

# Dagger Categories of Tame Relations

Bart Jacobs

**Abstract.** Within the context of an involutive monoidal category the notion of a comparison relation  $\text{cp}: \overline{X} \otimes X \rightarrow \Omega$  is identified. Instances are equality  $=$  on sets, inequality  $\leq$  on posets, orthogonality  $\perp$  on orthomodular lattices, non-empty intersection on powersets, and inner product  $\langle - | - \rangle$  on vector or Hilbert spaces. Associated with a collection of such (symmetric) comparison relations a dagger category is defined with “tame” relations as morphisms. Examples include familiar categories in the foundations of quantum mechanics, such as sets with partial injections, or with locally bifinite relations, or with formal distributions between them, or Hilbert spaces with bounded (continuous) linear maps. Of one particular example of such a dagger category of tame relations, involving sets and bifinite multirelations between them, the categorical structure is investigated in some detail. It turns out to involve symmetric monoidal dagger structure, with biproducts, and dagger kernels. This category may form an appropriate universe for discrete quantum computations, just like Hilbert spaces form a universe for continuous computation.

## 1. Introduction

So-called tame relations were introduced in [4] in the construction of a particular (monoidal) dagger category of formal distributions. The phrase ‘tame’ refers to finiteness restrictions in two directions, and is best illustrated in the context of relations. So suppose we have a relation  $r \subseteq X \times Y$ ; it can be described equivalently as a function  $X \rightarrow \mathcal{P}(Y)$ , where  $\mathcal{P}$  is powerset, or via reversal, as a function  $Y \rightarrow \mathcal{P}(X)$ . The relation is called tame, if both these functions factorise via the *finite* powerset  $\mathcal{P}_{fin}$ , as in  $X \rightarrow \mathcal{P}_{fin}(Y)$  and  $Y \rightarrow \mathcal{P}_{fin}(X)$ . Concretely, this means that for each  $x \in X$  there are only finitely many  $y \in Y$  with  $r(x, y)$ , and vice-versa. Such relations are often called (locally) bifinite. They may be used to model finitely non-deterministic reversible computations.

In [4] tameness is used in the context of polynomials. Let  $S[X]$  be the collection of (multivariate) polynomials, with variables in a set  $X$  and coefficients

in a semiring  $S$ ; similarly  $S[[X]]$  is used for possibly infinite such polynomials (or power series, or formal distributions). For certain analogues of relations, giving rise to mappings  $S[X] \rightarrow S[[Y]]$  and  $S[Y] \rightarrow S[[X]]$ , tameness means that these mappings factor via finite polynomials, as in  $S[X] \rightarrow S[Y]$  and  $S[Y] \rightarrow S[X]$ . For details, see Subsection 4.6 below.

This paper starts by identifying a general context in which this notion of tameness makes sense. It involves the notion of a comparison relation  $\text{cp}: \overline{X} \otimes X \rightarrow \Omega$ . Such a relation requires an ambient category with tensors  $\otimes$  and involution  $\overline{(-)}$ , as described for instance in [9, 2, 16]. A relation  $r: \overline{X} \otimes Y \rightarrow \Omega$  is then tame, if it factors via such comparisons, via appropriate maps  $r_*$  and  $r^*$ . It is shown that such categories of tame relations give rise to dagger categories, assuming the underlying comparison relation is symmetric. Section 4 illustrates how this general construction encompasses several known categories that are relevant in the foundations of quantum mechanics, such as orthomodular lattices with Galois connections, or sets with partial injections or with bifinite relations, or with bifinite multirelations, or with bistochastic relations. Some of these constructions are also described more abstractly, in terms of the monads involved, namely lift, finite powerset, multiset and distribution monads, see Subsection 4.3.

The formal distributions example from [4] is re-described in the present general setting. Additionally, bounded (or continuous) maps between Hilbert spaces are shown to correspond to tame relations (see Lemma 4.6).

