APPLICATIONS OF BAR CODE TO INVOLUTIVE DIVISIONS AND A GREEDY ALGORITHM FOR COMPLETE SETS.

MICHELA CERIA

Abstract. In this paper, we describe how to get Janet decomposition for a finite set of terms and detect completeness of that set by means of the associated Bar Code. Moreover, we explain an algorithm to find a variable ordering (if it exists) s.t. a given set of terms is complete according to that ordering. The algorithm is greedy and constructs a Bar Code from the maximal to the minimal variable, adjusting the variable ordering with a sort of backtracking technique, thus allowing to construct the desired ordering without trying all the $n!$ possible orderings.

1. Introduction

Let $\mathcal{P} := \mathbf{k}[x_1, ..., x_n]$ be the polynomial ring in n variables with coefficients in the field $\mathbf{k}$. The *semigroup of terms*, generated by $\{x_1, ..., x_n\}$ is: $\mathcal{T} := \{x^\gamma := x_1^{\gamma_1} \cdots x_n^{\gamma_n} | \gamma := (\gamma_1, ..., \gamma_n) \in \mathbb{N}^n\}$. Given a monomial/semigroup ideal $J \subset \mathcal{T}$ and its minimal set of generators $\mathsf{G}(J)$ Janet introduced in [32] the notion of *multiplicative variables* and the connected decomposition of J into disjoint *cones*, giving a procedure (*completion*) to construct such a decomposition. In particular, $\forall v \in \mathcal{T}$, there is a *unique* decomposition $v = tu$, with $t \in \mathsf{G}(J)$ and u a product of powers of t's multiplicative variables. While performing reduction w.r.t. an ideal whose initial ideal is J, the term w can be reduced by the only polynomial whose leading term generates the cone containing w. Involutive divisions date back to the works by Janet [32, 33, 34, 35] who, besides giving a cone decomposition for the monomial ideal J, in order to describe Riquier's [40] formulation of the description for the general solutions of a PDE problem, gave a similar decomposition also for the related *escalier* $\mathsf{N}(J) := \mathcal{T} \setminus J$. Later in [33, 34, 35], he gave a new decomposition (and an algorithm to produce it) which called *involutive* and which is behind both Gerdt-Blinkov [21, 22, 23] procedure to compute Gröbner bases and Seiler's [43] involutiveness theory. His aim was twofold: to reinterpret, in terms of multiplicative variables and cone decomposition, Cartan's solution to PDE problems [2, 3, 4] (whence the name *involutiveness*) and to re-evaluate within his theory the notion of *generic initial ideal* introduced by Delassus [15, 16, 17] and the correction of his mistake by Robinson [41, 42] and Gunther [27, 28], who remark that the notion requires J to be Borel-fixed (an equivalent modern reformulation was proposed by Galligo [20], who merged Hironaka and Grauert's ideas [31, 25]; see also [26, 18]). Janet remarked that all Borel-fixed ideals are involutive, but the converse is false. More precisely, in [34] Janet presents, as *nouvelle formes cannoniques*, Delassus, Robinson and Gunther's results and compares them with the one deductible from

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an involutive basis and in [35, p.62], given a homogeneous ideal $\mathcal{I} \triangleleft \mathcal{P}$ in generic coordinates, he restates Riquier's completion in terms of a Macaulay-like construction, iteratively computing the vector spaces $\mathcal{I}_d := \{f \in \mathcal{I} : \deg(f) = d\}$ until Cartan test grants that Castelnuovo-Mumford [37, pg.99] regularity D has been reached. This would allow him to consider the semigroup ideal $\mathsf{T}(\mathcal{I})$ of the leading terms w.r.t. deg-lex (in the sense of Gröbner basis theory) and get the *involutive reduction* required by Riquier's procedure. The formal definition of involutive division is due to Gerdt-Blinkov [21, 22].

Bar Codes, introduced in [6, 7], are a visual representation for finite sets of terms $M \subset \mathcal{T}$. In particular, if $M = \mathsf{N}(I)$ is the Groebner escalier of a zerodimensional ideal $I \triangleleft \mathcal{P}$, many of its properties can be directly deduced by its Bar Code. As an example, in [10], Bar Codes are employed to develop a combinatorial algorithm which, given a finite set of distinct points, computes the lexicographical Groebner escalier of its vanishing ideal. This algorithm is an alternative to those by Cerlienco-Mureddu [12, 13, 14] and by Felszeghy-Ráth-Rónyay [19], which keeps the former algorithm's iterativity, though reaching a complexity which is near to that of the latter one. In [6], we use Bar Codes as tools to define a bijection between zerodimensional (strongly) stable ideals in two or three variables and some partitions of their (constant) affine Hilbert polynomial.

Now, we are focusing on the properties of Bar Codes connected to involutive divisions. Bar Codes are a good technology to study involutive divisions. For example, it is trivial to compute the Pommaret [33] basis of I from the Bar Code. In [8], we exploit the Bar Code to compute by Moeller interpolation the Pommaret basis of the ideal of a finite set of distinct points. For a general overview of Bar Codes' applications see [7].

In this paper, we discuss some applications of the Bar Code to involutive divisions. In particular, we see how the Bar Code associated to a finite set of terms, which is non-necessarily an order ideal, allows to approach Janet decomposition [32] and decide whether that set is complete according to Janet's definition. Moreover, we give an algorithm to check whether there is a variables' ordering s.t. a given set $M \subset \mathcal{T}$ is complete. We need to remark that such a topic has some connections to the study of Stanley decompositions and Stanley depth. Indeed, Janet decomposition for a complete set is exactly a Stanley decomposition which can be easily read off from that set. Anyway, has stated by Herzog [30],

> *Janet decompositions from the viewpoint of Stanley depth are not optimal. They rarely give Stanley decompositions providing the Stanley depth of a monomial ideal. However one obtains the result that the Stanley depth of a monomial ideal is at least 1.*

and, actually, this paper places itself in the field of study mainly developed by Gerdt-Blinkov [21, 22, 23] and Seiler [43], which has aims and language that are different from those of Stanley depth.

After the next section, devoted to notation, we describe the Bar Code (section 3), as the fundamental tool for the following sections. Then, in section 4, we describe Janet decomposition into multiplicative/non-multiplicative variables and we explain how to use the Bar Code to get it from a finite set of terms. Moreover, we also deal with complete sets, explaining how also completeness can be read from a suitable Bar Code. In section 5, then, we explain an algorithm to detect a variable ordering (if it exists) s.t. a given set of terms is complete according to that ordering. The algorithm is greedy and constructs a Bar Code from the maximal to the minimal variable, adjusting the variable ordering with a sort of backtracking technique, and allowing to construct the desired ordering without trying all the $n!$ possible orderings.

2. Some general notation

Throughout this paper we mainly follow the notation of [36]. We denote by $\mathcal{P} := \mathbf{k}[x_1, ..., x_n]$ the ring of polynomials in n variables with coefficients in the field $\mathbf{k}$. The *semigroup of terms*, generated by the set $\{x_1, ..., x_n\}$ is:

$$\mathcal{T} := \{x^\gamma := x_1^{\gamma_1} \cdots x_n^{\gamma_n} \,|\, \gamma := (\gamma_1, ..., \gamma_n) \in \mathbb{N}^n\}.$$

If $\alpha \subseteq \{1, ..., n\}$ then $\mathcal{T}[\alpha] := \{x^\gamma := x_1^{\gamma_1} \cdots x_n^{\gamma_n} \in \mathcal{T} \,|\, \gamma_i \neq 0 \Rightarrow i \in \alpha\}$. If $t = x_1^{\gamma_1} \cdots x_n^{\gamma_n}$, then $\deg(t) = \sum_{i=1}^n \gamma_i$ is the *degree* of t and, for each $h \in \{1, ..., n\}$ $\deg_h(t) := \gamma_h$ is the *h-degree* of t. A *semigroup ordering* $<$ on $\mathcal{T}$ is a total ordering such that $t_1 < t_2 \Rightarrow st_1 < st_2$, $\forall s, t_1, t_2 \in \mathcal{T}$. For each semigroup ordering $<$ on $\mathcal{T}$, we can represent a polynomial $f \in \mathcal{P}$ as a linear combination of terms arranged w.r.t. $<$, with coefficients in the base field $\mathbf{k}$:

$$f = \sum_{t \in \mathcal{T}} c(f,t)t = \sum_{i=1}^{s} c(f,t_i)t_i : c(f,t_i) \in \mathbf{k} \setminus \{0\},\, t_i \in \mathcal{T},\, t_1 > ... > t_s,$$

