

Dichotomy Results for Fixed-Point Existence Problems for Boolean Dynamical Systems

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Abstract

A complete classification of the computational complexity of the fixed-point existence problem for boolean dynamical systems, i.e., finite discrete dynamical systems over the domain $\{0, 1\}$, is presented. For function classes \mathcal{F} and graph classes \mathcal{G} , an $(\mathcal{F}, \mathcal{G})$ -system is a boolean dynamical system such that all local transition functions lie in \mathcal{F} and the underlying graph lies in \mathcal{G} . Let \mathcal{F} be a class of boolean functions which is closed under composition and let \mathcal{G} be a class of graphs which is closed under taking minors. The following dichotomy theorems are shown: (1) If \mathcal{F} contains the self-dual functions and \mathcal{G} contains the planar graphs, then the fixed-point existence problem for $(\mathcal{F}, \mathcal{G})$ -systems with local transition function given by truth-tables is NP-complete; otherwise, it is decidable in polynomial time. (2) If \mathcal{F} contains the self-dual functions and \mathcal{G} contains the graphs having vertex covers of size one, then the fixed-point existence problem for $(\mathcal{F}, \mathcal{G})$ -systems with local transition function given by formulas or circuits is NP-complete; otherwise, it is decidable in polynomial time.

1 Introduction

Background on complex systems. A complex system, in a mathematical sense, can be viewed as a collection of highly interdependent variables. A discrete dynamical system is a complex system where variables update their values in discrete time. Though the interdependencies among the variables might have quite simple descriptions on a local level, the overall global behavior of the systems can be as complicated as unpredictable or undecidable (see, e.g., [12, 20]). This phenomenon has been widely studied in the theory of cellular automata [31, 33] and its applications (see, e.g., [16, 21]).

Finite discrete dynamical systems are characterized by finite sets of variables which can take values from a finite domain. In essence (see, e.g., [10, 9]), a finite discrete dynamical system (over a finite domain) consists of (1) a finite undirected graph, where vertices correspond to variables and edges correspond to an interdependence between the two connected variables, (2) a set of local transition functions, one for each vertex, that map values of variables depending on the values of all connected variables to new variable values, and (3) an update schedule that governs which variables are allowed to update their values in a certain time step. A formal definition is given in Sect. 2. Due to their structural simplicity and modelling flexibility, finite discrete dynamical systems are suitable for analyzing the behavior of real-world complex systems. In fact, the conception is motivated by

analysis and simulation issues in traffic flow (see, e.g., [2, 3]) and inter-domain routing [18]. It also has applications to two-species diffusion-reaction systems such as synchronous or asynchronous versions of the nearest neighbor coalescence reaction $A + A \rightarrow A$ on a lattice in the immobile reactant case [1].

A central problem in the study of discrete dynamical systems is the classification of systems according to the predictability of their behavior. In the finite setting, a certain behavioral pattern is considered predictable if it can be decided in polynomial time whether a given system will show the behavioral pattern [12]. In a rather strong sense, predictability and tractability are identified. It is not surprising that the reachability of patterns is, in general, an intractable problem, i.e., at least NP-hard (see, e.g., [17, 27, 7]). However, some restricted subclasses of finite discrete dynamical systems, i.e., systems given by restricted sets of local transition functions and network topologies, are known to possess easy-to-predict patterns (see, e.g., [5, 6, 7] and the discussion of related work below). For the purpose of analyzing and simulating real-world systems by finite discrete dynamical systems, it is highly desirable to have sharp boundaries between tractable and intractable cases.

A fundamental behavioral pattern for discrete dynamical systems are fixed points (homogeneous states, equilibria). A value assignment to the variables of a system is a fixed point if the values assigned to the variables are left unchanged if the system updates the values. A series of recent papers has been devoted to identification of finite systems with tractable/intractable fixed-point analyses [8, 30, 28, 29]. But although it is an old question how common intractability results are for discrete dynamical systems (see [32, Problem 19]), a precise characterization of the *islands of tractability* even for the fixed-point existence problem in the simplest case of the boolean domain $\{0, 1\}$ has remained an open problem (see [8]).

Contributions of the paper. We contribute to the problem of classifying boolean (discrete) dynamical systems, i.e., finite discrete dynamical systems over the domain $\{0, 1\}$, with regard to the computational complexity of the fixed-point existence problem, i.e., decide whether a given system has a fixed point, in two ways.

A first contribution is the proposal of a general analysis framework for systems. We say that a boolean dynamical system is an $(\mathcal{F}, \mathcal{G})$ -system if its local transition functions belong to the class \mathcal{F} of boolean functions and the underlying graph belongs to the graph class \mathcal{G} . We propose to consider two well-studied frameworks for functions and graphs (a formal introduction is given in Sect. 3):

- *Post classes* [23]: We assume that the function class \mathcal{F} is closed under composition (and some more reasonable operations). Equivalently, \mathcal{F} is a class of boolean functions that can be built by arbitrary circuits over gates from some finite logical basis [23]. Examples are the monotone functions, the linear functions, and the self-dual functions (i.e., functions f such that $f(x_1, \dots, x_n) = 1 - f(1 - x_1, \dots, 1 - x_n)$).
- *Graph minor classes* (see, e.g., [14]): We assume that the graph class \mathcal{G} is closed under taking minors, i.e., \mathcal{G} is closed under vertex deletions, edge deletions, and edge contractions. Equivalently, \mathcal{G} can be characterized by a finite set of forbidden minors [26]. An example is the class of planar graphs (together with the forbidden minors $K_{3,3}$ and K^5).

Certainly, other schemes can be devised for system classifications. In fact, many results in literature do not fit into this framework (see the discussion of related work below). However, the proposed scheme has strong features: first, it exhausts the class of all boolean dynamical systems; second, for fixed function classes \mathcal{F} and fixed graph classes \mathcal{G} , it is decidable in polynomial time whether a given system is in fact, an $(\mathcal{F}, \mathcal{G})$ -system (supposed the local transition functions are represented by lookup-tables); and third, it allows elegant proofs of dichotomy theorems (as exemplified by our second contribution). We mention that, as fixed points are invariant under changes of the update regime, a scheme for classifying update schedules is not needed for our study.

The main contribution of the paper is a complete complexity classification of the fixed-point existence problem with respect to our analysis framework. We make a distinction of the problem into three categories: systems for which the local transition functions are given by lookup-tables, by formulas (over logical bases), or by circuits (over logical bases). For each case we obtain a tractability/intractability dichotomy. Interestingly, the dichotomy theorems for formulas and circuits coincide. Let \mathcal{F} be a Post class of boolean functions and let \mathcal{G} be graph class closed under taking minors. The following is proved in Sect. 4:

- *Dichotomy for boolean dynamical systems based on lookup-tables:* If \mathcal{F} contains the self-dual functions and \mathcal{G} contains the planar graphs, then the fixed-point existence problem for $(\mathcal{F}, \mathcal{G})$ -systems with local transition functions given by lookup-tables is NP-complete; in all other cases, it is decidable in polynomial time (Theorem 4.1).
- *Dichotomy for boolean dynamical systems based on formulas or circuits:* If \mathcal{F} contains the self-dual functions and \mathcal{G} contains the graphs having vertex covers of size one, then the fixed-point existence problem for $(\mathcal{F}, \mathcal{G})$ -systems with local transition functions given by formulas/circuits (over the logical basis of \mathcal{F}) is NP-complete; in all other cases, it is decidable in polynomial time (Theorem 4.8).

