
Clustering-Based Quantisation for PDE-Based Image Compression

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Abstract Finding optimal data for inpainting is a key problem in the context of partial differential equation based image compression. The data that yields the most accurate reconstruction is real-valued. Thus, quantisation models are mandatory to allow an efficient encoding. These can also be understood as challenging data clustering problems. Although clustering approaches are well suited for this kind of compression codecs, very few works actually consider them. Each pixel has a global impact on the reconstruction and optimal data locations are strongly correlated with their corresponding colour values. These facts make it hard to predict which feature works best.

In this paper we discuss quantisation strategies based on popular methods such as k-means. We are lead to the central question which kind of feature vectors are best suited for image compression. To this end we consider choices such as the pixel values, the histogram or the colour map.

Our findings show that the number of colours can be reduced significantly without impacting the reconstruction quality. Surprisingly, these benefits do not directly translate to a good image compression performance. The gains in the compression ratio are lost due to increased storage costs. This suggests that it is integral to evaluate the clustering on both, the reconstruction error and the final file size.

Keywords Laplace interpolation · inpainting · compression · quantisation · clustering · partial differential equations

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1 Introduction

A major challenge in data analysis is the reconstruction of a signal from very few data points. In image processing this interpolation problem is called inpainting. Recent advances have shown that accurate reconstructions from a small sample of well chosen image points are possible by using methods based on partial differential equations (PDEs) [1, 4, 6, 12, 18, 21, 24, 31]. These efforts have also been exploited for lossy image compression schemes that can nowadays compete with other state-of-the-art approaches [28, 31]. Even simple methods that rely on the interpolation capabilities of the Laplacian may yield more accurate reconstructions than JPEG 2000 [26]. Unfortunately, the underlying optimisation problem concerned with selecting optimal point locations is computationally intensive. In addition, the resulting data is often difficult to store efficiently. The pixel locations may be scattered arbitrarily inside the image domain and the corresponding colour values are often computed in floating point precision. Storing this data as-is is prohibitively expensive and often unnecessary. The human visual system is only capable of distinguishing about thirty different shades of grey [20]. These thirty colours can be encoded with 5 bit instead of the 64 bit required for a floating point value in double precision. Finding the optimal number and distribution of these grey values is however a complicated task. Nevertheless, it is reasonable to expect that

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small changes to the colour values will only yield small changes in the reconstruction. This assumption allows us to replace real valued colours by their closest integer valued neighbours without causing too much damage.

This work extends the preliminary findings from [16]. There, the authors investigated clustering schemes and quality assessment measures to provide fast strategies to determine a good number of quantised colours. In [16], the clustering was not evaluated against the compression ratio of the final compressed file but only against the reconstruction quality of the clustered data. Nevertheless, the results indicated that simple clustering approaches yield good quantisations of the co-domain. In addition, the authors showed that these clusterings can be evaluated with quality measures such as the silhouette coefficient.

Our Contribution. In this paper we investigate how well different clustering optimisation techniques fare in a data compression context. Thus, we extend the results from [16] by providing a similar evaluation on the compression ratio. In addition we rank all considered clustering strategies with respect to their effectiveness and show that the quantisation must be done with the compression ratio in mind and not with respect to the reconstruction quality.

To this end we will discuss several algorithms as well as quality measures and show that not only the number of clusters, but also the final distribution of the labels is important. The distribution of the labels is mostly influenced by the underlying clustering algorithm, whereas the number of clusters can be optimised independently for any strategy. The latter task is also considered in detail. Several strategies are proposed. This discussion also extends the findings from [16] where only the silhouette coefficient was taken into consideration.

2 Partial Differential Equation Based Image Compression

Even though most diffusion-type partial differential equations can be used to reconstruct the data, previous works have shown that homogeneous, linear diffusion ranks among the best performing ones [26] in the compression context. We also refer to [2] for a more general overview on PDE-based inpainting methods. The simplicity of homogeneous diffusion inpainting allows a thorough mathematical analysis and an extensive optimisation [18, 19]. Since our sought optimal quantisation is likely to depend strongly on the underlying reconstruction process we focus exclusively on homogeneous diffusion in this work. Nevertheless, many of the presented strategies can be extended to other inpainting methods.