Finally, one particular example category of tame relations, arising via the multiset monad from the monad construction just mentioned, is further investigated in Section 5. We refer to this as the category **BifMRel** of sets and bifinite multirelations. Morphisms  $X \rightarrow Y$  are functions  $r: X \times Y \rightarrow S$ , into a semiring  $S$ , such that for each  $x \in X$  there are only finitely many  $y$  with  $r(x, y) \neq 0$ , and vice-versa. This means that the relation factors both as  $X \rightarrow \mathcal{M}_S(Y)$  and as  $Y \rightarrow \mathcal{M}_S(X)$ , where  $\mathcal{M}_S$  is the multiset monad which “counts in  $S$ ”. Such bifinite multirelations may be used to model finitely weighted, reversible computations. It is shown that this category **BifMRel** has, besides daggers, tensors  $\otimes$  and biproducts  $\oplus$ . Moreover, it has dagger kernels, as described in [11]. Thus, the category **BifMRel** resembles the category **Hilb** of Hilbert spaces. It is suggested that this category **BifMRel** is the discrete analogue of **Hilb**, useful for discrete quantum computations, such as usually occurring in a quantum computer science context (see *e.g.* [24, 25]). The quantum walks example from [14], formalised in **BifMRel**, supports this suggestion, but further evidence is required via more extensive investigation.

Thus, the contributions of the paper are two-fold: (1) identifying the uniformity in various models of quantum computation via a systematic exposition in terms of comparison relations, and (2) first investigation of one particular promising example of such a model for discrete quantum computation, namely the category of sets and bifinite multirelations.

## 2. Involutive categories, and comparisons therein

This section recalls the basics of involutive categories as presented in [16] (see also [2, 9]). Within such involutive categories the notion of ‘comparison’ is introduced.

A category  $\mathbf{A}$  will be called **involutive** if it comes with a ‘involution’ functor  $\mathbf{A} \rightarrow \mathbf{A}$ , written as  $X \mapsto \overline{X}$ , and a natural isomorphism  $\iota_X: X \xrightarrow{\cong} \overline{\overline{X}}$  satisfying  $\overline{\iota_X} = \iota_{\overline{X}}: \overline{X} \rightarrow \overline{\overline{\overline{X}}}$ .

Within such an involutive category a **self-conjugate** is an object  $X$  with a map  $j: \overline{X} \rightarrow X$  satisfying  $j \circ \overline{j} = \iota^{-1}: \overline{\overline{X}} \rightarrow X$ . Such a map  $j$  is necessarily an isomorphism. A self-conjugate is called a star-object in [2].

Each category is trivially involutive via the identity functor. The category  $\mathbf{PoSets}$  is involutive via order reversal  $(-)^{\text{op}}$ . This applies also to categories of, for instance, distributive lattices or Boolean algebras. Probably the most standard example of an involutive category is the category  $\mathbf{Vect}_{\mathbb{C}}$  of vector spaces over the complex numbers  $\mathbb{C}$ ; it is involutive via conjugation: for a vector space  $V \in \mathbf{Vect}_{\mathbb{C}}$  there is the ‘complex conjugate’ space  $\overline{V} \in \mathbf{Vect}_{\mathbb{C}}$  with the same vectors as  $V$ , but with adapted scalar multiplication  $s \cdot_{\overline{V}} v = \overline{s} \cdot_V v$ , for  $s \in \mathbb{C}$  and  $v \in V$ , where  $\overline{s} = a - ib$  is the conjugate of the complex number  $s = a + ib \in \mathbb{C}$ . This same involution exists on categories of Hilbert spaces (over  $\mathbb{C}$ ).

The negation map  $\neg: B^{\text{op}} \xrightarrow{\cong} B$  makes each Boolean algebra  $B$  self-conjugate, for the  $(-)^{\text{op}}$  involution on the category of Boolean algebras. The conjugation map  $(-)$  on the complex numbers makes  $\mathbb{C}$  a self-conjugate  $\overline{\mathbb{C}} \xrightarrow{\cong} \mathbb{C}$  in the category of vector (or Hilbert) spaces over  $\mathbb{C}$ .

