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## Research Article

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# Color image restoration with impulse noise based on fractional-order total variation and framelet transform

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## Abstract

Restoring images corrupted by noise and blur is a burgeoning subject in image processing and, despite the large number of proposed restoration algorithms, the effort to bring about some improvement is always of great interest. The definition of fractional derivatives in recent years has created a powerful tool for this purpose. In the present paper, using fractional-order total variation and framelet transform, the nonconvex model for image restoration with impulse noise problem is improved. Then by alternating direction method of multipliers (ADMM) and primal-dual problem, the proposed model is solved. The convergence of the proposed algorithm is studied and the proposed algorithm is evaluated using different types of tests. The output results show the effectiveness of the proposed method.

**Keywords:** Image deblurring; Fractional total variation; Framelet; Nonconvex.

## 1 Introduction

The image blurring and impulse noise process can be considered by using the following formula

$$F = N_{\epsilon}(h \circledast U),$$

where  $F$  and  $U \in \mathbb{R}^{n \times m}$  are observed and original images, respectively, and  $N_{\epsilon}$  shows noise. Also,  $h \in \mathbb{R}^{r \times s}$  is blurred kernel and  $\circledast$  denotes two-dimensional convolution operator. In this type of problem, unlike the blur kernel, the noise effect is severe, and the noise removal operation must be performed in the image restoration algorithm. The above relation can be rewritten as a linear equation as follows

$$f = n_{\epsilon}(Au)$$

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where  $f$  and  $u \in \mathbb{R}^{nm \times 1}$  denote the reshaped of observed and original images, respectively. Also  $A \in \mathbb{R}^{nm \times nm}$  is obtained according to the blurred kernel and boundary conditions for the blurred process. There are four main boundary conditions that include zero, periodic, reflexive and anti-reflexive boundary conditions. Depending on each of these boundary conditions, the matrix  $A$  has a special structure. For example in the zero boundary condition,  $A$  is a block toeplitz with toeplitz blocks (BTTB) matrix or in the periodic boundary condition,  $A$  is a block circulant with circulant blocks (BCCB) matrix. Due to the noise and large size of the unknown coefficients and ill-conditioned for this linear equation, it is not possible to solve this problem directly. The first attempts to solve this problem can be made on SVD decompositions. But in most cases, this method does not have the desired output. Other method, that can be used for this problem is the basis pursuit problem [3] by minimization problem as

$$\min_u \{\|u\|_1 : Au = f\},$$

which is studied in many papers as [29, 39]. One of the most popular methods that has attracted the most attention is the use of total variation (TV) based on regularization scheme (the ROF model). This model is introduced by Rudin, Osher, and Fatemi [27] as

$$\min_u \frac{\lambda}{2} \|Au - f\|_2^2 + \int_{\Omega} |Du|,$$

where  $\Omega$  shows a bounded open subset with Lipschitzian boundary in  $\mathbb{R}^2$  and  $Du$  denotes the derivative of  $u$ . The idea of using total variation has been considered in recent years and has been used in various articles, for example [15, 20, 25, 12, 30, 36]. One of the most important tools that has improved this method in recent years is the use of fractional derivatives. Despite the existence of different definitions for fractional derivatives, such as Riemann-Liouville (R-L) and Caputo definition, the Grünwald-Letnikov (G-L) definition is often used in image processing due to its lower computational complexity than other definitions. Based on these types of derivatives, fractional-order total variation (FTV) is introduced and used in image and signal processing [17, 23, 5]. With the development of the wavelet and framelet concept, an efficient tool for image and signal processing is developed that has been used in various articles as [2, 13, 1, 18]. These concepts are introduced in the following subsections. The reader can find more information about these concepts in [11, 31, 10]. The  $l_1$  norm based on frame transform for restoration problem is introduced by Dong, Ji and Shen [6]. Due to the structure of  $l_1$  norm as nonsmooth and nonseparable, solving a problem that includes these norms is hard. The split Bregman algorithm is a suitable tool for solving these types of problems [32, 40, 14]. In the proposed model a non-convex  $(l_1 - l_2)$ -norm is considered as regularization term.  $(l_1 - l_2)$ -norm is studied in many papers as [19, 22]. Also, this norm based on frame transform is used by Jingjing Liu et al in [19] as following model

$$\min_u \|Au - f\|_1 + \lambda_1 (\|Wu\|_1 - \beta \|u\|_2),$$

where  $W$  represent the matrix of framelet transform. As it is clear from this model, the total variation phrase is not observed in this model. Therefore, adding this expression can improve the algorithm. Total variation method can preserve edges very well in the restored image, then this helps to improve the restored image. Although the ordinary method of minimization is slow for this problem [35, 37], the optimal algorithm based on primal-dual is introduced for solving this problem in [4, 7, 8, 9, 26, 38].

The organization of this paper is as follows: In Section 2, the tools used in the proposed algorithm are briefly introduced. The details of the proposed method are given in Section 3. The convergence analysis of the proposed algorithm is given in Section 4. Simulation results and algorithm analysis are studied in Section 5. Also the summary of paper is given in Section 6.

## 2 Preliminaries

This section introduces some of the basic concepts used in the paper. Also in this section  $l_2(I)$  is considered as a set of all sequences in the complex space as  $\{x_i\}_{i \in I}$  such that  $\sum_{i \in I} |x_i|^2 < \infty$ .

**Definition 2.1.** Let  $\mathcal{H}$  be a separable Hilbert space. Then the sequence  $F = \{f_i\}_{i \in I} \subseteq \mathcal{H}$  is named a frame in  $\mathcal{H}$  if there exist two constants as  $\rho_1$  and  $\rho_2$  such that for all  $f \in \mathcal{H}$

$$\rho_1 \|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq \rho_2 \|f\|^2.$$

When  $\rho_1 = \rho_2 = 1$ , frame is called a Parseval frame in  $\mathcal{H} = L_2(\mathbb{R})$ .

If  $F$  be a frame in  $\mathcal{H}$ , the synthesis operator that is onto and well defined is considered as

$$T : l^2(I) \rightarrow \mathcal{H}, \quad T(\{v_i\}_{i \in I}) = \sum_{i \in I} v_i f_i.$$

The adjoint of the synthesis operator is named analysis operator and written as

$$T^* : \mathcal{H} \rightarrow l^2(I), \quad T^*(f) = \{\langle f, f_i \rangle\},$$

where  $\langle . \rangle$  denotes the inner product. Using these two operators, an operator on the frame can be introduced as follows

$$s : \mathcal{H} \rightarrow \mathcal{H}, \quad s(f) = \sum_{i \in I} \langle f, f_i \rangle f_i.$$

In frame theory, it is proved that  $f \in \mathcal{H}$  can written as

$$f = \sum_{i \in I} \langle f, s^{-1} f_i \rangle f_i = \sum_{i \in I} \langle f, f_i \rangle s^{-1} f_i.$$

This formula provides the main idea of image transfer that is studied in [11, 31, 10]. In this paper, particular Parseval framelet systems in  $\mathcal{H}$  that are constructed by B-spline whose refinement mask is  $h_0 = \frac{1}{4}[1, 2, 1]$ , with two corresponding framelet masks  $h_1 = \frac{\sqrt{2}}{4}[1, 0, -1]$  and  $h_2 = \frac{1}{4}[-1, 2, -1]$ . In Fig 1, framelet transform based on B-splines is shown.

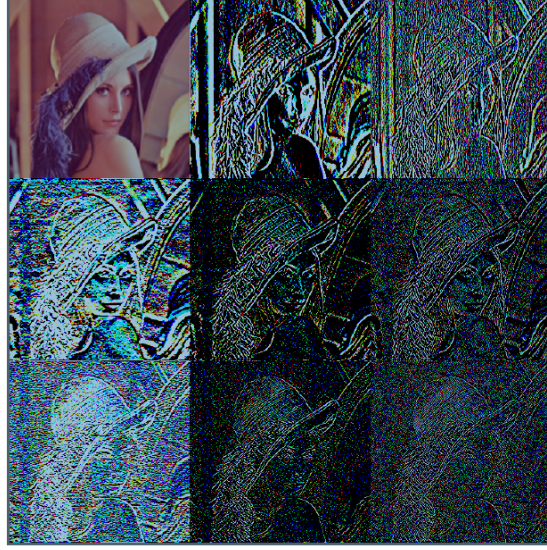


Figure 1: Framelet transform result for Lena ( $256 \times 256$ ).