A good image compression approach consists of several components. Besides the reconstruction method it is also necessary to optimise the considered data and to design an efficient encoding. Let us now briefly review the involved reconstruction and optimisation steps in the next paragraphs.

2.1 Inpainting with Homogeneous Diffusion

Inpainting with homogeneous diffusion (sometimes also called Laplace interpolation) is a rather simple reconstruction method that is well suited for highly scattered data in arbitrary dimensional settings. It can be modelled as follows. Let $f : \Omega \rightarrow \mathbb{R}$ be a smooth function on some bounded domain $\Omega \subset \mathbb{R}^2$ with a sufficiently regular boundary $\partial\Omega$. Throughout this work, we will restrict ourselves to the case of grey value images even though many of the results hold for arbitrary numbers of colour channels. Moreover, let us assume that there exists a closed nonempty set of known data $\Omega_K \subsetneq \Omega$ that will be interpolated by the underlying diffusion process. Homogeneous diffusion inpainting considers the following partial differential equation with mixed boundary conditions:

$$\begin{aligned} -\Delta u &= 0 && \text{on } \Omega \setminus \Omega_K \\ \text{with } \begin{cases} u = f & \text{on } \partial\Omega_K \\ \partial_n u = 0 & \text{on } \partial\Omega \setminus \partial\Omega_K \end{cases} && (1) \end{aligned}$$

where $\partial_n u$ denotes the derivative of u in the outer normal direction. We assume that both boundary sets $\partial\Omega_K$ and $\partial\Omega \setminus \partial\Omega_K$ are non-empty. Equations of this type are commonly referred to as mixed boundary value problems and sometimes also as Zaremba's problem [36]. The existence and uniqueness of solutions has been studied during the last century. Showing that (1) is indeed solvable is by no means a trivial feat. We refer to [13] for an extensive study of linear elliptic partial differential equations and to [17] for a more particular analysis of (1). A particularly simple case is the 1-D setting, where the solution can obviously be expressed using piecewise linear splines interpolating data on $\partial\Omega_K$.

2.2 Optimal Inpainting Positions

Finding good inpainting positions is a task related to the free knot problem from interpolation theory [3, 15]. In the special context of image inpainting and image compression the authors of [6, 9, 18, 24] suggested several strategies to find good locations. Mainberger et al. [24] propose stochastic optimisations that bear similarities to simulated annealing approaches. Chen et al. [6] use

fast bi-level optimisation schemes and in [18] Hoeltgen et al. present an optimal control model. Finally the works of Demaret et al. [8–10] use adaptive triangulations and mesh optimisation strategies, whereas Ochs et al. [25] suggest fast numerics for the task at hand.

2.3 Continuous Grey Value Optimisation

In [19, 24] the authors discuss approaches to find the best pixel values. These algorithms have in common, that they all yield real valued floating point results. If we denote the reconstruction operator that solves (1) for a given mask Ω_K by $M(\Omega_K)$, then this problem can be formulated as

$$\arg \min_u \left\{ \|M(\Omega_K)u - f\|^2 \right\} \quad (2)$$

Since our initial PDE is linear, the task stated in (2) corresponds to a linear least squares problem and can be solved efficiently. An alternative discrete optimisation has been suggested by Schmaltz et al. [31]. The latter approach iterates over the inpainting data and selects in a greedy way the currently best quantised grey value. The iterations continue until a convergent state is reached. While this approach comes conceptually close to clustering strategies, it has two notable drawbacks: (i) The approach requires that the distribution of the quantised values is fixed at the beginning, i.e. it does not change during the iterations; (ii) to test a certain grey value for its quality requires the solution of a large and sparse linear system, i.e. the computational costs are considerable as the algorithm requires the solution of thousands of linear systems. Nevertheless, we remark that it yields excellent qualitative results.

3 Clustering Inpainting Data

The following paragraphs discuss potential choices for the feature selection as well as for the algorithmic execution. Our motivation is to use models that come conceptually close to the continuous grey value optimisation and the quantisation aware approach of Schmaltz et al. [31]. Therefore, we focus on the squared Euclidean distance as cost function and seek the best set of grey values for the mask pixel locations. However, we will refrain from using the reconstruction inside our algorithms in order to obtain fast methods.

