# Convolution and Correlation Theorems for Wigner-Ville Distribution Associated with the Quaternion Offset Linear Canonical Transform 

Convolution and Correlation Theorems for WVD Associated with the QOLCT

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#### Abstract

The quaternion offset linear canonical transform(QOLCT) has gained much popularity in recent years because of its applications in many areas, including color image and signal processing. At the same time the applications of Wigner-Ville distribution (WVD) in signal analysis and image processing can not be excluded. In this paper we investigate the Winger-Ville Distribution associated with quaternion offset linear canonical transform (WVD-QOLCT). Firstly, we propose the definition of the WVD-QOLCT, and then several important properties of newly defined WVD-QOLCT, such as nonlinearity, bounded, reconstruction formula, orthogonality relation and Plancherel formula are derived. Secondly a novel canonical convolution operator and a related correlation operator for WVD-QOLCT are proposed. Moreover, based on the proposed operators, the corresponding generalized convolution, correlation theorems are studied.We also show that the convolution and correlation theorems of the QWVD and WVDQLCT can be looked as a special case of our achieved results.


Keywords Quaternion algebra • Offset linear canonical transform • Quaternion offset linear canonical transform • Wigner-Ville distribution • Convolution . Correlation • Modulation

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## 1 Introduction

In the time-frequency signal analysis the classical WignerVille distribution (WVD) or Wigner- Ville transform (WVT) has an important role to play. Eugene Wigner introduced the concept WVD while making his calculation of the quantum corrections. later on it was J. Ville who derived it independently as a quadratic representation of the local time-frequency energy of a signal in 1948. Many important properties of WVT has been studied by many authors. On replacing the kernel of the classical Fourier transform (FT) with the kernel of the LCT in the WVD domain, this transform can be extended to the domain of linear canonical transform [3]- [6], [13]- 19 .

On the other hand the quaternion Fourier transform (QFT) is of the interest in the present era. Many important properties like shift, modulation, convolution, correlation, differentiation, energy conservation, uncertainty principle of QFT have been found. Many generalized transforms are closely related to the QFTs, for example, the quaternion wavelet transform, fractional quaternion Fourier transform, quaternion linear canonical transform, and quaternionic windowed Fourier transform. Based on the QFTs, one also may extend the WVD to the quaternion algebra while enjoying similar properties as in the classical case. Many authors generalized the classical WVD to quaternion algebra, which they called as the quaternion Wigner-Ville distribution (QWVD). For more details we refer to [1], [2], [7]-[12].

The linear canonical transform (LCT) with four parameters $(a, b, c, d)$ has been generalized to a six parameter transform ( $a, b, c, d, u_{0}, w_{0}$ ) known as offset linear canonical transform (OLCT). Due to the time shifting $u_{0}$ and frequency modulation parameters, the OLCT has gained more flexibility over classical LCT. Hence
has found wide applications in image and signal processing. On the other side the convolution has some applications in various areas of Mathematics like linear algebra, numerical analysis and signal processing. Where as Correlation like convolution is an another important tool n signal processing, optics and detection applications. In the domains of LCT, WVD and OLCT the convolution and correlation operations have been studied [7]-[10].

The quaternion offset linear canonical transform (QOLCT) has gained much popularity in recent years because of its applications in many areas, including colour image and signal processing. At the same time the applications of Wigner-Ville distribution (WVD) in signal analysis and image processing can not be excluded. Motivated by QOLCT and WVD, we in this paper we investigate the Winger-Ville Distribution associated with quaternion offset linear canonical transform (WVD-QOLCT). Firstly, we propose the definition of the WVD-QOLCT, and then several important properties of newly defined WVD-QOLCT, such as nonlinearity, bounded, reconstruction formula, orthogonality relation and Plancherel formula are derived. Secondly a novel canonical convolution operator and a related correlation operator for WVD-QOLCT are proposed. Moreover, based on the proposed operators, the corresponding generalized convolution, correlation theorems are studied. We also show that the convolution and correlation theorems of the QWVD and WVD-QLCT can be looked as a special case of our achieved results.

The paper is organised as follows. In Section 2, we provide the definition of Wigner-Ville distribution associated with the quaternionic offset linear canonical transform (WVD-QOLCT). Then we will investigate several basic properties of the WVD-QOLCT which are important for signal representation in signal processing. In Section 3 we first define the convolution and correlation for the QOLCT. We then establish the new convolution and correlation for the WVD-QOLCT.We also show that the convolution theorems of the QWVD and WVD-QLCT can be looked as a special case of our achieved results I

## 2 Winger-ville Distribution associated with Quaternion Offset Linear Canonical Transform(WVD-QOLCT)

Since in practice most natural signals are non-stationary. In order to study a non-stationary signals the WignerVille distribution has become a suite tool for the analysis of the non stationary signals. In this section, we are going to give the definition of Wigner-Ville distribution
associated with the quaternionic offset linear canonical transform (WVD-QOLCT), then we will investigate several basic properties of the WVD-QOLCT which are important for signal representation in signal processing.

Definition 1 Let $A_{i}=\left[\begin{array}{ll|l}a_{i} & b_{i} \mid & r_{i} \\ c_{s} & d_{i} \mid & s_{i}\end{array}\right]$, be a matrix parameter such that $a_{s}, b_{i}, c_{i}, d_{i}, r_{i}, s_{i} \in \mathbf{R}$ and $a_{i} d_{i}-$ $b_{i} c_{i}=1$, for $i=1,2$. The Wigner-Ville distribution associated with the two-sided quaternionic offset linear canonical transform (WVD-QOLCT) of signals $f, g \in$ $L^{2}\left(\mathbf{R}^{2}, \mathbf{H}\right)$, is given by

$$
\left\{\begin{array}{l}
\int_{\mathbf{R}^{2}} K_{A_{1}}^{i}\left(n_{1}, u_{1}\right) f\left(t+\frac{n}{2}\right) \overline{g\left(t-\frac{n}{2}\right)} \\
K_{A_{2}}^{j}\left(n_{2}, u_{2}\right) d n, \\
b_{1}, b_{2} \neq 0, \\
\sqrt{d_{1}} e^{i\left(\frac{c_{1} d_{1}}{2}\left(u_{1}-r_{1}\right)^{2}+u_{1} r_{1}\right)} \\
f\left(t_{1}+\frac{d_{1}\left(u_{1}-r_{1}\right)}{2}, t_{2}+\frac{n_{2}}{2}\right) \\
\bar{g}\left(t_{1}-\frac{d_{1}\left(u_{1}-r_{1}\right)}{2}, t_{2}-\frac{n_{2}}{2}\right) \\
K_{A_{2}}^{j}\left(n_{2}, u_{2}\right), \\
b_{1}=0, b_{2} \neq 0 ; \\
\sqrt{d_{2}} K_{A_{1}}^{i}\left(n_{1}, u_{1}\right)  \tag{1}\\
f\left(t_{1}+\frac{n_{1}}{2}, t_{2}+\frac{d_{2}\left(u_{2}-r_{2}\right)}{2}\right) \\
\bar{g}\left(t_{1}-\frac{n_{1}}{2}, t_{2}-\frac{d_{2}\left(u_{2}-r_{2}\right)}{2}\right) \\
e^{j\left(\frac{c_{2} d_{2}}{2}\left(u_{2}-r_{2}\right)^{2}+u_{2} r_{2}\right)}, \\
b_{1} \neq 0, b_{2}=0 ; \\
\sqrt{d_{1} d_{2}} e^{i\left(\frac{c_{1} d_{1}}{2}\left(u_{1}-r_{1}\right)^{2}+u_{1} r_{1}\right)} \\
\left.\left.f\left(t_{1}+\frac{d_{1}\left(u_{1}-r_{1}\right.}{2}\right), t_{2}+\frac{d_{2}\left(u_{2}-r_{2}\right.}{2}\right)\right) \\
\left.\left.\bar{g}\left(t_{1}-\frac{d_{1}\left(u_{1}-r_{1}\right.}{2}\right), t_{2}-\frac{d_{2}\left(u_{2}-r_{2}\right.}{2}\right)\right) \\
e^{j\left(\frac{c_{2} d_{2}}{2}\left(u_{2}-r_{2}\right)^{2}+u_{2} r_{2}\right)}, \\
b_{1}=b_{2}=0 .
\end{array}\right.
$$

where $t=\left(t_{1}, t_{2}\right), u=\left(u_{1}, u_{2}\right), n=\left(n_{1}, n_{2}\right)$ and $K_{A_{1}}^{i}\left(n_{1}, u_{1}\right)$ and $K_{A_{2}}^{j}\left(n_{2}, u_{2}\right)$ are the quaternion kernels.