The concept of derivative was first introduced for natural numbers, and later the definition was extended to fractional numbers. Despite the different definitions that exist for the fraction derivative, due to the computational simplicity, the G-L fractional-order derivative definition is mostly used for image processing, which is defined below.

**Definition 2.2.** The G-L fractional-order derivatives  $D_x^\alpha$  and  $D_y^\alpha$  for input image  $u$  of order  $\alpha \in \mathbb{R}^+$  are defined as

$$D_x^\alpha u_{i,j} := \sum_{l=0}^{k-1} \phi_l^\alpha u_{i-k,j},$$

$$D_y^\alpha u_{i,j} := \sum_{l=0}^{k-1} \phi_l^\alpha u_{i,j-k},$$

where  $\phi_l^\alpha = (-1)^l \frac{\Gamma(\alpha+1)}{\Gamma(l+1)\Gamma(\alpha+1-l)}$  and  $k$  denotes the number of neighboring pixels. Also the discrete fractional-order gradient is considered as  $\nabla^\alpha u = [D_x^\alpha u, D_y^\alpha u]^T$ .

**Definition 2.3.** *The adjoint operators of the fractional-order derivatives are defined as*

$$\begin{aligned}(D_x^\alpha)^T u_{i,j} &:= \sum_{l=0}^{k-1} \phi_l^\alpha u_{i+k,j}, \\ (D_y^\alpha)^T u_{i,j} &:= \sum_{l=0}^{k-1} \phi_l^\alpha u_{i,j+k}.\end{aligned}$$

Based on above definitions for a vector function  $\mathbf{p}(x, y) = (p^1(x, y), p^2(x, y))$  the fractional divergence operator can be obtained as

$$\operatorname{div}^\alpha \mathbf{p} = (-1)^\alpha (\nabla^\alpha)^T \mathbf{p} = (-1)^\alpha ((D_x^\alpha)^T p_{i,j}^1 + (D_y^\alpha)^T p_{i,j}^2).$$

In the next section, the introduced concepts are used to improve the restoration algorithm.

### 3 Proposed minimization model and iterative method

In the first step of this section, the proposed model based on fractional-order total variation is introduced then a numerical algorithm is obtained for solving this model and in the last step, the proposed model is used for color image.

#### 3.1 Proposed model

In the proposed model, the following model based on FTV is introduced for restoration of images with impulse noise

$$\min_u \|Au - f\|_1 + \lambda_1 (\|Wu\|_1 - \beta \|u\|_2) + \lambda_2 \|u\|_{FTV}, \quad (3.1)$$

where  $\lambda_1, \lambda_2$  and  $\beta$  are positive given parameters. Also  $\|\cdot\|_{FTV}$  denotes the fractional-order total variation norm defined as

$$\|u\|_{FTV} = \sum_{i,j} \sqrt{(D_x^\alpha u_{i,j})^2 + (D_y^\alpha u_{i,j})^2}.$$

Adding this sentence improves the edges of the image in the algorithm. This model contains a nonconvex sentence. There are several methods for solving nonconvex optimization problems, for example see [24]. In the proposed method, alternating direction method of multipliers (ADMM) is used for solving this problem. By auxiliary variables  $m = \{m_i\}_{i=1}^3$ , (3.1) can be written as

$$\begin{aligned}\min_u & \|m_1\|_1 + \lambda_1 (\|m_2\|_1 - \beta \|u\|_2) + \lambda_2 \|m_3\|_{FTV}, \\ \text{s.t. } & m_1 = Au - f, \quad m_2 = Wu, \quad m_3 = u.\end{aligned} \quad (3.2)$$

The augmented Lagrangian for (3.2) is obtained as

$$\begin{aligned} \min_{u, m, n} L(u, m_1, m_2, m_3, n_1, n_2, n_3) = & \min_{u, m, n} \|m_1\|_1 + \lambda_1 (\|m_2\|_1 - \beta \|u\|_2) + \lambda_2 \|m_3\|_{FTV} \\ & + \mu_1 \langle Au - f - m_1, n_1 \rangle + \frac{\mu_1}{2} \|Au - f - m_1\|_2^2 + \mu_2 \langle Wu - m_2, n_2 \rangle \\ & + \frac{\mu_2}{2} \|Wu - m_2\|_2^2 + \mu_3 \langle u - m_3, n_3 \rangle + \frac{\mu_3}{2} \|u - m_3\|_2^2, \end{aligned} \quad (3.3)$$

where  $n = \{n_i\}_{i=1}^3$  and  $\mu_i > 0$ ,  $i = 1, 2, 3$  are the Lagrange multipliers and penalty parameters, respectively. The extended iterative algorithm for solving problem (3.3) based on ADMM is given as

$$\begin{aligned} u^{k+1} = \arg \min_u & -\lambda_1 \beta \|u\|_2 + \mu_1 \langle Au - f - m_1^k, n_1^k \rangle + \frac{\mu_1}{2} \|Au - f - m_1^k\|_2^2 \\ & + \mu_2 \langle Wu - m_2^k, n_2^k \rangle + \frac{\mu_2}{2} \|Wu - m_2^k\|_2^2 + \mu_3 \langle u - m_3^k, n_3^k \rangle \\ & + \frac{\mu_3}{2} \|u - m_3^k\|_2^2, \end{aligned} \quad (3.4)$$

$$m_1^{k+1} = \arg \min_{m_1} \|m_1\|_1 + \mu_1 \langle Au^{k+1} - f - m_1, n_1^k \rangle + \frac{\mu_1}{2} \|Au^{k+1} - f - m_1\|_2^2, \quad (3.5)$$

$$n_1^{k+1} = n_1^k + Au^{k+1} - f - m_1^{k+1}, \quad (3.6)$$

$$m_2^{k+1} = \arg \min_{m_2} \lambda_1 \|m_2\|_1 + \mu_2 \langle Wu^{k+1} - m_2, n_2^k \rangle + \frac{\mu_2}{2} \|Wu^{k+1} - m_2\|_2^2, \quad (3.7)$$

$$n_2^{k+1} = n_2^k + Wu^{k+1} - m_2^{k+1}, \quad (3.8)$$

$$m_3^{k+1} = \arg \min_{m_3} \lambda_2 \|m_3\|_{FTV} + \mu_3 \langle u^{k+1} - m_3, n_3^k \rangle + \frac{\mu_3}{2} \|u^{k+1} - m_3\|_2^2, \quad (3.9)$$

$$n_3^{k+1} = n_3^k + u^{k+1} - m_3^{k+1}. \quad (3.10)$$

### 3.2 Solve subproblems

In this subsection, the solution to each of the subproblems introduced in the previous subsection are studied. For (3.4) by using the optimal condition, we get

$$\begin{aligned} & (\mu_1 A^T A + \mu_3 I + \mu_2 I - \lambda_1 \beta / \|u^k\|_2 I) u \\ & = \mu_1 A^T (f + m_1^k - n_1^k) + \mu_2 W^T (m_2^k - n_2^k) + \mu_3 (m_3^k - n_3^k). \end{aligned} \quad (3.11)$$

Under the periodic condition for blurred processing, the blurred matrix can be generated as block circulant with circulant blocks (BCCB) matrix. It is well known in the field of linear algebra that these types of matrices are decomposed using Fourier transform. And due to the fast Fourier transform (FFT), the calculation is performed faster. So by considering the periodic condition, Eq. (3.11) is changed as

$$\begin{aligned} & (\mu_1 \Lambda^* \Lambda + (\mu_3 + \mu_2 - \lambda_1 \beta / \|u^k\|_2) I) F u^{k+1} \\ & = \mu_1 \Lambda^* F (f + m_1^k - n_1^k) + \mu_2 F W^T (m_2^k - n_2^k) + \mu_3 F (m_3^k - n_3^k), \end{aligned} \quad (3.12)$$

where  $*$  and  $F$  denote the complex conjugacy and fast Fourier transform, respectively. Also  $\Lambda$  is a diagonal matrix dependent on the eigenvalues of the blurred matrix. The equation (3.12) can be easily solved using Fourier calculations.

