with probabilistic uncertainty [32, 42] and specifying conditions on the power injections over which the power flow equations are guaranteed to have a solution [11, 40, 41]. However, many of these algorithms are based on sampling heuristics and either do not offer mathematical guarantees of robust feasibility or do not directly address the robust feasibility problem. More recently, Dvijotham, Nguyen, and Turitsyn [15] developed an approach to handle uncertainty which produced inner/lower bounds on the distance from the nominal values of the uncertain parameters for which the system can still be guaranteed to have solutions. This approach closely aligns with the methods describing our inner bound procedures, and further can produce a certificate of tightness under special conditions.

However, this method depends critically on the choice of norms, which is not straightforward to make.

Notation: We denote vectors by bold lowercase letters, e.g., $\mathbf{x}, \mathbf{u} \in \mathbb{R}^n$, and matrices by upper case letters, e.g., $A \in \mathbb{R}^{m \times n}$. The vectors of all zeros and all ones are denoted 0 and 1, respectively. Individual entries in a vector \mathbf{x} are denoted x_i , for instance. We let [n] denote the numbers $\{1, 2, \ldots, n\}$.

2 Problem Formulation

We study systems of quadratic equations of the form

$$Q(\mathbf{x}) + L\mathbf{x} = \mathbf{u} \tag{1}$$

where $Q : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a vector-valued quadratic function, that is, there exist matrices $Q_1, \ldots, Q_n \in \mathbb{R}^{n \times n}$, such that

$$[Q(\mathbf{x})]_i = \mathbf{x}^T Q_i \mathbf{x} \quad \forall i \in [n]$$

and $L \in \mathbb{R}^{n \times n}$, $\mathbf{u} \in \mathbb{R}^n$. We are interested in solutions to this system of equations under linear constraints of the form

$$(A\mathbf{x})_i \le b_i \quad \forall i \in [n] \tag{2}$$

where we assume that $(A\mathbf{x})_i \leq b_i$ for each *i* is free of redundant constraints and \mathbf{x} and \mathbf{u} have the same dimension. However, the parameter \mathbf{u} is uncertain and known only up to certain error bounds:

$$u_i^{\min} = u_i^* - e_i \le u_i \le u_i^* + e_i = u_i^{\max} \quad \forall i \in [n]$$
(3)

where \mathbf{u}^* is a forecast for \mathbf{u} and \mathbf{e} denotes the error bounds associated with the forecast. For example, in the case of quadratic equations appearing in infrastructure networks like the power grid, u_i represents uncertain power generation or consumption (for example uncertain weather-dependent power sources like solar or wind power). In the case of stochastic processes, \mathbf{u}^* represents an initial state distribution. Further, note that if the polyhedron given by $A\mathbf{x} \leq \mathbf{b}$ is not full dimensional, then there would exist at least a $i \in [1, \dots, n]$ such that x_i could be any number as long as satisfies (1). In this case, for the corresponding uncertain parameter u_i , there are no u_i^{min} and u_i^{max} .

Definition 2.1 (Robust Feasibility and Robustness Margin problem). Determine whether for all values of u satisfying (3), the system of equations (1) has a solution lying within the interior of the set of all x satisfying the constraints in (2). If this is true, the system comprised of (1),(2),(3) is said to be *robust feasible*. The largest r for which $e_i \ge r \forall i$ with $e_i > 0$ in (3) and such that the system is robust feasible is defined as the *robustness margin*. See Figure 1 for a pictorial depiction.

There are many practical uncertainty sets developed in previous studies such as ellipsoidal uncertainty [4], cardinality-constrained uncertainty [7], norm uncertainty [8], and some other types of uncertainty sets constructed based on probability theory [1, 2] and data driven approaches [29]. We have presented the definition for robustness margin in the most general form so as to capture



Figure 1: Illustration of robust feasibility. The system is robust feasible at the level of r (left), but is not robust feasible at r' > r (right).

most scenarios that may be considered by the researcher. Note that for a given choice of r, our uncertainty set specified as the ℓ_{∞} -ball of radius r forms the largest such set—many other commonly used uncertainty sets, e.g., ℓ_1 -ball, ℓ_2 -ball, ellipsoid (with major axis r), are subsets of the ℓ_{∞} -ball. For instance it very well may be the case that only some of the dimensions of u will have margins of uncertainty. Furthermore, one may have need of computing the robustness margin for only a subset of the dimensions of u which pertain to problem areas or nodes of particular interest to the research. This manual restriction will of course produce a robustness margin greater than or equal to that obtained by considering all dimensions, which certainly remains an option under the current setting.

3 Theoretical Results

We now describe the main technical results of this paper. In the first subsection we describe the setting under which the problem can be solved using the results which follow.

3.1 Topological Degree Theory

Our results take advantage of the well studied area of topological degree theory. For an introduction to topological degree theory see the works of [17], [33], and [34]. It suffices to say that should $\Omega \subset \mathbb{R}^n$ be open and bounded, $F : \Omega \to \mathbb{R}$ continuous, differentiable, and $F(\mathbf{x}) \neq \mathbf{y} \ \forall \mathbf{x} \in \partial \Omega$ for some $\mathbf{y} \in \mathbb{R}^n$, then the degree of F at \mathbf{y} over Ω , denoted $d(\Omega, F, \mathbf{y}) \in \mathbb{Z}$, is defined. For the purposes of this article we utilize the following property of degree as our definition of the topological degree of a function F at \mathbf{y} over a set Ω . See O'Regan et al. [34] for details.

$$d(\Omega, F, \mathbf{y}) = \sum_{\mathbf{x} \in F^{-1}(\mathbf{y})} \operatorname{sign} \left(J_F(\mathbf{x}) \right)$$
(4)

where sign $(J_F(\mathbf{x}))$ denotes the sign of the determinant of the Jacobian of F at x, i.e.,

$$\operatorname{sign} \left(J_F(\mathbf{x}) \right) = \begin{cases} -1 & \text{if } J_F(\mathbf{x}) < 0, \\ 0 & \text{if } J_F(\mathbf{x}) = 0, \\ 1 & \text{if } J_F(\mathbf{x}) > 0. \end{cases}$$

We assume that the sum in Equation (4) evaluates to 0 if $F^{-1}(\mathbf{y}) = \emptyset$. Additionally we utilize the following common properties of the topological degree. Again, see O'Regan et al. [34] for details. If $H : [0, 1] \times \overline{\Omega} \to \mathbb{R}^n$ is continuous such that $H(t, \mathbf{x}) \neq \mathbf{y} \ \forall t \in [0, 1], \mathbf{x} \in \partial\Omega$, then