Note 1 If $f=g$ then $\mathcal{W}_{f, f}^{A_{1}, A_{2}}(t, u)$ we call it the Auto WVD-QOLCT.Otherwise is is called Cross WVD-QOLCT

Without loss of generality we will deal with the case $b_{i} \neq 0, i=1,2$, as in other cases proposed transform reduces to a chrip multiplications. Thus for any $f, g \in$ $L^{2}\left(\mathbf{R}^{2}, \mathbf{H}\right)$ we have

$$
\begin{align*}
\mathcal{W}_{f, g}^{A_{1}, A_{2}}(t, u) & =\int_{\mathbf{R}^{2}} K_{A_{1}}^{i}\left(n_{1}, u_{1}\right) f\left(t+\frac{n}{2}\right) \overline{g\left(t-\frac{n}{2}\right)} K_{A_{2}}^{j}\left(n_{2}, u_{2}\right) d n \\
& =\mathcal{O}_{A_{1}, A_{2}}^{i, j}\left\{f\left(t+\frac{n}{2}\right) \overline{g\left(t-\frac{n}{2}\right)}\right\} \\
& =\mathcal{O}_{A_{1}, A_{2}}^{i, j}\left\{h_{f, g}(t, n)\right\} \tag{2}
\end{align*}
$$

Where $h_{f, g}(t, n)=f\left(t+\frac{n}{2}\right) \overline{g\left(t-\frac{n}{2}\right)}$ is known as quaternion correlation product. Applying the inverse QOLCT
to (2), we get

$$
\left\{h_{f, g}(t, n)\right\}=\left\{\mathcal{O}_{A_{1}, A_{2}}^{i, j}\right\}^{-1}\left\{\mathcal{W}_{f, g}^{A_{1}, A_{2}}(t, u)\right\}
$$

which implies

$$
\begin{align*}
f\left(t+\frac{n}{2}\right) \overline{g\left(t-\frac{n}{2}\right)}= & \left\{\mathcal{O}_{A_{1}, A_{2}}^{i, j}\right\}^{-1}\left\{\mathcal{W}_{f, g}^{A_{1}, A_{2}}(t, u)\right\} \\
= & \int_{\mathbf{R}^{2}} K_{A_{1}}^{-i}\left(t_{1}, u_{1}\right) \mathcal{W}_{f, g}^{A_{1}, A_{2}}(t, u) \\
& K_{A_{2}}^{-j}\left(t_{2}, u_{2}\right) d w \tag{3}
\end{align*}
$$

Now, we discuss several basic properties of the WVDQOLCT given by (11). These properties play important roles in signal representation.

Theorem 1 (Boundedness) Let $f, g \in L^{2}\left(\mathbf{R}^{2}, \mathbf{H}\right)$. Then
$\left|\mathcal{W}_{f, g}^{A_{1}, A_{2}}(t, u)\right| \leq \frac{2}{\pi \sqrt{b_{1} b_{2}}}\|f\|_{L^{2}\left(R^{2}, H\right)}\|g\|_{L^{2}\left(R^{2}, H\right)}$
Proof By the virtue of Cauchy-Schwarz inequality in quaternion domain, we have

$$
\begin{aligned}
& \left|\mathcal{W}_{f, g}^{A_{1}, A_{2}}(t, u)\right|^{2} \\
& =\left|\int_{\mathbf{R}^{2}} K_{A_{1}}^{i}\left(n_{1}, u_{1}\right) f\left(t+\frac{n}{2}\right) \overline{g\left(t-\frac{n}{2}\right)} K_{A_{2}}^{j}\left(n_{2}, u_{2}\right) d n\right|^{2} \mathrm{~N} \\
& \leq\left(\int_{\mathbf{R}^{2}}\left|K_{A_{1}}^{\mathbf{i}}{ }_{\mathrm{N}}^{\mathrm{w}}\left(n_{1}, u_{1}\right) f\left(t+\frac{n}{2}\right) \overline{g\left(t-\frac{n}{2}\right)} K_{A_{2}}^{\mathbf{j}}\left(n_{2}, u_{2}\right)\right| \mathrm{dn} \mathbf{I}\right. \\
& =\left(\frac{1}{\sqrt{4 \pi^{2}\left|b_{1} b_{2}\right|}} \int_{\mathbf{R}^{2}}\left|f\left(t+\frac{n}{2}\right) \overline{g\left(t-\frac{n}{2}\right)}\right| \mathrm{dn}\right)^{2} \\
& \leq \frac{1}{4 \pi^{2}\left|b_{1} b_{2}\right|}\left(\int_{\mathbf{R}^{2}}\left|f\left(t+\frac{n}{2}\right)\right|^{2} \mathrm{dn}\right)\left(\int_{\mathbf{R}^{2}}\left|\overline{g\left(t-\frac{n}{2}\right)}\right|^{2} \mathrm{ds}\right) \\
& =\frac{1}{4 \pi^{2}\left|b_{1} b_{2}\right|}\left(4 \int_{\mathbf{R}^{2}}|f(w)|^{2} \mathrm{dw}\right)\left(4 \int_{\mathbf{R}^{2}}|\overline{g(y)}|^{2} \mathrm{dy}\right) \\
& =\frac{4}{\pi^{2}\left|b_{1} b_{2}\right|}\|f\|_{L^{2}\left(R^{2}, H\right)}^{2}\|g\|_{L^{2}\left(R^{2}, H\right)}^{2}
\end{aligned}
$$

where applying the change of variables $w=t+\frac{n}{2}$ and $y=t-\frac{n}{2}$ in the last second step. Then we have
$\left|\mathcal{W}_{f, g}^{A_{1}, A_{2}}(t, u)\right| \leq \frac{2}{\pi \sqrt{\left|b_{1} b_{2}\right|}}\|f\|_{L^{2}\left(R^{2}, H\right)}\|g\|_{L^{2}\left(R^{2}, H\right)}$
which completes the proof of Theorem.
Theorem 2 (Nonlinearity) Let $f$ and $g$ be two quaternion functions in $L^{2}\left(\mathbf{R}^{2}, \mathbf{H}\right)$. Then
$\mathcal{W}_{f+g}^{A_{1}, A_{2}}=\mathcal{W}_{f, f}^{A_{1}, A_{2}}+\mathcal{W}_{f, g}^{A_{1}, A_{2}}+\mathcal{W}_{g, f}^{A_{1}, A_{2}}+\mathcal{W}_{g, g}^{A_{1}, A_{2}}$