Subproblem (3.5) can be rewritten as

$$m_1^{k+1} = \arg \min_{m_1} \|m_1\|_1 + \frac{\mu_1}{2} \|m_1 - (Au^{k+1} - f + n_1^k)\|_2^2.$$

The solution for the above problem based on proximal mapping for  $l_1$ -norm can be obtained as

$$m_1^{k+1} = \Psi_{1/\mu_1}(Au^{k+1} - f + n_1^k), \quad (3.13)$$

where  $\Psi$  is defined as

$$\Psi_a(x) = \text{sign}(x) \max(|x| - a, 0).$$

In a similar way, (3.7) can be written as

$$m_2^{k+1} = \arg \min_{m_2} \lambda_1 \|m_2\|_1 + \frac{\mu_2}{2} \|m_2 - (Wu^{k+1} + n_2^k)\|_2^2,$$

and the solution by proximal mapping is earned as

$$m_2^{k+1} = \Psi_{\lambda_1/\mu_2}(Wu^{k+1} + n_2^k). \quad (3.14)$$

In order to avoid complex calculations for Eq. (3.9), the dual problem is used to find the solution. In the first step, Eq. (3.9) is changed as

$$m_3^{k+1} = \arg \min_{m_3} \lambda_2 \|m_3\|_{FTV} + \frac{\mu_3}{2} \|m_3 - (u^{k+1} - n_3^k)\|_2^2, \quad (3.15)$$

then in the next step, the dual problem is obtained and solved for this problem.

**Lemma 3.1.** *If we consider  $J(u) = \min_u \|u\|_{FTV}$  then the dual problem of  $J(u)$  is obtained as [7]*

$$J(u) = \sup_{\mathbf{p}} \langle \mathbf{p}, \nabla^\alpha u \rangle - J^*(\mathbf{p}),$$

where

$$J^*(\mathbf{p}) = \begin{cases} 0, & \text{if } |\mathbf{p}| \leq 1, \\ \infty, & \text{if } |\mathbf{p}| > 1. \end{cases}$$

Based on Lemma 3.1, the corresponding primal-dual problem of (3.15) is written as follows

$$(m_3^{k+1}, \mathbf{p}^{k+1}) = \arg \min_{m_3} \max_{\mathbf{p} \in \chi} \frac{\lambda_2}{\mu_3} \langle \mathbf{p}, \nabla^\alpha m_3 \rangle + \frac{1}{2} \|m_3 - (u^{k+1} - n_3^k)\|_2^2, \quad (3.16)$$

where  $\chi = \{\mathbf{p} \in \mathbb{R}^{2nm} | \mathbf{p}_i \in \mathbb{R}^2, \|\mathbf{p}_i\|_2 \leq 1, \forall i \in \{1, \dots, nm\}\}$ . By the iterative scheme, the solution of the primal-dual problem (3.16) can be written as

$$\mathbf{p}^{k+1} = \arg \max_{\mathbf{p} \in \chi} \frac{\lambda_2}{\mu_3} \langle \mathbf{p}, \nabla^\alpha \hat{m}_3^k \rangle - \frac{1}{2\gamma} \|\mathbf{p} - \mathbf{p}^k\|_2^2, \quad (3.17)$$

$$m_3^{k+1} = \arg \min_{m_3} \frac{\lambda_2}{\mu_3} \langle \mathbf{p}^{k+1}, \nabla^\alpha m_3 \rangle + \frac{1}{2} \|m_3 - (u^{k+1} - n_3^k)\|_2^2, \quad (3.18)$$

$$\hat{m}_3^{k+1} = 2m_3^{k+1} - m_3^k. \quad (3.19)$$

After simplifying (3.17)-(3.18), the following statements are obtained

$$\mathbf{p}^{k+1} = \frac{\mathbf{p}^k + \frac{\lambda_2 \gamma}{\mu_3} \nabla^\alpha \hat{m}_3^k}{\max(\|\mathbf{p}^k + \frac{\lambda_2 \gamma}{\mu_3} \nabla^\alpha \hat{m}_3^k\|, 1)}, \quad (3.20)$$

$$m_3^{k+1} = m_3^k - \tau \left( \frac{\lambda_2 \gamma}{\mu_3} (\nabla^\alpha)^T \mathbf{p}^{k+1} + m_3^k - (u^{k+1} + n_3^k) \right), \quad (3.21)$$

where  $\tau$  is step size and  $\gamma$  is a positive constant.

The general structure of the proposed method for restoration of blurred images with impulse noise is summarized in Algorithm 1.

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Algorithm 1: Restoration proposed algorithm.

---

**Initialization:**  $\alpha, \beta, \tau, \gamma, \{\lambda_i\}_{i=1}^2, \{\mu_i\}_{i=1}^3, m^0, n^0, u^0$ .

**for**  $k=1, \dots$  **do**

    Compute  $u^k$  by solving (3.12),

    Compute  $m_1^k$  by solving (3.13),

    Compute  $n_1^k$  by solving (3.6),

    Compute  $m_2^k$  by solving (3.14),

    Compute  $n_2^k$  by solving (3.8),

    Compute  $m_3^k$  by solving (3.19)-(3.21),

    Compute  $n_3^k$  by solving (3.10),

**end for**

    If the stop condition is met in the above step, stop the loop.

---

### 3.3 Restoration algorithm for color image

The proposed restoration algorithm for grayscale image is introduced in the previous subsection. But this algorithm can be extended to color images. In the following, the details of the proposed algorithm for color images are explained. The blurred matrix for color image is given in  $\mathbb{R}^{3nm \times 3nm}$  as

$$A = \begin{bmatrix} A_{rr} & A_{rg} & A_{rb} \\ A_{gr} & A_{gg} & A_{gb} \\ A_{br} & A_{bg} & A_{bb} \end{bmatrix}.$$

Also, in the calculations related to this subsection, for different layers of the color image, we expand the values that is calculated in the previous subsection as  $u = [u_r; u_g; u_b]$ ,  $f = [f_r; f_g; f_b]$ ,  $m_i = [m_{i,r}; m_{i,g}; m_{i,b}]$ ,  $n_i = [n_{i,r}; n_{i,g}; n_{i,b}]$  for  $i = 1, 2, 3$ . Using the same procedure mentioned in the pervious section the following phrase is obtained

$$\begin{aligned} & (\mu_1 \Lambda^* \Lambda + (\mu_3 + \mu_2 - \lambda \beta / \|u\|_2) I) \hat{F} u^{k+1} \\ & = \mu_1 \Lambda^* \hat{F} (f + m_1^k - n_1^k) + \mu_2 \hat{F} W^T (m_2^k - n_2^k) + \mu_3 \hat{F} (m_3^k - n_3^k), \end{aligned} \quad (3.22)$$

where  $\hat{F} = I \otimes F$ , here  $\otimes$  indicates the Kronecker product and  $I$  denotes the identity matrix. Moreover,  $\Lambda$  is obtained after the Fourier transform decomposition as

$$\Lambda = \begin{bmatrix} \Lambda_{rr} & \Lambda_{rg} & \Lambda_{rb} \\ \Lambda_{gr} & \Lambda_{gg} & \Lambda_{gb} \\ \Lambda_{br} & \Lambda_{bg} & \Lambda_{bb} \end{bmatrix},$$

where  $\Lambda_{ij}$ ,  $i, j = r, g, b$  are diagonal matrices depending on the eigenvalues of the  $A_{ij}$ ,  $i, j = r, g, b$ . Other parts of the proposed algorithm are similar to those described in the pervious section, and the calculation details are given in Algorithm 2. The simulation results of the algorithms described in this section are studied in the simulation results section.

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Algorithm 2: Color image restoration proposed algorithm.