 $d(\Omega, H(t, \cdot), \mathbf{y})$ does not depend on t. (5)

If
$$d(\Omega, F, \mathbf{y}) \neq 0$$
, then there exists $\mathbf{x} \in \Omega$ such that $F(\mathbf{x}) = \mathbf{y}$. (6)

3.2 New Theoretical Results

In this section we will take full advantage of properties (4), (5), and (6) as they apply to the Robust Feasibility Problem. We begin by assuming there is a unique solution to the forecasted system at which point the Jacobian is non-zero. We conclude by property (4) that the degree is non-zero at u^* for the forecasted system. We then utilize property (5) to equate the degree of u to the degree of u^* for all u satisfying the limits specified in Equation (3) (under a proposed robustness margin), which by property (6) allows us to guarantee solutions to the system under all realizations of u satisfying the limits (in 3), i.e., verify the system is robust feasible for a given robustness margin. Invoking property (5), however, requires us to develop a homotopy that captures the system under all possible realizations of u satisfying the limits in (3). Once we define such a homotopy we reduce the Robust Feasibility Problem to the problem of verifying the hypothesis of property (5).

To that end let $F(\mathbf{x}) = Q(\mathbf{x}) + L\mathbf{x}$, $\Omega = {\mathbf{x} | A\mathbf{x} \leq \mathbf{b}}$ and $\hat{\mathbf{x}} \in \text{Int}(\Omega)$ be a solution to the forecasted system $F(\mathbf{x}) = \mathbf{u}^*$ given in Equation (1), such that $\operatorname{sign}(J_F(\hat{\mathbf{x}})) \neq 0$. For a review of efficient methods of verification that could be used here, see the work of Griewank [25]. If no solution exists, then certainly the system is not robust feasible. We define $\Omega_u = {\mathbf{u} | \mathbf{u} \text{ satisfies limits in Equation (3)}}$. Our task is then to verify using existing methods or those we propose in this paper that no other solutions exist in $\operatorname{Int}(\Omega)$. This step may require further restricting the domain or even a slight perturbation of the forecasted \mathbf{u} . Thus by property (4) we have verified that $d(\Omega, F(\mathbf{x}), \mathbf{u}^*) \neq 0$. Note that this is not the only method for verification, but in some sense is the easiest to carry out for our purposes.

We now introduce the homotopy we use to invoke property (5). Let $\ell_{\mathbf{u}^*}$ represent an arbitrary line passing through \mathbf{u}^* and let \mathbf{l}_{\min} and \mathbf{l}_{\max} be the two points of intersection of $\partial \Omega_u$ and $\ell_{\mathbf{u}^*}$. We define a homotopy $H_{\ell_{\mathbf{u}^*}} : [0, 1] \times \overline{\Omega} \to \mathbb{R}^n$ as

$$H_{\ell_{\mathbf{u}^*}}(t, \mathbf{x}) = F(\mathbf{x}) - \left[(1-t)\mathbf{l}_{\min} + t\mathbf{l}_{\max} \right].$$
(7)

Based on this homotopy, we present the key result on verification of robust solvability problem.

Lemma 3.1. Let $\Omega = {\mathbf{x} | A\mathbf{x} \leq \mathbf{b}}$, $\Omega_u = {\mathbf{u} | \mathbf{u} \text{ satisfies limits in Equation (3)} and <math>F(\mathbf{x}) = Q(\mathbf{x}) + L\mathbf{x}$, as described in Equations (1), (2), and (3). If $d(\Omega, H_{\ell}(\frac{1}{2}, \mathbf{x}), \mathbf{0}) \neq 0$ for each choice of $\ell = \ell_{\mathbf{u}^*}$, then the system is robust solvable if and only if the following statement holds:

$$\exists \mathbf{x} \in \partial \Omega, \mathbf{u} \in \Omega_u \quad such \ that \ F(\mathbf{x}) - \mathbf{u} = \mathbf{0}.$$
(8)

Proof. Since $d(\Omega, F(\mathbf{x}), \mathbf{u}^*) \neq 0$ is verified by property (4), to show the problem is robust solvable, it follows by the fact that there exists a unique solution $\hat{\mathbf{x}} \in \text{Int}(\Omega)$ to the forecasted system given in Equation (1) such that $H_{\ell_{\mathbf{u}^*}}(\frac{1}{2}, \hat{\mathbf{x}}) = F(\hat{\mathbf{x}}) - \mathbf{u}^* = \mathbf{0}$ and $\operatorname{sign}(J_{H_{\ell_{\mathbf{u}^*},\frac{1}{2}}}(\hat{\mathbf{x}})) \neq 0$. And this condition holds if and only if $d(\Omega, H_{\ell}(\frac{1}{2}, \mathbf{x}), \mathbf{0}) \neq 0$ according to property (6). Note that this property holds for all such lines passing through \mathbf{u}^* since for each $\hat{\mathbf{u}} \in \Omega_u \setminus \{\mathbf{u}^*\}$, there exists a line $\hat{\ell}_{\mathbf{u}^*}$ passing through \mathbf{u}^* and $t \in [0, 1]$, such that $\hat{\mathbf{u}} = (1 - t)\hat{\mathbf{l}}_{\min} + t\hat{\mathbf{l}}_{\max}$. Thus, when $d(\Omega, H_{\ell}(\frac{1}{2}, \mathbf{x}), \mathbf{0}) \neq 0$ for each choice of $\ell_{\mathbf{u}^*}$, we have

$$d(\Omega, H_{\ell}\left(\frac{1}{2}, \mathbf{x}\right), \mathbf{0}) \neq 0 \iff F(\mathbf{x}) - \mathbf{u} \neq \mathbf{0}, \forall \mathbf{x} \in \partial\Omega, \mathbf{u} \in \Omega_{u}.$$

Hence the system is robust solvable if and only if the statement (8) holds.

Note that the statement (8) is equivalent to property (5) holding. From here on we will assume $d(\Omega, H(\frac{1}{2}, \mathbf{x}), \mathbf{0}) \neq 0$ and focus our efforts on the development of methods for validating or invalidating the statement (8).

Lemma 3.2. Let $X \subset \mathbb{R}^n$ be full-dimensional and compact. If $F : \mathbb{R}^n \to \mathbb{R}^n$ is continuous, and

$$\min_{\|\boldsymbol{\lambda}\|=1} \max_{\mathbf{x}\in X} \boldsymbol{\lambda}^T F(\mathbf{x})$$

obtains its optimal value at $\hat{\mathbf{x}}$ and $\lambda_{\hat{\mathbf{x}}}$ then $F(\hat{\mathbf{x}}) \in \partial F(X)$.