Proof By definition 1 we have

$$
\begin{aligned}
& \mathcal{W}_{f+g}^{A_{1}, A_{2}}(t, u) \\
& =\int_{\mathbf{R}^{2}} K_{A_{1}}^{i}\left(n_{1}, u_{1}\right)\left[f\left(t+\frac{n}{2}\right)+g\left(t+\frac{n}{2}\right)\right] \\
& {\left[f\left(t-\frac{n}{2}\right)+g\left(t-\frac{n}{2}\right)\right] K_{A_{2}}^{j}\left(n_{2}, u_{2}\right) d n} \\
& =\int_{\mathbf{R}^{2}} K_{A_{1}}^{i}\left(n_{1}, u_{1}\right)\left[f\left(t+\frac{n}{2}\right) \overline{f\left(t-\frac{n}{2}\right)}\right. \\
& +f\left(t+\frac{n}{2}\right) \overline{g\left(t-\frac{n}{2}\right)} \\
& \left.+g\left(t+\frac{n}{2}\right) \overline{f\left(t-\frac{n}{2}\right)}+g\left(t+\frac{n}{2}\right) \overline{g\left(t-\frac{n}{2}\right)}\right] K_{A_{2}}^{j}\left(n_{2}, u_{2}\right) d n \\
& =\int_{\mathbf{R}^{2}} K_{A_{1}}^{i}\left(n_{1}, u_{1}\right) f\left(t+\frac{n}{2}\right) \overline{f\left(t-\frac{n}{2}\right)} K_{A_{2}}^{j}\left(n_{2}, u_{2}\right) d n \\
& +\int_{\mathbf{R}^{2}} K_{A_{1}}^{i}\left(n_{1}, u_{1}\right) f\left(t+\frac{n}{2}\right) \overline{g\left(t-\frac{n}{2}\right)} K_{A_{2}}^{j}\left(n_{2}, u_{2}\right) d n \\
& +\int_{\mathbf{R}^{2}} K_{A_{1}}^{i}\left(n_{1}, u_{1}\right) g\left(t+\frac{n}{2}\right) \overline{f\left(t-\frac{n}{2}\right)} K_{A_{2}}^{j}\left(n_{2}, u_{2}\right) d n \\
& +\int_{\mathbf{R}^{2}} K_{A_{1}}^{i}\left(n_{1}, u_{1}\right) g\left(t+\frac{n}{2}\right) \overline{g\left(t-\frac{n}{2}\right)} K_{A_{2}}^{j}\left(n_{2}, u_{2}\right) d n \\
& =\mathcal{W}_{f, f}^{A_{1}, A_{2}}+\mathcal{W}_{f, g}^{A_{1}, A_{2}}+\mathcal{W}_{g, f}^{A_{1}, A_{2}}+\mathcal{W}_{g, g}^{A_{1}, A_{2}}
\end{aligned}
$$

which completes the proof of Theorem.

Note the properties like Shift,Modulation,Dilation are
similar to the classical QOLCT so we avoided them. 2
Theorem 3 (Reconstruction formula). For $f, g \in$
$L^{2}\left(\mathbf{R}^{2}, \mathbf{H}\right)$ where $g$ does not vanish at 0 . We get the following inversion formula of the WVD-QOLCT:

$$
\begin{align*}
f(t)= & \frac{1}{\overline{g(0)}} \int_{\mathbf{R}^{2}} K_{A_{1}}^{-i}\left(u_{1}, n_{1}\right) \mathcal{W}_{f, g}^{A_{1}, A_{2}}\left(\frac{t}{2}, u\right)  \tag{6}\\
& K_{A_{2}}^{-j}\left(u_{2}, n_{2}\right) d u
\end{align*}
$$

Proof By (3), we have

$$
\left\{h_{f, g}(t, n)\right\}=\left\{\mathcal{O}_{A_{1}, A_{2}}^{i, j}\right\}^{-1}\left\{\mathcal{W}_{f, g}^{A_{1}, A_{2}}(t, u)\right\}
$$

which implies

$$
\begin{aligned}
& f\left(t+\frac{n}{2}\right) \overline{g\left(t-\frac{n}{2}\right)} \\
& =\int_{\mathbf{R}^{2}} K_{A_{1}}^{-i}\left(t_{1}, u_{1}\right) \mathcal{W}_{f, g}^{A_{1}, A_{2}}(t, u) K_{A_{2}}^{-j}\left(t_{2}, u_{2}\right) d w
\end{aligned}
$$

Now let $t=\frac{n}{2}$ and taking change of variable $w=2 t$, we get
$f(w)=\frac{1}{\overline{g(0)}} \int_{\mathbf{R}^{2}} K_{A_{1}}^{-i}\left(u_{1}, n_{1}\right) \mathcal{W}_{f, g}^{A_{1}, A_{2}}\left(\frac{w}{2}, u\right) K_{A_{2}}^{-j}\left(u_{2}, n_{2}\right) d u$
which completes the proof of Theorem.

Theorem 4 (Orthogonality relation). If $f_{1}, f_{2}, g_{1}, g_{2} \in$ which completes the proof theorem.
$L^{2}\left(\mathbf{R}^{2}, \mathbf{H}\right)$ are quaternion-valued signals. Then $\left\langle\mathcal{W}_{f_{1}, g_{1}}^{A_{1}, A_{2}}(t, u), \mathcal{W}_{f_{2}, g_{2}}^{A_{1}, A_{2}}(t, u)\right\rangle=\left[\left\langle f_{1}, f_{2}\right\rangle\left\langle g_{2}, g_{1}\right\rangle\right]_{\mathbf{H}}$
Proof By the definition of Winger-ville distribution associated with quaternion OLCT and innear product relation we have

$$
\begin{align*}
& \left\langle W_{f_{1}, g_{1}}^{A_{1}, A_{2}}(t, u), \mathcal{W}_{f_{2}, g_{2}}^{A_{1}, A_{2}}(t, u)\right\rangle \\
& =\int_{\mathbf{R}^{4}}\left[\mathcal{W}_{f_{1}, g_{1}}^{A_{1}, A_{2}}(t, u) \overline{\mathcal{W}_{f_{2}, g_{2}}^{A_{1}, A_{2}}(t, u)}\right]_{\mathbf{H}} d u d t \\
& =\int_{\mathbf{R}^{4}}\left[\mathcal{W}_{f_{1}, g_{1}}^{A_{1}, A_{2}}(t, u)\right. \\
& \overline{\int_{\mathbf{R}^{2}} K_{A_{1}}^{i}\left(n_{1}, u_{1}\right) f_{2}\left(t+\frac{n}{2}\right) \overline{g_{2}\left(t-\frac{n}{2}\right)}} \\
& \left.\overline{K_{A_{2}}^{j}\left(n_{2}, u_{2}\right) d n}\right]_{\mathbf{H}} d u d t \\
& =\int_{\mathbf{R}^{6}}\left[\mathcal{W}_{f_{1}, g_{1}}^{A_{1}, A_{2}}(t, u) K_{A_{2}}^{-j}\left(n_{2}, u_{2}\right) g_{2}\left(t-\frac{n}{2}\right)\right. \\
& \left.\overline{f_{2}\left(t+\frac{n}{2}\right)} K_{A_{1}}^{-i}\left(n_{1}, u_{1}\right)\right]_{\mathbf{H}} d u d t d n \\
& =\int_{\mathbf{R}^{6}}\left[K_{A_{1}}^{-i}\left(n_{1}, u_{1}\right) \mathcal{W}_{f_{1} g_{1}}^{A_{1}, A_{2}}(t, u) K_{A_{2}}^{-j}\left(n_{2}, u_{2}\right)\right. \\
& \left.g_{2}\left(t-\frac{n}{2}\right) \overline{f_{2}\left(t+\frac{n}{2}\right)}\right]_{\mathbf{H}} d u d t d n \\
& =\int_{\mathbf{R}^{4}}\left[\int_{\mathbf{R}^{2}} K_{A_{1}}^{-i}\left(n_{1}, u_{1}\right) \mathcal{W}_{f_{1}, g_{1}}^{A_{1}, A_{2}}(t, u) K_{A_{2}}^{-j}\left(n_{2}, u_{2}\right) d u\right. \\
& \left.g_{2}\left(t-\frac{n}{2}\right) \overline{f_{2}\left(t+\frac{n}{2}\right)}\right]_{\mathbf{H}} d t d n \tag{8}
\end{align*}
$$