with $\mathsf{T}(f) := t_1$ the *leading term* of f, $Lc(f) := c(f, t_1)$ the *leading coefficient* of f and $tail(f) := f - c(f, \mathsf{T}(f))\mathsf{T}(f)$ the *tail* of f. A *term ordering* is a semigroup ordering such that 1 is lower than every variable or, equivalently, it is a *well ordering*. In all paper, we consider the *lexicographical ordering* induced by $x_1 < ... < x_n$, i.e: $x_1^{\gamma_1} \cdots x_n^{\gamma_n} <_{Lex} x_1^{\delta_1} \cdots x_n^{\delta_n} \Leftrightarrow \exists j \,|\, \gamma_j < \delta_j,\, \gamma_i = \delta_i,\, \forall i > j$, which is a term ordering. Since we do not consider any term ordering other than Lex, we drop the subscript and denote it by $<$ instead of $<_{Lex}$.
A subset $J \subseteq \mathcal{T}$ is a *semigroup ideal* if $t \in J \Rightarrow st \in J$, $\forall s \in \mathcal{T}$; a subset $\mathsf{N} \subseteq \mathcal{T}$ is an *order ideal* if $t \in \mathsf{N} \Rightarrow s \in \mathsf{N}\, \forall s|t$. We have that $\mathsf{N} \subseteq \mathcal{T}$ is an order ideal if and only if $\mathcal{T} \setminus \mathsf{N} = J$ is a semigroup ideal. Given a semigroup ideal $J \subset \mathcal{T}$ we define $\mathsf{N}(J) := \mathcal{T} \setminus J$. The minimal set of generators $\mathsf{G}(J)$ of J is called the *monomial basis* of J. For all subsets $G \subset \mathcal{P}$, $\mathsf{T}\{G\} := \{\mathsf{T}(g),\, g \in G\}$ and $\mathsf{T}(G)$ is the semigroup ideal of leading terms defined as $\mathsf{T}(G) := \{t\mathsf{T}(g),\, t \in \mathcal{T}, g \in G\}$. Fixed a term order $<$, for any ideal $I \triangleleft \mathcal{P}$ the monomial basis of the semigroup ideal $\mathsf{T}(I) = \mathsf{T}\{I\}$ is called *monomial basis* of I and denoted again by $\mathsf{G}(I)$, whereas the ideal $In(I) := (\mathsf{T}(I))$ is called *initial ideal* and the order ideal $\mathsf{N}(I) := \mathcal{T} \setminus \mathsf{T}(I)$ is called *Groebner escalier* of I.

3. Bar Code for monomial ideals

In this section, referring to [6, 7], we summarize the main definitions and properties about Bar Codes, which will be used in what follows. First of all, we recall the general definition of Bar Code.

Definition 3.1. A Bar Code B is a picture composed by segments, called *bars*, superimposed in horizontal rows, which satisfies conditions $a., b.$ below. Denote by

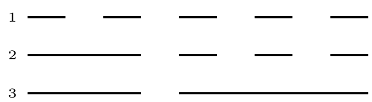
- $\mathsf{B}_j^{(i)}$ the j-th bar (from left to right) of the i-th row (from top to bottom), $1 \leq i \leq n$, i.e. the *j-th i-bar*;
- $\mu(i)$ the number of bars of the i-th row

- $l_1(\mathsf{B}_j^{(1)}) := 1$, $\forall j \in \{1, 2, ..., \mu(1)\}$ the $(1-)$*length* of the 1-bars;
- $l_i(\mathsf{B}_j^{(k)})$, $2 \le k \le n$, $1 \le i \le k-1$, $1 \le j \le \mu(k)$ the *i-length* of $\mathsf{B}_j^{(k)}$, i.e. the number of i-bars lying over $\mathsf{B}_j^{(k)}$

a. $\forall i, j$, $1 \le i \le n-1$, $1 \le j \le \mu(i)$, $\exists! \overline{j} \in \{1, ..., \mu(i+1)\}$ s.t. $\mathsf{B}_{\overline{j}}^{(i+1)}$ lies under $\mathsf{B}_j^{(i)}$

b. $\forall i_1, i_2 \in \{1, ..., n\}$, $\sum_{j_1=1}^{\mu(i_1)} l_1(\mathsf{B}_{j_1}^{(i_1)}) = \sum_{j_2=1}^{\mu(i_2)} l_1(\mathsf{B}_{j_2}^{(i_2)})$; we will then say that *all the rows have the same length*.

Example 3.2. An example of Bar Code B is



The 1-bars have length 1. As regards the other rows, $l_1(\mathsf{B}_1^{(2)}) = 2$, $l_1(\mathsf{B}_2^{(2)}) = l_1(\mathsf{B}_3^{(2)}) = l_1(\mathsf{B}_4^{(2)}) = 1$, $l_2(\mathsf{B}_1^{(3)}) = 1$,$l_1(\mathsf{B}_1^{(3)}) = 2$ and $l_2(\mathsf{B}_2^{(3)}) = l_1(\mathsf{B}_2^{(3)}) = 3$, so $\sum_{j_1=1}^{\mu(1)} l_1(\mathsf{B}_{j_1}^{(1)}) = \sum_{j_2=1}^{\mu(2)} l_1(\mathsf{B}_{j_2}^{(2)}) = \sum_{j_3=1}^{\mu(3)} l_1(\mathsf{B}_{j_3}^{(3)}) = 5$.

We outline now the construction of the Bar Code associated to a finite set of terms. For more details, see [7], while for an alternative construction, see [6].
First of all, given a term $t = x_1^{\gamma_1} \cdots x_n^{\gamma_n} \in \mathcal{T} \subset \mathbf{k}[x_1, ..., x_n]$, for each $i \in \{1, ..., n\}$, we take $\pi^i(t) := x_i^{\gamma_i} \cdots x_n^{\gamma_n} \in \mathcal{T}$. Taken a finite set of terms $M \subset \mathcal{T}$, for each $i \in \{1, ..., n\}$, we then define $M^{[i]} := \pi^i(M) := \{\pi^i(t) | t \in M\}$.
Now we take $M \subseteq \mathcal{T}$, with $|M| = m < \infty$ and we order its elements increasingly w.r.t. Lex, getting the list $\overline{M} = [t_1, ..., t_m]$. Then, we construct the sets $M^{[i]}$, and the corresponding lexicographically ordered lists[1] $\overline{M}^{[i]}$, for $i = 1, ..., n$.
We define the $n \times m$ matrix of terms $\mathcal{M}$ s.t. its i-th row is $\overline{M}^{[i]}$, $i = 1, ..., n$, i.e.

$$\mathcal{M} := \begin{pmatrix} \pi^1(t_1) & ... & \pi^1(t_m) \\ \pi^2(t_1) & ... & \pi^2(t_m) \\ \vdots & & \vdots \\ \pi^n(t_1) & ... & \pi^n(t_m) \end{pmatrix}$$

Definition 3.3. The *Bar Code diagram* B associated to M (or, equivalently, to $\overline{M}$) is a $n \times m$ diagram, made by segments s.t. the i-th row of B, $1 \le i \le n$ is constructed as follows:

(1) take the i-th row of $\mathcal{M}$, i.e. $\overline{M}^{[i]}$
(2) consider all the sublists of repeated terms, i.e. $[\pi^i(t_{j_1}), \pi^i(t_{j_1+1}), ..., \pi^i(t_{j_1+h})]$ s.t. $\pi^i(t_{j_1}) = \pi^i(t_{j_1+1}) = ... = \pi^i(t_{j_1+h})$, noting that[2] $0 \le h < m$
(3) underline each sublist with a segment
(4) delete the terms of $\overline{M}^{[i]}$, leaving only the segments (i.e. the *i-bars*).

We usually label each 1-bar $\mathsf{B}_j^{(1)}$, $j \in \{1, ..., \mu(1)\}$ with the term $t_j \in \overline{M}$.

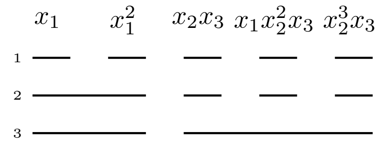
A Bar Code diagram is a Bar Code in the sense of Definition 3.1.

[1] $\overline{M}$ cannot contain repeated terms, while the $\overline{M}^{[i]}$, for $1 < i \le n$, can. In case some repeated terms occur in $\overline{M}^{[i]}$, $1 < i \le n$, they clearly have to be adjacent in the list, due to the lexicographical ordering.