The results provide easy criteria for deciding whether $(\mathcal{F}, \mathcal{G})$ -systems have tractable or intractable fixed-point existence problems. For instance, tractability follows for systems with linear or monotone local transition function on arbitrary networks (see also [8]). The tractability regions with respect to arbitrary local transition functions and restricted graphs correspond to bounded treewidth (in the case of lookup-tables) and bounded degree (in the case of formulas or circuits). Tractable network classes (for lookup-table-based systems) are, e.g., trees, outer-planar graphs, or series-parallel graphs.

Related work. There is a series of work regarding the complexity of certain computational problems for discrete dynamical systems (see, e.g., [17, 27, 8, 5, 6, 30, 28, 7] and the references therein).

Detailed studies of computational problems related to fixed-point existence have been reported in [8, 30, 28, 29]. As shown in [8], tractable cases for fixed-point existence are constituted by systems with linear, generalized local transition functions, systems with monotone local transition functions and systems where local transition function are computed by gates (of unbounded fan-in) from $\{\text{AND}, \text{OR}, \text{NAND}, \text{NOR}\}$; intractable cases are boolean dynamical systems having local transition functions computed by gates (of unbounded fan-in) from the sets $\{\text{NAND}, \text{XNOR}\}$, $\{\text{NAND}, \text{XOR}\}$, $\{\text{NOR}, \text{XNOR}\}$, or $\{\text{NOR}, \text{XOR}\}$. Moreover, in [30, 28, 29], the problem of enumerating fixed points of boolean

dynamical systems has been studied: counting the number of fixed points is in general $\#P$ -complete, even counting the number of fixed points for boolean dynamical systems with monotone local transition functions over planar bipartite graphs or over uniformly sparse graphs is $\#P$ -complete. We note that all system classes considered are based on formula or circuit representations, i.e., the intractability results fall into the scope of Theorem 4.8.

Mainly, tractability and intractability results have been shown for various version of pattern reachability problems such as garden of Eden existence (e.g., [8, 30, 28]), predecessor existence (e.g., [27, 6]), parameterized and unparameterized reachability (e.g., [17, 27, 6, 4, 7]). To summarize, the system subclasses considered restrict the local transition functions to linear functions, monotone functions, various types of threshold functions, or symmetric functions. Except the linear and monotone functions none of these classes is closed under composition. Restrictions to the dependency networks involve planar graphs, regular graphs, bounded-degree graphs, star networks, and bounded pathwidth.

In the theory of finite discrete dynamical systems, tight dichotomy results have been shown for very restricted classes of systems (as, e.g., in [12]). Exhausting results similar to those of this paper are standard for constraint satisfaction problems (see, e.g., [11] for a survey). In fact, our results rely on close relationships between fixed-point and constraint satisfaction problems (as in the proof of Theorem 4.3).

2 The Dynamical Systems Framework

In this section, we describe our model of dynamical systems. We follow the approach given by [9] with a marginal generalization regarding update schedules.

Dynamical systems. The underlying network structure of a dynamical system is given by an undirected graph $G = (V, E)$ without multi-edges and loops. We suppose that the set V of vertices is ordered. So, without loss of generality, we assume $V = \{1, 2, \dots, n\}$. For any vertex set $U \subseteq V$, let $N_G(U)$ denote the neighbors of U in G , i.e.,

$$N_G(U) =_{\text{def}} \{ j \mid j \notin U \text{ and there is an } i \in U \text{ such that } \{i, j\} \in E \}.$$

If $U = \{i\}$ for some vertex i , then we use $N_G(i)$ as a shorthand for $N_G(\{i\})$. Define $N_G^0(i) =_{\text{def}} \{i\} \cup N_G(i)$. The degree d_i of a vertex i is the number of its neighbors, i.e., $d_i =_{\text{def}} \|N_G(i)\|$.

A *dynamical system* S over a domain \mathcal{D} is a pair (G, F) where $G = (V, E)$ is an undirected graph (the *network*) and $F = \{f_i \mid i \in V\}$ is a set of *local transition functions* $f_i : \mathcal{D}^{d_i+1} \rightarrow \mathcal{D}$. The intuition of the definition is that each vertex i corresponds to an active element (entity, agent, actor etc.) which is always in some state x_i and which is capable to change its state, if necessary. The domain of S formalizes the set of possible states of all vertices of the network, i.e., for all $i \in V$, it always holds that $x_i \in \mathcal{D}$. A vector $\vec{x} = (x_i)_{i \in V}$ such that $x_i \in \mathcal{D}$ for all $i \in V$ is called a *configuration* of S . If it is more convenient, then we also say that a mapping $I : V \rightarrow \mathcal{D}$ is a configuration. A *subconfiguration* of I with respect to $A \subseteq V$ is a mapping $I[A] : A \rightarrow \mathcal{D}$ such that $I[A](i) = I(i)$ for all $i \in A$. The local transition function f_i for some vertex i describes how i changes its state depending on the states of its neighbors $N_G(i)$ in the network and its own state.

Discrete dynamical systems. We are particularly interested in dynamical system operating on a discrete time-scale. A *discrete dynamical system* $\mathcal{S} = (S, \alpha)$ consists of a dynamical

system S and a mapping $\alpha : \{1, \dots, T\} \rightarrow \mathcal{P}(V)$, where V is a set of vertices of the network of S and $T \in \mathbb{N}$. The mapping α is called the *update schedule* and specifies which state updates are realized at certain time-steps: for $t \in \{1, \dots, T\}$, $\alpha(t)$ specifies those vertices that simultaneously update their states in step t .

Global maps. A discrete dynamical system $\mathcal{S} = (S, \alpha)$ over domain \mathcal{D} induces a global map $\mathbf{F}_{\mathcal{S}} : \mathcal{D}^n \rightarrow \mathcal{D}^n$ where n is the number of vertices of S . For each vertex $i \in V$, define an *activity function* φ_i for a set $U \subseteq V$ and $\vec{x} = (x_1, \dots, x_n) \in \mathcal{D}^n$ by

$$\varphi_i[U](\vec{x}) =_{\text{def}} \begin{cases} f_i(x_{i_1}, \dots, x_{i_{d_i+1}}) & \text{if } i \in U \\ x_i & \text{if } i \notin U \end{cases}$$

where $\{i_1, i_2, \dots, i_{d_i+1}\} = N_G^0(i)$. For a set $U \subseteq V$, define the *global transition function* $\mathbf{F}_{\mathcal{S}}[U] : \mathcal{D}^n \rightarrow \mathcal{D}^n$ for all $\vec{x} \in \mathcal{D}^n$ by

$$\mathbf{F}_{\mathcal{S}}[U](\vec{x}) =_{\text{def}} (\varphi_1[U](\vec{x}), \dots, \varphi_n[U](\vec{x})).$$

Note that the global transition function does not refer to the update schedule, i.e., it only depends on the dynamical system S and not on \mathcal{S} . The function $F_S : \mathcal{D}^n \rightarrow \mathcal{D}^n$ computed by the discrete dynamical system \mathcal{S} , the *global map* of \mathcal{S} , is defined by

$$\mathbf{F}_{\mathcal{S}} =_{\text{def}} \prod_{k=1}^T \mathbf{F}_{\mathcal{S}}[\alpha(k)].$$

Fixed points. The central notion for our study of dynamical systems is the concept of a fixed point, i.e., a configuration which does not change under any global behavior of the system. Let $S = (G, \{f_i \mid i \in V\})$ be a dynamical system over domain \mathcal{D} . A configuration $\vec{x} \in \mathcal{D}^n$ is said to be a *local fixed point of S for $U \subseteq V$* if and only if $\mathbf{F}_{\mathcal{S}}[U](\vec{x}) = \vec{x}$. A configuration $\vec{x} \in \mathcal{D}^n$ is said to be a *fixed point of S* if and only if \vec{x} is a local fixed point of S for V .