3.1 Feature Selection

Our setup provides us the full image data f as well as an optimised inpainting mask Ω_K . There exist many

possible choices to extract interesting features from this data. The following eight feature choices seem reasonable:

1. position and value of all image pixels
2. position and value of all mask pixels
3. grey values values of all image pixels (i.e. discarding the positional information)
4. grey values of all mask pixels (i.e. discarding the positional information)
5. the histogram of all image pixels
6. the histogram of all mask pixels
7. the colour map (i.e. histogram without frequency count) of all image pixels
8. the colour map (i.e. histogram without frequency count) of all mask pixels

The following observations can be made with respect to the feature choices.

- We remark that including the pixel position into the feature vector renders each sample unique. In that case it makes no sense to track their frequency as it will always be 1.
- Approaches that exploit the full image information have better chances at adapting to features in the image that are not covered by the mask pixels alone. On the other hand, optimised positions for the inpainting with homogeneous diffusion are usually in the vicinity of edges and other important structures. Therefore, it is likely that the image information provided by the mask pixels is already sufficient to determine a good clustering. In addition, for compression purposes the inpainting masks are usually very sparse. Thus, we also benefit from a smaller feature set and a higher performance.
- Strategies that focus exclusively on the grey values of an image are likely to perform fastest since they can be carried out on the histogram. As soon as we include the spatial position into the feature set we obtain as many unique feature values as pixels. For large images we may encounter memory or run time restrictions.

Finally let us remark that from a clustering perspective it is irrelevant whether we consider grey values or colour images. Only the mask optimisation and inpainting steps are more difficult to carry out for colour images.

3.2 Clustering Approaches for Image Quantisation

In this work we opt for some of the most popular and best understood approaches to classify our data. Since clustering approaches are commonly subdivided into partitional and hierarchical methods we select the following representatives:

1. *partitional clustering*: We use the k-means++ variant of the venerable k-means algorithm of Lloyd [22].
2. *hierarchical clustering*: We employ a bottom up strategy where initially each feature represents a single cluster. These clusters are merged step-by-step using Ward’s criterion [35].
3. *probabilistic clustering*: As an alternative to the “hard” labelling performed by the k-means approach we also consider a “softer” variant using a Gaussian mixture model and employ the popular expectation maximisation algorithm of Sundberg [32,33] to compute the probabilities that a feature belongs to a certain cluster.

We also refer to [14] for a discussion on the viability of various clustering techniques.

3.3 Quality Measures and Optimal Number of Clusters

When evaluating a partitioning of the data we need to consider several criteria. First, the quality of the clustering itself: Features in the same group should indeed be similar and yet also distinct from observations from other clusters. For our purposes it is also important to assess the quality of the reconstruction as well as the compression ratio of the final file. Especially the latter quantity is difficult to estimate. Our quantised data should have few different grey values, (i.e. a small number of clusters) such that the file compression can be efficient. However, it is also essential that the reconstruction quality is fair, which is likely to require many different grey values (i.e. a large number of clusters) in floating point precision. Thus, there is a trade-off between reconstruction quality and compression ratio. This influence can be steered by optimising the number of clusters. From the large pool of quality measures available in the literature ([11] lists more than 50) we chose the following to help us identify an optimal grouping of our data. In this description, k denotes always the number of clusters and N the number of features.

1. The Calinsky Harabasz (CH) criterion [5], also called *variance ratio criterion*, considers the quantity

$$\frac{\sum_{j=1}^k \|m_j - m\|^2}{\sum_{j=1}^k \sum_{x \in C_j} \|x - m_j\|^2} \frac{N - k}{k - 1} \quad (3)$$

where m_j is the centroid of the cluster C_j and m the overall mean of all features. Higher values indicate better clusterings.