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**Initialization:**  $\alpha, \beta, \tau, \gamma, \{\lambda_i\}_{i=1}^2, \{\mu_i\}_{i=1}^3, m^0, n^0, u^0$ .  
**for**  $k=1, \dots$  **do**  
    Compute  $u^k$  by solving (3.22),  
    **for**  $l=r, g, b$  **do**  
        Compute  $m_{1,l}^k$  by solving:  $m_{1,l}^k = \Psi_{1/\mu_1} (Au_l^k - f_l + n_{1,l}^{k-1})$ ,  
        Compute  $n_{1,l}^k$  by solving:  $n_{1,l}^k = n_{1,l}^{k-1} + Au_l^k - f - m_{1,l}^k$ ,  
        Compute  $m_{2,l}^k$  by solving:  $m_{2,l}^k = \Psi_{\lambda_1/\mu_2} (Wu_l^k + n_{2,l}^{k-1})$ ,  
        Compute  $n_{2,l}^k$  by solving:  $n_{2,l}^k = n_{2,l}^{k-1} + Wu_l^k - m_{2,l}^k$ ,  
        Compute  $m_{3,l}^k$  by solving:  
            
$$\mathbf{p}_l^k = \frac{\mathbf{p}_l^{k-1} + \frac{\lambda_2 \gamma}{\mu_3} \nabla^\alpha \hat{m}_{3,l}^{k-1}}{\max(|\mathbf{p}_l^{k-1} + \frac{\lambda_2 \gamma}{\mu_3} \nabla^\alpha \hat{m}_{3,l}^{k-1}|, 1)},$$
  
            
$$m_{3,l}^k = m_{3,l}^{k-1} - \tau \left( \frac{\lambda_2 \gamma}{\mu_3} (\nabla^\alpha)^T \mathbf{p}_l^k + m_{3,l}^{k-1} - (u_l^k + n_{3,l}^{k-1}) \right),$$
  
            
$$\hat{m}_{3,l}^k = 2m_{3,l}^k - m_{3,l}^{k-1},$$
  
        Compute  $n_{3,l}^k$  by solving:  $n_{3,l}^k = n_{3,l}^{k-1} + u_l^k - m_{3,l}^k$ ,  
    **end for**  
**end for**  
    If the stop condition is met in the above step, stop the loop.

---

## 4 Convergence analysis

In this section, the convergence of the proposed method is studied. The convergence analysis for proposed method is based on the described method in [19, 22].

**Lemma 4.1.** *Let the objective function be coercive, that is  $\|Au - f\|_1 + \lambda_1(\|Wu\|_1 - \beta\|u\|_2) + \lambda_2\|u\|_{FTV} \rightarrow \infty$  when  $\|u\|_2 \rightarrow \infty$ . Also assume that  $\{u^k, m_1^k, m_2^k, m_3^k, n_1^k, n_2^k, n_3^k\}$  be the sequence generated by the proposed method, then the following statements hold:*

a)

$$\begin{aligned} & L(u^{k+1}, m_1^{k+1}, m_2^{k+1}, m_3^{k+1}, n_1^{k+1}, n_2^{k+1}, n_3^{k+1}) - L(u^k, m_1^k, m_2^k, m_3^k, n_1^k, n_2^k, n_3^k) \\ & \leq \sum_{i=1}^3 C_i \|m_i^{k+1} - m_i^k\|_2^2. \end{aligned}$$

b) If there exists a  $p \in \partial_u L(u^{k+1}, m_1^{k+1}, m_2^{k+1}, m_3^{k+1}, n_1^{k+1}, n_2^{k+1}, n_3^{k+1})$  then

$$\begin{aligned} & \|p\|_2 + \sum_i^3 (\partial_{m_i} + \partial_{n_i}) L(u^{k+1}, m_1^{k+1}, m_2^{k+1}, m_3^{k+1}, n_1^{k+1}, n_2^{k+1}, n_3^{k+1}) \\ & \leq \sum_{i=1}^3 C_{i+3} \|m_i^{k+1} - m_i^k\|_2, \end{aligned}$$

where in conditions (a) and (b),  $\{C_i\}_{i=1}^6$  are constant.

*Proof.* a) By problem (3.4), we get

$$L(u^{k+1}, m_1^k, m_2^k, m_3^k, n_1^k, n_2^k, n_3^k) - L(u^k, m_1^k, m_2^k, m_3^k, n_1^k, n_2^k, n_3^k) \leq 0.$$

Also by using (3.5)-(3.10), we obtain

$$\begin{aligned} & L(u^{k+1}, m_1^{k+1}, m_2^{k+1}, m_3^{k+1}, n_1^{k+1}, n_2^{k+1}, n_3^{k+1}) - L(u^k, m_1^k, m_2^k, m_3^k, n_1^k, n_2^k, n_3^k) \\ & \leq \|m_1^{k+1}\|_1 - \|m_1^k\|_1 + \lambda_1(\|m_2^{k+1}\|_1 - \|m_2^k\|_1) + \lambda_2(\|m_3^{k+1}\|_{FTV} - \|m_3^k\|_{FTV}) \\ & + \sum_{i=3}^3 \mu_i (\|n_i^{k+1} - n_i^k\|_2^2 - \langle n_i^k, m_i^{k+1} - m_i^k \rangle - \frac{1}{2} \|m_i^{k+1} - m_i^k\|_2^2 \\ & - \langle m_i^{k+1} - m_i^k, n_i^{k+1} - n_i^k \rangle). \end{aligned}$$

Based on optimality conditions for (3.5)-(3.10), we have  $\partial_{m_1} \|m^{k+1}\|_1 = \mu_1 n_1^{k+1}$ ,  $\lambda_1 \partial_{m_2} \|m_2^{k+1}\|_1 = \mu_2 n_2^{k+1}$  and  $\lambda_2 \partial_{m_3} \|m_3^{k+1}\|_{FTV} = \mu_3 n_3^{k+1}$ , then by using Lipschitz continuous gradient and Young's inequality, the following inequalities for any positive

$\{c_i\}_{i=1}^3$  hold

$$\begin{aligned} \|m_1^{k+1}\|_1 - \|m_1^k\|_1 - \mu_1 \langle n_1^k, m_1^{k+1} - m_1^k \rangle &\leq \frac{L_1}{2} \|m_1^{k+1} - m_1^k\|_2^2, \\ \lambda_1 (\|m_2^{k+1}\|_1 - \|m_2^k\|_1) - \mu_2 \langle n_2^k, m_2^{k+1} - m_2^k \rangle &\leq \frac{L_2}{2} \|m_2^{k+1} - m_2^k\|_2^2, \\ \lambda_2 (\|m_3^{k+1}\|_{FTV} - \|m_3^k\|_{FTV}) - \mu_3 \langle n_3^k, m_3^{k+1} - m_3^k \rangle &\leq \frac{L_3}{2} \|m_3^{k+1} - m_3^k\|_2^2, \\ -\mu_i \langle m_i^{k+1} - m_i^k, n_i^{k+1} - n_i^k \rangle &\leq c_i \mu_i \|n_3^{k+1} - n_3^k\|_2^2 + \frac{\mu_i}{4c_i} \|m_3^{k+1} - m_3^k\|_2^2, \quad i = 1, 2, 3, \end{aligned}$$

if  $\{c_i\}_{i=1}^3$  is chosen as  $\{\mu_i/2L_i\}_{i=1}^3$ , we get

$$\begin{aligned} &L(u^{k+1}, m_1^{k+1}, m_2^{k+1}, m_3^{k+1}, n_1^{k+1}, n_2^{k+1}, n_3^{k+1}) - L(u^k, m_1^k, m_2^k, m_3^k, n_1^k, n_2^k, n_3^k) \\ &\leq \sum_{i=1}^3 \left( \frac{L_i}{2} - \frac{\mu_i}{2} + \frac{\mu_i}{4c_i} \right) \|m_i^{k+1} - m_i^k\|_2^2 + (1 + c_i) \mu_i \|n_i^{k+1} - n_i^k\|_2^2 \\ &= \sum_{i=1}^3 C_i \|m_i^{k+1} - m_i^k\|_2^2, \end{aligned}$$

where  $C_i = (3L_i - \mu_i)/2 + L_i^2/\mu_i$ . Therefore, the first part of the lemma is proved.