[2] Clearly if a term $\pi^i(t_{\overline{j}})$ is not repeated in $\overline{M}^{[i]}$, the sublist containing it will be only $[\pi_i(t_{\overline{j}})]$, i.e. $h = 0$.

Example 3.4. Given $M = \{x_1, x_1^2, x_2x_3, x_1x_2^2x_3, x_2^3x_3\} \subset \mathbf{k}[x_1, x_2, x_3]$, we have: the 3×5 table on the left and then to the Bar Code on the right:

x_1	x_1^2	x_2x_3	$x_1x_2^2x_3$	$x_2^3x_3$
1	1	x_2x_3	$x_2^2x_3$	$x_2^3x_3$
1	1	x_3	x_3	x_3



Now we recall the vice versa, i.e. how to associate a finite set of terms M_{B} to a given Bar Code B. In [6] we first give a more general procedure to do so and then we specialize it in order to have a *unique* set of terms for each Bar Code. Here we give only the specialized version, so we follow the steps below:

$\mathfrak{B}1$ consider the n-th row, composed by the bars $B_1^{(n)}, ..., B_{\mu(n)}^{(n)}$. Let $l_1(B_j^{(n)}) = \ell_j^{(n)}$, for $j \in \{1, ..., \mu(n)\}$. Label each bar $B_j^{(n)}$ with $\ell_j^{(n)}$ copies of x_n^{j-1}.

$\mathfrak{B}2$ For each $i = 1, ..., n-1$, $1 \leq j \leq \mu(n-i+1)$ consider the bar $B_j^{(n-i+1)}$ and suppose that it has been labelled by $\ell_j^{(n-i+1)}$ copies of a term t. Consider all the $(n-i)$-bars $B_{\overline{j}}^{(n-i)}, ..., B_{\overline{j}+h}^{(n-i)}$ lying immediately above $B_j^{(n-i+1)}$; note that h satisfies $0 \leq h \leq \mu(n-i) - \overline{j}$. Denote the 1-lengths of $B_{\overline{j}}^{(n-i)}, ..., B_{\overline{j}+h}^{(n-i)}$ by $l_1(B_{\overline{j}}^{(n-i)}) = \ell_{\overline{j}}^{(n-i)}$,..., $l_1(B_{\overline{j}+h}^{(n-i)}) = \ell_{\overline{j}+h}^{(n-i)}$. For each $0 \leq k \leq h$, label $B_{\overline{j}+k}^{(n-i)}$ with $\ell_{\overline{j}+k}^{(n-i)}$ copies of tx_{n-i}^k.

Definition 3.5. A Bar Code B is *admissible* if the set M obtained by applying $\mathfrak{B}1$ and $\mathfrak{B}2$ to B is an order ideal.

By definition of order ideal, using $\mathfrak{B}1$ and $\mathfrak{B}2$ is the only way an order ideal can be associated to an admissible Bar Code.

Definition 3.6. Given a Bar Code B, let us consider a 1-bar $B_{j_1}^{(1)}$, with $j_1 \in \{1, ..., \mu(1)\}$. The *e-list* associated to $B_{j_1}^{(1)}$ is the n-tuple $e(B_{j_1}^{(1)}) := (b_{j_1,n}, ..., b_{j_1,1})$, defined as follows:

- consider the n-bar $B_{j_n}^{(n)}$, lying under $B_{j_1}^{(1)}$. The number of n-bars on the left of $B_{j_n}^{(n)}$ is $b_{j_1,n}$.
- for each $i = 1, ..., n-1$, let $B_{j_{n-i+1}}^{(n-i+1)}$ and $B_{j_{n-i}}^{(n-i)}$ be the $(n-i+1)$-bar and the $(n-i)$-bar lying under $B_{j_1}^{(1)}$. Consider the $(n-i+1)$-block associated to $B_{j_{n-i+1}}^{(n-i+1)}$, i.e. $B_{j_{n-i+1}}^{(n-i+1)}$ and all the bars lying over it. The number of $(n-i)$-bars of the block, which lie on the left of $B_{j_{n-i}}^{(n-i)}$ is $b_{j_1,n-i}$.

Remark 3.7. Given a Bar Code B, fix a 1-bar $B_j^{(1)}$, with $j \in \{1, ..., \mu(1)\}$. Comparing Definition 3.6 and the steps $\mathfrak{B}1$ and $\mathfrak{B}2$ described above, we can observe that the values of the e-list $e(B_j^{(1)}) := (b_{j,n}, ..., b_{j,1})$ are exactly the exponents of the term labelling $B_j^{(1)}$, obtained applying $\mathfrak{B}1$ and $\mathfrak{B}2$ to B (compare Example 3.4).

Proposition 3.8 (Admissibility criterion)**.** A Bar Code B is admissible if and only if, for each 1-bar $\mathsf{B}_j^{(1)}$, $j \in \{1, ..., \mu(1)\}$, the e-list $e(\mathsf{B}_j^{(1)}) = (b_{j,n}, ..., b_{j,1})$ satisfies the following condition: $\forall k \in \{1, ..., n\}$ s.t. $b_{j,k} > 0$, $\exists \overline{j} \in \{1, ..., \mu(1)\} \setminus \{j\}$ s.t.

$$e(\mathsf{B}_{\overline{j}}^{(1)}) = (b_{j,n}, ..., b_{j,k+1}, (b_{j,k}) - 1, b_{j,k-1}, ..., b_{j,1}).$$

□

Consider the sets $\mathcal{A}_n := \{\mathsf{B} \in \mathcal{B}_n \text{ s.t. } \mathsf{B} \text{ admissible}\}$ and $\mathcal{N}_n := \{\mathsf{N} \subset \mathcal{T},\, |\mathsf{N}| < \infty \text{ s.t. } \mathsf{N} \text{ is an order ideal}\}$. We can define the map $\eta : \mathcal{A}_n \to \mathcal{N}_n;\quad \mathsf{B} \mapsto \mathsf{N}$, where N is the order ideal obtained applying $\mathfrak{B}1$ and $\mathfrak{B}2$ to B, and it can be easily proved that η is a bijection.
Up to this point, we have discussed the link between Bar Codes and order ideals, i.e. we focused on the link between Bar Codes and Groebner escaliers of monomial ideals. We show now that, given a Bar Code B and the order ideal $\mathsf{N} = \eta(\mathsf{B})$ it is possible to deduce a very specific generating set for the monomial ideal I s.t. $\mathsf{N}(I) = \mathsf{N}$.

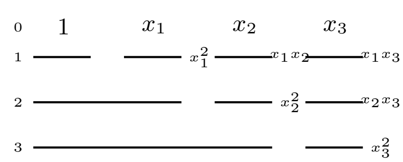
Definition 3.9. The *star set* of an order ideal N and of its associated Bar Code $\mathsf{B} = \eta^{-1}(\mathsf{N})$ is a set $\mathcal{F}_{\mathsf{N}}$ constructed as follows:

a) $\forall 1 \leq i \leq n$, let t_i be a term which labels a 1-bar lying over $\mathsf{B}^{(i)}_{\mu(i)}$, then $x_i \pi^i(t_i) \in \mathcal{F}_{\mathsf{N}}$;

b) $\forall 1 \leq i \leq n-1$, $\forall 1 \leq j \leq \mu(i)-1$ let $\mathsf{B}^{(i)}_j$ and $\mathsf{B}^{(i)}_{j+1}$ be two consecutive bars not lying over the same $(i+1)$-bar and let $t^{(i)}_j$ be a term which labels a 1-bar lying over $\mathsf{B}^{(i)}_j$, then $x_i \pi^i(t^{(i)}_j) \in \mathcal{F}_{\mathsf{N}}$.

We usually represent $\mathcal{F}_{\mathsf{N}}$ within the associated Bar Code B, inserting each $t \in \mathcal{F}_{\mathsf{N}}$ on the right of the bar from which it is deduced. Reading the terms from left to right and from the top to the bottom, $\mathcal{F}_{\mathsf{N}}$ is ordered w.r.t. Lex.

Example 3.10.
For $\mathsf{N} = \{1, x_1, x_2, x_3\} \subset \mathbf{k}[x_1, x_2, x_3]$, we have $\mathcal{F}_{\mathsf{N}} = \{x_1^2, x_1x_2, x_2^2, x_1x_3, x_2x_3, x_3^2\}$; looking at Definition 3.9, we can see that the terms x_1x_3, x_2x_3, x_3^2 come from a), while the terms x_1^2, x_1x_2, x_2^2 come from b).



In [11], given a monomial ideal I, the authors define the following set, calling it *star set*: $\mathcal{F}(I) = \left\{ x^\gamma \in \mathcal{T} \setminus \mathsf{N}(I) \;\middle|\; \frac{x^\gamma}{\min(x^\gamma)} \in \mathsf{N}(I) \right\}$.