2. The Davies Bouldin (DB) criterion [7] is based on a ratio of within-cluster and between-cluster dis-

tances. The Davies-Bouldin index is defined as

$$\frac{1}{k} \sum_{j=1}^k \max_{i \neq j} \left\{ \frac{\bar{d}_j + \bar{d}_i}{d_{j,i}} \right\} \quad (4)$$

with \bar{d}_ℓ being the average distance of the centroid to each element in the cluster C_ℓ and where $d_{r,s}$ is the distance between the centroids of the clusters C_r and C_s . Lower values are better.

3. The GAP criterion [34] formalises the well known *L-term heuristic* by estimating the “elbow” location. The “elbow” occurs at the most dramatic decrease in error measurement, i.e. the largest gap value. To this end, the within cluster sum of squares is compared to its expectation under an appropriate null reference distribution of the data. This leads to the following formula for the GAP value:

$$GAP_N(k) := E_N^*[\log(W_k)] - \log(W_k) \quad (5)$$

$$W_k := \sum_{r=1}^k \frac{1}{2|C_r|} \sum_{c_i, c_j \in C_r} \|c_i - c_j\|$$

where E_N^* denotes the expectation under a sample of size N from the reference distribution. The optimal number of clusters is given by k which maximises the GAP value.

4. The Silhouette index [29] of a cluster C is defined to be the average silhouette value of all its elements. The silhouette value of a single element $x \in C$ is defined as

$$\begin{cases} 0, & \text{if } \text{dist}(x, C) = 0 \\ \frac{\text{dist}(x, B) - \text{dist}(x, C)}{\max\{\text{dist}(x, C), \text{dist}(x, B)\}}, & \text{else} \end{cases} \quad (6)$$

where $\text{dist}(x, C)$ is the distance of x to the cluster C and where B is the next closest cluster. Comparing the silhouette indices for all possible clusters gives an estimate on how well structured the clustering is. These indices should be as close to 1 as possible.

We remark that the GAP criterion is one of the most flexible approaches but also the most expensive measure in terms of computational effort.

4 Numerical Experiments

4.1 Performance of the Clustering Algorithms

In total the proposed 8 features, 3 algorithms and 6 quality criteria allow 144 possible combinations to be evaluated. An exhaustive testing would require the evaluation of all these approaches on several test images, thus increasing the data to be analysed even more. Certainly not all of these combinations are reasonable



Fig. 1 *Left*: The considered test image *trui*. The image is 256×256 in size. *Right*: A corresponding inpainting mask with a density of 5%. Locations used for the reconstruction are marked in white.

and some strategies also require considerable run times such that they become impractical for most applications. Thus, we have decided to restrict ourselves to the following choices. This list is also motivated by the findings from [16] where it has been noticed that the k-means and hierarchical clustering are more reliable and usually perform better than the probabilistic clustering with a Gaussian mixture model.

1. k-means on the position and value of all mask pixels
2. hierarchical clustering on the position and value of all mask pixels
3. probabilistic clustering on the position and value of all mask pixels
4. k-means on the pixel values of all mask pixels
5. hierarchical clustering on the pixel values of all mask pixels
6. k-means on the histogram of all mask pixels
7. k-means on the histogram of all image pixels
8. k-means on the unique grey values of all mask pixels
9. k-means on the unique grey values of all image pixels

Our experiments were carried out on the data set presented in Fig. 1. The considered test image is of size 256×256 and has 170 unique grey values. These are byte-wise coded (i.e. the maximal amount of different grey values is 256) and the corresponding binary mask has a density of 5%. It has been obtained with the optimal control approach of Hoeltgen et al. [18]. The histograms of the image and of the pixels indicated by the mask are depicted in Fig. 2. The considered test image and its histogram are representative for a large class of images. The image has smooth regions as well as highly textured areas. The corresponding histograms are also very generic. They do not show any particular patterns. As a such, we believe that the presented findings would be similar for other images. The methods presented in this work are also agnostic towards how the mask was obtained.