Proof. We get the result by arriving at a contradiction. Assume $F(\hat{\mathbf{x}}) \in F(X) \setminus \partial F(X)$. Let θ the angle between $\lambda_{\hat{\mathbf{x}}}$ and $F(\hat{\mathbf{x}})$. Thus

$$\min_{\|\boldsymbol{\lambda}\|=1} \max_{\mathbf{x}\in X} \boldsymbol{\lambda}^T F(\mathbf{x}) = \boldsymbol{\lambda}_{\hat{\mathbf{x}}}^T F(\hat{\mathbf{x}}) = \|F(\hat{\mathbf{x}})\|\cos(\theta).$$

Since $X \subset \mathbb{R}^n$ is full-dimensional and compact, it has nonempty interior [30]. And since F is continuous, F(X) is also nonempty and compact. Hence there exists an r > 0 such that $B_r(F(\hat{\mathbf{x}}))$, the ball of radius r centered at $F(\hat{\mathbf{x}})$, is in $F(X) \setminus \partial F(X)$. Let \mathbf{y} be the antipodal point on $\partial B_r(F(\hat{\mathbf{x}}))$ to the point of intersection between the line segment connecting the origin to $F(\hat{\mathbf{x}})$ and $B_r(F(\hat{\mathbf{x}}))$. It follows then that $\|\mathbf{y}\| > \|F(\hat{\mathbf{x}})\|$ and θ is the angle between $\lambda_{\hat{\mathbf{x}}}$ and \mathbf{y} . Let $\mathbf{x}^* \in X$ be such that $F(\mathbf{x}^*) = \mathbf{y}$. Such an \mathbf{x}^* exists since \mathbf{y} lies inside F(x) which is nonepty and F is continuous. Therefore $\lambda_{\hat{\mathbf{x}}}^T F(\mathbf{x}^*) = \|F(\mathbf{x}^*)\| \cos(\theta) > \|F(\hat{\mathbf{x}})\| \cos(\theta) = \lambda_{\hat{\mathbf{x}}}^T F(\hat{\mathbf{x}})$, which is a contradiction. The lemma now follows.

We illustrate Lemma 3.2 in Figure 2. The compact set $X \subset \mathbb{R}^2$ is shown in pink with boundary shown in blue. For continuous $F : \mathbb{R}^2 \to \mathbb{R}^2$, let $\hat{\mathbf{x}}$ be the point and $\lambda_{\hat{\mathbf{x}}}$ the unit vector that give the optimal value of $\min_{\|\boldsymbol{\lambda}\|=1} \max_{\mathbf{x}\in X} \boldsymbol{\lambda}^T F(\mathbf{x})$ as $\boldsymbol{\lambda}_{\hat{\mathbf{x}}}^T F(\hat{\mathbf{x}})$ based on Lemma 3.2. These vectors are shown in red. In the case when the origin is contained in F(X), as it happens here, we observe that there exists another point \mathbf{x} and unit vector $\boldsymbol{\lambda}_{\mathbf{x}}$ satisfying $F(\mathbf{x}) \in \partial F(X)$ (and thus $\mathbf{x} \in \partial X$) such that we can get the lower bound of $\min_{\|\boldsymbol{\lambda}\|=1} \max_{\mathbf{x}\in X} \boldsymbol{\lambda}^T F(\mathbf{x})$ as $\min_{\mathbf{x}\in\partial X} \max_{\|\boldsymbol{\lambda}\|=1} \boldsymbol{\lambda}^T F(\mathbf{x})$ (shown in green).

We formalize the last observation for the general case in our main theorem, which characterizes the structure of the function F.



Figure 2: Illustration of Lemma 3.2. The unit vectors $\lambda_{\hat{x}}$ and λ_{x} are shown as the red and green arrows, respectively.

Theorem 3.3. Let $X \subset \mathbb{R}^n$ be full-dimensional and compact, $F : \mathbb{R}^n \to \mathbb{R}^n$ be continuous such that F(X) contains the origin. If F is injective over X then

$$\min_{\|\boldsymbol{\lambda}\|=1} \max_{\mathbf{x} \in X} \, \boldsymbol{\lambda}^T F(\mathbf{x}) \geq \min_{\mathbf{x} \in \partial X} \max_{\|\boldsymbol{\lambda}\|=1} \, \boldsymbol{\lambda}^T F(\mathbf{x}).$$

Proof. Consider the origin lies on the boundary of F(X), and let $\hat{\mathbf{x}} \in \partial X$ such that $F(\hat{\mathbf{x}}) = \mathbf{0}$. Such an $\hat{\mathbf{x}}$ exists since $\partial F(X) = F(\partial X)$ by the Invariance of Domain theorem, as F is continuous and injective over a compact set (so, an interior point cannot get mapped to a boundary point). Then clearly

$$\min_{\|\boldsymbol{\lambda}\|=1} \max_{\mathbf{x} \in X} \boldsymbol{\lambda}^T F(\mathbf{x}) \geq \min_{\|\boldsymbol{\lambda}\|=1} \boldsymbol{\lambda}^T F(\hat{\mathbf{x}}) = 0 = \max_{\|\boldsymbol{\lambda}\|=1} \boldsymbol{\lambda}^T F(\hat{\mathbf{x}}) \geq \min_{\mathbf{x} \in \partial X} \max_{\|\boldsymbol{\lambda}\|=1} \boldsymbol{\lambda}^T F(\mathbf{x}).$$

Hence assume the origin lies in the interior of F(X). Let $\hat{\mathbf{x}}$ be the point and $\lambda_{\hat{\mathbf{x}}}$ the unit vector at which

$$\min_{\|\boldsymbol{\lambda}\|=1} \max_{\mathbf{x} \in X} \, \boldsymbol{\lambda}^T F(\mathbf{x})$$

obtains its optimal value. We consider two cases.