Proposition 3.11 ([6])**.** With the above notation $\mathcal{F}_{\mathsf{N}} = \mathcal{F}(I)$.

The star set $\mathcal{F}(I)$ of a monomial ideal I is strongly connected to Janet's theory [32, 33, 34, 35] and to the notion of Pommaret basis [38, 39, 43], as explicitly pointed out in [11]. In particular, for quasi-stable ideals, the star set is finite and coincides with Pommaret bases.

4. Janet decomposition and completeness.

Given a monomial/semigroup ideal $J \subset \mathcal{T}$ and its monomial basis $\mathsf{G}(J)$, Janet introduced in [32] both the notion of *multiplicative variables* and the connected decomposition of J into disjoint *cones*, characterizing, according to Gerdt-Blinkov notation, an *involutive division.*

Definition 4.1. [32, ppg.75-9] Let $U \subset \mathcal{T}$ be a set of terms and $t = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ be an element of U. A variable x_j is called *multiplicative* for t with respect to U if there is no term in U of the form $t' = x_1^{\beta_1} \cdots x_j^{\beta_j} x_{j+1}^{\alpha_{j+1}} \cdots x_n^{\alpha_n}$ with $\beta_j > \alpha_j$. We denote by $M_J(t, U)$ the set of multiplicative variables for t with respect to U.

The variables that are not multiplicative for t w.r.t. U are called *non-multiplicative* and we denote by $NM_J(t,U)$ the set containing them.

It is clear that the above definition depends on the order of the variables.

Example 4.2. Consider the set $U = \{x_1, x_2\} \subset \mathbf{k}[x_1, x_2]$. If $x_1 < x_2$, then $M_J(x_1,U) = \{x_1\}$, $NM_J(x_1,U) = \{x_2\}$, $M_J(x_2,U) = \{x_1,x_2\}$, $NM_J(x_2,U) = \emptyset$. If, instead $x_2 < x_1$, then $M_J(x_1,U) = \{x_1,x_2\}$, $NM_J(x_1,U) = \emptyset$, $M_J(x_2,U) = \{x_2\}$, $NM_J(x_2,U) = \{x_1\}$.

Definition 4.3. With the previous notation, the *cone* of t with respect to U is the set $C_J(t,U) := \{tx_1^{\lambda_1} \cdots x_n^{\lambda_n} \,|\, \text{where } \lambda_j \neq 0 \text{ only if } x_j \text{ is multiplicative for } t \text{ w.r.t. } U\}$.

Example 4.4. Consider the set $J = \{x_1^3, x_2^3, x_1^4x_2x_3, x_3^2\} \subseteq \mathbf{k}[x_1,x_2,x_3]$; suppose $x_1 < x_2 < x_3$. Let $t = x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3} = x_1^3$, so $\alpha_1 = 3$, $\alpha_2 = \alpha_3 = 0$. The variable x_1 is multiplicative for t w.r.t J since there are no terms $t' = x_1^{\beta_1}x_2^{\beta_2}x_3^{\beta_3} \in J$ satisfying both conditions $\beta_1 > 3$and $\beta_2 = \beta_3 = 0$. On the other hand, x_2 is not multiplicative for t since $t'' = x_2^3 \in U$ satisfies $t'' = x_1^{\gamma_1}x_2^{\gamma_2}x_3^{\gamma_3}$ with $\gamma_2 = 3 > 0 = \alpha_2$, $\gamma_3 = \alpha_3 = 0$. Similarly, x_3 is not multiplicative since $x_3^2 \in U$. In conclusion, we have $M_J(t,U) = \{x_1\}$, $NM_J(t,U) = \{x_2,x_3\}$; $C_J(t,U) = \{x_1^h | h \in \mathbb{N},\, h \geq 3\}$.

Remark 4.5. Observe that, by definition of multiplicative variable, the only element in $C_J(t,U) \cap U$ is t itself. Indeed, if $t \in U$ and also $ts \in U$ for a non constant term s, then $\max(s)$ cannot be multiplicative for t, hence. $ts \notin C_J(t,U)$.

Janet introduced then the concept of *complete system* and gave a procedure (*completion*) to produce the decomposition in cones.

Definition 4.6. [32, ppg.75-9] A set of terms $U \subset \mathcal{T}$ is called *complete* if for every $t \in U$ and $x_j \in NM_J(t,U)$, there exists $t' \in U$ such that $x_jt \in C_J(t',U)$. The term t' is called *involutive divisor* of x_jt w.r.t. Janet division.

Depending on the notion of multiplicative variable, then also completeness depends on the variables' ordering.

Remark 4.7. If $U = \{t\} \subseteq \mathbf{k}[x_1..., x_n]$ is a singleton, it is complete, since $M_J(t,u) = \{x_1, ..., x_n\}$.

In the same paper, in order to describe Riquier's [40] formulation of the description for the general solutions of a PDE problem, Janet gave a similar decomposition in terms of disjoint cones, generated by multiplicative variables, also for the related normal set/order ideal/*escalier* $\mathbf{N}(J)$.
The construction of a Bar Code can help to assign to each element t of a finite set of terms $U \subset \mathcal{T}$ its multiplicative variables, according to Janet's Definition 4.1.
Let $U \subset \mathcal{T} \subset \mathbf{k}[x_1, ..., x_n]$ be a finite set of terms and suppose $x_1 < x_2 < ... < x_n$. As explained in section 3, we can associate a Bar Code B to it. Once B is constructed, even if it is not necessary that B is an admissible Bar Code, we can mimick on it the set up we generally perform to construct the star set. In particular:

a) $\forall 1 \leq i \leq n$, place a star symbol $*$ on the right of $\mathsf{B}^{(i)}_{\mu(i)}$;
b) $\forall 1 \leq i \leq n-1$, $\forall 1 \leq j \leq \mu(i)-1$ let $\mathsf{B}^{(i)}_j$ and $\mathsf{B}^{(i)}_{j+1}$ be two consecutive bars not lying over the same $(i+1)$-bar; place a star symbol $*$ between them.

Now, given a term $t \in U$, to detect its multiplicative variables it is enough to check the bars over which it lies, as stated in the following proposition (see [9]).

Proposition 4.8. Let $U \subseteq \mathcal{T}$ be a finite set of terms and let us denote by B_U its Bar Code. For each $t \in U$ x_i, $1 \leq i \leq n$ is multiplicative for t if and only if, in B_U, the i-bar $\mathsf{B}_j^{(i)}$, over which t lies, is followed by a star.

Example 4.9. For the set $U = \{x_1^3, x_2^3, x_1^4x_2x_3, x_3^2\} \subseteq \mathbf{k}[x_1, x_2, x_3]$, $x_1 < x_2 < x_3$, of example 4.4, we have the following Bar Code

$$\begin{array}{lllll} {}_0 & x_1^3 & x_2^3 & x_1^4x_2x_3 & x_3^2 \\ {}_1 & \text{—}\,* & \text{—}\,* & \text{—}\,* & \text{—}\,* \\ {}_2 & \text{—} & \text{—}\,* & \text{—}\,* & \text{—}\,* \\ {}_3 & \multicolumn{2}{l}{\text{————}} & \text{—} & \text{—}\,* \end{array}$$

Then, looking at the stars, we can desume that:

- $M_J(x_1^3, U) = \{x_1\}$, $NM_J(x_1^3, U) = \{x_2, x_3\}$;
- $M_J(x_2^3, U) = \{x_1x_2\}$, $NM_J(x_2^3, U) = \{x_3\}$;
- $M_J(x_1^4x_2x_3, U) = \{x_1, x_2\}$,$NM_J(x_1^4x_2x_3, U) = \{x_3\}$;
- $M_J(x_3^2, U) = \{x_1, x_2, x_3\}$, $NM_J(x_3^2, U) = \emptyset$.

Remark 4.10. The Bar Code we are using to detect multiplicative variables is a reformulation of Gerdt-Blinkov-Yanovich *Janet trie* [24], but in the (equivalent) presentation given by Seiler [43]. However, given a finite set of terms, the algorithms for producing its Janet decomposition which can be deduced from both the formulations above of the Janet tree, are different from the algorithm naturally arising from the previous proposition.

In [32], starting from his definition of multiplicative variable (Definition 4.1), Janet deduces the following straightforward corollary, whose proof is reported in [35].