The following useful proposition is easily seen.

Proposition 2.1. *Let $S = (G, \{f_i \mid i \in V\})$ be a dynamical system over domain \mathcal{D} .*

1. *If the configuration $\vec{x} \in \mathcal{D}^n$ is a local fixed point of S for $U' \subseteq V$ and \vec{x} is a local fixed point of S for $U'' \subseteq V$, then \vec{x} is a local fixed point of S for $U' \cup U''$.*
2. *A configuration $\vec{x} \in \mathcal{D}^n$ is a local fixed point of S for $U \subseteq V$ if and only if \vec{x} is a local fixed point for all $U' \subseteq U$.*

Notice that the second item of Proposition 2.1 shows that the concept of fixed points is independent of update schedules.

Corollary 2.2. *Let $S = (G, \{f_i \mid i \in V\})$ be a dynamical system over domain \mathcal{D} . A configuration $\vec{x} \in \mathcal{D}^n$ is a fixed point of S if and only if for all update schedules $\alpha : \{1, \dots, T\} \rightarrow \mathcal{P}(V)$, it holds that $\mathbf{F}_{(S, \alpha)}(\vec{x}) = \vec{x}$.*

3 The Analysis Framework

In this section, we give a formal description of classification schemes for dynamical systems. Local transition functions are classified by Post classes (i.e., closures under composition) and networks are classified by forbidden minors (i.e., closures under taking minors).

Post classes. We adopt notation from [11]. An n -ary boolean function f is a mapping $f : \{0, 1\}^n \rightarrow \{0, 1\}$. Let BF denote the class of all boolean functions. There are two 1-ary boolean functions: $\text{id}(x) =_{\text{def}} x$ and $\text{not}(x) =_{\text{def}} 1 - x$ (which are denoted in formulas by x for $\text{id}(x)$ and \bar{x} for $\text{not}(x)$). For $b \in \{0, 1\}$, a boolean function f is said to be *b-reproducing* if and only if $f(b, \dots, b) = b$. For binary n -tuples $\vec{a} = (a_1, \dots, a_n)$ and $\vec{b} = (b_1, \dots, b_n)$, we say that $(a_1, \dots, a_n) \leq (b_1, \dots, b_n)$ if and only if for all $i \in \{1, \dots, n\}$, it holds that $a_i \leq b_i$. An n -ary boolean function f is said to be *monotone* if and only if for all $\vec{x}, \vec{y} \in \{0, 1\}^n$, $\vec{x} \leq \vec{y}$ implies $f(\vec{x}) \leq f(\vec{y})$. An n -ary boolean function f is said to be *self-dual* if and only if for all $(x_1, \dots, x_n) \in \{0, 1\}^n$, it holds that $f(x_1, \dots, x_n) = \text{not}(f(\text{not}(x_1), \dots, \text{not}(x_n)))$. A boolean function f is linear if and only if there are constants $a_1, \dots, a_n \in \{0, 1\}$ such that $f(x_1, \dots, x_n) = a_0 \oplus a_1 x_1 \oplus \dots \oplus a_n x_n$. Here, \oplus is understood as addition modulo 2 and xy is understood as multiplication modulo 2.

We say that a function class \mathcal{F} is *Post* if and only if \mathcal{F} contains the function id and \mathcal{F} is closed under introduction of fictive variables, permutations of variables, identification of variables, and superposition (see, e.g., [11] for a formal definition). It is a famous theorem by Post [23] that the family of all Post classes of boolean functions is a countable lattice with respect to set inclusion (see, e.g., [23, 34] for a proof). The maximal meet-irreducible classes are the following (see, e.g., [11]):

R_0	$=_{\text{def}} \{ f \in \text{BF} \mid f \text{ is 0-reproducing} \}$	with logical basis {AND, XOR}
R_1	$=_{\text{def}} \{ f \in \text{BF} \mid f \text{ is 1-reproducing} \}$	with logical basis {OR, XNOR}
L	$=_{\text{def}} \{ f \in \text{BF} \mid f \text{ is linear} \}$	with logical basis {XOR, 0, 1}
M	$=_{\text{def}} \{ f \in \text{BF} \mid f \text{ is monotone} \}$	with logical basis {AND, OR, 0, 1}
D	$=_{\text{def}} \{ f \in \text{BF} \mid f \text{ is self-dual} \}$	with logical basis $\{(x \wedge \bar{y}) \vee (x \wedge \bar{z}) \vee (\bar{y} \wedge \bar{z})\}$

From the structure of Post's lattice follows that each other class which does not contain all boolean functions is included in an intersection of two of the classes listed.

Graph minor classes. We adopt notation from [14]. Let X and Y be two undirected graphs. We say that X is minor of Y if and only if there is an isomorphic subgraph Y' of Y such that X is obtained by contracting edges of Y' . Let \preceq be the relation on graphs defined by $X \preceq Y$ if and only if X is a minor of Y . A class \mathcal{G} of graphs is said to be *closed under taking minors* if and only if for all graphs G and G' , if $G \in \mathcal{G}$ and $G' \preceq G$, then $G' \in \mathcal{G}$. Let \mathcal{X} be any set of graphs. $\text{Forb}_{\preceq}(\mathcal{X})$ denotes the class of all graphs without a minor in \mathcal{X} (and which is closed under isomorphisms). More specifically, $\text{Forb}_{\preceq}(\mathcal{X}) =_{\text{def}} \{G \mid G \not\preceq X \text{ for all } X \in \mathcal{X}\}$. The set \mathcal{X} is called the set of *forbidden minors*. Note that $\text{Forb}_{\preceq}(\emptyset)$ is the class of all graphs. As usual, we write $\text{Forb}_{\preceq}(X_1, \dots, X_n)$ instead of $\text{Forb}_{\preceq}(\{X_1, \dots, X_n\})$. A useful property of the forbidden-minor classes is the monotonicity with respect to \preceq , i.e., $X \preceq Y$ implies $\text{Forb}_{\preceq}(X) \subseteq \text{Forb}_{\preceq}(Y)$.

The celebrated Graph Minor Theorem, due to Robertson and Seymour [26], shows that there are only countably many graph classes closed under taking minors: A class \mathcal{G}

of graphs is closed under taking minors if and only if there is a finite set \mathcal{X} such that $\mathcal{G} = \text{Forb}_{\preceq}(\mathcal{X})$. An important consequence of this theorem is that for each graph class \mathcal{G} which is closed under taking minors there exists a polynomial-time algorithm to decide whether a given graph belongs to \mathcal{G} [25].

The most prominent example of a characterization of a class closed under taking minors in terms of forbidden sets are the planar graphs. Let K^n denote the complete graphs on n vertices and let $K_{n,m}$ denote the complete bipartite graph having n vertices in one component and m vertices in the other component. The well-known Kuratowski-Wagner theorem (see, e.g., [14]) states that a graph G is planar if and only if G belongs to $\text{Forb}_{\preceq}(K_{3,3}, K^5)$. Planar graphs have an algorithmically important property: a graph X is planar if and only if $\text{Forb}_{\preceq}(X)$ has bounded treewidth [24]. As we use treewidth of a graph only in a black-box fashion, we refer to, e.g., [14] for a definition of treewidth. A class \mathcal{G} of graphs is said to have *bounded treewidth* if and only if there is a $k \in \mathbb{N}$ such that all graphs in the class have treewidth at most k .