Let us in a first step evaluate our considered criteria on the *trui* test image from Fig. 1. The results of

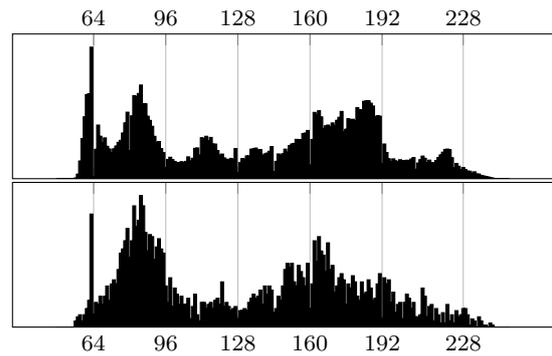


Fig. 2 *Top*: Histogram of the *trui* test image. *Bottom*: Histogram of the data at the mask positions.

our experiments are listed in Table 1. We see that integrating the position into the feature vectors significantly deteriorates the reconstruction quality (rows 1-3 in Table 1). The suggested optimal numbers of clusters however look reasonable, except for the GAP criterion which tends to suggest rather large numbers of clusters. The other experiments (rows 4-9 in Table 1) yield, at least in terms of mean squared error, good results. Also here the optimal number of clusters is often overestimated. Notable exceptions are the GAP criterion for the setups 4, 5, 8 and 9 where the suggested number of clusters was around 12 and the reconstruction error around 52. For the setup 6, the GAP criterion suggested 36 colours with a corresponding mean squared error of 48.91. Considering that the reconstruction with the original 170 colours has an mean squared error of 46.96, this is actually a very satisfying result. Also the suggested number of clusters coincides with our expectations.

Figure 3 plots the evolution of the reconstruction error as a function of the number of clusters for the k-means method as well as for the hierarchical clustering. The k-means method performs slightly better than the hierarchical scheme. For 35 or more clusters there is hardly any improvement in the reconstruction anymore and the errors converge towards the mean squared error of the original data. The steady and rather stable decrease of the error might explain why most clustering methods tend to overestimate the number of clusters. In terms of error, the findings for 40 clusters are almost as good as those with 72 clusters. Without an explicit requirement that the optimal number of clusters should be small, these results are difficult to discern.

4.2 Application to Image Compression

In order to evaluate the impact of quantisation by clustering on actual compression applications, we combine our approach with the state-of-the-art codec of Peter

Table 1 Clustering results for the trui test image. We tested all cluster sizes ranging from 12 to 72. The mean squared error is always given for the corresponding suggested optimal number of clusters. As we can see, most methods tend to return cluster numbers close to the maximal or minimal allowed value. Features that include the position of a pixel yield rather bad results. The suggested optimal number of clusters from the GAP criterion coincides best with our expectations. The corresponding reconstruction errors are also quite good, considering that the reconstruction error with the original 170 colours had an error of 46.96

Feature	mean squared error				Best number of clusters			
	Silhouette	DB	CH	GAP	Silhouette	DB	CH	GAP
1	404	246	350	125	16	25	15	66
2	380	199	271	147	13	38	27	68
3	418	316	295	152	12	15	19	71
4	47.15	47.08	46.95	54.88	63	72	70	12
5	47.26	47.32	47.26	52.63	72	69	72	14
6	63.17	47.51	47.66	48.91	12	72	72	36
7	52.50	51.65	53.01	54.86	72	72	72	66
8	51.30	46.63	46.86	50.34	12	63	69	12
9	52.94	46.57	46.62	49.88	12	72	67	12

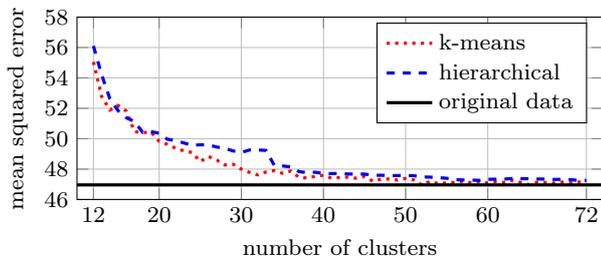


Fig. 3 Evolution of the mean squared error of the reconstruction from the clustered pixel values on the mask position. The solid black line marks the reconstruction error for the unaltered data. As we can see, the k-means method performs slightly better. For 35 or more clusters there is hardly any improvement in the reconstruction anymore. Also the mean squared error converges towards the error obtained by using the original data. This suggests, that the optimal number of clusters should actually be somewhere between 30 and 40 clusters.

et al. [27]. It stores binary masks with free choice of locations on the pixel grid by employing a block coding scheme [37]. In the original codec, the corresponding grey values are subject to an equidistant quantisation and both the locations and values are finally stored with the context-mixing scheme PAQ by Mahoney [23]. In the following, we investigate how replacing the equidistant quantisation with our clustering approach influences the performance of this codec. Since kmeans++ has yielded the most consistent results in the previous sections, we apply this strategy in the following.