Case 1: If the angle θ between $F(\hat{\mathbf{x}})$ and $\lambda_{\hat{\mathbf{x}}}$ is 0, then $\max_{\|\boldsymbol{\lambda}\|=1} \boldsymbol{\lambda}^T F(\hat{\mathbf{x}}) = \boldsymbol{\lambda}_{\hat{\mathbf{x}}}^T F(\hat{\mathbf{x}})$. Furthermore by Lemma 3.2, $F(\hat{\mathbf{x}}) \in \partial F(X)$ and thus $\hat{\mathbf{x}} \in \partial X$ since F is injective, i.e., F maps ∂X to $\partial F(X)$, by hypothesis. It follows now that

$$\min_{\mathbf{x}\in\partial X}\max_{\|\boldsymbol{\lambda}\|=1} \boldsymbol{\lambda}^T F(\mathbf{x}) \leq \max_{\|\boldsymbol{\lambda}\|=1} \boldsymbol{\lambda}^T F(\hat{\mathbf{x}}) = \boldsymbol{\lambda}_{\hat{\mathbf{x}}}^T F(\hat{\mathbf{x}}) = \min_{\|\boldsymbol{\lambda}\|=1}\max_{\mathbf{x}\in X} \boldsymbol{\lambda}^T F(\mathbf{x}).$$

Case 2: Assume the angle $\theta \neq 0$ (between $F(\hat{\mathbf{x}})$ and $\lambda_{\hat{\mathbf{x}}}$). Let \mathbf{x}^* be a point on the boundary of X such that the angle between $\lambda_{\hat{\mathbf{x}}}$ and $F(\mathbf{x}^*)$ is 0. Such a point must exist as F(X) is compact and by hypothesis F(X) contains the origin, is injective, and by assumption the origin lies in the interior of F(X). It follows then that

$$\min_{\mathbf{x}\in\partial X}\max_{\|\boldsymbol{\lambda}\|=1}\boldsymbol{\lambda}^{T}F(\mathbf{x}) \leq \max_{\|\boldsymbol{\lambda}\|=1}\boldsymbol{\lambda}^{T}F(\mathbf{x}^{*}) = \boldsymbol{\lambda}_{\hat{\mathbf{x}}}^{T}F(\mathbf{x}^{*}) \leq \max_{\mathbf{x}\in X}\boldsymbol{\lambda}_{\hat{\mathbf{x}}}^{T}F(\mathbf{x}) = \min_{\|\boldsymbol{\lambda}\|=1}\max_{\mathbf{x}\in X}\boldsymbol{\lambda}^{T}F(\mathbf{x}).$$

The theorem now follows.

Theorem 3.3 and Lemma 3.2 provide us with the theoretical tools we need to develop procedures for approximating the robustness margin. The hypothesis of Theorem 3.3 does however require us to assume the system is injective under the constraints in Equation (2). However, injectivity is only required to ensure $\partial F(X) = F(\partial X)$, and thus we can generalize to systems that are not necessarily injective if they yet retain $\partial F(X) = F(\partial X)$ as an applicable property. With this in mind we carry with us the necessary property $\partial F(X) = F(\partial X)$ throughout the rest of the article.

We will use the terminology "inner bound procedures" to describe the processes of verifying robust feasibility while expanding the uncertainty box centered at u^* in order to compute the lower bounds on the robustness margin, which these procedures undertake. We use the terminology "outer bound procedures" to capture in a similar fashion the procedures used to compute the upper bounds on the robustness margin by contracting the uncertainty box until the system may be robust feasible. As such we dedicate the next two sections to the development of these inner and outer bound formulations.

4 Computing Lower Bounds on the Robustness Margin

In this section we will derive procedures for computing a lower bound on the robustness margin. We start with an exact formulation, which turns out to be hard to implement efficiently in practice. Hence we relax the procedures until they become computationally tractable. We end the section by providing three different implementations of our final derived, relaxed, computationally tractable procedure specified in Equation (10). Each of these three practical implementations brings a unique set of attributes, which makes none of them the clearly preferred candidate.

Theorem 4.1. Let $\Omega = {\mathbf{x} | A\mathbf{x} \leq \mathbf{b}}$, $\Omega_u = {\mathbf{u} | u_i^{\min} \leq u_i \leq u_i^{\max} \forall i}$, and $F(\mathbf{x}) = Q(\mathbf{x}) + L\mathbf{x}$ as described in Equations (1), (2), and (3). Let

$$z = \min_{\mathbf{x} \in \partial \Omega, \mathbf{u} \in \Omega_u} \max_{\|\mathbf{\lambda}\| = 1} \mathbf{\lambda}^T \left(F(\mathbf{x}) - \mathbf{u} \right)$$

If there is an r > 0 such that $r \le e_i \forall i$ with $e_i > 0$, where e_i denotes the error bounds associated with u_i^{\min} and u_i^{\max} , and if z > 0 then the system is robust feasible and has a robustness margin of at least r.

Proof. If Equation (8) is invalidated then there exists $\hat{\mathbf{x}} \in \partial \Omega$ such that $F(\hat{\mathbf{x}}) = \hat{\mathbf{u}}$ for some $\hat{\mathbf{u}} \in \Omega_u$ and thus

$$z = \min_{\mathbf{x} \in \partial \Omega, \mathbf{u} \in \Omega_u} \max_{\|\boldsymbol{\lambda}\|=1} \, \boldsymbol{\lambda}^T \left(F(\mathbf{x}) - \mathbf{u} \right) \leq \max_{\|\boldsymbol{\lambda}\|=1, \mathbf{x} = \hat{\mathbf{x}}} \, \boldsymbol{\lambda}^T \left(F(\mathbf{x}) - \hat{\mathbf{u}} \right) = 0.$$

Hence if z > 0, Equation (8) is validated, and the system is robust feasible. It follows then by definition that the system has a robustness margin of at least r.

Theorem 4.2. Let $\partial \Omega_i = {\mathbf{x} | (A\mathbf{x})_i = b_i, A\mathbf{x} \leq \mathbf{b}}, \ \Omega_u = {\mathbf{u} | u_i^{\min} \leq u_i \leq u_i^{\max} \forall i}, and F(\mathbf{x}) = Q(\mathbf{x}) + L\mathbf{x} as described in Equations (1) to (3). Define$

$$z_i = \min_{\mathbf{x} \in \partial \Omega_i, \mathbf{u} \in \Omega_u} \|F(\mathbf{x}) - \mathbf{u}\|.$$
(9)

The system is robust feasible if and only if $z_i > 0$ for each i = 1, ..., m, where m is the number of rows of A.

Proof.