A similar but much less subtle behavior to planar graphs show graphs with a vertex cover of size one. Let $G = (V, E)$ be a graph. We say that a subset $U \subseteq V$ is a *vertex cover* of G if and only if for all edges $\{u, v\} \in E$, it holds that $\{u, v\} \cap U \neq \emptyset$. It is known that the class of graphs having a vertex cover of size at most k is closed under taking minors [13]. Moreover, G has a vertex cover of size one if and only if G belongs to $\text{Forb}_{\preceq}(K^3, K^2 \oplus K^2)$ [13], where for graphs G and G' , $G \oplus G'$ denotes the graph obtained by the disjoint union of G and G' . A class of graphs is said to have *bounded degree* if and only if there is a $k \in \mathbb{N}$ such that all graphs in the class have a maximum vertex-degree of at most k .

Proposition 3.1. *Let X be a graph. Then, X has a vertex cover of size one if and only if $\text{Forb}_{\preceq}(X)$ has bounded degree.*

Proof. For (\Rightarrow) , suppose that X has a vertex cover of size one. Then, X consists of some star graph $K_{1,k}$ and some isolated vertices u_1, \dots, u_r . Assume that $\text{Forb}_{\preceq}(X)$ does not have bounded degree. Thus, there exists a graph $G \in \text{Forb}_{\preceq}(X)$ of maximum vertex-degree at least $k + r$. So, G contains a subgraph $K_{1,k+r}$. Hence, $X \preceq G$, contradicting $G \in \text{Forb}_{\preceq}(X)$. Therefore, $\text{Forb}_{\preceq}(X)$ has bounded degree.

For (\Leftarrow) , suppose that X does not have any vertex cover of size one. It is easily seen that in this case, X contains a triangle or two non-incident edges. First, suppose X contains a triangle K^3 . As $\text{Forb}_{\preceq}(K^3)$ contains the class of all trees, which certainly does not have bounded degree, we easily obtain from $K^3 \preceq X$, that $\text{Forb}_{\preceq}(X)$ does not have bounded degree as well. Second, suppose X contains at least two non-incident edges, i.e., $K^2 \oplus K^2 \preceq X$. It follows that $\text{Forb}_{\preceq}(X)$ contains for all $k \in \mathbb{N}$, the star graph $K_{1,k}$. Thus, $\text{Forb}_{\preceq}(X)$ does not have bounded degree. This completes the proof of the direction from left to right. \square

4 Islands of Tractability for Fixed-Point Existence

In this section we are interested in the computational complexity of the following problem. Let \mathcal{F} be a class of boolean functions and let \mathcal{G} be a class of graphs.

Problem: $\text{FIXEDPOINTS}(\mathcal{F}, \mathcal{G})$
Input: A boolean dynamical system $S = (G, \{f_1, \dots, f_n\})$ such that $G \in \mathcal{G}$ and
 for all $i \in \{1, \dots, n\}$, $f_i \in \mathcal{F}$
Question: Does S have a fixed point?

The complexity of the problem depends on how transition functions are represented. We consider the cases of look-up table, formula, and circuit representations. The corresponding problems are denoted by FIXEDPOINTS_T , FIXEDPOINTS_F , and FIXEDPOINTS_C . It is obvious that all problem versions belong to NP. We say that a problem is intractable if it is NP-hard, and it is tractable if it is solvable in polynomial time.

4.1 The Case of Local Transition Functions Given by Look-up Tables

The main result of this subsection is the following dichotomy result.

Theorem 4.1. *Let \mathcal{F} be a Post class of boolean functions and let \mathcal{G} be a class of graphs closed under taking minors. If $\mathcal{F} \supseteq \text{D}$ and $\mathcal{G} \supseteq \text{Forb}_{\leq}(K_{3,3}, K^5)$ then $\text{FIXEDPOINTS}_T(\mathcal{F}, \mathcal{G})$ is intractable. Otherwise, $\text{FIXEDPOINTS}_T(\mathcal{F}, \mathcal{G})$ is tractable.*

We postpone the proof to the end of this paragraph when we have proved a number of complexity results for several classes of systems.

Dynamical systems with a tractable fixed-point analysis. We first summarize the tractable classes of local transition functions of systems on arbitrary underlying networks. The results presented next are well-known or follow easily from the definitions.

- Proposition 4.2.**
1. $\text{FIXEDPOINTS}_T(\text{R}_1, \text{Forb}(\emptyset))$ is solvable in polynomial time.
 2. $\text{FIXEDPOINTS}_T(\text{R}_0, \text{Forb}(\emptyset))$ is solvable in polynomial time.
 3. [8] $\text{FIXEDPOINTS}_T(\text{M}, \text{Forb}(\emptyset))$ is solvable in polynomial time.
 4. [8] $\text{FIXEDPOINTS}_T(\text{L}, \text{Forb}(\emptyset))$ is solvable in polynomial time.

Notice that D is the only remaining class which does not contain all boolean functions.

Next we restrict network classes. To identify tractable cases, we express the fixed-point existence problem as a constraint satisfaction problem. A constraint satisfaction problem (CSP) consists of triples $(X, \mathcal{D}, \mathcal{C})$, where $X = \{x_1, \dots, x_n\}$ is the set of variables, \mathcal{D} is the domain of the variables, \mathcal{C} is a set of constraints Rx_{i_1}, \dots, x_{i_k} having associated corresponding relations R_{i_1, \dots, i_k} , such that there exists an assignment $I : X \rightarrow \mathcal{D}$ satisfying $(I(x_{i_1}), \dots, I(x_{i_k})) \in R_{i_1, \dots, i_k}$ for all constraints $Rx_{i_1}, \dots, x_{i_k} \in \mathcal{C}$. We suppose that \mathcal{C} is listed by pairs $\langle Rx_{i_1}, \dots, x_{i_k}, R_{i_1, \dots, i_k} \rangle$.

Theorem 4.3. *Let X be a planar graph. Then, $\text{FIXEDPOINTS}_T(\text{BF}, \text{Forb}_{\leq}(X))$ is solvable in polynomial time.*

Proof. We describe a general reduction from $\text{FIXEDPOINTS}_T(\mathcal{F}, \mathcal{G})$ to constraint satisfaction problems. Suppose we are given a dynamical system $S = (G, \{f_1, \dots, f_n\})$. Let $G = (V, E)$ be the underlying network. Without loss of generality, we may assume that G

does not have isolated vertices, i.e., $d_i \geq 1$ for all $i \in V$. Define $\text{CSP}(S) = (X, \mathcal{D}, \mathcal{C})$ to be the constraint satisfaction problem specified as follows:

$$\begin{aligned} X &=_{\text{def}} \{x_1, \dots, x_n\} \\ \mathcal{D} &=_{\text{def}} \bigcup_{i \in V} \mathcal{D}_i \text{ where for all } i \in V, \\ &\quad \mathcal{D}_i =_{\text{def}} \{(I, i) \mid I : N_G^0(i) \rightarrow \{0, 1\} \text{ such that } f_i(I(i_0), \dots, I(i_k)) = I(i) \\ &\quad \text{where } \{i_0, \dots, i_k\} = N_G^0(i)\} \\ \mathcal{C} &=_{\text{def}} \{Ex_i x_j \mid \{i, j\} \in E(G) \text{ and } i \leq j\} \text{ where for all } i \leq j, \\ &\quad E_{ij} =_{\text{def}} \{(I_i, i), (I_j, j)\} \mid I_i(k) = I_j(k) \text{ for all } k \in N_G^0(i) \cap N_G^0(j)\} \end{aligned}$$