b) By using optimality condition for (3.4), there is a  $q \in \partial_u(-\lambda_1 \beta_1 \|u\|_2)$  such that

$$\begin{aligned} &q + \mu_1 A^T (Au^{k+1} - f + n_1^k - m_1^k) + \mu_2 W^T (Wu^{k+1} + n_2^k - m_2^k) \\ &+ \mu_3 (u^{k+1} + n_3^k - m_3^k) = 0, \end{aligned}$$

now let

$$\begin{aligned} p &= q + \mu_1 A^T (Au^{k+1} - f + n_1^{k+1} - m_1^{k+1}) + \mu_2 W^T (Wu^{k+1} + n_2^{k+1} - m_2^{k+1}) \\ &+ \mu_3 (u^{k+1} + n_3^{k+1} - m_3^{k+1}) \in \partial_u L(u^{k+1}, m_1^{k+1}, m_2^{k+1}, m_3^{k+1}, n_1^{k+1}, n_2^{k+1}, n_3^{k+1}), \end{aligned}$$

then we obtain

$$\|p\|_2 \leq \sum_{i=1}^3 c_{i+3} (\mu_i + L_i) \|m_i^{k+1} - m_i^k\|_2, \quad (4.1)$$

where  $\{c_i\}_{i=4}^6$  are constant. From the optimality condition of (3.4)-(3.10), the following inequalities for  $i = 1, 2, 3$  are obtained

$$\begin{aligned} \|\partial_{m_i} L(u^{k+1}, m_1^{k+1}, m_2^{k+1}, m_3^{k+1}, n_1^{k+1}, n_2^{k+1}, n_3^{k+1})\|_2 &\leq L_i \|m_i^{k+1} - m_i^k\|_2, \\ \|\partial_{n_i} L(u^{k+1}, m_1^{k+1}, m_2^{k+1}, m_3^{k+1}, n_1^{k+1}, n_2^{k+1}, n_3^{k+1})\|_2 &\leq L_i \|m_i^{k+1} - m_i^k\|_2. \end{aligned}$$

Therefore, by combining the above inequalities with (4.1), the relation (b) is obtained.  $\square$

Using the above lemma, the final theorem for convergence is considered as follows. Due to the similarity of the proof with [22], the proof is omitted.

**Theorem 4.2.** *Considering the assumptions of Lemma 4.1 and let  $\mu_i > (3 + \sqrt{17})L_i/2$  for  $i = 1, 2, 3$ , then*

- a) the sequence  $(u^k, \{m_i^k\}_{i=1}^3, \{n_i^k\}_{i=1}^3)$  generated by proposed method is bounded and has at least one limit point.*
- b)  $\|u^{k+1} - u^k\|_2 \rightarrow 0$ ,  $\|m_i^{k+1} - m_i^k\|_2 \rightarrow 0$  and  $\|n_i^{k+1} - n_i^k\|_2 \rightarrow 0$  for  $i = 1, 2, 3$ .*
- c) each limit point  $(u^*, m_1^*, m_2^*, m_3^*, n_1^*, n_2^*, n_3^*)$  is a stationary point of  $L(u, m_1, m_2, m_3, n_1, n_2, n_3)$ , and  $u^*$  is a stationary point of the proposed model (3.1).*

## 5 Simulation results

The simulation results of the proposed algorithm that we introduced in the previous sections are studied in this section. For simulation results, a computer based on Windows 10-64bit, Intel(R) Core(TM) i3-5005U CPU @2.00GHz, by matlab 2014b is used. Also stopping criterion in this section is considered as

$$\frac{\|u^{k+1} - u^k\|_2}{\|u^{k+1}\|_2} \leq tol,$$

where  $u^k$  is the restored image at the  $k$ th iteration. In all examples, the  $tol$  equal to  $10^{-4}$  and  $10^{-3}$  are selected for grayscale and color images, respectively. In the analysis of the proposed algorithm various tests including signal-to-noise ratio (SNR), peak signal to noise ratio (PSNR) which is used to measure the differences and similarities between the original and restored image and obtained by

$$PSNR = 10 \log_{10} \frac{p^2}{MSE(u^o, u^r)},$$

where  $u^o$  and  $u^r$  show original and restored image, respectively,  $p$  denotes the maximum pixel value in an 8-bit grayscale image (i.e, 255).  $MSE$  is mean squared error and find by

$$MSE = \frac{1}{n \times m} \sum_{i=1}^n \sum_{j=1}^m |u_{i,j}^o - u_{i,j}^r|^2.$$

Structural similarity (SSIM) is used to focus on the detail preservation characteristic, this value is obtained by

$$SSIM = \frac{(\mu_o \mu_r + C_1)(2\sigma_{or} + C_2)}{(\mu_o^2 + \mu_r^2 + C_1)(\sigma_o + \sigma_r + C_2)},$$

where  $\mu_i, \sigma_i$ ,  $i = o, r$  are the means and variances of the original and restored images, respectively, and  $\sigma_{or}$  denotes covariance between the original and restored images. Also  $C_1$  and  $C_2$  are two variables to stabilize the division with weak denominator. More information on these criteria value can be found in [34, 28, 16]. In the following examples,  $M(len, \theta)$  denotes the linear motion blur of a camera by  $len$  pixels, with an angle of  $\theta$  degrees in a counterclockwise direction,  $A([r_1, r_2])$  shows the average blur with a size  $[r_1, r_2]$ ,  $G(size, \sigma)$  denotes the Gaussian blur of size

$hsize$  with standard deviation  $\sigma$  (positive). The following values are considered in the simulation results:  $\lambda_1 = \lambda_2 = \{0.005, 0.02, 0.025\}$ ,  $\mu_1 = \{3, 20, 50, 75, 100\}$ ,  $\mu_2 = \mu_3 = \{0.01, 0.2, 0.7, 1.0, 1.5\}$ ,  $\alpha = \{0.25, 0.5, 0.75, 1, 1.5\}$ ,  $\beta = \{1, 1.1, 1.3, 1.5\}$ ,  $\tau = 1/40$  and  $\gamma = \{0.005, 0.0005\}$ . After obtaining the results with these values, we consider the best result.

**Example 5.1.** (Grayscale images) In this example, Algorithm 1 is used for different images from the USC-SIPI images database<sup>1</sup>, i.e., “5.1.12 (256 × 256)”, “5.2.10 (512 × 512)”, “4.2.05 (512 × 512)” and “7.1.09 (512 × 512)”. Also “7.1.09” and “4.2.05” are changed to images with size (256 × 256) and grayscale by Matlab internal functions and used in the simulation. For images “5.1.12” and “5.2.10”,  $A([11, 11])$  with “salt & pepper” and “Random-valued” noise, respectively, with different density are used, for image “7.1.09”, blur kernel  $G(7, 4)$  with “salt & pepper” noise with different densities are used. The blurred and noisy images for “5.1.12”, “7.1.09” and “5.2.10” are shown in Figs 2, 3 and 3, respectively. Also, in these figures, the restored images corresponding to each image are given below each image. Table 1 compares numerical results for different densities value by proposed algorithm and method in [19]. In the simulation of the algorithm [19],  $\lambda = \{0.005, 0.02, 0.025\}$ ,  $\alpha = \{1, 1.1, 1.3, 1.5\}$ ,  $\mu_1 \{3, 20, 50, 75, 100\}$  and  $\mu_2 = \{0.01, 0.2, 0.7, 1.0, 1.5\}$  are considered and after the calculation, the best results are shown in this table. These results indicate an improvement in the output of the proposed algorithm. Also to investigate the effect of the proposed algorithm on the edges of the image, the restored results for the “Lena” image (256 × 256) are given in Fig. 5. In this case,  $G(7, 4)$  and “salt & pepper” noise with 0.3 density are used. Based on these results, we can see that the proposed method has a better effect on restoring the edges of the image. In order to show the convergence of the proposed method, the output of the absolute error based on  $l_2$ -norm and PSNR values for 500 repetitions are given in Fig. 6. Also in order to show the effect of blurred kernel and noise on the image, in Figure 7 (a-d), each of the steps of blurring by  $M(35, 135)$  and adding noise by “random-valued” noise with 10% density and restoring image are given separately. In this case we have  $PSNR = 33.031$ ,  $SNR = 30.241$  and  $SSIM = 0.92600$ . The results in this example show a good representation of the proposed algorithm.