In our first experimental setup, we use the same binary mask to compare the compression performance of equidistant and clustering-based quantisation: We quantise the original grey values with each approach

respectively and apply the same coding techniques afterwards. Fig. 4 shows that for low compression ratios, equidistant and clustering-based quantisation lead to similar results with a slight benefit for clustering. However, for higher compression ratios, the equidistant quantisation yields clearly superior results. In order to understand this outcome, we analyse the dependencies of the reconstruction error and the ratio on the number of quantisation levels for both methods in Fig. 5 and Fig. 6.

Fig. 5 shows that the reconstruction error of clustering-based quantisation fluctuates much less depending on the number of quantised grey values than in the case of equidistant quantisation. However, it does not offer a distinct qualitative advantage. On the contrary, for very coarse quantisations, the equidistant quantisation can also outperform the results of clustering in isolated cases.

More importantly, the non-equidistant quantisation does not only change the reconstruction error, but also the entropy of the sequence of grey values that need to be stored. Fig. 6 reveals that this yields a consistent increase in storage costs compared to an equidistant approach. Consequently, clustering needs to use a coarser quantisation at the cost of reconstruction quality in order to reach the same compression ratio as an equidistant method. Overall, this leads to the superior performance of the equidistant quantisation.

Moreover, there is another factor that needs to be considered for our evaluation: State-of-the-art compression codecs do not only carefully select the location of known data, but also optimise the pixel values under the constraint of the quantisation [27, 30]. Data optimisation schemes, such as the method of Hoeltgen

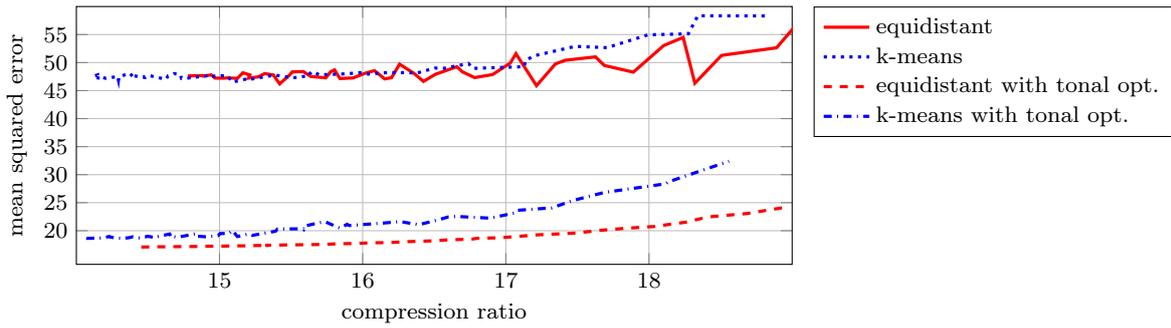


Fig. 4 Comparison of compression performance on *trui* over different compression ratios. The solid and dotted curves show the results with quantised original grey values whereas the dashed and dash-dotted curves present the findings for the compression with quantisation-aware tonal optimisation.