Because
$\overline{K_{A_{1}}^{i}\left(n_{1}, u_{1}\right)}=K_{A_{1}}^{-i}\left(u_{1}, n_{1}\right)=K_{A_{1}^{-1}}^{i}\left(u_{1}, n_{1}\right)$
$\overline{K_{A_{2}}^{j}\left(n_{2}, u_{2}\right)}=K_{A_{2}}^{-j}\left(n_{2}, u_{2}\right)=K_{A_{2}^{-1}}^{j}\left(u_{2}, n_{2}\right)$
Now by using (3) in (8), we have

$$
\begin{aligned}
& \left\langle\mathcal{W}_{f_{1}, g_{1}}^{A_{1}, A_{2}}(t, u), \mathcal{W}_{f_{2}, g_{2}}^{A_{1}, A_{2}}(t, u)\right\rangle \\
& =\int_{\mathbf{R}^{4}}\left[\int_{\mathbf{R}^{2}} K_{A_{1}^{-1}}^{i}\left(u_{1}, n_{1}\right) \mathcal{W}_{f_{1}, g_{1}}^{A_{1}, A_{2}}(t, u) K_{A_{2}^{-1}}^{j}\left(u_{2}, n_{2}\right) d u\right. \\
& \left.g_{2}\left(t-\frac{n}{2}\right) \overline{f_{2}\left(t+\frac{n}{2}\right)}\right]_{\mathbf{H}} d t d n \\
& =\int_{\mathbf{R}^{4}}\left[f_{1}\left(t+\frac{n}{2}\right) \overline{g_{1}\left(t-\frac{n}{2}\right)} g_{2}\left(t-\frac{n}{2}\right) \overline{f_{2}\left(t+\frac{n}{2}\right)}\right]_{\mathbf{H}}
\end{aligned}
$$

Using the change of variables $t+\frac{n}{2}=\omega$, and $t-\frac{n}{2}=\xi$ the equation becomes

$$
\begin{aligned}
& \left\langle\mathcal{W}_{f_{1}, g_{1}}^{A_{1}, A_{2}}(t, u), \mathcal{W}_{f_{2}, g_{2}}^{A_{1}, A_{2}}(t, u)\right\rangle \\
& =\int_{\mathbf{R}^{4}}\left[f_{1}(\omega) \overline{g_{1}(\xi)} g_{2}(\xi) \overline{f_{2}(\omega)}\right]_{\mathbf{H}} d \omega d \xi \\
& =\left[\int_{\mathbf{R}^{2}} f_{1}(\omega) \overline{f_{2}(\omega)} d \omega \int_{\mathbf{R}^{2}} g_{2}(\xi) \overline{g_{1}(\xi)} d \xi\right]_{\mathbf{H}} \\
& =\left[\left\langle f_{1}, f_{2}\right\rangle\left\langle g_{2}, g_{1}\right\rangle\right]_{\mathbf{H}}
\end{aligned}
$$

## Consequences of Theorem 4.

1. If $g_{1}=g_{2}=g$, then

$$
\left\langle\mathcal{W}_{f_{1}, g}^{A_{1}, A_{2}}(t, u), \mathcal{W}_{f_{2}, g}^{A_{1}, A_{2}}(w, u)\right\rangle=\|g\|_{L^{2}\left(\mathbf{R}^{2}\right)}^{2}\left\langle f_{1}, f_{2}\right\rangle(9)
$$

2. If $f_{1}=f_{2}=f$, then

$$
\begin{equation*}
\left\langle\mathcal{W}_{f, g_{1}}^{A_{1}, A_{2}}(t, u), \mathcal{W}_{f, g_{2}}^{A_{1}, A_{2}}(w, u)\right\rangle=\|f\|_{L^{2}\left(\mathbf{R}^{2}\right)}^{2}\left\langle g_{1}, g_{2}\right\rangle .( \tag{10}
\end{equation*}
$$

3. If $f_{1}=f_{2}=f$ and $g_{1}=g_{2}=g$, then

$$
\begin{align*}
& \left\langle\mathcal{W}_{f, g}^{A_{1}, A_{2}}(t, u), \mathcal{W}_{f, g}^{A_{1}, A_{2}}(w, u)\right\rangle \\
& =\int_{\mathbf{R}^{2}} \int_{\mathbf{R}^{2}}\left|\mathcal{W}_{f, g}^{A_{1}, A_{2}}(t, u)\right|^{2} d u d t \\
& =\|f\|_{L^{2}\left(\mathbf{R}^{2}\right)}^{2}\|g\|_{L^{2}\left(\mathbf{R}^{2}\right)}^{2} \tag{11}
\end{align*}
$$

## Theorem 5 (Plancherel's theorem for

WVD-QOLCT).For $f, g \in L^{2}\left(\mathbf{R}^{2}, \mathbf{H}\right)$, we have the equality

$$
\begin{align*}
& \int_{\mathbf{R}^{2}} \int_{\mathbf{R}^{2}}\left|\mathcal{W}_{f, g}^{A_{1}, A_{2}}(t, u)\right|^{2} d u d t \\
& =\left\|\mathcal{W}_{f, g}^{A_{1}, A_{2}}\right\|_{L^{2}\left(\mathbf{R}^{2}, \mathbf{H}\right)}^{2} \\
& =\|f\|_{L^{2}\left(\mathbf{R}^{2}, \mathbf{H}\right)}^{2} \|\left. g\right|_{L^{2}\left(\mathbf{R}^{2}, \mathbf{H}\right)} ^{2} \tag{12}
\end{align*}
$$

Proof If we look at (11), the proof of the theorem follows.

Now we move forward towards our main section that is convolution and correlation theorems for wingerville distribution associated with quaternion offset linear canonical transform.

## 3 Convolution and Correlation theorem for WVD-QOLCT

The convolution and correlation are fundamental signal processing algorithms in the theory of linear timeinvariant(LTI) systems. In engineering, they have been widely used for various template matchings. In the following we first define the convolution and correlation for the QOLCT. They are extensions of the convoludtion definition from the OLCT (see [16]) to the QOLCT domain. We then establish the new convolution and correlation for the WVD-QOLCT. We also show that the convolution theorems of the QWVD and WVD-QLCT can be looked as a special case of our achieved results.

Definition 2 For any two quaternion functions $f, g \in$ $L^{2}\left(\mathbf{R}^{2}, \mathbf{H}\right)$, we define the convolution operator of the QOLCT as
$(f \star g)(t)=\int_{\mathbf{R}^{2}} \Psi\left(z_{1}, t_{1}\right) f(z) g(t-z) \Psi\left(z_{2}, t_{2}\right) d z$

Where $\Psi\left(z_{1}, t_{1}\right)$ and $\Psi\left(z_{2}, t_{2}\right)$ are known as weight functions.
We assume
$\Psi\left(z_{1}, t_{1}\right)=e^{-i \frac{a_{1}}{b_{1}} 2 z_{1}\left(t_{1}-z_{1}\right)}$
and
$\Psi\left(z_{2}, t_{2}\right)=e^{-j \frac{a_{2}}{b_{2}} 2 z_{2}\left(t_{2}-z_{2}\right)}$
As a consequence of the above definition, we get the following important theorem.