Corollary 4.11 ([32]). *Let $U = \{t_1, ..., t_m\} \subseteq \mathcal{T}$ be a finite set of terms, $t_i = x_1^{\alpha_1^{(i)}} \cdots x_n^{\alpha_n^{(i)}}$ and $t_i' = x_1^{\alpha_1^{(i)}} \cdots x_{n-1}^{\alpha_{n-1}^{(i)}} = \frac{t_i}{x_n^{\alpha_n^{(i)}}}$, for $i = 1, ..., m$. Let $U' = \{t_1', ..., t_m'\}$, $\alpha = \max\{\alpha_n^{(i)},\ 1 \leq i \leq m\}$. For each $\lambda \leq \alpha$, we define $I_\lambda := \{i : 1 \leq i \leq m | \alpha_n^{(i)} = \lambda\}$, the set indexing the terms in U with n-th degree equal to λ, and $U_\lambda' := \{t_i' | i \in I_\lambda\}$. Then U is complete if and only if the two conditions below hold:*

1. *For each $\lambda \in \{\alpha_n^{(i)}, 1 \leq i \leq m\}$, U_λ' is a complete set;*
2. *$\forall t_i' \in U_\lambda'$, $\lambda < \alpha$, there exists $j \in \{1, ..., m\}$ such that*
   - *$t_i' \in C_J(t_j', U')$;*
   - *$t_j' \in U_{\lambda+1}'$.*

Completeness of a given finite set U can be detected by exploiting the Bar Code, as stated in the following proposition.

Proposition 4.12. Let $U \subseteq \mathcal{T}$ be a finite set of terms and B be its Bar Code. Let $t \in U$, $x_i \in NM_J(t, U)$ and $\mathsf{B}_j^{(i)}$ the i-bar under t.
Let $s \in U$; it holds $s \mid_J x_it$ if and only if

1. $s \mid x_it$
2. s lies over $\mathsf{B}_{j+1}^{(i)}$ and
3. $\forall j'$ appearing with nonzero exponent in $\frac{x_it}{s}$ there is a star after the j'-bar under s.

Proof. "$\Leftarrow$" It is an obvious consequence of Proposition 4.8; indeed, by 1. $s \mid x_it$. Thanks to 3., all the variables in $\frac{x_it}{s}$ are multiplicative. Note that x_i is not a variable of $w := \frac{x_it}{s}$ and it does not need to be multiplicative for s, since, s lies over $\mathsf{B}_{j+1}^{(i)}$, so $\deg_i(s) = \deg_i(t) + 1$. So $sw = x_it$ and w contains only multiplicative variables for s; therefore $s \mid_J x_it$.
"$\Rightarrow$" Let $s \in U$, $s \mid_J x_it$; $s \mid x_it$ by definition of Janet division[3].

[3]And actually of involutive division.

If s would lie over $\mathsf{B}_j^{(i)}$, then $\deg_l(s) = \deg_l(t)$ for $l = i, ..., n$, i.e. in s and t the variables $x_i, ..., x_n$ appear with the same exponent. Then, being $s \mid x_i t$, $x_i \in V_s := \{x_j, 1 \le j \le n : x_j \mid w := \frac{x_i t}{s}\}$, so x_i should be multiplicative for s, this meaning having a star after $\mathsf{B}_j^{(i)}$, which is impossible by hypothesis, since in this case $x_i t \in C_J(s,U) \cap C_J(t,U)$.
If s would lie over $\mathsf{B}_l^{(i)}$, $l > j+1$, there exists $h \in \{i, ..., n\}$ s.t. $\deg_h(s) > \deg_h(x_i t)$, so $s \nmid x_i t$, which is again a contradiction.
If s would lie over $\mathsf{B}_l^{(i)}$, $l < j$, then $s <_{Lex} t$ and it cannot happen that $\deg_{l'}(s) = \deg_{l'}(t)$ for $l' = i, ..., n$ (since otherwise s would have been over $\mathsf{B}_j^{(i)}$). Let $x_k := \max\{x_h,\, h = 1, ..., n \mid \deg_h(s) < \deg_h(t)\}$; then, since $t \in U$ and $\deg_n(t) = \deg_n(s), ...,$ $\deg_{k+1}(t) = \deg_{k+1}(s)$ and $\deg_k(t) > \deg_k(s)$, by definition of multiplicative variable according to Janet division, $x_k \in NM_J(s,U)$. Then s must lie over $\mathsf{B}_{j+1}^{(i)}$. For being $s \mid x_i t$, all the variables appearing with nonzero exponent in $\frac{x_i t}{s}$ must be multiplicative for s, and this implies that $\forall j'$ appearing with nonzero exponent in $\frac{x_i t}{s}$ there is a star after the j'-bar under s, by Proposition 4.8. □

From Proposition 4.12 we finally get the following

Theorem 4.13. *Let $U \subseteq \mathcal{T}$ be a finite set of terms and B be its Bar Code. Then U is a complete set if and only if $\forall t \in U$, $\forall x_i \in NM_J(t,U)$, called $\mathsf{B}_j^{(i)}$ the i-bar under t, there exists a term $s \in U$ satisfying conditions 1, 2, 3 of Proposition 4.12.*

According to Proposition 4.12 and Theorem 4.13, given a finite set of term $U \subseteq \mathcal{T}$, to check its completeness we take, $\forall t \in U$, $\forall x_i \in NM_J(t,U)$, the i-bar $\mathsf{B}_j^{(i)}$, $1 \le j \le \mu(i)$ under t and we look for an involutive divisor among the terms over $\mathsf{B}_{j+1}^{(i)}$, checking conditions 1,3 above. We see now two simple examples of this procedure.

Example 4.14. Coming back to examples 4.4 and 4.9, we consider again $U = \{x_1^3, x_2^3, x_1^4 x_2 x_3,\, x_3^2\} \subseteq \mathbf{k}[x_1, x_2, x_3]$ and its Bar Code

$$\begin{array}{lllll}
0 & x_1^3 & x_2^3 & x_1^4x_2x_3 & x_3^2 \\
1 & \text{---}\, * & \text{---}\, * & \text{---}\, * & \text{---}\, * \\
2 & \text{---} & \text{---}\, * & \text{---}\, * & \text{---}\, * \\
3 & \multicolumn{2}{l}{\text{------------}} & \text{---} & \text{---}\, *
\end{array}$$

Take $t = x_1^3$ and $x_2 \in NM_J(t,U) = \{x_2, x_3\}$; t lies over $\mathsf{B}_1^{(2)}$ and the only term over $\mathsf{B}_2^{(2)}$ is $x_2^3 \nmid x_1^3 x_2 = t x_2$, so $t x_2$ has no involutive divisor on U and this implies that our set is actually non-complete.

Example 4.15. Consider the set $U = \{x^2, xy\} \subset \mathbf{k}[x,y]$, $x < y$. Its Bar Code is

$$\begin{array}{lll}
0 & x^2 & xy \\
1 & \text{---}\, * & \text{---}\, * \\
2 & \text{---} & \text{---}\, *
\end{array}$$

so, looking at the stars, we can desume $M_J(x^2,U) = \{x\}$, $NM_J(x^2,U) = \{y\}$, $M_J(xy,U) = \{x,y\}$, $NM_J(xy,U) = \emptyset$. Now, $t = x^2$ lies over $\mathsf{B}_1^{(2)}$ and over $\mathsf{B}_2^{(2)}$ there is only xy s.t. $xy \mid x^2 y$. Since $x \in M_J(xy,U)$, $xy \mid_J x^2 y$ and we can conclude that U is complete, w.r.t. the given ordering on the variables.

5. A greedy algorithm for complete sets.

In this section, given a finite set of terms $U = \{t_1, ..., t_m\} \subseteq \mathcal{T}$, we try to find out whether there exists an *ordering on the variables* $x_1, ..., x_n$ such that U is *complete*. As explained in section 4, the Bar Code allows to detect the completeness of U. Clearly such a construction depends on the variables' ordering, so if we want to solve the problem, in principle, we should draw and check $n!$ different Bar Codes,

which turns out to be rather tedious and time consuming. Exploiting again the Bar Code and Corollary 4.11, we can look for the solution of our problem in a “greedy” way, so that most of the tests can be skipped. The idea consists in constructing the Bar Code B of the set $U = \{t_1, ..., t_m\} \subset \mathcal{T}$ from the maximal variable to the minimal one, checking if, with the choice made up to the current point on the variables' ordering, the conditions of Proposition 4.12 hold for each term in U, and going back retracting our steps in case of failure, so modifying previous choices.
Let $X = \{x_1, ..., x_n\}$ be the set of all variables. In the first step we look for the subset $Y \subseteq X$ of good candidates for being the maximal variable, scanning the elements of X (see Corollary 4.11). For $i = 1, ..., n$, we compute the sets $D_i := \{\beta \in \mathbb{N} | \exists t \in U, \deg_i(t) = \beta\}$. If for some $\gamma \in D_i$ $\gamma < \max(D_i)$, $\gamma + 1 \notin D_i$, then x_i cannot be the maximal variable. Indeed, if x_i would be the maximal variable, then there would exist $t'_1 = \frac{t_1}{x_1^\gamma} \in U'_\gamma$ and by Corollary 4.11, we would need a term in $U'_{\gamma+1}$, which is actually the empty set. Clearly, if $Y = \emptyset$, no variable is suitable for being the maximal one, making U complete and this implies that U is not complete for any variables' ordering.