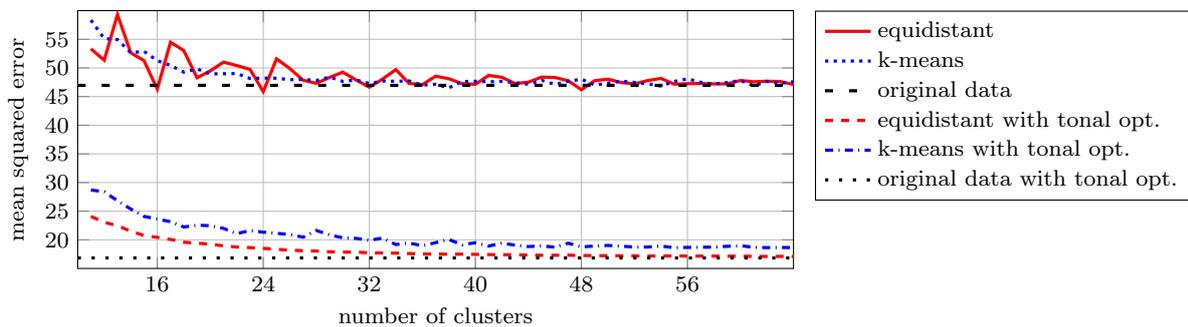


Fig. 5 Performance evaluation of our quantisation schemes in terms of reconstruction error. The solid and dotted curves denote the findings for an equidistant quantisation and a k-means quantisation without a tonal optimisation as post processing. The other two curves depict the findings with an additional tonal optimisation. The black lines denote the error when using the original data and tonal optimised data.

et al. [18] are unable to handle quantisations of the co-domain. Nevertheless, there is a clear mutual dependency between optimal data positions and values that should be respected in the quantisation process. Such a so-called quantisation-aware tonal optimisation efficiently corrects suboptimal data locations for the quantised colour values in a post-processing scheme and leads to a much more distinct advantage of equidistant quantisations (see Fig. 4). The clustering limits the ability of the tonal optimisation to adjust the behaviour of the inpainting algorithm locally. This is due to an inherent property of non-equidistant quantisations: By definition, the clustering leads to sparse regions in the grey value domain, where only few cluster centres are located while other regions are densely populated. In isolated regions, quantisation-aware tonal optimisation can only apply large changes for grey values while much smaller changes are possible in dense regions. This introduces a bias to the tonal optimisation that does not exist for equidistant quantisation: Here, each pixel value can be tuned by the same step size between quantisation levels, and thereby allows to diverge more significantly from the original distribution of grey values.

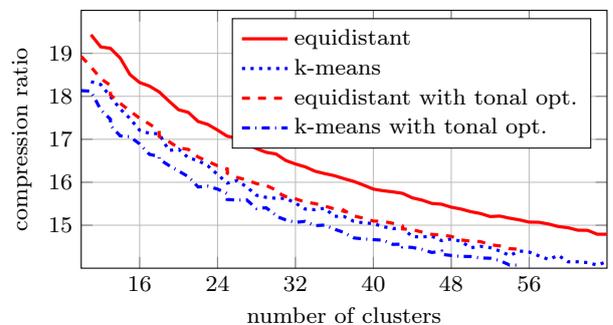


Fig. 6 Performance evaluation of our quantisation schemes in terms of compression ratio. The solid and dotted curves denote the findings for an equidistant quantisation and a k-means quantisation without a tonal optimisation as post processing. The other two curves depict the findings with an additional tonal optimisation.

Our evaluation shows that clustering on its own is not a suitable replacement for the simple equidistant quantisation in PDE-based compression. More complex quantisation techniques would require a more complex integration in the full compression pipeline, taking into account the balance of storage cost and reconstruction quality. While such an approach to non-equidistant

quantisation would be promising, it is beyond the scope of our publication.

5 Summary and Conclusions

Clustering makes a lot of sense for pure quantisation of existing data. Our experiments show that we can reduce the number of colours used for PDE-based inpainting by as much as 80% without encountering any significant loss in the reconstruction quality. In addition, the presented strategies are fast and easy to implement.

However, the cost of the associated data and the limitations imposed on tonal optimisation make this concept difficult to apply to compression if it is treated as an isolated component in the compression pipeline. Our findings show that a good quantisation with respect to the reconstruction error does not necessarily imply an efficient encoding of the data.

Therefore, successful data optimisation strategies in the context of image compression must include the encoding costs in their framework. It is not enough to consider the reconstruction error for the quantisation.

Such potential new models are challenging in several ways. Not only are the encoding costs difficult to predict, but also their optimisation is likely to be costly. For practical applications it is however necessary to devise strategies that are very fast since state-of-the-art data compression codecs have run times in the millisecond range.

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