Theorem 6 (WVD-QOLCT Convolution). For any two quaternion functions $f, g \in L^{2}\left(\mathbf{R}^{2}, \mathbf{H}\right)$, the following result holds

$$
\begin{align*}
& \mathcal{W}_{f \star g}^{A_{1}, A_{2}}(t, u) \\
& =\sqrt{2 \pi b_{1} i} e^{\frac{-i}{2 b_{1}}\left[d_{1}\left(u_{1}^{2}+r_{1}^{2}\right)-2 u_{1}\left(d_{1} r_{1}-b_{1} s_{1}\right)\right]} \\
& \times\left\{\int_{\mathbf{R}^{2}} e^{-i \frac{a_{1}}{b_{1}}\left(4 w_{1}\left(t_{1}-w_{1}\right)\right)} \mathcal{W}_{f, f}^{A_{1}, A_{2}}(w, u) \mathcal{W}_{g, g}^{A_{1}, A_{2}}(t-w, u)\right. \\
& \left.e^{-j \frac{a_{2}}{b_{2}}\left(4 w_{2}\left(t_{2}-w_{2}\right)\right)} d w\right\} \\
& \times \sqrt{2 \pi b_{2} j} e^{\frac{-j}{2 b_{2}}\left[d_{2}\left(u_{2}^{2}+r_{2}^{2}\right)-2 u_{2}\left(d_{2} r_{2}-b_{2} s_{2}\right)\right]} \tag{15}
\end{align*}
$$

Proof Applying the definition of the WVD-QOLCT we have

$$
\begin{align*}
\mathcal{W}_{f \star g}^{A_{1}, A_{2}}(t, u)= & \int_{\mathbf{R}^{2}} K_{A_{1}}^{i}\left(n_{1}, u_{1}\right)\left[(f \star g)\left(t+\frac{n}{2}\right)\right] \\
& {\left[\bar{f} \star \bar{g}\left(t-\frac{n}{2}\right)\right] K_{A_{2}}^{j}\left(n_{2}, u_{2}\right) d n } \tag{16}
\end{align*}
$$

Now using Definition 2 in (16) we have

$$
\begin{align*}
& \mathcal{W}_{f \star g}^{A_{1}, A_{2}}(t, u) \\
& =\int_{\mathbf{R}^{2}} K_{A_{1}}^{i}\left(n_{1}, u_{1}\right)\left\{\int_{\mathbf{R}^{2}} \Psi_{1}\left(z_{1}, t_{1}+\frac{n_{1}}{2}\right) f(z)\right. \\
& \quad g\left(t+\frac{n}{2}-z\right) \Psi_{2}\left(z_{2}, t_{2}+\frac{n_{2}}{2}\right) d z \\
& \quad \times \int_{\mathbf{R}^{2}} \Psi_{1}\left(\gamma_{1}, t_{1}-\frac{n_{1}}{2}\right) \overline{f(\gamma) g\left(t-\frac{n}{2}-\gamma\right)} \\
& \left.\quad \Psi_{2}\left(\gamma_{2}, t_{2}-\frac{n_{2}}{2}\right) d \gamma\right\} K_{A_{2}}^{j}\left(n_{2}, u_{2}\right) d n \\
& =\int_{\mathbf{R}^{2}} K_{A_{1}}^{i}\left(n_{1}, u_{1}\right)\left\{\int_{\mathbf{R}^{2}} e^{-i \frac{a_{1}}{b_{1}} 2 z_{1}\left(\left(t_{1}+\frac{n_{1}}{2}\right)-z_{1}\right)} f(z)\right. \\
& \quad g\left(t+\frac{n}{2}-z\right) e^{-j \frac{a_{2}}{b_{2}} 2 z_{2}\left(\left(t_{2}+\frac{n_{2}}{2}\right)-z_{2}\right)} d z \\
& \quad \times \int_{\mathbf{R}^{2}} e^{-i \frac{a_{1}}{b_{1}} 2 \gamma_{1}\left(\left(t_{1}-\frac{n_{1}}{2}\right)-\gamma_{1}\right)} \frac{f(\gamma) g\left(t-\frac{n}{2}-\gamma\right)}{\left.e^{-j \frac{a_{2}}{b_{2}} 2 \gamma_{2}\left(\left(t_{2}-\frac{n_{2}}{2}\right)-\gamma_{2}\right)} d \gamma\right\} K_{A_{2}}^{j}\left(n_{2}, u_{2}\right) d n}
\end{align*}
$$

For simplicity let us denote

$$
\begin{gather*}
K_{A_{1}}^{i}\left(t_{1}, u_{1}\right)=K_{A_{1}}^{i} e^{\frac{i}{2 b_{1}}\left[a_{1} t_{1}^{2}+2 t_{1}\left(r_{1}-u_{1}\right)-2 u_{1}\left(d_{1} r_{1}-b_{1} s_{1}\right)+d_{1} u_{1}^{2}\right]}, \\
K_{A_{1}}^{i}=\frac{1}{\sqrt{2 \pi b_{1} i}} e^{i \frac{d_{1}}{2 b_{1}} r_{1}^{2}} \tag{18}
\end{gather*}
$$

and

$$
\begin{gather*}
K_{A_{2}}^{j}\left(t_{2}, u_{2}\right)=K_{A_{2}}^{j} e^{\frac{j}{2 b_{2}}\left[a_{2} t_{2}^{2}+2 t_{2}\left(r_{2}-u_{2}\right)-2 u_{2}\left(d_{2} r_{2}-b_{2} s_{2}\right)+d_{2} u_{2}^{2}\right]} \\
K_{A_{2}}^{j}=\frac{1}{\sqrt{2 \pi b_{2} j}} e^{j \frac{d_{2}}{2 b_{2}} r_{2}^{2}} \tag{19}
\end{gather*}
$$

Now using (18) and (19) in (17), we have

$$
\begin{aligned}
& \mathcal{W}_{f \star g}^{A_{1}, A_{2}}(t, u) \\
& =\int_{\mathbf{R}^{6}} K_{A_{1}}^{i} e^{\frac{i}{2 b_{1}}\left[a_{1} n_{1}^{2}+2 n_{1}\left(r_{1}-u_{1}\right)-2 u_{1}\left(d_{1} r_{1}-b_{1} s_{1}\right)+d_{1} u_{1}^{2}\right]} \\
& e^{-i \frac{a_{1}}{b_{1}} 2 z_{1}\left(\left(t_{1}+\frac{n_{1}}{2}\right)-z_{1}\right)} \\
& \times f(z) g\left(t+\frac{n}{2}-z\right) e^{-j \frac{a_{2}}{b_{2}} 2 z_{2}\left(\left(t_{2}+\frac{n_{2}}{2}\right)-z_{2}\right)} \\
& e^{-i \frac{a_{1}}{b_{1}} 2 \gamma_{1}\left(\left(t_{1}-\frac{n_{1}}{2}\right)-\gamma_{1}\right)} \\
& \times f(\gamma) g\left(t-\frac{n}{2}-\gamma\right) e^{-j \frac{a_{2}}{b_{2}} 2 \gamma_{2}\left(\left(t_{2}-\frac{n_{2}}{2}\right)-\gamma_{2}\right)} \\
& \times K_{A_{2}}^{j} e^{\frac{j}{2 b_{2}}\left[a_{2} n_{2}^{2}+2 n_{2}\left(r_{2}-u_{2}\right)-2 u_{2}\left(d_{2} r_{2}-b_{2} s_{2}\right)+d_{2} u_{2}^{2}\right]} d z d \gamma d n
\end{aligned}
$$

Setting $z_{i}=w_{i}+\frac{p_{i}}{2}, \gamma_{i}=w_{i}-\frac{p_{i}}{2}, i=1,2$ we get

$$
\begin{aligned}
& \mathcal{W}_{f \star g}^{A_{1}, A_{2}}(t, u) \\
& =\int_{\mathbf{R}^{6}} K_{A_{1}}^{i} e^{\frac{i}{2 b_{1}}\left[a_{1} n_{1}^{2}+2 n_{1}\left(r_{1}-u_{1}\right)-2 u_{1}\left(d_{1} r_{1}-b_{1} s_{1}\right)+d_{1} u_{1}^{2}\right]} \\
& \quad e^{-i \frac{a_{1}}{b_{1}} 2\left(w_{1}+\frac{p_{1}}{2}\right)\left(\left(t_{1}+\frac{n_{1}}{2}\right)-\left(w_{1}+\frac{p_{1}}{2}\right)\right)} \\
& \quad \times f\left(w+\frac{p}{2}\right) g\left(t+\frac{n}{2}-\left(w+\frac{p}{2}\right)\right) \\
& \quad e^{-j \frac{a_{2}}{b_{2}} 2\left(w_{2}+\frac{p_{2}}{2}\right)\left(\left(t_{2}+\frac{n_{2}}{2}\right)-\left(w_{2}+\frac{p_{2}}{2}\right)\right)} \\
& \quad \times e^{-i \frac{a_{1}}{b_{1}} 2\left(w_{1}-\frac{p_{1}}{2}\right)\left(\left(t_{1}-\frac{n_{1}}{2}\right)-\left(w_{1}-\frac{p_{1}}{2}\right)\right)} \\
& \quad f\left(w-\frac{p}{2}\right) g\left(t-\frac{n}{2}-\left(w-\frac{p}{2}\right)\right) \\
& \quad \times e^{-j \frac{a_{2}}{b_{2}} 2\left(w_{2}-\frac{p_{2}}{2}\right)\left(\left(t_{2}-\frac{n_{2}}{2}\right)-\left(w_{2}-\frac{p_{2}}{2}\right)\right)} \\
& K_{A_{2}}^{j} e^{\frac{j}{2 b_{2}}\left[a_{2} n_{2}^{2}+2 n_{2}\left(r_{2}-u_{2}\right)-2 u_{2}\left(d_{2} r_{2}-b_{2} s_{2}\right)+d_{2} u_{2}^{2}\right]} d p d q d w
\end{aligned}
$$