Example 5.1. Consider $U = \{x_1x_2^3, x_1^3x_2\} \subset \mathbf{k}[x_1, x_2]$. Such a set is not complete since $D_1 = D_2 = \{1, 3\}$. As a confirmation, we can see that, if $x_1 < x_2$, we have

$$\begin{array}{lcc}
{}_0 & x_1^3x_2 & x_1x_2^3 \\
{}_1 & \rule{1.2em}{0.4pt}\,* & \rule{1.2em}{0.4pt}\,* \\
{}_2 & \rule{1.2em}{0.4pt} & \rule{1.2em}{0.4pt}\,*
\end{array}$$

Then $M_J(x_1^3x_2, U) = \{x_1\}$, $M_J(x_1x_2^3, U) = \{x_1, x_2\}$ and $x_1^3x_2^2$ does not belong neither to $C_J(x_1^3x_2, U)$ nor to $C_J(x_1x_2^3, U)$.

On the other hand, if $x_2 < x_1$, we have

$$\begin{array}{lcc}
{}_0 & x_1x_2^3 & x_1^3x_2 \\
{}_2 & \rule{1.2em}{0.4pt}\,* & \rule{1.2em}{0.4pt}\,* \\
{}_1 & \rule{1.2em}{0.4pt} & \rule{1.2em}{0.4pt}\,*
\end{array}$$

Thus $M_J(x_1x_2^3, U) = \{x_2\}$, $M_J(x_1^3x_2, U) = \{x_1, x_2\}$ and $x_1^2x_2^3$ does not belong neither to $C_J(x_1^3x_2, U)$ nor to $C_J(x_1x_2^3, U)$.

Suppose now $\emptyset \neq Y = \{x_{j_1}, ..., x_{j_l}\} \subseteq X$; we start picking $x_{j_1} \in Y$ as maximal variable. We reorder the elements of U, increasingly w.r.t. their j_1-degree; we pose the only condition $t < t'$ when $t \mid t'$ for some $t, t' \in U$ with $deg_{j_1}(t) = deg_{j_1}(t')$. Then, we write the corresponding j_1-bars $\mathsf{B}_1^{(j_1)}, ..., \mathsf{B}_{\mu(j_1)}^{(j_1)}$ under the terms (one bar under each terms of some j_1-degree).

Example 5.2. Let us consider the set $U = \{x_1, x_1^2, x_2, x_1x_3\} \subset \mathbf{k}[x_1, x_2, x_3]$; we first compute $D_1 = \{0, 1, 2\}$, $D_2 = D_2 = \{0, 1\}$. All the variables are good candidates for being the maximal one. We pick, for example, x_3, so we have

$$\begin{array}{lcccc}
{}_0 & x_1 & x_1^2 & x_2 & x_1x_3 \\
\\
\\
{}_3 & \multicolumn{3}{c}{\rule{10em}{0.4pt}} & \rule{1.2em}{0.4pt}\,*
\end{array}$$

We remark that, we could also have picked another variables, obtaining a different Bar Code, for example, picking x_1, we would have got:

$$\begin{array}{lcccc}
{}_0 & x_2 & x_1 & x_1x_3 & x_1^2 \\
\\
\\
{}_1 & \rule{1.2em}{0.4pt} & \multicolumn{2}{c}{\rule{6em}{0.4pt}} & \rule{1.2em}{0.4pt}\,*
\end{array}$$

Now, with the routine Friends, we look for candidate terms for having condition 2 of Corollary 4.11 satisfied, so for $1 \leq j < \mu(j_1)$, we fix $\mathsf{B}_j^{(j_1)}$ and, for each t over $\mathsf{B}_j^{(j_1)}$ we define the set $U(t, x_j) = \{(u, \alpha) | u \in B_{j+1}^{(j_1)}$ and $\alpha : tx_j = um,\, m \in \mathcal{T}[\alpha]\}$ of the candidate involutive divisors for tx_{j_1} (notice that x_{j_1} is the maximal variable, so $x_{j_1} \in M_J(v, U) \Leftrightarrow v$ lies over $\mathsf{B}_{\mu(j_1)}^{(j_1)}$ by Proposition 4.8). If exists $1 \leq j < \mu(j_1)$ and $\exists t'$ over $\mathsf{B}_j^{(j_1)}$ with $U(t', x_{j_1}) = \emptyset$, then x_{j_1} is not a good candidate for being the maximal variable, so we come back to Y and we start again with a new maximal variable.

Example 5.3. Coming back to example 5.2, we have $U(x_1, x_3) = \{(x_1x_3, \emptyset)\}$, $U(x_1^2, x_3) = \{(x_1x_3, \{x_1\})\}$ and $U(x_2, x_3) = \emptyset$, so x_3 was a bad choice for being the maximal variable and we try with x_1, getting

$$\begin{array}{lcccc}
{\scriptstyle 0} & x_2 & x_1 & x_1x_3 & x_1^2 \\
 & & & & \\
{\scriptstyle 1} & \rule{1em}{0.4pt} & \multicolumn{2}{c}{\rule{4em}{0.4pt}} & \rule{1em}{0.4pt}\, *
\end{array}$$

Now, $U(x_2, x_1) = \{(x_1, \{x_2\})\}$, $U(x_1, x_1) = \{(x_1^2, \emptyset)\}$ and $U(x_1x_3, x_1) = \{(x_1^2, \{x_3\})\}$, so x_1, at least for now, is a good choice for the maximal variable.

Suppose to be in the non-failure case; if for $1 \leq j \leq \mu(j_1)$ there is only one term over $\mathsf{B}_j^{(j_1)}$, all the bars are *unitary* so we say that we are in the *unitary case*. In this case, each variable ordering s.t. x_{j_1} is the maximal variable makes U a complete set of terms. Indeed, in this case, for each choice of the following variables, their corresponding bars will be unitary again and, by the construction of the stars, all of them will be followed by a star. In other words, for each $t \in U$, and for each $x_j \neq x_{j_1}$, $x_j \in M_J(t, U)$. Moreover $\forall t \in U$, $|U(t, x_{j_1})| = 1$, so let (u, α) be the only element of $U(t, x_{j_1})$, then all variables in α differ from x_{j_1}, so they are multiplicative for u and this makes u the required involutive divisor of $x_{j_1}t$, ensuring the completeness of U.

Example 5.4. For $U = \{x^3, xy, y^2\} \subset \mathbf{k}[x, y]$, $D_1 = \{0, 1, 3\}$ and $D_2 = \{0, 1, 2\}$, so $Y = \{y\}$. Chosing y as maximal variable we have:

$$\begin{array}{lccc}
{\scriptstyle 0} & x^3 & xy & y^2 \\
{\scriptstyle 1} & & & \\
{\scriptstyle 2} & \rule{1em}{0.4pt} & \rule{1em}{0.4pt} & \rule{1em}{0.4pt}
\end{array}$$

All the 2-bars are unitary, so, completing the Bar Code we get

$$\begin{array}{lccc}
{\scriptstyle 0} & x^3 & xy & y^2 \\
{\scriptstyle 1} & \rule{1em}{0.4pt}\, * & \rule{1em}{0.4pt}\, * & \rule{1em}{0.4pt}\, * \\
{\scriptstyle 2} & \rule{1em}{0.4pt} & \rule{1em}{0.4pt} & \rule{1em}{0.4pt}\, *
\end{array}$$

We can easily read from the diagram that $(x^3) \cdot y \in C_J(xy, U)$ and $(xy) \cdot y \in C_J(y^2, U)$, making U complete.