Let $n = \|V\|$ and $m = \|E\|$. By construction of $\text{CSP}(S)$ we obtain that $\|X\| = n$ and that the number of constraints in \mathcal{C} is just m . The size of the domain \mathcal{D} is at most proportional to $\sum_{i \in V} (1 + d_i) \cdot 2^{1+d_i}$, and the size of the set of constraint relations can be bounded by

$$c \cdot \sum_{\{i, j\} \in E} (2 + d_i + d_j) \cdot 2^{1+d_i} \cdot 2^{1+d_j} \leq c \cdot \left(\sum_{i \in V} (1 + d_i) \cdot 2^{1+d_i} \right)^2$$

for some constant $c > 0$. The latter holds because $d_i \geq 1$ for all $i \in V$. All in all, this easily implies that $|\text{CSP}(S)| = O(|S|^2)$. Hence, $\text{CSP}(S)$ is computable in polynomial time in the size of S . We have to show that

$$S \text{ has a fixed point} \iff \text{CSP}(S) \text{ has a satisfying assignment.}$$

We prove both directions of the equivalence separately.

For (\Rightarrow) , suppose the configuration $I : V \rightarrow \{0, 1\}$ is a fixed point of S . Define an assignment $I' : \{x_1, \dots, x_n\} \rightarrow \mathcal{D}$ by $I'(x_i) =_{\text{def}} I[N_G^0(i)]$. We show that I' satisfies all constraints. For all $i \in V$, let I_i denote $I[N_G^0(i)]$. Let $Ex_i x_j$ be any constraint of $\text{CSP}(S)$ and let E_{ij} be the relation associated with $Ex_i x_j$. By definition, $\{i, j\} \in E$. Since I is a fixed point, it holds that (I_i, i) and (I_j, j) belong to \mathcal{D} . For all $k \in N_G^0(i) \cap N_G^0(j)$ we obtain $I_i(k) = I'(x_i)(k) = I'(x_j)(k) = I_j(k)$. Thus, $\{(I_i, i), (I_j, j)\}$ lies in E_{ij} . This proves the direction from left to right.

For (\Leftarrow) , suppose I is a satisfying assignment for $\text{CSP}(S)$. Let $(I_i, i) \in \mathcal{D}$ be the pair which I assigns to the variable $x_i \in V$. Define a configuration $I' : V \rightarrow \{0, 1\}$ by $I'(i) =_{\text{def}} I_i(i)$. We are done if we are able to show that the following is true for all $i \in V$:

$$I'[N_G^0(i)] \equiv I_i \tag{4.1}$$

Since for all $i \in V$, $f_i(I_i(i_0), \dots, I_i(i_k)) = I_i(i)$ where $\{i_0, \dots, i_k\} = N_G^0(i)$, Eq. (4.1) implies that I' is a fixed point of S . To verify the equation, we consider i and all neighbors of i individually. Let $j \in N_G^0(i)$. If $j = i$, there is nothing to show. So let $j \neq i$. Since $\{i, j\} \in E(G)$ and since I is a satisfying assignment, $I_i(k) = I_j(k)$ holds for all $k \in N_G^0(i) \cap N_G^0(j)$. Thus, we obtain $I'(j) = I_j(j) = I_i(j)$. This shows the correctness of Eq. (4.1) for all $i \in V$. Hence, the direction from right to left is proved.

We thus have established a polynomial-time many-one reduction between the fixed-point existence problem and constraint satisfaction problems. Moreover, it is easy to see that the graph G of a given dynamical system S is isomorphic to $\text{CSP}(S)$'s constraint graph which consists of the vertex set $\{x_1, \dots, x_n\}$ and the edge set $\{\{x_i, x_j\} \mid Ex_i x_j \in \mathcal{C}\}$.

Thus, if X is planar, then $\text{Forb}_{\preceq}(X)$ has bounded treewidth, and so the constraint graph of $\text{CSP}(S)$ has bounded treewidth for each dynamical system $S = (G, \{f_1, \dots, f_n\})$ such that $G \in \text{Forb}_{\preceq}(X)$. The theorem follows from the well-known polynomial-time algorithms for constraint satisfaction problems having constraint graphs of bounded treewidth [15]. \square

Dynamical systems with an intractable fixed-point analysis. We turn to the intractable cases of the fixed-point existence problem. As a first step, $\text{FIXEDPOINTS}_T(\mathcal{F}, \mathcal{G})$ is shown to be NP-complete for arbitrary boolean, local transition functions and planar networks. Recall that similar results have only been shown for formula representation.

We use PLANAR 3SAT to prove NP-hardness. Let $H = C_1 \wedge \dots \wedge C_m$ be a propositional formula in conjunctive normal form where each clause C_j consists of three literals (for short, 3CNF) where positive literals x_i and negative literals $\neg x_i$ are taken from the set of variables $\{x_1, \dots, x_n\}$. The graph representation $\Gamma(H)$ of H is a bipartite graph consisting of the vertex set $\{x_1, \dots, x_n, C_1, \dots, C_m\}$ and all edges $\{x_i, C_j\}$ such that variable x_i appears as a literal in the clause C_j . A planar 3CNF is a 3CNF such that its graph representation is planar. PLANAR 3SAT is the problem to decide whether a given planar 3CNF is satisfiable. It is well known that PLANAR 3SAT is an NP-complete problem [19].

Theorem 4.4. $\text{FIXEDPOINTS}_T(\text{BF}, \text{Forb}(K_{3,3}, K^5))$ is NP-complete, even restricted to underlying networks having maximum vertex degree three.

Note that for graphs having maximum degree at most two the problem is tractable.

Proof. Let $H = C_1 \wedge C_2 \wedge \dots \wedge C_m$ be a 3CNF having variables x_1, \dots, x_n and a planar graph representation $\Gamma(H)$ where $V(\Gamma(H)) = U \cup W$, $U = \{x_1, \dots, x_n\}$, and $W = \{C_1, \dots, C_m\}$. Suppose that $V(\Gamma(H))$ is totally ordered such that the clause vertices of W come before the variable vertices of U .

We construct the following system. For the underlying network $G = (V, E)$, compute an embedding of $\Gamma(H)$ in the plane (in linear time). Now replace the vertices of U in the following way. Let $x_i \in U$ and suppose that x_i 's neighbors C_{j_1}, \dots, C_{j_r} in $\Gamma(H)$ are clockwise ordered with respect to the planar embedding, such that j_1 is minimal in $\{j \mid x_i \text{ appears as a literal in } C_j\}$. Replace the vertex $x_i \in U$ by a cycle $\{x_{i,j_1}, \dots, x_{i,j_r}\}$ which is connected to the neighbors of x_i in $\Gamma(H)$ by having an edge $\{x_{i,j}, C_j\}$ whenever x_i appears as a literal in C_j . Let $G = (V, E)$ be the graph obtained after all replacements are made in this way. Clearly, G is planar, i.e., $G \in \text{Forb}_{\preceq}(K_{3,3}, K^5)$, and can be computed in time polynomial in the size of H . Note that the maximum degree of a vertex in G is three. To complete the construction, we specify the local transition functions. Let $C_i \in W$ be clause vertex. Let $x_{i_1, j_1}, x_{i_2, j_2}$, and x_{i_3, j_3} be the neighbors of C_i in G . Define the local transition function f_{C_i} by

$$f_{C_i}(I(C_i), I(x_{i_1, j_1}), I(x_{i_2, j_2}), I(x_{i_3, j_3})) \\ =_{\text{def}} \begin{cases} 1 & \text{if } (I(x_{i_1, j_1}), I(x_{i_2, j_2}), I(x_{i_3, j_3})) \text{ is a satisfying assignment of } C_i \\ \text{not}(I(C_i)) & \text{otherwise} \end{cases}$$