and $n_{i}=p_{i}+q_{i}, i=1,2$ we obtain

$$
\begin{aligned}
& \mathcal{W}_{f \star g}^{A_{1}, A_{2}}(t, u) \\
& =\int_{\mathbf{R}^{6}} \\
& K_{A_{1}}^{i} e^{\frac{i}{2 b_{1}}\left[a_{1}\left(p_{1}+q_{1}\right)^{2}+2\left(p_{1}+q_{1}\right)\left(r_{1}-u_{1}\right)-2 u_{1}\left(d_{1} r_{1}-b_{1} s_{1}\right)+d_{1} u_{1}^{2}\right]} \\
& e^{-i \frac{a_{1}}{b_{1}}\left(4 w_{1}\left(t_{1}-w_{1}\right)\right)} e^{-i \frac{a_{1}}{b_{1}} p_{1} q_{1}} \\
& \times f\left(w+\frac{p}{2}\right) \overline{f\left(w-\frac{p}{2}\right)} g\left(t-w+\frac{q}{2}\right) \overline{g\left(t-w-\frac{q}{2}\right)} \\
& e^{-j \frac{a_{2}}{b_{2}}\left(4 w_{2}\left(t_{2}-w_{2}\right)\right)} e^{-j \frac{a_{2}}{b_{2}} p_{2} q_{2}} \\
& \times K_{A_{2}}^{j} e^{\frac{j}{2 b_{2}}\left[a_{2} n_{2}^{2}+2 n_{2}\left(r_{2}-u_{2}\right)-2 u_{2}\left(d_{2} r_{2}-b_{2} s_{2}\right)+d_{2} u_{2}^{2}\right]} d p d q d w \\
& =\int_{\mathbf{R}^{2}}\left\{\left[\int_{\mathbf{R}^{2}}\right.\right. \\
& K_{A_{1}}^{i} e^{\frac{i}{2 b_{1}}\left[a_{1} p_{1}^{2}+2 p_{1}\left(r_{1}-u_{1}\right)-2 u_{1}\left(d_{1} r_{1}-b_{1} s_{1}\right)+d_{1} u_{1}^{2}\right]} \\
& f\left(w+\frac{p}{2}\right) \overline{f\left(w-\frac{p}{2}\right)} \\
& \left.\times K_{A_{2}}^{j} e^{\frac{j}{22_{2}}\left[a_{2} p_{2}^{2}+2 p_{2}\left(r_{2}-u_{2}\right)-2 u_{2}\left(d_{2} r_{2}-b_{2} s_{2}\right)+d_{2} u_{2}^{2}\right]} d p\right]
\end{aligned}
$$

$$
\begin{align*}
& \times \int_{\mathbf{R}^{2}} e^{\frac{i}{2 b_{1}}\left[a_{1} q_{1}^{2}-2 q_{1}\left(r_{1}-u_{1}\right)\right]} g\left(t-w+\frac{q}{2}\right) \\
& \times \overline{g\left(t-w-\frac{q}{2}\right)} e^{\frac{j}{2 b_{2}}\left[a_{2} q_{2}^{2}-2 q_{2}\left(r_{2}-u_{2}\right)\right]} d q \\
& \times e^{-i \frac{a_{1}}{b_{1}}\left(4 w_{1}\left(t_{1}-w_{1}\right)\right)} e^{-j \frac{a_{2}}{b_{2}}\left(4 w_{2}\left(t_{2}-w_{2}\right)\right)} d w \tag{20}
\end{align*}
$$

Now multiply (20) both sides by $K_{A_{1}}^{i} \frac{i}{2 b_{1}}\left[d_{1} u_{1}^{2}-2 u_{1}\left(d_{1} r_{1}-b_{1} s_{1}\right)\right]$ Definition 3 For any two quaternion functions $f, g \in$ and $K_{A_{2}}^{j} e^{\frac{j}{2 b_{2}}\left[d_{2} u_{2}^{2}-2 u_{2}\left(d_{2} r_{2}-b_{2} s_{2}\right)\right]}$, we get

$$
\begin{align*}
& K_{A_{1}}^{i} e^{\frac{i}{2 b_{1}}\left[d_{1} u_{1}^{2}-2 u_{1}\left(d_{1} r_{1}-b_{1} s_{1}\right)\right]} \\
& K_{A_{2}}^{j} e^{\frac{j}{2 b_{2}}\left[d_{2} u_{2}^{2}-2 u_{2}\left(d_{2} r_{2}-b_{2} s_{2}\right)\right]} \mathcal{W}_{f \star g}^{A_{1}, A_{2}}(t, u) \\
& =\int_{\mathbf{R}^{2}} e^{-i \frac{a_{1}}{b_{1}}\left(4 w_{1}\left(t_{1}-w_{1}\right)\right)} \mathcal{W}_{f, f}^{A_{1}, A_{2}}(w, u) \\
& \mathcal{W}_{g, g}^{A_{1}, A_{2}}(t-w, u) e^{-j \frac{a_{2}}{b_{2}}\left(4 w_{2}\left(t_{2}-w_{2}\right)\right)} d w \tag{21}
\end{align*}
$$

Now using (18) and (19) in (21) we get,

$$
\begin{aligned}
& \mathcal{W}_{f \star g}^{A_{1}, A_{2}}(t, u) \\
& =\sqrt{2 \pi b_{1} i} e^{\frac{-i}{2 b_{1}}\left[d_{1}\left(u_{1}^{2}+r_{1}^{2}\right)-2 u_{1}\left(d_{1} r_{1}-b_{1} s_{1}\right)\right]} \\
& \times\left\{\int_{\mathbf{R}^{2}} e^{-i \frac{a_{1}}{b_{1}}\left(4 w_{1}\left(t_{1}-w_{1}\right)\right)} \mathcal{W}_{f, f}^{A_{1}, A_{2}}(w, u) \mathcal{W}_{g, g}^{A_{1}, A_{2}}(t-w, u)\right. \\
& \left.e^{-j \frac{a_{2}}{b_{2}}\left(4 w_{2}\left(t_{2}-w_{2}\right)\right)} d w\right\} \\
& \times \sqrt{2 \pi b_{2} j} e^{\frac{-j}{2 b_{2}}\left[d_{2}\left(u_{2}^{2}-r_{2}^{2}\right)-2 u_{2}\left(d_{2} r_{2}-b_{2} s_{2}\right)\right]}
\end{aligned}
$$

which completes the proof of theorem.