**Definition 2.1.** An **involutive (symmetric) monoidal category** is a category  $\mathbf{A}$  which is both involutive and (symmetric) monoidal in which involution  $(-): \mathbf{A} \rightarrow \mathbf{A}$  is a (symmetric) monoidal functor—via maps  $\zeta: I \rightarrow \overline{I}$  and  $\xi: \overline{X} \otimes \overline{Y} \rightarrow \overline{X \otimes Y}$  commuting with the monoidal isomorphisms—and  $\iota: \text{id} \Rightarrow (-)$  is a monoidal natural transformation; this means that the following diagrams commutes.

$$\begin{array}{ccc}
 I & \xlongequal{\quad} & I \\
 \parallel & & \downarrow \iota \\
 I & \xrightarrow{\zeta} \overline{I} \xrightarrow{\overline{\zeta}} & \overline{\overline{I}}
 \end{array}
 \qquad
 \begin{array}{ccc}
 X \otimes Y & \xlongequal{\quad} & X \otimes Y \\
 \iota \otimes \iota \downarrow & & \downarrow \iota \\
 \overline{\overline{X}} \otimes \overline{\overline{Y}} & \xrightarrow{\xi} \overline{\overline{X \otimes Y}} \xrightarrow{\overline{\xi}} & \overline{\overline{\overline{X \otimes Y}}}
 \end{array}
 \tag{1}$$

One can show (see [16]) that the involution functor  $(-)$  is automatically strong monoidal: the maps  $\zeta: I \rightarrow \overline{I}$  and  $\xi: \overline{X} \otimes \overline{Y} \rightarrow \overline{X \otimes Y}$  are necessarily isomorphisms.

In the symmetric case, with symmetry  $\gamma: X \otimes Y \xrightarrow{\cong} Y \otimes X$ , we often use the ‘twist’  $\tau$  defined by:

$$\tau \stackrel{\text{def}}{=} \left( \overline{\overline{X \otimes Y}} \xrightarrow{\cong} \overline{\overline{X}} \otimes \overline{\overline{Y}} \xrightarrow{\cong} \overline{\overline{X}} \otimes \overline{\overline{Y}} \xrightarrow{\iota^{-1} \otimes \text{id}} X \otimes Y \xrightarrow{\cong} \overline{\overline{Y \otimes X}} \right). \tag{2}$$

For  $Y = X$  this map makes the object  $\overline{\overline{X}} \otimes X$  self-conjugate.

## 2.1. Comparison relations

The equality relation on a set  $X$  can be described as a map  $=: X \times X \rightarrow 2$  in **Sets**, where  $2 = \{0, 1\}$ . We wish to capture such maps more generally under the name ‘comparison relation’.

**Definition 2.2.** Assume an involutive monoidal category with a special object  $\Omega$ . A **comparison relation** is a map of the form  $\text{cp}: \overline{X} \otimes X \rightarrow \Omega$  satisfying:

$$f = g \quad \text{follows from either} \quad \begin{cases} \text{cp} \circ (\text{id} \otimes f) = \text{cp} \circ (\text{id} \otimes g) & \text{or} \\ \text{cp} \circ (f \otimes \text{id}) = \text{cp} \circ (g \otimes \text{id}) \end{cases}$$

In presence of exponents  $\multimap$ , these ‘mono requirements’ mean that the two associated abstraction maps  $X \rightarrow (\overline{X} \multimap \Omega)$  and  $\overline{X} \rightarrow (X \multimap \Omega)$  are monic.

In a symmetric monoidal setting such a comparison relation is called **symmetric** if the following diagram commutes,

$$\begin{array}{ccc} \overline{\overline{X} \otimes X} & \xrightarrow[\cong]{\tau} & \overline{X} \otimes X \\ \text{cp} \downarrow & & \downarrow \text{cp} \\ \overline{\Omega} & \xrightarrow[\cong]{j} & \Omega \end{array} \quad (3)$$

where a self-conjugate structure  $\overline{\Omega} \xrightarrow{j} \Omega$  is assumed, and where  $\tau$  is the twist map from (2).

In the symmetric case the two mono requirements—for each argument separately—can be reduced to a single requirement—in one argument only: if  $\text{cp} \circ (\text{id} \otimes f) = \text{cp} \circ (\text{id} \otimes g)$  implies  $f = g$ , then one can deduce that also  $\text{cp} \circ (f \otimes \text{id}) = \text{cp} \circ (g \otimes \text{id})$  implies  $f = g$  (and vice-versa).

An equality relation  $=: X \times X \rightarrow 2 = \{0, 1\}$  on a set  $X$  is given by  $(x = x) = 1$  and  $(x = x') = 0$  for  $x \neq x'$ . This is a symmetric comparison relation in the category **Sets**, with trivial (identity) involution. More interestingly, for a poset  $(X, \leq)$ , the order forms a non-symmetric comparison relation  $\leq: X^{\text{op}} \times X \rightarrow 2$  in **PoSets**. The involution  $(-)^{\text{op}}$  in the type of the first argument is needed for monotonicity, since:  $x \geq x'$  and  $x \leq y$  and  $y \leq y'$  implies  $x' \leq y'$ . The mono requirement translates (in one argument) to:  $x = y$  follows from  $x \leq z$  iff  $y \leq z$  for all  $z$ .