**Example 5.2.** (Color image) In this example, Algorithm 2 is studied for different color images. In the simulation results, “Lena” (512 × 512) image is blurred by following kernel

$$A = \begin{bmatrix} 0.7A([15, 15]) & 0.15G(11, 9) & 0.15G(31, 13) \\ 0.1G(21, 11) & 0.8A([17, 17]) & 0.1A([13, 13]) \\ 0.0M(41, 90) & 0.2M(21, 45) & 0.6M(61, 135) \end{bmatrix},$$

“House” (512 × 512) image is blurred by

$$A = \frac{1}{3} \text{diag}(A([5, 5]), A([7, 7]), A([9, 9])),$$

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<sup>1</sup><http://sipi.usc.edu/database/>

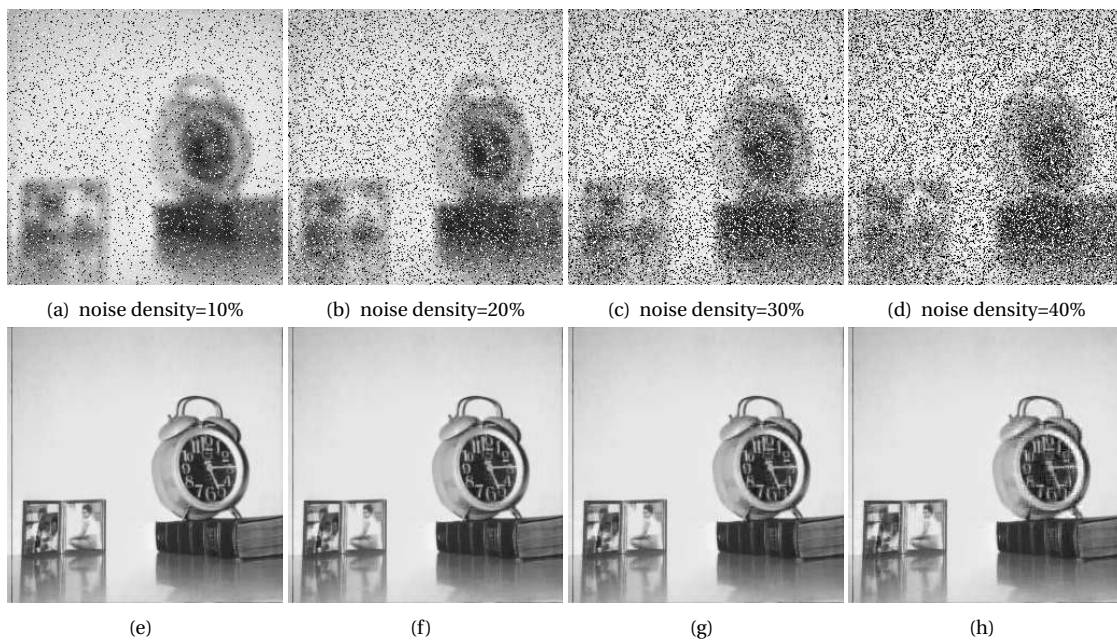


Figure 2: (a-d) Blurred and noisy images, (e-h) restored images for 5.1.12

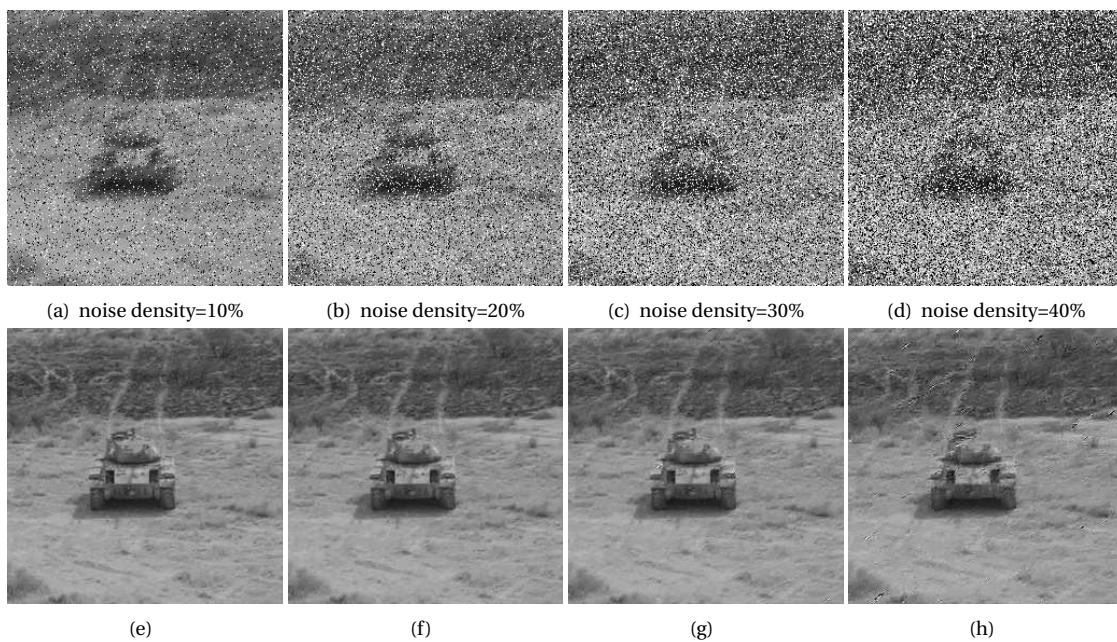


Figure 3: (a-d) Blurred and noisy images, (e-h) restored images for 7.1.09

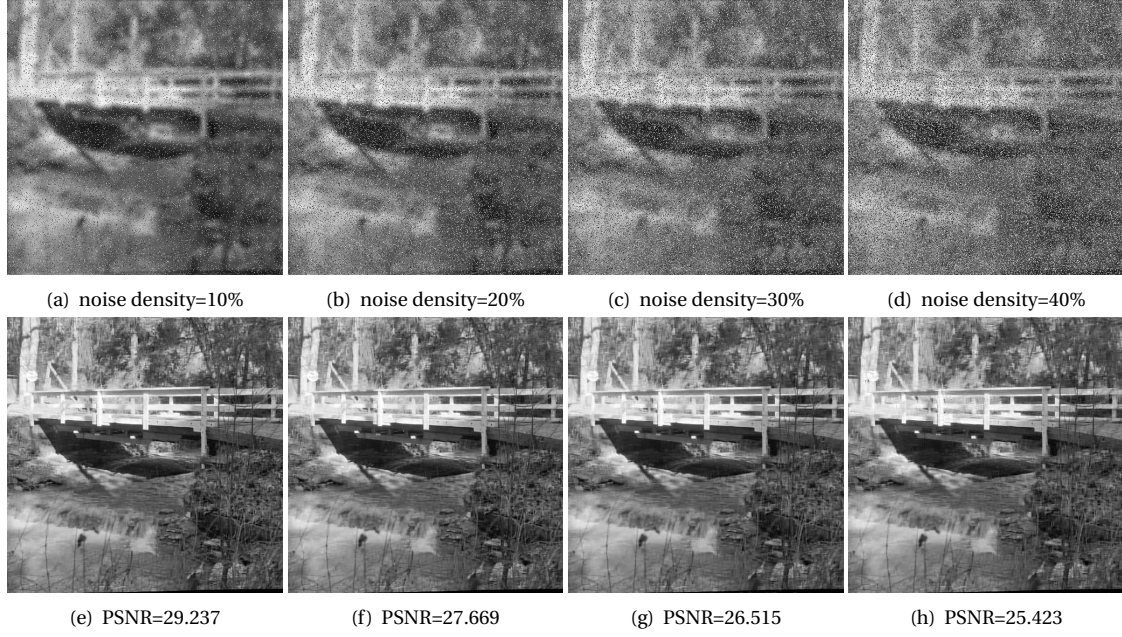


Figure 4: (a-d) Blurred and noisy images, (e-h) restored images for 5.2.10

Table 1: Test results for different grayscale images.