 \Rightarrow

If the system is not robust feasible then there exists $\hat{\mathbf{u}}' \in \Omega_u$ such that $F(\mathbf{x}) = \hat{\mathbf{u}}'$ has no interior solution. Since $F(\mathbf{x}) = \hat{\mathbf{u}}'$ has a solution and F is continuous over a compact domain, there must exist an $\hat{\mathbf{x}} \in \Omega = {\mathbf{x} | A\mathbf{x} \leq \mathbf{b}}$ such that $F(\hat{\mathbf{x}}) = \hat{\mathbf{u}}$ for some $\hat{\mathbf{u}} \in \Omega_u \cap \partial F(X)$. Since $\partial F(X) = F(\partial X)$ we have that $\hat{\mathbf{x}} \in \partial \Omega$, which implies that there is an *i* such that $A\hat{\mathbf{x}}_i = b_i$, and thus $0 \leq z_i \leq ||F(\hat{\mathbf{x}}) - \hat{\mathbf{u}}|| = 0$.

$$\Leftarrow$$

If there exists a *i* such that $z_i = 0$, then it follows that there must be the $\hat{\mathbf{x}} \in \partial \Omega$ which contains \hat{x}_i as an element such that $F(\hat{\mathbf{x}}) = \hat{\mathbf{u}}$ for some $\hat{\mathbf{u}} \in \Omega_u$. Thus there is no *interior* point \mathbf{x} to make the equation $F(\mathbf{x}) = \hat{\mathbf{u}}$ hold. Then based on Definition 2.1, the system is not robust feasible.

The optimization problems presented in Theorem 4.2 are nonlinear and nonconvex. Hence it may be difficult to solve them in general. In fact, since $F(\mathbf{x})$ is quadratic in \mathbf{x} , they are quadratically constrained quadratic programs (QCQPs), which are NP-hard in general [35]. At the same time, we can use well known *semidefinite programming relaxations for QCQPs* to obtain lower bounds on the optimal values [39]. Since Q is quadratic, it can also be written as a linear function of $\mathbf{x}\mathbf{x}^T$. More concretely, we can write

$$Q(\mathbf{x})_i = \mathbf{x}^T Q_i \mathbf{x} = \operatorname{Trace}(Q_i \mathbf{x} \mathbf{x}^T)$$

where each Q_i is as defined in Equation (1). Since x should satisfy $Ax \leq b$, we get

$$(\mathbf{b} - A\mathbf{x})(\mathbf{b} - A\mathbf{x})^T \ge 0 \implies \mathbf{b}\mathbf{b}^T - A\mathbf{x}\mathbf{b}^T - \mathbf{b}(A\mathbf{x})^T + A(\mathbf{x}\mathbf{x}^T)A^T \ge 0$$

If we allow a symmetric positive semidefinite matrix X to take the place of $\mathbf{x}\mathbf{x}^T$ and drop the rank constraint (rank(X) = 1)) we can construct the following relaxation for the optimization problem presented in Theorem 4.2:

$$\hat{z}_{i} = \min_{\mathbf{x}} b_{i} - (A\mathbf{x})_{i}$$
subject to Trace $(Q_{i}X) + L_{i}\mathbf{x} \ge \mathbf{u}_{i}^{\min} \quad \forall i$
Trace $(Q_{i}X) + L_{i}\mathbf{x} \le \mathbf{u}_{i}^{\max} \quad \forall i$
 $A\mathbf{x} \le \mathbf{b}$
 $\mathbf{b}\mathbf{b}^{T} - A\mathbf{x}\mathbf{b}^{T} - \mathbf{b}(A\mathbf{x})^{T} + AXA^{T} \ge O$
X is symmetric and positive semidefinite.
(10)

Here L_i denotes the i^{th} rows of L, and O denotes the $2n \times 2n$ matrix of zeros with the constraints understood to be component-wise inequalities. Note that if we impose $\operatorname{rank}(X) = 1$, we do get an exact formulation. At the same time, it may be difficult to impose this constraint. But the system (in Equation (10)) without the rank constraint is a convex optimization problem (a semidefinite program, in fact) and can be solved efficiently. Since this is a relaxation of the procedure described in Theorem 4.2, if $\hat{z}_i > 0$ for each i, the condition in Theorem 4.2 (i.e., $z_i > 0 \forall i$) is satisfied. We can further relax the formulation in Equation (10) by dropping the condition that X be positive semidefinite, which transforms the problem from a semidefinite program to a linear program. We now present three formulations for this new, relaxed program, and provide numerical results for each of them in Section 7.2. We also describe the advantages and drawbacks of each formulation.

LP Feasibility Procedure

subject to
Find an
$$\mathbf{x}$$

subject to
 $(A\mathbf{x})_i = b_i$
Trace $(Q_iX) + L_i\mathbf{x} \ge \mathbf{u}_i^{\min} \quad \forall i$
Trace $(Q_iX) + L_i\mathbf{x} \le \mathbf{u}_i^{\max} \quad \forall i$
 $A\mathbf{x} \le \mathbf{b}$
 $\mathbf{b}\mathbf{b}^T - A\mathbf{x}\mathbf{b}^T - \mathbf{b}(A\mathbf{x})^T + AXA^T \ge O$
X is symmetric.
(11)

This procedure has the advantage of being a linear program and hence can be solved efficiently. But as noted by the objective z_i , one must iterate over each dimension of Ax checking feasibility of the procedure. Of course, should the procedure prove feasible then we have found a solution on the boundary and thus invalidating the Statement in (Equation (8)). If the procedure proves infeasible then we are free to push the robustness margin higher and test again. An alternative approach is to consider all of the dimensions of Ax simultaneously by introducing extra binary variables and creating a MIP as follows.

MIP Procedure

$$\begin{array}{ll} \max & z \\ \text{subject to} & \operatorname{Trace} \left(Q_i X\right) + L_i \mathbf{x} \ge \mathbf{u}_i^{\min} \ \forall i \\ & \operatorname{Trace} \left(Q_i X\right) + L_i \mathbf{x} \le \mathbf{u}_i^{\max} \ \forall i \\ & A \mathbf{x} \le \mathbf{b} \\ & \mathbf{b} \mathbf{b}^T - A \mathbf{x} \mathbf{b}^T - \mathbf{b} (A \mathbf{x})^T + A X A^T \ge O \\ & X \text{ is symmetric} \\ & z \le (A \mathbf{x})_i - b_i + R(1 - d_i) \ \forall i \ \text{ for some large enough } R \\ & \sum_i d_i = 1 \\ & d_i \in \{0, 1\} \ \forall i. \end{array}$$
(12)

The MIP and LP Feasibility procedures are similar and should theoretically give the same results. However, as we will show in our numerical studies, the LP Feasibility procedure could outperform the MIP procedure by producing higher robustness margins and running faster in higher dimensions. In both procedures, the process ends after a boundary solution is found. This solution

may not be a boundary solution to the actual system, but may be an artifact of the relaxations used to create the procedures. One way of tackling this issue is by updating the constraints using an iterative process as outlined in the following procedure.