A non-trivial symmetric example is the inner product  $\langle - | - \rangle: \overline{H} \otimes H \rightarrow \mathbb{C}$  on a Hilbert space  $H$  (over  $\mathbb{C}$ ). The bilinearity and antilinearity requirements of an inner product are captured via tensor and conjugation in the input type of the operation: it yields  $\langle s \cdot x | y \rangle = \overline{s} \cdot \langle x | y \rangle$  and  $\langle x_1 + x_2 | y \rangle = \langle x_1 | y \rangle + \langle x_2 | y \rangle$ , and similarly,  $\langle x | s \cdot y \rangle = s \cdot \langle x | y \rangle$  and  $\langle x | y_1 + y_2 \rangle = \langle x | y_1 \rangle + \langle x | y_2 \rangle$ . The symmetry requirement for a comparison relation says that  $\overline{\langle y | x \rangle} = \langle x | y \rangle$ . The mono requirement holds, since if  $\langle x | z \rangle = \langle y | z \rangle$  for all  $z$ , then  $0 = \langle x | z \rangle - \langle y | z \rangle = \langle x - y | z \rangle$ . By taking  $z = x - y$  we get  $\langle x - y | x - y \rangle = 0$ , from which we conclude  $x - y = 0$  and thus  $x = y$ .

*Remark 2.3.* Notice that our notion of comparison does not involve the usual inner product requirements  $\langle x | x \rangle \geq 0$  and  $\langle x | x \rangle = 0 \Rightarrow x = 0$  for Hilbert spaces. Such requirements are not needed for what we wish to achieve (in the next section) and involve additional assumptions, namely the presence of zero objects (or maps). The kind of inner product that is captured via a comparison relation corresponds to a Minkowski inner product.

Although we do not pursue this here, we would like to mention that in presence of such a zero one can introduce complementation with respect to a comparison relation: for  $U \subseteq X$ , take  $U^\perp = \{x \in X \mid \forall x' \in U. \text{cp}(x, x') = 0\}$ . For sets this gives ordinary complement, and for Hilbert spaces it yields orthocomplementation of closed subsets.

Another point not pursued here is the similarity with profunctors [3, 23], commonly understood as ‘categorified’ relations. Taking opposites  $(-)^{\text{op}}$  forms an involution on the category **Cat** of (small) categories and functors between them. One can think of taking homsets  $\text{Hom}: \mathbf{C}^{\text{op}} \times \mathbf{C} \rightarrow \mathbf{Sets}$  as a comparison relation in **Cat**. The tame relations discussed in the next section then correspond to adjunctions. In order to obtain a symmetric comparison relation we need to replace **Sets** by a self-dual category, like the category **Rel** of sets and relations, or a groupoid.

### 3. Tame relations

This section introduces the setting in which one can define tameness for relations, leading to the first result, namely that such tame relations give rise to a dagger category (Proposition 3.5).

**Definition 3.1.** A **comparison cluster** consists of a collection  $(\overline{X}_i \otimes X_i \xrightarrow{\text{cp}_i} \Omega)_i$  of comparison maps in an involutive monoidal category (with a shared target object  $\Omega$ ). This cluster is called **symmetric** if each of the comparison relations  $\text{cp}_i$  is symmetric.

In the category **Sets** each object  $X$  carries equality  $=$  as a comparison relation  $X \times X \rightarrow 2$ . But there also situations where only specific objects in a category carry such a relation. For instance, in the category **JSL** of (finite) join semilattices the free objects carry such comparisons. Recall that free semilattices are given by finite powersets  $\mathcal{P}_{\text{fin}}(X) = \{U \subseteq X \mid U \text{ is finite}\}$ . They carry a comparison relation  $\text{cp}_X: \mathcal{P}_{\text{fin}}(X) \otimes \mathcal{P}_{\text{fin}}(X) \rightarrow 2$  in **JSL**, where  $\text{cp}_X(U \otimes V) = 1$  iff  $U \cap V \neq \emptyset$ . The tensor  $\otimes$  in **JSL** arises because of bilinearity:  $\emptyset \cap V \neq \emptyset$  never holds, and  $(U_1 \cup U_2) \cap V \neq \emptyset$  iff either  $U_1 \cap V \neq \emptyset$  or  $U_2 \cap V \neq \emptyset$ . Hence these  $\text{cp}_X: \mathcal{P}_{\text{fin}}(X) \otimes \mathcal{P}_{\text{fin}}(X) \rightarrow 2$  form a comparison cluster in **JSL**, indexed by sets  $X$ .