Let $x_{i,j} \in V$ be a variable vertex. Suppose that $\{C_j, x_{i,k_0}, \dots, x_{i,k_r}\} = N_G^0(x_{i,j})$ where r is the degree of $x_{i,j}$ in G . Define the local transition function $f_{x_{i,j}}$ by

$$f_{x_{i,j}}(I(C_j), I(x_{i,k_0}), \dots, I(x_{i,k_r})) =_{\text{def}} \begin{cases} I(x_{i,j}) & \text{if } I(x_{i,k_0}) = \dots = I(x_{i,k_r}) \\ \text{not}(I(x_{i,j})) & \text{otherwise} \end{cases}$$

Let S_H denote the dynamical system $(G, \{f_v | v \in V\})$ constructed from any planar 3CNF H in the way just specified. Note that, since the maximum degree of any vertex in G is three, we can compute S_H in time polynomial in the size of H .

By construction of S_H , a configuration I is a fixed point of S_H if and only if I satisfies $I(x_{i,j_1}) = \dots = I(x_{i,j_r})$ for all $i \in \{1, \dots, n\}$ and, furthermore, $I(C_j) = 1$ for all $j \in \{1, \dots, m\}$. Hence, it is easily seen that H has a satisfying assignment if and only if S_H has a fixed point. \square

We extend the NP-completeness of the fixed-point existence problem to systems with self-dual transition functions and planar graphs. First, observe that the transition functions for clause variables are not self-dual. This implies that the construction used in the last theorem is not fully appropriate. However, arbitrary boolean functions can easily be embedded into a self-dual function of larger arity.

Proposition 4.5. *Let $n \in \mathbb{N}_+$. For each k -ary boolean function $f : \{0, 1\}^k \rightarrow \{0, 1\}$, the function $\text{sd}_n(f) : \{0, 1\}^{k+n+1} \rightarrow \{0, 1\}$ defined for all $x_1, \dots, x_k, y_1, \dots, y_n, z \in \{0, 1\}$ by*

$$\text{sd}_n(f)(x_1, \dots, x_k, y_1, \dots, y_n, z) =_{\text{def}} \begin{cases} f(x_1, \dots, x_k) & \text{if } y_1 = \dots = y_n = 0 \\ \text{not}(f(\text{not}(x_1), \dots, \text{not}(x_k))) & \text{if } y_1 = \dots = y_n = 1 \\ \text{not}(z) & \text{otherwise} \end{cases}$$

is self-dual.

Proof. Follows from the definitions by case analysis. \square

The usage of Proposition 4.5 introduces ambiguity to the set of fixed points.

Proposition 4.6. *Let $S = (G, \{f_i | i \in V\})$ be a dynamical system over $\{0, 1\}$ so that all local transition functions f_i are self-dual. Let $U \subseteq V$. Then, a configuration $I : V \rightarrow \{0, 1\}$ is a local fixed point of S for U if and only if the configuration $\bar{I} : V \rightarrow \{0, 1\}$ defined by $\bar{I}(i) = \text{not}(I(i))$, is a local fixed point of S for U .*

Proof. Immediate from the definitions. \square

Theorem 4.7. $\text{FIXEDPOINTS}_T(\text{D}, \text{Forb}(K_{3,3}, K^5))$ *is NP-complete.*

Proof. We reduce from $\text{FIXEDPOINTS}_T(\text{BF}, \text{Forb}(K_{3,3}, K^5))$. Let $S = (G, \{f_i | i \in V\})$ be a dynamical system such that the underlying network G is planar. We construct another system $S' = (G', \{f'_i | i \in V'\})$ as follows. The network $G' = (V', E')$ is defined by $V' =_{\text{def}} V \cup E$ and $E' =_{\text{def}} E \cup \{\{i, e\} | i \in V, e \in E, i \in e\}$. It is easily seen that G' is planar as well. Suppose V' is ordered such that V is in the same ordering as for S , E is arbitrarily ordered, and E comes completely after V . The local transition functions of the vertices of V' are specified as follows. Suppose $i \in V$. Let $\{i_0, \dots, i_k\} = N_{G'}^0(i) \cap V$ and let $\{e_1, \dots, e_k\} = N_{G'}^0(i) \cap E$. We define the local transition function f'_i by

$$f'_i(x_{i_0}, \dots, x_{i_k}, x_{e_1}, \dots, x_{e_k}) =_{\text{def}} \text{sd}_k(f_i)(x_{i_0}, \dots, x_{i_k}, x_{e_1}, \dots, x_{e_k}, x_i).$$

By Proposition 4.5 and since D is closed under identification of variables, f'_i is self-dual. Moreover, as the degree of i in V' is $2k$ where k is the degree of i in G , the function table

can be computed in polynomial time. Now consider a vertex $e = \{i, j\} \in E$. Define the local transition function f'_e by

$$f'_{\{i,j\}}(x_i, x_j, x_{\{i,j\}}) =_{\text{def}} x_{\{i,j\}}.$$

Clearly, f'_e is self-dual. As the degree of $e \in E$ is two, the function table is trivially computable in polynomial time. It remains to show that

$$S \text{ has a fixed point} \iff S' \text{ has a fixed point.}$$

We prove both direction individually.

For (\Rightarrow) , suppose the configuration $I : V \rightarrow \{0, 1\}$ is a fixed point of S , i.e., for all $i \in V$, it holds that $f_i(I(i_0), \dots, I(i_k)) = I(i)$ where $\{i_0, \dots, i_k\} = N_G^0(i)$. Define a configuration $I' : V' \rightarrow \{0, 1\}$ for $i \in V$ by $I'(i) =_{\text{def}} I(i)$ and for $e \in E$ by $I'(e) =_{\text{def}} 0$. Consider $i \in V$. Assume $\{i_0, \dots, i_k\} = N_{G'}^0(i) \cap V$ and $\{e_1, \dots, e_k\} = N_{G'}^0(i) \cap E$. We obtain

$$\begin{aligned} f'_i(I'(i_0), \dots, I'(i_k), I'(e_1), \dots, I'(e_k)) \\ = \text{sd}_k(f_i)(I(i_0), \dots, I(i_k), 0, \dots, 0, I(i)) = f_i(I(i_1), \dots, I(i_k)) = I(i) = I'(i). \end{aligned}$$

Thus, I' is a local fixed point for $i \in V$. Suppose $e = \{i, j\} \in E$. By definition, $f'_{\{i,j\}}(I'(i), I'(j), I'(\{i, j\})) = I'(\{i, j\})$. Thus, I' is a local fixed point for $e \in E$. Proposition 2.1 implies that I' is a fixed point of S' .