If we are not in the unitary case, we have to choose the next variable and continue drawing the Bar Code, using the routine Common.
To get the candidates for being the next variable, we execute the procedure CandidateVar to each j_1-bar and (procedure Candidate) we intersect the results. If the intersection is empty then x_{j_1} was not a good choice for being the maximal variable and we have to come back and repeat the whole procedure for another

maximal variable. Otherwise, we choose some x_{j_2} among the variables in the intersection, and for each $1 \leq j \leq \mu(j_1)$, we order the terms over $\mathsf{B}_j^{(j_1)}$ exactly as done for constructing the j_1-bars and we draw all the j_2-bars. Employing again the routine Friends, separately for each j_1-bar, we look for candidate involutive divisors when x_{j_2} is not multiplicative. Moreover, we check whether the choice of x_{j_2} is suitable to the candidates found in the previous step. Indeed, for each t over $\mathsf{B}_j^{(j_1)}$, $1 \leq j < \mu(j_1)$, we have constructed a set of candidates $U(t, x_{j_1})$. Given $(u, \alpha) \in U(t, x_{j_1})$, if $x_{j_2} \notin \alpha$, then the multiplicativity of x_{j_2} is irrelevant for u, so (u, α) still remains a good candidate for being an involutive divisor. It is still a good candidate also if $x_{j_2} \in \alpha$ and the j_2-bar of u is in one of the conditions for being followed by a star (see section 3), since it means that x_{j_2} is multiplicative for u. Otherwise we remove (u, α) from the candidates. If for some t its candidate list is empty we have to revoke the choice of x_{j_2} and come back with another candidate. If the procedure gives a positive outcome, then a new variable has been chosen and the routine Common keeps calling itself until

- all variables have been placed (positive outcome)
- the unitary case is reached (positive outcome)
- continue revocations of choices lead to failure (negative outcome).

Example 5.5. We conclude now examples 5.2 and 5.3. From

$$\begin{array}{ccccccccc}
{\scriptstyle 0} & x_2 & & x_1 & & x_1x_3 & & x_1^2 & \\
 & & & & & & & & \\
{\scriptstyle 1} & \rule[0.5ex]{2em}{0.4pt} & & \multicolumn{3}{c}{\rule[0.5ex]{6em}{0.4pt}} & & \rule[0.5ex]{2em}{0.4pt} & *
\end{array}$$

we choose now x_2 as following variable and we get

$$\begin{array}{ccccccccc}
{\scriptstyle 0} & x_2 & & x_1 & & x_1x_3 & & x_1^2 & \\
 & & & & & & & & \\
{\scriptstyle 2} & \rule[0.5ex]{2em}{0.4pt} & * & \multicolumn{3}{c}{\rule[0.5ex]{6em}{0.4pt}} & * & \rule[0.5ex]{2em}{0.4pt} & * \\
{\scriptstyle 1} & \rule[0.5ex]{2em}{0.4pt} & & \multicolumn{3}{c}{\rule[0.5ex]{6em}{0.4pt}} & & \rule[0.5ex]{2em}{0.4pt} & *
\end{array}$$

Since x_2 is multiplicative for all terms, and since over each x_1-bar there is only one x_2-bar, Friends gives a positive outcome. Finally choosing x_3, we get

$$\begin{array}{ccccccccc}
{\scriptstyle 0} & x_2 & & x_1 & & x_1x_3 & & x_1^2 & \\
{\scriptstyle 3} & \rule[0.5ex]{2em}{0.4pt} & * & \rule[0.5ex]{2em}{0.4pt} & & \rule[0.5ex]{2em}{0.4pt} & * & \rule[0.5ex]{2em}{0.4pt} & * \\
{\scriptstyle 2} & \rule[0.5ex]{2em}{0.4pt} & * & \multicolumn{3}{c}{\rule[0.5ex]{6em}{0.4pt}} & * & \rule[0.5ex]{2em}{0.4pt} & * \\
{\scriptstyle 1} & \rule[0.5ex]{2em}{0.4pt} & & \multicolumn{3}{c}{\rule[0.5ex]{6em}{0.4pt}} & & \rule[0.5ex]{2em}{0.4pt} & *
\end{array}$$

Now, x_3 is multiplicative for x_1^2 as required by $U(x_1x_3, x_1)$ and we have $U(x_1, x_3) = \{(x_1x_3, \emptyset)\}$, so U turns out to be complete with the variables' ordering $x_3 < x_2 < x_1$. We point out that this is not the only ordering making U complete, in particular, for $x_1 < x_3 < x_2$ U is complete again:

$$\begin{array}{ccccccccc}
{\scriptstyle 0} & x_1 & & x_1^2 & & x_1x_3 & & x_2 & \\
{\scriptstyle 1} & \rule[0.5ex]{2em}{0.4pt} & & \rule[0.5ex]{2em}{0.4pt} & * & \rule[0.5ex]{2em}{0.4pt} & * & \rule[0.5ex]{2em}{0.4pt} & * \\
{\scriptstyle 3} & \multicolumn{3}{c}{\rule[0.5ex]{6em}{0.4pt}} & & \rule[0.5ex]{2em}{0.4pt} & * & \rule[0.5ex]{2em}{0.4pt} & * \\
{\scriptstyle 2} & \multicolumn{5}{c}{\rule[0.5ex]{10em}{0.4pt}} & & \rule[0.5ex]{2em}{0.4pt} & *
\end{array}$$

Indeed

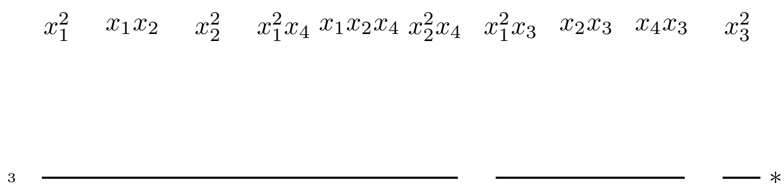
- $M_J(x_1,U)=\emptyset$, $NM_J(x_1,U)=\{x_1,x_2,x_3\}$, with $x_1^2\in C_J(x_1^2,U)$, $x_1x_2\in C_J(x_2,U)$, $x_1x_3\in C_J(x_1x_3,U)$;
- $M_J(x_1^2,U)=\{x_1\}$, $NM_J(x_1^2,U)=\{x_2,x_3\}$, with $x_1^2x_2\in C_J(x_2,U)$, $x_1^2x_3\in C_J(x_1x_3,U)$;
- $M_J(x_1x_3,U)=\{x_1,x_3\}$, $NM_J(x_1x_3,U)=\{x_2\}$, with $x_1x_2x_3\in C_J(x_2,U)$;
- $M_J(x_2,U)=\{x_1,x_2,x_3\}$, $NM_J(x_2,U)=\emptyset$.

The pseudocode of all mentioned routines is displayed in Appendix A. We see now a complete example for the whole procedure.

Example 5.6. Consider the set

$$M=\{x_2x_3,x_1^2,x_3^2,x_2^2,x_1x_2,x_1x_2x_4,x_1^2x_4,x_4x_3,x_2^2x_4,x_1^2x_3\}\subset \mathbf{k}[x_1,x_2,x_3,x_4].$$

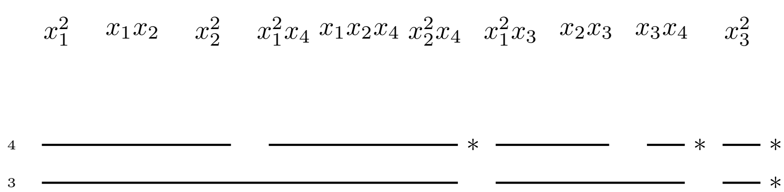
First, we compute $D_1=D_2=D_3=\{0,1,2\}$, $D_4=\{0,1\}$, desuming that each variable is a good candidate for being the maximal one, so $Y=\{x_1,x_2,x_3,x_4\}$. We choose, for example, x_3, getting



Now, running Friends for the first time, we get

- $U(x_1^2,x_3)=\{(x_1^2x_3,\emptyset)\}$;
- $U(x_1x_2,x_3)=\{(x_2x_3,\{x_1\})\}$;
- $U(x_2^2,x_3)=\{(x_2x_3,\{x_2\})\}$;
- $U(x_1^2x_4,x_3)=\{(x_1^2x_3,\{x_4\}),(x_3x_4,\{x_1\})\}$;
- $U(x_1x_2x_4,x_3)=\{(x_2x_3,\{x_1,x_4\}),(x_3x_4,\{x_1,x_2\})\}$;
- $U(x_2^2x_4,x_3)=\{(x_2x_3,\{x_2,x_4\}),(x_3x_4,\{x_2\})\}$;
- $U(x_1^2x_3,x_3)=\{(x_3^2,\{x_1\})\}$;
- $U(x_2x_3,x_3)=\{(x_3^2,\{x_2\})\}$;
- $U(x_3x_4,x_3)=\{(x_3^2,\{x_4\})\}$.