Similarly, we may consider the collection of Hilbert spaces with their inner products  $(\overline{H} \otimes H \xrightarrow{\langle - | - \rangle_H} \mathbb{C})_{H \in \text{Hilb}}$  as a comparison cluster in the category **Vect** $_{\mathbb{C}}$  of vector spaces over  $\mathbb{C}$ .

More formally, we understand the index elements  $i$  in Definition 3.1 as objects of a discrete category (no arrows except identities). The mapping

$i \mapsto X_i$  then forms a functor, like the finite powerset  $\mathcal{P}_{fin}$  above. We do not need morphisms between these index elements. This functorial view is sometimes convenient, so we may describe a comparison cluster in a category  $\mathbf{A}$  as a collection  $(\overline{F(X)} \otimes F(X) \xrightarrow{\text{cp}_X} \Omega)_{X \in \mathbf{D}}$ , where  $\mathbf{D}$  is a discrete category and  $F: \mathbf{D} \rightarrow \mathbf{A}$  is a functor.

**Definition 3.2.** Assume a comparison cluster  $(\overline{F(X)} \otimes F(X) \xrightarrow{\text{cp}_X} \Omega)_{X \in \mathbf{D}}$ , as described above, in an involutive monoidal category  $\mathbf{A}$ . A map in  $\mathbf{A}$  of the form  $r: \overline{F(X)} \otimes F(Y) \rightarrow \Omega$  is called a relation. Such a relation is called **tame** if there are necessarily unique maps  $r_*: F(X) \rightarrow F(Y)$  and  $r^*: F(Y) \rightarrow F(X)$  in  $\mathbf{A}$  for which the following diagram commutes.

$$\begin{array}{ccccc}
 \overline{F(Y)} \otimes F(Y) & \xleftarrow{\overline{r_*} \otimes \text{id}} & \overline{F(X)} \otimes F(Y) & \xrightarrow{\text{id} \otimes r^*} & \overline{F(X)} \otimes F(X) \\
 & \searrow \text{cp}_Y & \downarrow r & \swarrow \text{cp}_X & \\
 & & \Omega & & 
 \end{array} \tag{4}$$

Given another (tame) relation  $s: \overline{F(Y)} \otimes F(Z) \rightarrow \Omega$  we define a composition  $s \bullet r$  as:

$$s \bullet r = \left( \overline{F(X)} \otimes F(Z) \xrightarrow{\overline{r_*} \otimes \text{id}} \overline{F(Y)} \otimes F(Z) \xrightarrow{s} \Omega \right).$$

Notice that if  $r$  is a tame relation,  $r_*$  and  $r^*$  determine each other:  $r_*$  determines  $r$ , as  $r = \text{cp} \circ (\overline{r_*} \otimes \text{id})$ , and thus  $r^*$  via the mono-property of  $\text{cp}$ . As we shall see in the examples below, commutation of the triangles (4) amounts to an adjointness correspondence.

Recall the symmetric comparison cluster  $(X \times X \xrightarrow{\cong} 2)_{X \in \mathbf{Sets}}$  given by equality. A relation  $r: X \times Y \rightarrow 2$  is tame (wrt. this cluster) if there are functions  $r_*: X \rightarrow Y$  and  $r^*: Y \rightarrow X$  such that, for all  $x \in X$  and  $y \in Y$ ,

$$r_*(x) = y \iff r(x, y) = 1 \iff x = r^*(y).$$

This means that  $r_*$  and  $r^*$  are each other's inverses. One can interpret this as: set-theoretic reversible computation requires isomorphisms (bijections).

Before we can form a category of tame relations, we need the following results.