For (\Leftarrow) , suppose the configuration $I' : V' \rightarrow \{0, 1\}$ is a fixed point of S' . Observe that for all $e, e' \in E \subseteq V'$, there is a walk in G' from e to e' alternating between vertices in V and E , i.e., there are vertices $p_0, \dots, p_{2\ell} \in V'$ such that $\{p_i, p_{i+1}\} \in E'$ for all $0 \leq i < 2\ell$, $p_0 = e$, $p_{2\ell} = e'$, and for all $0 \leq j < \ell$, it holds that $p_{2j} \in E$ and $p_{2j+1} \in V$. Consider a vertex $p_{2j+1} \in V$. Let $\{i_0, \dots, i_k\} = N_{G'}^0(p_{2j+1}) \cap V$ and $\{e_1, \dots, e_k\} = N_{G'}^0(p_{2j+1}) \cap E$. Since I' is a fixed point of S , we have

$$\begin{aligned} I'(p_{2j+1}) &= f'_{p_{2j+1}}(I'(i_0), \dots, I'(i_k), I'(e_1), \dots, I'(e_k)) \\ &= \text{sd}_k(f_{p_{2j+1}})(I'(i_0), \dots, I'(i_k), I'(e_1), \dots, I'(e_k), I'(i)) \end{aligned}$$

By definition of $\text{sd}_k(f_{p_{2j+1}})$, it follows that $I'(e_1) = \dots = I'(e_k)$. In particular, $I'(p_{2j}) = I'(p_{2j+2})$. This implies that for all $e, e' \in E$, it holds that $I'(e) = I'(e')$. By Proposition 4.6, we may assume that $I'(e) = 0$ for all $e \in E$. Define a configuration $I : V \rightarrow \{0, 1\}$ of S by $I(i) = I'(i)$ for all $i \in V$. It is easily seen that I is a fixed point of S . \square

Composing the big picture. We come back to the proof of the main result of the subsection. For convenience, we state it once more.

Theorem 4.1. *Let \mathcal{F} be a Post class of boolean functions and let \mathcal{G} be a class of graphs closed under taking minors. If $\mathcal{F} \supseteq \text{D}$ and $\mathcal{G} \supseteq \text{Forb}_{\prec}(K_{3,3}, K^5)$ then $\text{FIXEDPOINTS}_{\text{T}}(\mathcal{F}, \mathcal{G})$ is intractable. Otherwise, $\text{FIXEDPOINTS}_{\text{T}}(\mathcal{F}, \mathcal{G})$ is tractable.*

Proof. If $\mathcal{F} \supseteq \text{D}$ and $\mathcal{G} \supseteq \text{Forb}(K_{3,3}, K^5)$, then $\text{FIXEDPOINTS}(\mathcal{F}, \mathcal{G})$ is NP-complete by Theorem 4.7. Suppose $\mathcal{F} \not\supseteq \text{D}$ or $\mathcal{G} \not\supseteq \text{Forb}(K_{3,3}, K^5)$. The maximal classes \mathcal{F} that do not contain D are R_1 , R_0 , M , and L . For all these classes, by Proposition 4.2, the fixed-point existence problem is solvable in polynomial time. It remains to consider the case that $\mathcal{G} \not\supseteq \text{Forb}(K_{3,3}, K^5)$. Suppose $\mathcal{G} = \text{Forb}(X_1, \dots, X_n)$. Since $\mathcal{G} \not\supseteq \text{Forb}_{\prec}(K_{3,3}, K^5)$, there is an i such that X_i is planar. Since $\mathcal{G} \subseteq \text{Forb}_{\prec}(X_i)$, Theorem 4.3 shows that $\text{FIXEDPOINTS}_{\text{T}}(\text{BF}, \mathcal{G})$ is solvable in polynomial time. \square

4.2 Succinctly Represented Local Transition Functions

In this section we prove a dichotomy theorem for the fixed-point existence problem when transitions are given by formulas or circuits. As usual, the size of formula is the number of symbols from the basis used to encode the formula, the size of a circuit is the number of gates it consists of (including the input gates). Both succinct representations of functions lead to the same result.

Theorem 4.8. *Let \mathcal{F} be a Post class of boolean functions and let \mathcal{G} be a class of graphs closed under taking minors. If $\mathcal{F} \supseteq \text{D}$ and $\mathcal{G} \supseteq \text{Forb}_{\preceq}(K^3, K^2 \oplus K^2)$, then $\text{FIXEDPOINTS}_{\text{F}}(\mathcal{F}, \mathcal{G})$ is intractable. Otherwise, $\text{FIXEDPOINTS}_{\text{F}}(\mathcal{F}, \mathcal{G})$ is tractable. Moreover, the same statement holds for $\text{FIXEDPOINTS}_{\text{C}}$.*

Again we postpone the proof until we have proved a number of special results.

On the side of tractable cases, first note that Proposition 4.2 still holds for formulas and circuits. Actually, the results in [8] were stated for formulas. Furthermore, notice that circuits over the basis $\{\text{XOR}, 0, 1\}$ can easily be transformed in polynomial time into equivalent formulas over the same basis. The following result provides the tractability limit for restricted network classes.

Theorem 4.9. *Let X be a graph having a vertex cover of size one. Then, $\text{FIXEDPOINTS}_{\text{F}}(\text{BF}, \text{Forb}_{\preceq}(X))$ and $\text{FIXEDPOINTS}_{\text{C}}(\text{BF}, \text{Forb}_{\preceq}(X))$ are solvable in polynomial time.*

Proof. Suppose X has a vertex cover of size one. Then, there is an $r \in \mathbb{N}$ such that all graphs in $\text{Forb}_{\preceq}(X)$ have maximum vertex degree r . Thus, if we compute lookup tables from formulas or circuits, then each lookup table has at most 2^{r+1} entries. Hence, in polynomial time, we can transform each dynamical system S with a network in $\text{Forb}_{\preceq}(X)$ and local transition functions given by formulas or circuits into a dynamical system S' with the same networks and local transition functions given by lookup tables such that S and S' have the same fixed-point configurations. Moreover, since X is planar (note that $K^3 \preceq K_{3,3}$ and $K^2 \oplus K^2 \preceq K_{3,3}$ as well as $K^3 \preceq K_{3,3}$ and $K^2 \oplus K^2 \preceq K^5$), Theorem 4.3 implies polynomial-time solvability of $\text{FIXEDPOINTS}_{\text{F}}(\text{BF}, \text{Forb}_{\preceq}(X))$ and $\text{FIXEDPOINTS}_{\text{C}}(\text{BF}, \text{Forb}_{\preceq}(X))$. \square

Theorem 4.10. $\text{FIXEDPOINTS}_{\text{F}}(\text{D}, \text{Forb}_{\preceq}(K^3, K^2 \oplus K^2))$ is NP-complete.

Proof. The reduction is from 3SAT. We start with a description of a reduction to dynamical systems where each transition function is computed by a 3CNF. Suppose we are given a 3CNF $H = C_1 \wedge \dots \wedge C_m$ having variables x_1, \dots, x_n . Note that the formula $H' =_{\text{def}} H \vee \overline{x_0}$, where x_0 is a new variable, satisfies for any assignment $I : \{x_0, x_1, \dots, x_n\} \rightarrow \{0, 1\}$ that $f_{H'}(I(x_0), I(x_1), \dots, I(x_n)) = I(x_0)$ if and only if $I(H) = 1$ and $I(x_0) = 1$. Moreover, H' is equivalent to a 4CNF which can be transformed in a 3CNF \hat{H} with variables $x_0, x_1, \dots, x_n, x_{n+1}, \dots, x_{n+m}$ such that \hat{H} is satisfiable with 1 assigned to x_0 if and only if H is satisfiable with 1 assigned to x_0 . Define S_H to be the dynamical system $(G, \{f_0, \dots, f_{n+m}\})$ consisting of the network $G = (V, E)$ where $V =_{\text{def}} \{0, 1, \dots, n+m\}$ and $E =_{\text{def}} \{ \{0, i\} \mid i \in \{1, \dots, n+m\} \}$. Thus, G is a star $K_{1, n+m}$, i.e., $G \in \text{Forb}_{\preceq}(K^3, K^2 \oplus K^2)$. The local transition functions are given as follows. For a vertex $i \in \{1, \dots, n+m\}$, the local transition function f_i is defined to be computed by the formula $H(x_0, x_i) =_{\text{def}} x_i$. For the central vertex, the local transition function f_0 is given by

the formula $\hat{H}(x_0, x_1, \dots, x_{n+m})$ where the variable x_i stands for a vertex $i \in V$. Clearly, S_H can be computed in time polynomial in the size of H and we have that H is satisfiable if and only if S_H has a fixed point.