The procedure gives a positive outcome, so, since we are not in the unitary case, we apply Common. All the variables are good candidates for being the second in order of magnitude and, for example, we choose x_4, getting:



We have:

- $U(x_1^2,x_4)=\{(x_1^2x_4,\emptyset),(x_4,\{x_1\})\}$;
- $U(x_1x_2,x_4)=\{(x_1x_2x_4,\emptyset)\}$;
- $U(x_2^2,x_4)=\{(x_2^2x_4,\emptyset)\}$;
- $U(x_2,x_4)=\{(x_4,\{x_2\})\}$.

We check that the choice of x_4 is suitable for the condition imposed in the previous step

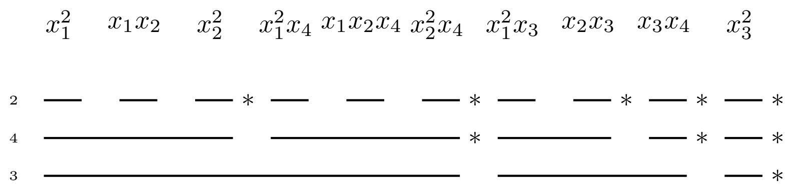
- for $U(x_1^2x_4,x_3)=\{(x_1^2x_3,\{x_4\}),(x_3x_4,\{x_1\})\}$ e notice that $x_1^2x_3$ does not lie on the rightmost 4-bar, so x_4 is not multiplicative. Since we have more than one term associated to $x_1^2x_4$, we only delete $x_1^2x_3$ and keep x_3x_4.
  The same argument holds for $x_1x_2x_4,x_2^2x_4$.

- For $U(x_3x_4, x_3) = \{(x_3^2, \{x_4\})\}$, since x_3^2 lies on the rightmost 4-bar, x_3^2 passes the test, remaining a good candidate for being an involutive divisor.

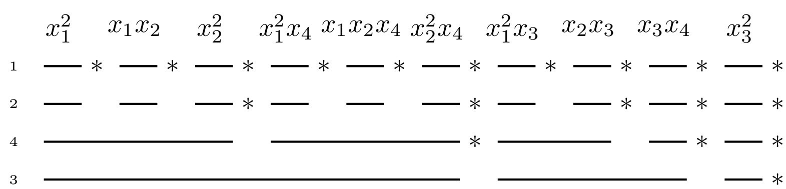
So we have

- $U(x_1^2, x_3) = \{(x_1^2x_3, \emptyset)\}$;
- $U(x_1x_2, x_3) = \{(x_2x_3, \{x_1\})\}$;
- $U(x_2^2, x_3) = \{(x_2x_3, \{x_2\})\}$;
- $U(x_1^2x_4, x_3) = \{(x_3x_4, \{x_1\})\}$;
- $U(x_1x_2x_4, x_3) = \{(x_3x_4, \{x_1, x_2\})\}$;
- $U(x_2^2x_4, x_3) = \{(x_3x_4, \{x_2\})\}$;
- $U(x_1^2x_3, x_3) = \{(x_3^2, \{x_1\})\}$;
- $U(x_2x_3, x_3) = \{(x_3^2, \{x_2\})\}$;
- $U(x_3x_4, x_3) = \{(x_3^2, \{x_4\})\}$;
- $U(x_1^2, x_4) = \{(x_1^2x_4, \emptyset), (x_4, \{x_1\})\}$;
- $U(x_1x_2, x_4) = \{(x_1x_2x_4, \emptyset)\}$;
- $U(x_2^2, x_4) = \{(x_2^2x_4, \emptyset)\}$;
- $U(x_2, x_4) = \{(x_4, \{x_2\})\}$.

We continue choosing x_2 as next variable and we get:



This way, all the 2-bars are unitary. We check on the 2-bars to have nonincreasing exponents for x_1 and this is true. Moreover, we check that x_2 is multiplicative where it is marked, i.e. for x_2x_3, x_3x_4 but it clearly holds. The set M is complete for $x_1 < x_2 < x_4 < x_3$ and its final Bar Code w.r.t. the chosen ordering is



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Appendix A. Pseudocode of all procedures

Algorithm 1 Procedure to generate the candidate list for the current maximal variable (subroutine).

procedure CandidateVar(M, C) ▷ M is a set of terms; C is a set of variables.
  for $i = 1, ..., |C|$ **do**
  $D_i := \{\beta \in \mathbb{N} | \exists t \in M, \deg_{C[i]}(t) = \beta\}$
  end for
  if for some $\gamma_1 \in D_i$, $\gamma_1 < \max(D_i)$, $\gamma_1 + 1 \notin D_i$ **then**
    Delete $C[i]$ from C
  return C

Algorithm 2 Procedure to generate the candidate list for the current maximal variable.

procedure Candidates(M, C) ▷ M is a list of lists of terms; C is a set of variables.
  for $i = 1, ..., |M|$ **do**
  $Y[i]$:=CandidateVar($M[i], C$);
  end for
  return $\bigcap_i Y[i]$

Algorithm 3 Friends

procedure FRIENDS(A, Y, x_j, T) ▷ T is the output of a previous execution of Friends (or it is empty), so it is formed by sets of the form $T(t, x_j)$, t terms in the given set, and x_j variables.

  for $B \in A$ **do**

  B' =NextB(B, A) ▷ Subroutine taking as input a bar B and the (partial) Bar Code A containing it and giving as output the bar in A that is just on the right of B or error if there is not such bar.

  for $t \in B$ **do**

  $U(t, x_j) = \{(u, \alpha) | u \in B' \text{ and } \alpha : tx_j = um,\, m \in \mathcal{T}[\alpha]\}$

  end for

  if $U(t, x_j) = \emptyset$ **then return** fail

  end for

  if $T \neq \emptyset$ **then**

    for $B \in A$ **do**

    for $t \in B$ **do**

    for $y \in X \setminus Y$ **do**

    $U(t, y) = \emptyset$

    for $(U, \alpha) \in T(t, y)$ **do**

    if $x_j \notin \alpha$ **then**

      $U(t, y) = \{(U, \alpha)\} \cup U(t, y)$

    else

      if $x_j \in \alpha$ and Star(x_j, t) =true **then**

        $U(t, y) = \{(U, \alpha)\} \cup U(t, y)$

    end for

    if $U(t, y) = \emptyset$ **then**

      return fail

    end for

    end for

    end for

  return U

Algorithm 4 Common

procedure COMMON(A, X, x_i, T) ▷ T is the output of a previous execution of Friends (or it is empty), so it is formed by sets of the form $T(t, x_j)$, t terms in the given set, and x_j variables.
 $Y = X \setminus \{x_i\}$
 Y' =Candidate(A, Y)
 if $|Y'| = 0$ **then return** $\emptyset$
 for $x_j \in Y'$ **do**
 for $B \in A$ **do**
 construct $B^j = \{B^{(j}\}$; $C = \{B^{(j)} | B \in A\}$; U =Friends(C, Y, x_j, T)
 if U = failure **then continue**
 if $|B^{(j)}| = 1, \forall B^{(j)} \in C$ **then** $ord = Y \cup \{x_j\}$ **return** ord
 if $Y \neq \emptyset$ **then**
  C =Common(C, Y, x_j, U)
 else
  if $Y = \emptyset$ **then** $ord = ord \cup \{x_j\}$
   return ord
  if $ord \neq \emptyset$ **then** $ord = ord \cup \{x_j\}$
  else **continue**
 end for
 end for
 return $\emptyset$

Algorithm 5 Ordering

procedure ORDERING(M, X) ▷ M is a given set of terms; X is the set of all variables
 Y =Candidate(M, X)
 for $x_i \in Y$ **do**
 $A = \{A^{(i)}\}$ ▷ The bars according to $\deg_i$ of the Bar Code we construct.
 T =Friends($A, X, x_i, \emptyset$)
 if T = failure **then**
  continue
 if $|A^{(i)}| = 1, \forall A^{(i)} \in A$ **then** ▷ Unitary case.
  ord = Append($X \setminus \{x_i\}, x_i$) ▷ We append x_i to the set $X \setminus \{x_i\}$.
  return ord
 C =Common(A, X, x_i, T)
 return the variable ordering.
 if $C \neq \emptyset$ **then**
  $ord = C \cup \{x_i\}$
  return ord
 else **continue**
 end for
 return $\emptyset$

DEPARTMENT OF COMPUTER SCIENCE, UNIVERSITY OF MILAN, MILAN, ITALY.
E-mail address: michela.ceria@gmail.com