## Consequences of theorem 6.

1. Changing parameter $A_{i}=\left[\begin{array}{ll|l}a_{i} & b_{i} \mid & r_{i} \\ c_{i} & d_{i} \mid & s_{i}\end{array}\right], i=1,2$ to $A_{i}=\left[\begin{array}{ll|l}a_{i} & b_{i} & 0 \\ c_{i} & d_{i} & 0\end{array}\right], i=1,2$, then the Theorem 4 reduces to convolution theorem of the WVD-QLCT as follows:

$$
\begin{aligned}
& \mathcal{W}_{f \star g}^{A_{1}, A_{2}}(t, u) \\
& =\sqrt{2 \pi b_{1} i} e^{\frac{-i}{b_{1}} d_{1} u_{1}^{2}} \sqrt{2 \pi b_{2} j} e^{\frac{-j}{2 b_{2}} d_{2} u_{2}^{2}} \\
& \times\left\{\int_{\mathbf{R}^{2}} e^{-i \frac{a_{1}}{b_{1}}\left(4 w_{1}\left(t_{1}-w_{1}\right)\right)} \mathcal{W}_{f, f}^{A_{1}, A_{2}}(w, u) \mathcal{W}_{g, g}^{A_{1}, A_{2}}(t-w, u)\right. \\
& \left.e^{-j \frac{a_{2}}{b_{2}}\left(4 w_{2}\left(t_{2}-w_{2}\right)\right)} d w\right\}
\end{aligned}
$$

where $\mathcal{W}_{f, f}^{A_{1}, A_{2}}$ and $\mathcal{W}_{g, g}^{A_{1}, A_{2}}$ is the WVD in the QLCT domain of a signal $f$ and $g$, respectively.
2. Changing parameter $A_{i}=\left[\begin{array}{ll|l}a_{i} & b_{i} \mid & r_{i} \\ c_{i} & d_{i} \mid & s_{i}\end{array}\right], i=1,2$ to $A_{i}=\left[\begin{array}{cc|c}0 & 1 & 0 \\ -1 & 0 & 0\end{array}\right], i=1,2$, then the Theorem 4 reduces to convolution theorem of the WVD in Quaternion Domain as follows:
$\mathcal{W}_{f \star g}(t, u)=\sqrt{2 \pi i}\left\{\int_{\mathbf{R}^{2}} \mathcal{W}_{f, f}^{i, j}(w, u) \mathcal{W}_{g, g}^{i, j}(t-w, u) d w\right\}$

$$
\begin{equation*}
\sqrt{2 \pi j} \tag{23}
\end{equation*}
$$

where $\mathcal{W}_{f, f}^{i, j}$ and $\mathcal{W}_{g, g}^{i, j}$ is the WVD in the Quaternion domain of a signal $f$ and $g$, respectively.
Next, we will derive the correlation theorem in the WVDQOLCT. Let us define the correlation for the QOLCT. $L^{2}\left(\mathbf{R}^{2}, \mathbf{H}\right)$, we define the correlation operator of the QOLCT as
$(f \circ g)(t)=\int_{\mathbf{R}^{2}} e^{i \frac{a_{1}}{b_{1}} 2 z_{1}\left(z_{1}+t_{1}\right)} \overline{f(z)} g(z+t) e^{j \frac{a_{2}}{b_{2}} 2 z_{2}\left(z_{2}+t_{2}\right)} d z$
Now, we reap a consequence of the above definition .
Theorem 7 (WVD-QOLCT Correlation). For any two quaternion functions $f, g \in L^{2}\left(\mathbf{R}^{2}, \mathbf{H}\right)$, the following result holds

$$
\begin{align*}
& \mathcal{W}_{f \circ g}^{A_{1}, A_{2}}(t, u) \\
& =\sqrt{2 \pi b_{1} i} e^{\frac{-i}{2 b_{1}}\left[d_{1}\left(u_{1}^{2}+r_{1}^{2}\right)+2 u_{1}\left(d_{1} r_{1}-b_{1} s_{1}\right)\right]} \\
& \times\left\{\int_{\mathbf{R}^{2}} e^{i \frac{a_{1}}{b_{1}}\left(4 w_{1}\left(t_{1}+w_{1}\right)\right)} \mathcal{W}_{f, f}^{A_{1}, A_{2}}(w,-u)\right. \\
& \left.\mathcal{W}_{g, g}^{A_{1}, A_{2}}(t+w, u) e^{j \frac{a_{2}}{b_{2}}\left(4 w_{2}\left(t_{2}+w_{2}\right)\right)} d w\right\} \\
& \times \sqrt{2 \pi b_{2} j} e^{\frac{-j}{2 b_{2}}\left[d_{2}\left(u_{2}^{2}+r_{2}^{2}\right)+2 u_{2}\left(d_{2} r_{2}-b_{2} s_{2}\right)\right]} \tag{25}
\end{align*}
$$

Proof Applying the definition of the WVD-QOLCT we have

$$
\begin{align*}
& \mathcal{W}_{f \circ g}^{A_{1}, A_{2}}(t, u) \\
& =\int_{\mathbf{R}^{2}} K_{A_{1}}^{i}\left(n_{1}, u_{1}\right)\left[(f \circ g)\left(t+\frac{n}{2}\right)\right] \\
& {\left[\bar{f} \circ \bar{g}\left(t-\frac{n}{2}\right)\right] K_{A_{2}}^{j}\left(n_{2}, u_{2}\right) d n} \tag{26}
\end{align*}
$$

Now using definition 3 in (26) we have

$$
\begin{align*}
& \mathcal{W}_{f \circ g}^{A_{1}, A_{2}}(t, u) \\
& =\int_{\mathbf{R}^{2}} K_{A_{1}}^{i}\left(n_{1}, u_{1}\right)\left\{\int_{\mathbf{R}^{2}} e^{i \frac{a_{1}}{b_{1}} 2 z_{1}\left(z_{1}+\left(t_{1}+\frac{n_{1}}{2}\right)\right)} \overline{f(z)}\right. \\
& g\left(z+t+\frac{n}{2}\right) e^{j \frac{a_{2}}{b_{2}} 2 z_{2}\left(z_{2}+\left(t_{2}+\frac{n_{2}}{2}\right)\right)} d z \\
& \times \int_{\mathbf{R}^{2}} e^{i \frac{a_{1}}{b_{1}} 2 \gamma_{1}\left(\gamma_{1}+\left(t_{1}-\frac{n_{1}}{2}\right)\right)} \overline{\overline{f(\gamma)}} g\left(\gamma+\left(t-\frac{n}{2}\right)\right)  \tag{27}\\
& \left.2) e^{j \frac{a_{2}}{b_{2}} 2 \gamma_{2}\left(\gamma_{2}+\left(t_{2}-\frac{n_{2}}{2}\right)\right)} d \gamma\right\} K_{A_{2}}^{j}\left(n_{2}, u_{2}\right) d n \tag{22}
\end{align*}
$$

Now with the help of (18) and (19), we have from (27)

$$
\begin{aligned}
& \mathcal{W}_{f \circ g}^{A_{1}, A_{2}}(t, u) \\
& =\int_{\mathbf{R}^{6}} K_{A_{1}}^{i} e^{\frac{i}{2 b_{1}}}\left[a_{1} n_{1}^{2}+2 n_{1}\left(r_{1}-u_{1}\right)-2 u_{1}\left(d_{1} r_{1}-b_{1} s_{1}\right)+d_{1} u_{1}^{2}\right] \\
& e^{i \frac{a_{1}}{b_{1}} 2 z_{1}\left(z_{1}+\left(t_{1}+\frac{n_{1}}{2}\right)\right)}
\end{aligned}
$$

$$
\times \overline{f(z)} g\left(z+\left(t+\frac{n}{2}\right)\right) e^{j \frac{a_{2}}{b_{2}} 2 z_{2}\left(z_{2}+\left(t_{2}+\frac{n_{2}}{2}\right)\right)} e^{i \frac{a_{1}}{b_{1}} 2 \gamma_{1}\left(\gamma_{1}+\left(t_{1}-\frac{n_{1}}{2}\right)\right)}
$$

$$
\times \overline{\overline{f(\gamma)} g\left(\gamma+\left(t-\frac{n}{2}\right)\right)} e^{j \frac{a_{2}}{b_{2}} 2 \gamma_{2}\left(\gamma_{2}+\left(t_{2}-\frac{n_{2}}{2}\right)\right)}
$$

$$
\times K_{A_{2}}^{j} e^{\frac{j}{2 b_{2}}\left[a_{2} n_{2}^{2}+2 n_{2}\left(r_{2}-u_{2}\right)-2 u_{2}\left(d_{2} r_{2}-b_{2} s_{2}\right)+d_{2} u_{2}^{2}\right]} d z d \gamma d n
$$