LP Bound Tightening Procedure

$$\max z_{i} = (A\mathbf{x})_{i}$$
subject to
$$\operatorname{Trace} (Q_{i}X) + L_{i}\mathbf{x} \ge \mathbf{u}_{i}^{\min} \quad \forall i$$

$$\operatorname{Trace} (Q_{i}X) + L_{i}\mathbf{x} \le \mathbf{u}_{i}^{\max} \quad \forall i$$

$$A\mathbf{x} \le \mathbf{b}$$

$$\mathbf{b}\mathbf{b}^{T} - A\mathbf{x}\mathbf{b}^{T} - \mathbf{b}(A\mathbf{x})^{T} + AXA^{T} \ge O$$

$$X \text{ is symmetric.} \qquad (13)$$

What distinguishes the LP Bound Tightening procedure from the other two procedures is the ability to use it iteratively by updating the constraints of Equation (13), replacing b with z, the vector of z_i 's found after running the procedure over all dimensions of Ax. Since clearly $z \leq b$, we have that the polytope $\{x \mid Ax \leq z\} \subseteq \{x \mid Ax \leq b\}$. Thus it follows that any system deemed robust feasible using Equation (11) or Equation (12) will certainly be found robust feasible using Equation (13), but a system found robust feasible using Equation (13) may not be found robust feasible using Equation (11) or Equation (12). The drawback of the bound tightening procedure, as we shall see in the Computational Results (Section 7.2), is choice of parameters to be manually set in order to tell the procedure when to stop.

5 Computing Upper Bounds on the Robustness Margin

Due to the hardness of solving the problem $\min_{\mathbf{x}\in\partial\Omega,\mathbf{u}\in\Omega_u}\max_{\|\boldsymbol{\lambda}\|=1} \boldsymbol{\lambda}^T (F(\mathbf{x}) - \mathbf{u})$, we derived a sequence of relaxations in order to arrive at a method that was computationally efficient for non-trivial data sets. We can similarly work with $\min_{\|\boldsymbol{\lambda}\|=1}\max_{\mathbf{x}\in\overline{\Omega},\mathbf{u}\in\Omega_u}\boldsymbol{\lambda}^T (F(\mathbf{x}) - \mathbf{u})$ to produce computationally efficient procedures to compute upper bounds for the robustness margin.

Theorem 5.1. Let $\overline{\Omega} = {\mathbf{x} | A\mathbf{x} \leq \mathbf{b}}$, $\Omega_u = {\mathbf{u} | u_i^{\min} \leq u_i \leq u_i^{\max} \forall i}$, and $F(\mathbf{x}) = Q(\mathbf{x}) + L\mathbf{x}$ as described in Equations (1) to (3). Let

$$z = \min_{\|\boldsymbol{\lambda}\|=1} \max_{\mathbf{x} \in \bar{\Omega}, \mathbf{u} \in \Omega_u} \boldsymbol{\lambda}^T \left(F(\mathbf{x}) - \mathbf{u} \right).$$

If there is an r > 0 such that $r \le e_i$ for all *i* with $e_i > 0$, where e_i is the error bound associated with u_i^{\min} and u_i^{\max} , and if z = 0 then the system has robustness margin of no more than *r*.

Proof. Observe by Lemma 3.2, if $z_i = \min_{\|\lambda\|=1} \max_{\mathbf{x} \in \overline{\Omega}, \mathbf{u} \in \Omega_u} \lambda^T (F(\mathbf{x}) - \mathbf{u}^*) = 0$, then we have $F(\hat{\mathbf{x}}) \in \partial F(X) = F(\partial X)$. Thus, it's followed by the statement that there exist a $\mathbf{x} \in \partial X$ such that $F(\mathbf{x}) - \mathbf{u} = 0$ for some $u \in \Omega_u$ which invalids Equation (8).

We can relax the computation suggested in Theorem 5.1 by utilizing the same techniques as before, replacing $\mathbf{x}\mathbf{x}^T$ with a positive semidefinite matrix X, with the option of dropping the condition that X be positive semidefinite. This step transforms the computation from a semidefinite program to a linear or mixed integer program depending on how one deals with the constraint $\|\boldsymbol{\lambda}\| = 1$. In our tests we use the ℓ_1 norm and introduce variables that capture the absolute value of each λ_i . We obtain the following procedure.

Outer Bound Procedure: Formulation

$$z = \min_{\|\boldsymbol{\lambda}\|=1} \max_{\mathbf{x}} \boldsymbol{\lambda}^{T} \left(F(\mathbf{x}) - \mathbf{u}^{*} \right)$$

subject to $A\mathbf{x} \leq \mathbf{b}$
 $\mathbf{b}\mathbf{b}^{T} - A\mathbf{x}\mathbf{b}^{T} - \mathbf{b}(A\mathbf{x})^{T} + AXA^{T} \geq O$
 X is symmetric. (14)

In order to implement this procedure, we use linear programming duality to write the problem in Equation (14) as direct minimization LP (in place of a min-max problem). To construct the dual of the inner maximum objective function, we first write the constraints as $M\hat{\mathbf{x}} \leq \mathbf{B}$ and the objective as $g(Q, L, \lambda)\hat{\mathbf{x}} - \lambda^T \mathbf{u}^*$, where $\hat{\mathbf{x}}^T = [\mathbf{x}^T \hat{X}^T]$, with \hat{X} being the vector form of the upper triangular (including diagonal) portion of X. We enforce the constraint that X is symmetric by utilizing only the upper triangular portion of X. We can write $M\hat{\mathbf{x}} \leq \mathbf{B}$ using only the upper triangular entries including the diagonal as follows.

$$\begin{bmatrix} A_{11} & \dots & A_{1n} & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ A_{m1} & \dots & A_{mn} & 0 & \dots & 0 \\ \mathbf{b}_{1}A_{11} + \mathbf{b}_{1}A_{11} & \dots & \mathbf{b}_{1}A_{1n} + \mathbf{b}_{1}A_{1n} & -A_{11}A_{11} & \dots & -A_{1n}A_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{b}_{r}A_{q1} + \mathbf{b}_{q}A_{r1} & \dots & \mathbf{b}_{r}A_{qn} + \mathbf{b}_{q}A_{rn} & -A_{q1}A_{r1} & \dots & -A_{1n}A_{2n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{b}_{m}A_{m1} + \mathbf{b}_{m}A_{m1} & \dots & \mathbf{b}_{m}A_{mn} + \mathbf{b}_{m}A_{mn} & -A_{m1}A_{m1} & \dots & -A_{mn}A_{mn} \end{bmatrix} \begin{bmatrix} x_{1} \\ \vdots \\ x_{n} \\ x_{n} \\ \vdots \\ x_{n} \\ \vdots \\ x_{n} \\ \vdots \\ x_{n} \end{bmatrix} \leq \begin{bmatrix} b_{1} \\ \vdots \\ b_{m} \\ b_{m} \\ b_{1}b_{1} \\ \vdots \\ b_{q}b_{r} \\ \vdots \\ b_{m}b_{m} \end{bmatrix}$$