| imgae  | method   | density | PSNR  | SNR   | ISNR  | SSIM   |
|--------|----------|---------|-------|-------|-------|--------|
| 5.1.12 | Proposed | 10%     | 38.55 | 36.20 | 24.65 | 0.9814 |
|        |          | 20%     | 36.80 | 34.45 | 25.54 | 0.9765 |
|        |          | 30%     | 34.73 | 32.38 | 25.12 | 0.9678 |
|        |          | 40%     | 31.79 | 29.44 | 23.41 | 0.9973 |
|        | [19]     | 10%     | 38.37 | 36.02 | 24.55 | 0.9810 |
|        |          | 20%     | 36.68 | 34.33 | 25.43 | 0.9763 |
|        |          | 30%     | 34.52 | 32.18 | 24.96 | 0.9675 |
|        |          | 40%     | 31.44 | 29.09 | 23.02 | 0.9466 |
| 7.1.09 | Proposed | 10%     | 35.63 | 29.80 | 20.15 | 0.9371 |
|        |          | 20%     | 32.98 | 27.16 | 20.42 | 0.8868 |
|        |          | 30%     | 30.80 | 24.96 | 19.96 | 0.8260 |
|        |          | 40%     | 26.61 | 20.79 | 16.98 | 0.6917 |
|        | [19]     | 10%     | 35.56 | 29.73 | 20.11 | 0.9362 |
|        |          | 20%     | 32.81 | 26.98 | 20.20 | 0.8843 |
|        |          | 30%     | 30.71 | 24.89 | 19.83 | 0.8253 |
|        |          | 40%     | 25.76 | 19.94 | 16.12 | 0.6891 |



Figure 5: (a & d) Original image, (b & e) restored image without fractional-order total variation [19], (c & f)-(g & j)-(h & k)-(i & l) restored images by the proposed method with  $\alpha = 0.1$ ,  $\alpha = 0.75$ ,  $\alpha = 1$  and  $\alpha = 2$ , respectively

blurred kernel for “Peppers” image ( $256 \times 256$ ) is considered as

$$A = \begin{bmatrix} 0.8A([11, 11]) & 0.1G(11, 5) & 0.1M(21, 135) \\ 0.15A([11, 11]) & 0.7G(11, 5) & 0.15M(21, 135) \\ 0.2A([11, 11]) & 0.2G(11, 5) & 0.6M(21, 135) \end{bmatrix},$$

and finally blurred kernel  $A = B \otimes M(41, 135)$  is used for “Plate” with size ( $256 \times 256$ ), where

$$B = \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.15 & 0.7 & 0.15 \\ 0.2 & 0.2 & 0.6 \end{bmatrix}.$$

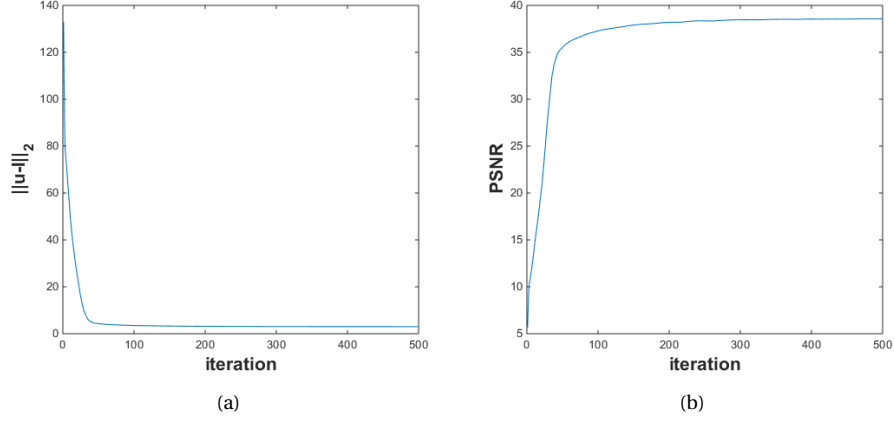


Figure 6: (a) Absolute error and (b) PSNR values for 5.1.12 with 0.1 noise density.

Also for the noise process, “salt & pepper” noise is used for “Lena”, “House” and “Plate” images and “random-valued” noise is used for “Peppers” image. Table 2 compares the results of the proposed method with the results of the algorithms in [21] and [33]. Other results are also given in Table 3 for test images. The results show the efficiency of the proposed algorithm in comparison with these algorithms. Also, the proposed algorithm outputs for different images with different noise densities are given in Figs 7-10. Figure 7 (e-h) shows the effect of the blurred kernel and noise on the color image. In this results, “salt & pepper” noise with 30% density is used. Test results for this case are included  $PSNR = 28.871$ ,  $SNR = 27.049$ ,  $ISNR = 20.636$  and  $SSIM = 0.9535$ . Also, the convergence of the proposed method for color images can be seen in Fig. 11.

Table 2: Test results for different color images.

| density |          | 10%   |        | 20%   |        | 30%   |        | 40%   |        |
|---------|----------|-------|--------|-------|--------|-------|--------|-------|--------|
| imgae   | method   | SNR   | SSIM   | SNR   | SSIM   | SNR   | SSIM   | SNR   | SSIM   |
| House   | Proposed | 27.23 | 0.9620 | 26.83 | 0.9591 | 26.04 | 0.9524 | 24.04 | 0.9348 |
|         | [21]     | 25.54 | 0.9185 | 24.57 | 0.9071 | 23.36 | 0.9024 | 22.58 | 0.8876 |
|         | [33]     | 22.70 | 0.9103 | 22.33 | 0.9045 | 22.08 | 0.9027 | 21.61 | 0.8904 |
| Lena    | Proposed | 25.15 | 0.9744 | 24.96 | 0.9734 | 24.66 | 0.9717 | 24.33 | 0.9699 |
|         | [21]     | 25.12 | 0.9035 | 24.48 | 0.8941 | 23.88 | 0.8860 | 23.90 | 0.8856 |
|         | [33]     | 23.50 | 0.8831 | 23.48 | 0.8841 | 23.87 | 0.8907 | 23.71 | 0.8864 |
| Peppers | Proposed | 26.34 | 0.9895 | 25.80 | 0.9877 | 24.49 | 0.9826 | 20.43 | 0.9547 |
|         | [21]     | 23.85 | 0.8681 | 22.88 | 0.8504 | 21.73 | 0.8302 | 20.39 | 0.7982 |
|         | [33]     | 22.34 | 0.8530 | 21.79 | 0.8379 | 20.88 | 0.8208 | 19.88 | 0.7941 |

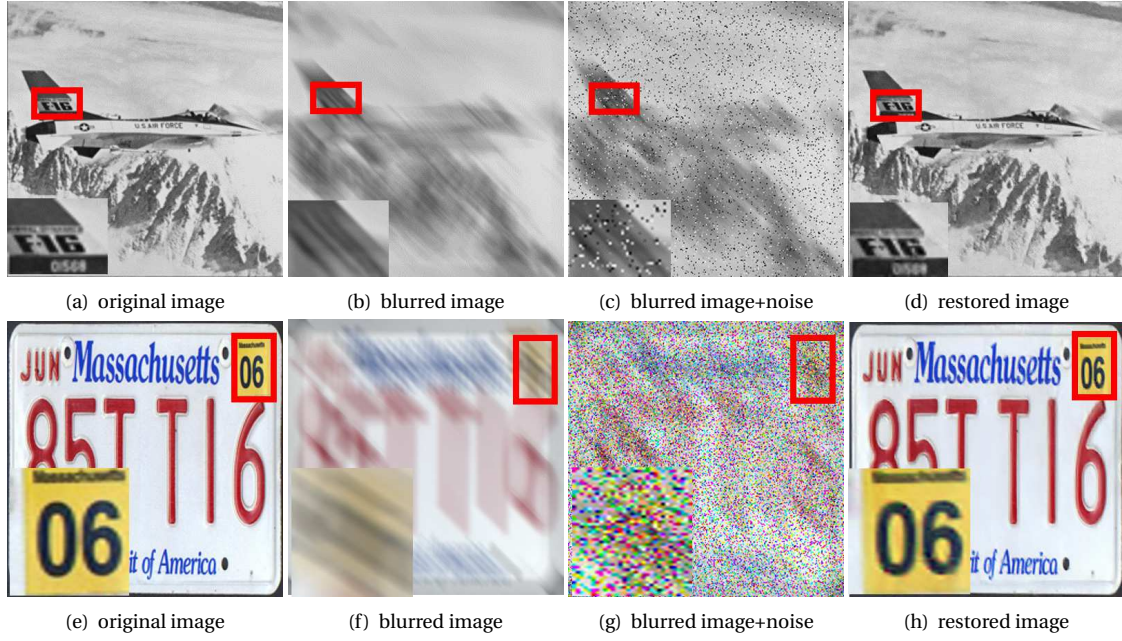


Figure 7: (a & e) Original images, (b & f) blurred images, (c & g) blurred and noisy images, (d & h) restored images.