Setting $z_{i}=w_{i}+\frac{p_{i}}{2}, \gamma_{i}=w_{i}-\frac{p_{i}}{2}, i=1,2$, we get

$$
\begin{align*}
& \mathcal{W}_{f \circ g}^{A_{1}, A_{2}}(t, u) \\
& =\int_{\mathbf{R}^{6}} K_{A_{1}}^{i}{ }^{\frac{i}{2 b_{1}}\left[a_{1} n_{1}^{2}+2 n_{1}\left(r_{1}-u_{1}\right)-2 u_{1}\left(d_{1} r_{1}-b_{1} s_{1}\right)+d_{1} u_{1}^{2}\right]} \\
& e^{i \frac{a_{1}}{b_{1}} 2\left(w_{1}+\frac{p_{1}}{2}\right)\left(\left(t_{1}+\frac{n_{1}}{2}\right)+\left(w_{1}+\frac{p_{1}}{2}\right)\right)} \\
& \times f\left(w+\frac{p}{2}\right) g\left(\left(t+\frac{n}{2}\right)+\left(w+\frac{p}{2}\right)\right) \\
& e^{j \frac{a_{2}}{b_{2}} 2\left(w_{2}+\frac{p_{2}}{2}\right)\left(\left(t_{2}+\frac{n_{2}}{2}\right)+\left(w_{2}+\frac{p_{2}}{2}\right)\right)} \\
& \times e^{i \frac{a_{1}}{b_{1}} 2\left(w_{1}-\frac{p_{1}}{2}\right)\left(\left(t_{1}-\frac{n_{1}}{2}\right)+\left(w_{1}-\frac{p_{1}}{2}\right)\right)} f\left(w-\frac{p}{2}\right) \\
& g\left(\left(t-\frac{n}{2}\right)+\left(w-\frac{p}{2}\right)\right) \\
& \times e^{j \frac{a_{2}}{b_{2}} 2\left(w_{2}-\frac{p_{2}}{2}\right)\left(\left(t_{2}-\frac{n_{2}}{2}\right)+\left(w_{2}-\frac{p_{2}}{2}\right)\right)} K_{A_{2}}^{j} \\
& e^{\frac{j}{2 b_{2}}\left[a_{2} n_{2}^{2}+2 n_{2}\left(r_{2}-u_{2}\right)-2 u_{2}\left(d_{2} r_{2}-b_{2} s_{2}\right)+d_{2} u_{2}^{2}\right]} d p d q d w \quad(2 \tag{28}
\end{align*}
$$

Now put $n_{i}=q_{i}-p_{i}, i=1,2$ and on following the same procedure as followed in previous Theorem 6, we have from (28)

$$
\begin{align*}
& \mathcal{W}_{f \circ g}^{A_{1}, A_{2}}(t, u) \\
& =\int_{\mathbf{R}^{2}}\left[\int_{\mathbf{R}^{2}} e^{\frac{i}{2 b_{1}}\left[a_{1} p_{1}^{2}-2 p_{1}\left(r_{1}-u_{1}\right)\right]} \overline{f\left(w+\frac{p}{2}\right)} f\left(w-\frac{p}{2}\right)\right. \\
& \left.e^{\frac{j}{2 b_{2}}\left[a_{2} p_{2}^{2}-2 p_{2}\left(r_{2}-u_{2}\right)\right]} d p\right] \\
& \times\left[\int_{\mathbf{R}^{2}} K_{A_{1}}^{i} e^{\frac{i}{2 b_{1}}\left[a_{1} q_{1}^{2}+2 q_{1}\left(r_{1}-u_{1}\right)-2 u_{1}\left(d_{1} r_{1}-b_{1} s_{1}\right)+d_{1} u_{1}^{2}\right]}\right. \\
& \times g\left(t+w+\frac{q}{2}\right) \overline{g\left(t+w-\frac{q}{2}\right)} \\
& \left.\times K_{A_{2}}^{j} e^{\frac{j}{2 b_{2}}\left[a_{2} q_{2}^{2}+2 q_{2}\left(r_{2}-u_{2}\right)-2 u_{2}\left(d_{2} r_{2}-b_{2} s_{2}\right)+d_{2} u_{2}^{2}\right]} d q\right] \\
& \times e^{i \frac{a_{1}}{b_{1}}\left(4 w_{1}\left(t_{1}+w_{1}\right)\right)} e^{j \frac{a_{2}}{b_{2}}\left(4 w_{2}\left(t_{2}+w_{2}\right)\right)} d w
\end{align*}
$$

On multiplying (29) both sides by $K_{A_{1}}^{i} e^{\frac{i}{2 b_{1}}\left[d_{1} u_{1}^{2}-2 u_{1}\left(d_{1} r_{1}-b_{1} s_{1}\right)+4 p_{1}\left(r_{1}-u_{1}\right)\right]}$ and $K_{A_{2}}^{j} e^{\frac{j}{2 b_{2}}\left[d_{2} u_{2}^{2}-2 u_{2}\left(d_{2} r_{2}-b_{2} s_{2}\right)+4 p_{2}\left(r_{2}-u_{2}\right)\right]}$, we get

$$
\begin{align*}
& K_{A_{1}}^{i} e^{\frac{i}{2 b_{1}}\left[d_{1} u_{1}^{2}-2 u_{1}\left(d_{1} r_{1}-b_{1} s_{1}\right)+4 p_{1}\left(r_{1}-u_{1}\right)\right]} \\
& K_{A_{2}}^{j} \frac{j}{e^{b_{2}}\left[d_{2} u_{2}^{2}-2 u_{2}\left(d_{2} r_{2}-b_{2} s_{2}\right)+4 p_{2}\left(r_{2}-u_{2}\right)\right]} \mathcal{W}_{f \circ g}^{A_{1}, A_{2}}(t, u) \\
& =\int_{\mathbf{R}^{2}} e^{-i \frac{a_{1}}{b_{1}}\left(4 w_{1}\left(t_{1}-w_{1}\right)\right)} \mathcal{W}_{f, f}^{A_{1}, A_{2}}(w, u) \\
& \mathcal{W}_{g, g}^{A_{1}, A_{2}}(t-w, u) e^{-j \frac{a_{2}}{b_{2}}\left(4 w_{2}\left(t_{2}-w_{2}\right)\right)} d w \tag{30}
\end{align*}
$$

Now using (18) and (19) in (30) we obtain,

$$
\begin{aligned}
& \mathcal{W}_{f \star g}^{A_{1}, A_{2}}(t, u) \\
& =\sqrt{2 \pi b_{1}} e^{\frac{-i}{2 b_{1}}\left[d_{1}\left(u_{1}^{2}+r_{1}^{2}\right)+2 u_{1}\left(d_{1} r_{1}-b_{1} s_{1}\right)\right]} \\
& \times\left\{\int_{\mathbf{R}^{2}} e^{i \frac{a_{1}}{b_{1}}\left(4 w_{1}\left(t_{1}+w_{1}\right)\right)} \mathcal{W}_{f, f}^{A_{1}, A_{2}}(w,-u) \mathcal{W}_{g, g}^{A_{1}, A_{2}}(t+w, u)\right. \\
& \left.\times e^{j \frac{a_{2}}{b_{2}}\left(4 w_{2}\left(t_{2}+w_{2}\right)\right)} d w\right\} \sqrt{2 \pi b_{2} j} e^{\frac{-j}{2 b_{2}}\left[d_{2}\left(u_{2}^{2}+r_{2}^{2}\right)+2 u_{2}\left(d_{2} r_{2}-b_{2} s_{2}\right)\right]}
\end{aligned}
$$

which completes the proof of theorem.

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