We can write the objective function as $g(Q, L, \lambda)\hat{\mathbf{x}} - \lambda^T \mathbf{u}^*$ in the following way.

$$\begin{bmatrix} \sum_{j=1}^{n} L_{j,1}\lambda_j & \dots & \sum_{j=1}^{n} L_{j,n}\lambda_j & \sum_{j=1}^{n} \lambda_j Q_{11}^j & \dots & \sum_{j=1}^{n} \lambda_j Q_{nn}^j \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ X_{11} \\ \vdots \\ X_{nn} \end{bmatrix} - \begin{bmatrix} \lambda_1 & \dots & \lambda_n \end{bmatrix} \begin{bmatrix} u_1^* \\ \vdots \\ u_n^* \end{bmatrix}.$$

If $\mathbf{u}^* = \mathbf{0}$, then the objective function reduces to just $g(Q, L, \lambda)\hat{\mathbf{x}}$.

On the other hand, if $\mathbf{u}^* \neq \mathbf{0}$ then we can add an extra dummy variable \mathbf{x}_{n+1} with the constraint

 $\mathbf{x}_{n+1} = 1$, to obtain the following system.

$$\begin{bmatrix} A_{11} & \dots & A_{1n} & A_{1,n+1} & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{m1} & \dots & A_{mn} & A_{mn+1} & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & -1 & 0 & \dots & 0 \\ \mathbf{b}_{1}A_{11} + \mathbf{b}_{1}A_{11} & \dots & \mathbf{b}_{1}A_{1n} + \mathbf{b}_{1}A_{1n} & 0 & -A_{11}A_{11} & \dots & -A_{1n}A_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{b}_{r}A_{q1} + \mathbf{b}_{q}A_{r1} & \dots & \mathbf{b}_{r}A_{qn} + \mathbf{b}_{q}A_{rn} & 0 & -A_{q1}A_{r1} & \dots & -A_{1n}A_{2n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{b}_{m}A_{m1} + \mathbf{b}_{m}A_{m1} & \dots & \mathbf{b}_{m}A_{mn} + \mathbf{b}_{m}A_{mn} & 0 & -A_{m1}A_{m1} & \dots & -A_{mn}A_{mn} \end{bmatrix} \begin{bmatrix} x_{1} \\ \vdots \\ x_{n} \\ x_{n} \\ x_{n+1} \\ \vdots \\ X_{qr} \\ \vdots \\ X_{nn} \end{bmatrix} \leq \begin{bmatrix} b_{1} \\ \vdots \\ b_{m} \\ 1 \\ -1 \\ b_{1}b_{1} \\ \vdots \\ \vdots \\ b_{q}b_{r} \\ \vdots \\ b_{m}b_{m} \end{bmatrix}.$$

We can then write the objective function $g(Q, L, \lambda)\hat{\mathbf{x}}$ as follows.

$$\begin{bmatrix} \sum_{j=1}^{n} L_{j,1}\lambda_j & \dots & \sum_{j=1}^{n} L_{j,n}\lambda_j & \sum_{j=1}^{n} -\lambda_j u_j^* & \sum_{j=1}^{n} \lambda_j Q_{11}^j & \dots & \sum_{j=1}^{n} \lambda_j Q_{nn}^j \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ x_{n+1} \\ \vdots \\ \vdots \\ X_{nn} \end{bmatrix}.$$

In either case, the final optimization problem, and hence the procedure we use in our tests, is given as follows.

Outer Bound Procedure

$$z = \min_{\|\boldsymbol{\lambda}\|=1, \mathbf{y}} \mathbf{B}^T \mathbf{y}$$

subject to $M^T \mathbf{y} = g(Q, L, \lambda)^T$
 $\mathbf{y} \ge \mathbf{0}.$ (15)

F

Illustrative Example 6

We present a toy example to illustrate the lower bound feasibility and bound tightening procedures as well as the upper bound approximation method. We take the problem data as follows.

$$A = \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} -0.5 \\ 3 \\ -0.5 \\ 3 \end{bmatrix}, \ Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ Q_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \ L = \begin{bmatrix} 1 & -3 \\ 2 & -1 \end{bmatrix}, \ \text{and} \ \mathbf{u}^* = \begin{bmatrix} -2 \\ 4 \end{bmatrix}.$$

These components of data correspond to the following quadratic system.

$$F(\mathbf{x}) = \begin{bmatrix} x_1^2 + x_1 - 3x_2 \\ x_2^2 + 2x_1 - x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix} \quad \text{with} \quad \Omega_{\mathbf{x}} = \left\{ \mathbf{x} \mid \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} \le \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} 3 \\ 3 \end{bmatrix} \right\}.$$

This system has a unique solution in Ω_x given by

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cong \begin{bmatrix} 1.36 \\ 1.74 \end{bmatrix}.$$

Running the Outer Bound Procedure (Equation (15)) gives an upper bound on the robustness margin of 2.63462. Running the LP feasibility procedure results in an lower bound on the robustness margin of 1.20454, while the bound tightening yields a lower bound of 1.706649. We illustrate $F(\Omega_x)$ with u^{*} in Figure 3. Notice that the bound tightening procedure produces a better lower bound approximation as theoretically predicted.



Figure 3: Illustration of $F(\Omega_x)$ (dark blue surface), \mathbf{u}^* (center of the boxes), and Ω_{u^*} with radii given by the upper bound procedure (light blue box), bound tightening procedure (turquoise box), and feasibility procedure (yellow box).

7 Implementation on Power Systems

We now present results from computational studies that demonstrate the efficiency of the bounding procedures we have introduced. For all the numerical results presented, we apply the LP and MIP feasibility procedures (Equations (11) and (12)), the LP bound tightening procedure (Equation (13)), and the Outer bound procedure (Equation (15)) to find lower and upper bounds for the robustness margins with respect to the optimal power flow equations derived using datasets obtained from the MatPower package found in the MATLAB software [44] and the NICTA Energy System Test Case Archive (NESTA) [13]. We specifically show results for tests conducted on cases 5, 9, 14, 29, 30, 39 and 57. In every case the power flow equations were converted into the form of a system of quadratic equations we study (as described in Equations (1) to (3)). To simulate real life scenarios we allowed the first 5 dimensions of u to represent renewable energy, and the others representing no variation. We then slowly increased the variation of the first 5 dimensions of u, while utilizing procedures for the upper and lower bound verification to verify robust feasibility. We first detail specifically how this transformation was conducted.