We now transform the dynamical system S_H into another system S'_H with self-dual local transition functions given by formulas over the corresponding basis. Note that D has a single basis function of arity three. Let D denote the corresponding ternary function symbol, i.e., the semantics of D is defined by $D(x, y, z) \equiv_{\text{def}} (x \wedge \bar{y}) \vee (x \wedge \bar{z}) \vee (\bar{y} \wedge \bar{z})$. Note that $D(x, x, y) \equiv \bar{y}$. We embed a 3CNF into a self-dual function, similarly to Proposition 4.5. That is, for an arbitrary 3CNF $H = C_1 \wedge \dots \wedge C_m$ having variables x_1, \dots, x_n , define the formula $\text{dual}(H)(x_1, \dots, x_n) = \overline{H(\bar{x}_1, \dots, \bar{x}_n)}$. Define $\text{sd}(H)(x_1, \dots, x_n, z) \equiv_{\text{def}} (H \wedge z) \vee (\text{dual}(H) \wedge \bar{z})$. By induction over the number of clauses, we show that $\text{sd}(H)$ is equivalent to a formula built using D which is of polynomial size:

1. For the base of induction, suppose $m = 1$. Since we know how to express negation using D , without loss of generality, we assume that $H = (x_1 \vee x_2 \vee x_3)$. So, $\text{sd}(H) \equiv ((x_1 \vee x_2 \vee x_3) \wedge z) \vee ((x_1 \wedge x_2 \wedge x_3) \wedge \bar{z})$. By truth-table inspection we obtain that $\text{sd}(H) \equiv \overline{D(\bar{z}, D(z, \bar{x}_1, \bar{x}_2), D(z, \bar{x}_1, \bar{x}_3))}$.
2. For the induction step, suppose $m > 1$. Let $H' = C_1 \wedge \dots \wedge C_m$ be a 3CNF over the variables x_1, \dots, x_n . Define $H' \equiv_{\text{def}} C_1 \wedge \dots \wedge C_{\lfloor m/2 \rfloor}$ and $H'' \equiv_{\text{def}} C_{\lfloor m/2 \rfloor + 1} \wedge \dots \wedge C_m$. Some equivalent transformations show that $\text{sd}(H) \equiv \overline{D(\bar{z}, \text{sd}(H'), \text{sd}(H''))}$. By induction hypothesis, $\text{sd}(H')$ and $\text{sd}(H'')$ can be expressed using D . Replacing $\text{sd}(H')$ and $\text{sd}(H'')$ (and the negation) gives the appropriate formula for $\text{sd}(H)$. Note that the recursion depth for formula replacement is logarithmic. It follows that the size of the formula for $\text{sd}(H)$ is $O(|H|^2)$.

Finally, we define the dynamical system S'_H to be specified as follows. The network $G' = (V', E')$ consists of the vertex set $V' \equiv_{\text{def}} \{\underline{0}, 0, \dots, n+m\}$ and the edge set $E' \equiv_{\text{def}} \{\{\underline{0}, i\} \mid i \in \{0, \dots, n+m\}\}$. Thus, G' is a star $K_{1, n+m+1}$, i.e., $G' \in \text{Forb}_{\leq}(K^3, K^2 \oplus K^2)$. The local transition function f_i for a vertex $i \in \{0, \dots, n+m\}$ is given by $H(x_{\underline{0}}, x_i) = D(x_{\underline{0}}, x_{\underline{0}}, D(x_{\underline{0}}, x_{\underline{0}}, x_i))$. Notice that $H(x_{\underline{0}}, x_i) \equiv x_i$. The local transition function $f_{\underline{0}}$ for the vertex $\underline{0}$ is given as follows. Recall that \hat{H} is the 3CNF associated with the local transition function of vertex 0 in the system S_H . Then $f_{\underline{0}}$ is represented by the D -formula equivalent to $\text{sd}(\hat{H})(x_0, \dots, x_{n+m}, x_{\underline{0}})$. Clearly, S'_H can be computed in time polynomial in the size of H . Moreover, it is easy to verify that H is satisfiable if and only if S'_H has a fixed point. \square

We combine Theorem 4.9 and Theorem 4.10 to prove Theorem 4.8.

Proof. (Theorem 4.8) If $\mathcal{F} \supseteq D$ and $\mathcal{G} \supseteq \text{Forb}(K^3, K^2 \oplus K^2)$, then $\text{FIXEDPOINTS}(\mathcal{F}, \mathcal{G})$ is NP-complete by Theorem 4.10. Suppose $\mathcal{F} \not\supseteq D$ or $\mathcal{G} \not\supseteq \text{Forb}(K^3, K^2 \oplus K^2)$. The maximal classes \mathcal{F} that do not contain D are R_1, R_0, M , and L . For all these classes, by Proposition 4.2, the fixed-point existence problem is solvable in polynomial time. It remains to consider the case that $\mathcal{G} \not\supseteq \text{Forb}(K^3, K^2 \oplus K^2)$. Suppose $\mathcal{G} = \text{Forb}(X_1, \dots, X_n)$. Since $\mathcal{G} \not\supseteq \text{Forb}_{\leq}(K^3, K^2 \oplus K^2)$, there is an i such that X_i has a vertex cover of size one. Since $\mathcal{G} \subseteq \text{Forb}_{\leq}(X_i)$, Theorem 4.9 shows that $\text{FIXEDPOINTS}_T(\text{BF}, \mathcal{G})$ is solvable in polynomial time. \square

5 Conclusion

We characterized the islands of tractability for the fixed-point existence problem for boolean dynamical systems with respect to transition classes \mathcal{F} closed under composition and network classes \mathcal{G} closed under taking minors: If \mathcal{F} contains the self-dual functions and \mathcal{G} contains the planar graphs, then $\text{FIXEDPOINTS}_{\mathbb{T}}(\mathcal{F}, \mathcal{G})$ is intractable, otherwise it is tractable. Replacing “planar graphs” by “graphs having a vertex cover of size one” yields the same dichotomy theorem for the succinct representations of local transition functions by formulas or circuits. The linear and monotone functions have been shown to be tractable cases in [8]. There, the authors suggested to find more tractable classes of local transition functions. Over the boolean domain our results show that, aside from two obvious exceptions (the 0- and 1-reproducing functions), there are no more such function classes.

Although the proposed analysis framework allows elegant dichotomy theorems for fixed-point existence, it is certainly necessary to examine its usefulness for other computational problems for discrete dynamical systems and to refine it appropriately. Another important open problem is the extension of the dichotomy theorems to larger domains. This seems a challenging issue as even in the case of a ternary domain the number of Post classes is continuum (see, e.g., [22]).

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