Figure 8: (a-d) Blurred and noisy images, (e-h) restored images for Lena (512×512).

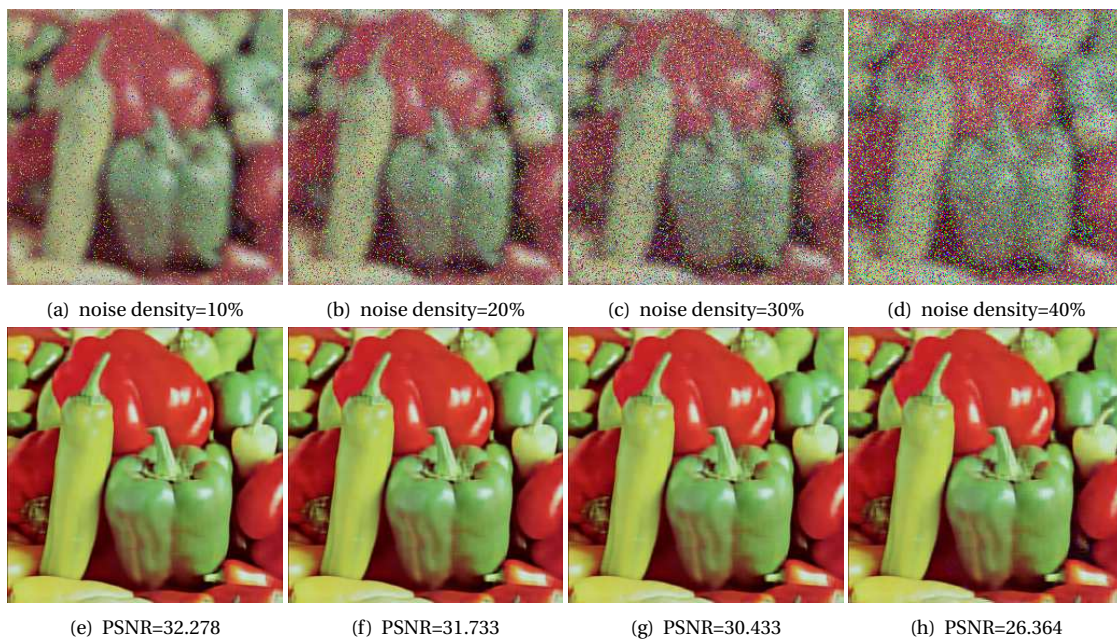


Figure 9: (a-d) Blurred and noisy images, (e-h) restored images for Peppers ( $256 \times 256$ ).

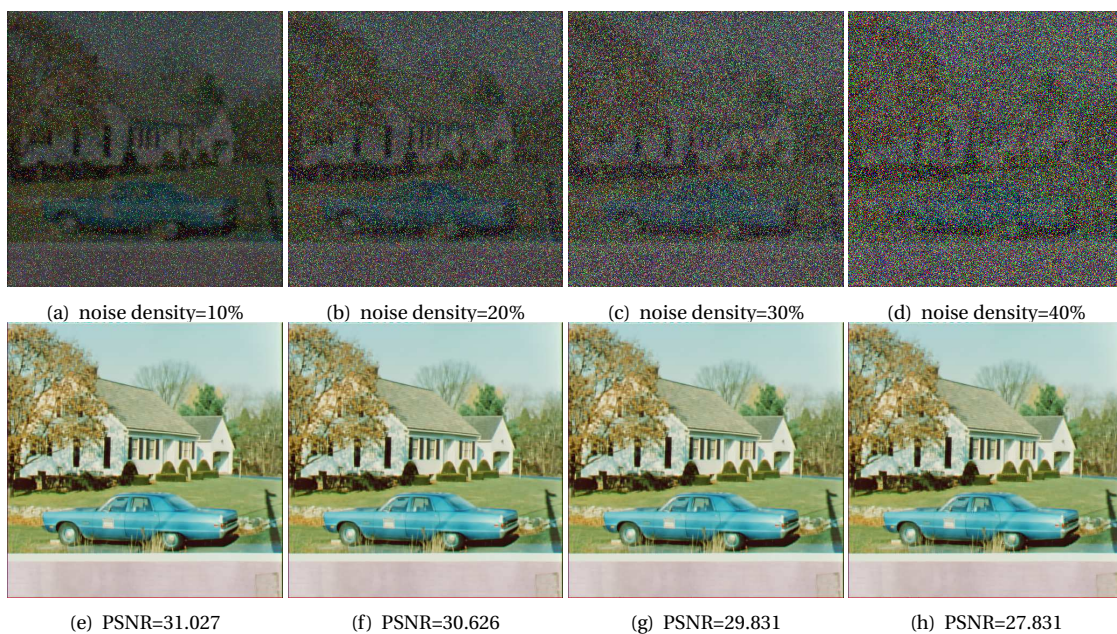


Figure 10: (a-d) Blurred and noisy images, (e-h) restored images for Househ ( $512 \times 512$ ).

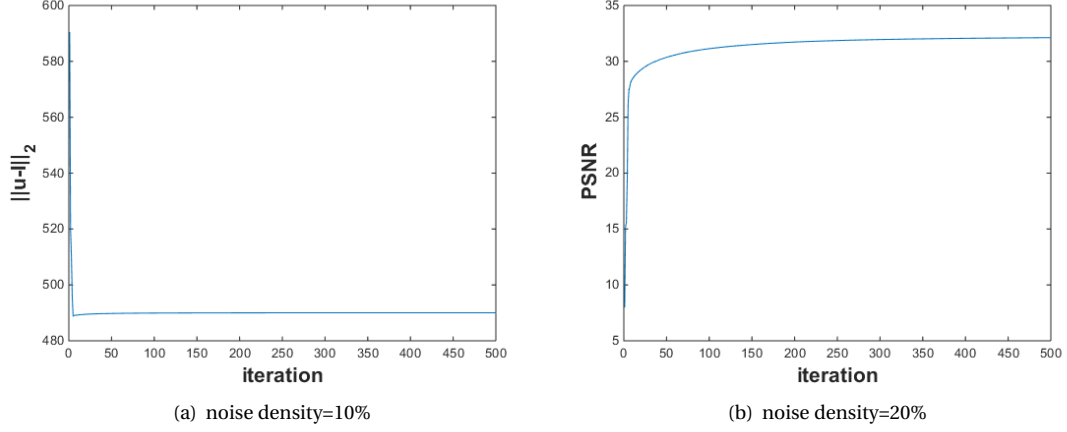


Figure 11: (a) Absolute error and (b) PSNR values for Lena (512×512) with 0.1 noise density.

Table 3: Test results for different grayscale images with different noise densities.

| imgae   | density | ISNR   | RE        | FSIM    |
|---------|---------|--------|-----------|---------|
| House   | 10%     | 24.054 | 0.0018923 | 0.99976 |
|         | 20%     | 23.907 | 0.0020754 | 0.99973 |
|         | 30%     | 23.342 | 0.0024927 | 0.99965 |
|         | 40%     | 21.567 | 0.0039502 | 0.99947 |
| Lena    | 10%     | 16.234 | 0.0030536 | 0.99914 |
|         | 20%     | 18.433 | 0.0031943 | 0.99908 |
|         | 30%     | 19.703 | 0.0034223 | 0.99898 |
|         | 40%     | 20.507 | 0.0036912 | 0.99886 |
| Peppers | 10%     | 16.785 | 0.0023246 | 0.99724 |
|         | 20%     | 17.983 | 0.0026354 | 0.99716 |
|         | 30%     | 17.946 | 0.0035549 | 0.99689 |
|         | 40%     | 14.841 | 0.0090734 | 0.99597 |

## 6 Conclusion

In this paper the nonconvex model based on fractional-order total variation and framelet transfer is introduced for image restoration. The proposed model is solved by ADMM and primal-dual methods. In the analysis of the proposed algorithm convergence analysis is studied and simulation results are evaluated. The results have been compared with other methods and these results show the efficiency of the pro-

posed algorithm in restoring images that have been damaged due to blur and noise.

#### **Conflict of interest**

Authors declare that they have no conflict of interest.

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