7.1 OPF to Quadratic System

As described by Dvijotham et al. [15], the AC power flow equations can be written as follows.

$$\operatorname{Re}\left(\sum_{k=1}^{n} V_{i}\left(\overline{Y_{ik}V_{k}} + \overline{Y_{i0}V_{0}}\right)\right) = p_{i}, \forall i \in PQ$$

$$\operatorname{Im}\left(\sum_{k=1}^{n} V_{i}\left(\overline{Y_{ik}V_{k}} + \overline{Y_{i0}V_{0}}\right)\right) = q_{i}, \forall i \in PQ$$

$$\operatorname{Re}\left(\sum_{k=1}^{n} V_{i}\left(\overline{Y_{ik}V_{k}} + \overline{Y_{i0}V_{0}}\right)\right) = p_{i}, \forall i \in PV$$

$$|V_{i}|^{2} = v_{i}, \forall i \in PV,$$

$$(16)$$

where V_i denotes the complex voltage phasor, p_i the active and q_i the reactive power injection, and Y the admittance matrix at node *i*. *PV* denotes the set of PV or *Generator* nodes/buses, *PQ* denotes the set of PQ or *Load* buses, and v_i denotes the squared voltage magnitude setpoints at the *PV* buses. We can then rewrite Equation (16) into the system outlined in Equations (1) to (3) by setting

$$\mathbf{x} = \begin{bmatrix} \operatorname{Re}(V_1) & \dots & \operatorname{Re}(V_n) & \operatorname{Im}(V_1) & \dots & \operatorname{Im}(V_n) \end{bmatrix}^T \text{ and } \\ \mathbf{u} = \begin{bmatrix} p_1 & \dots & p_n & q_1 & \dots & q_n & v_1 & \dots & v_n \end{bmatrix}^T.$$

7.2 Computational Results

The procedures were computed with $A\mathbf{x} \leq \mathbf{b} = B\mathbf{1}$ for $B \in \{0.001, 0.005, 0.01\}$. A in these cases is a matrix such that $A\mathbf{x} \leq \mathbf{b}$ controls the flow between nodes in the power grid, i.e., each row of $A\mathbf{x} \leq \mathbf{b}$ has the form $x_i - x_j \leq b_k$. All computations were performed on a laptop running the 64bit MacOS Catalina operating system containing an 2.3GHz dual-core Intel Core i5, Turbo Boost up to 3.6GHz, with 64MB of eDRAM. Details on the computation are given in Table 1, for determining the practical scaling properties of these procedures with the fixed B = 0.001. We display the data on a case by case basis to highlight the effect of allowing more fluctuation between the nodes, i.e., as B increases, in Figure 4.

Case #	BdTgtLower			MIPLower			LPLower			LPUpper		
	Time(s)	Var	Cons#	Time(s)	Var#	Cons#	Time(s)	Var#	Cons#	Time(s)	Var#	Cons#
5	10.91	44	616	0.32	69	641	0.43	44	616	0.52	615	88
9	35.82	152	1364	6.88	189	1401	2.82	152	1364	2.95	1347	288
14	743.11	377	6532	93.31	458	6613	468.55	377	6532	35.73	6495	718
29	9436.28	1536	25477	3164.68	1703	25816	9236.42	1536	25477	483.72	24377	3126
30	9888.07	1769	27176	3674.53	1934	27341	9484.48	1769	27176	531.31	27075	3438
39	21764.43	3248	39265	9436.08	3771	41026	18329.58	3248	39265	1442.36	41342	4855

Table 1: Solution times, number of variables, and number of constraints for the procedures we considered on the Cases 5, 9, 14, 29, 30, and 39.



Figure 4: Lower and upper bound estimates on the robustness margin for Case 5 (Top Left), Case 9 (Top Right), Case 14 (Middle Left), Case 29 (Middle Right), Case 30 (Bottom left), and Case 39 (Bottom right). 17

Case 57 is not included in the table and graphs as the only inner bound procedure to run in a reasonable time was the Feasibility procedure that produced a maximum robustness margin of ≈ 0.003 with a computational time of ≈ 48924 seconds, and a gap to the outer bound procedure of ≈ 0.030 when B = 0.001, which had a running time of ≈ 27080 seconds.

As evident from the graphs, the LP Bound Tightening Procedure produces a better approximation of the lower bound on the robustness margin as the complexity of the data set increases. The outer bound procedure (in Equation (15)) is derived using looser relaxations of the original problem compared to any of those used to derive lower bounds (in Equations (11) to (13)). Hence we expect that the upper bound produced is not as tight as the lower bounds. Certainly one would expect the bound tightening procedure to out perform the other inner bound procedures for all cases, but the choice of procedure parameters has a big effect on the efficiency and capability of the procedure. For instance, setting a low tolerance for a minimal sufficient change in the dimensions of b will result in a better lower bound approximation, but an extremely long running time for most cases. Thus in the low as well as marginally high complexity cases, it should be expected that the other procedures will out perform the bound tightening procedure as these manually set parameters will have more of an impact.

8 Discussion

We have proposed novel and efficient techniques for determining robust solvability of quadratic systems with uncertainty. It is worth mentioning that the same machinery can be used to determine robustness margins of solutions to static systems by taking A to be the identity, $e_i = 0 \forall i$, and simply adjusting b accordingly.

We have employed results on the computational complexity of QCQPs to shed some light on the hardness of the optimization problems in the key Theorem 4.2. It would be interesting to prove NP-hardness of the optimization problems in Equation (9) using direct arguments.

Our implementation on power systems in Section 7 is just one direct application of the general (theoretical) framework we have developed. We are exploring other avenues for applying our framework including gas and water flow networks.

In the context of our application to power systems, we must point out that finding the robustness margin of an OPF instance is inherently harder than solving the original OPF instance itself. Hence it is expected that larger sized instances of the default OPF problem are solved in practice than ones for which our inner and outer bound procedures are run efficiently.

We have presented (in Section 7.2) bounds on the robustness margins for solutions of OPF instances. One generalization we could consider is that of combining measures of robustness and optimality. In practice, a near optimal solution with a large robustness margin might be more desirable than an optimal solution with a small robustness margin.

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Data Availability

The datasets analyzed in the current study are available in the MATPOWER repository [44] at https://matpower.org/ and the NESTA repository [13], the instances of which are available as part of the Library of IEEE PES Power Grid Benchmark repository at https://github.com/power-grid-lib/. No new datasets were created as part of this study.

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