**Lemma 3.3.** *In the context of the previous definition:*

1. comparison relations are tame, with  $(\text{cp}_X)_* = \text{id}_{F(X)} = (\text{cp}_X)^*$ ;
2. for tame relations  $r: \overline{F(X)} \otimes F(Y) \rightarrow \Omega$  and  $s: \overline{F(Y)} \otimes F(Z) \rightarrow \Omega$ , the relation composition  $s \bullet r$  is tame, with  $(s \bullet r)_* = s_* \circ r_*$  and  $(s \bullet r)^* = r^* \circ s^*$ .

**Proof** The first point is immediate, and for the second point we show that the maps  $s_* \circ r_*$  and  $r^* \circ s^*$  satisfy the appropriate equations, making the

relation  $s \bullet r$  tame:

$$\begin{aligned}
\text{cp} \circ (\overline{(s_* \circ r_*)}) \otimes \text{id} &= \text{cp} \circ (\overline{s_*} \otimes \text{id}) \circ (\overline{r_*} \otimes \text{id}) \\
&= s \circ (\overline{r_*} \otimes \text{id}) \\
&= s \bullet r \\
\text{cp} \circ (\text{id} \otimes (r^* \circ s^*)) &= \text{cp} \circ (\text{id} \otimes r^*) \circ (\text{id} \otimes s^*) \\
&= r \circ (\text{id} \otimes s^*) \\
&= \text{cp} \circ (\overline{r_*} \otimes \text{id}) \circ (\text{id} \otimes s^*) \\
&= \text{cp} \circ (\text{id} \otimes s^*) \circ (\overline{r_*} \otimes \text{id}) \\
&= s \circ (\overline{r_*} \otimes \text{id}) \\
&= s \bullet r.
\end{aligned}$$

□

The comparison cluster  $(X^{\text{op}} \times X \xrightarrow{\leq} 2)_{X \in \mathbf{PoSets}}$  from the previous section is non-symmetric. A relation  $r: X \times Y \rightarrow 2$  in  $\mathbf{PoSets}$  is tame if there are monotone functions  $r_*: X \rightarrow Y$  and  $r^*: Y \rightarrow X$  such that for all  $x \in X$  and  $y \in Y$ ,

$$r_*(x) \leq y \iff r(x, y) = 1 \iff x \leq r^*(y).$$

Thus a tame relation  $r$  comes from a Galois connection. As is well-known, Galois connections are closed under composition, in the obvious manner. But exchanging  $r_*$  and  $r^*$  does (in general) not yield another Galois connection—but see Subsection 4.1 for a remedy for orthomodular lattices. In the next result we shall use symmetry to obtain such reversals, in the form of daggers.

**Lemma 3.4.** *For a tame relation  $r: X \otimes Y \rightarrow \Omega$  we define a swapped version:*

$$r^\dagger = \left( \overline{F(Y)} \otimes F(X) \xrightarrow{\overline{r^*} \otimes \text{id}} \overline{F(X)} \otimes F(Y) \xrightarrow{\text{cp}} \Omega \right).$$

Assuming that the comparison cluster is symmetric, we get:

1.  $r^\dagger$  is the same as the composite:

$$\overline{F(Y)} \otimes F(X) \xrightarrow[\cong]{\tau^{-1}} \overline{F(X)} \otimes F(Y) \xrightarrow{\overline{r}} \overline{\Omega} \xrightarrow[\cong]{j} \Omega,$$

where  $\tau$  is the twist map from (2);

2.  $(r^\dagger)_* = r^*$  and  $(r^\dagger)^* = r_*$ , making also  $r^\dagger$  tame.

**Proof** For the first point we obtain, by symmetry (3):

$$\begin{aligned}
j \circ \overline{r} \circ \tau^{-1} &= j \circ \overline{\text{cp}} \circ \overline{(\text{id} \otimes r^*)} \circ \tau^{-1} \\
&= j \circ \overline{\text{cp}} \circ \tau^{-1} \circ (\overline{r^*} \otimes \text{id}) \quad \text{by naturality of } \tau \\
&\stackrel{(3)}{=} \text{cp} \circ (\overline{r^*} \otimes \text{id}) \\
&= r^\dagger.
\end{aligned}$$

By construction of  $r^\dagger$  as  $\text{cp} \circ (\overline{r^*} \otimes \text{id})$ , the map  $r^*$  plays the role of  $(r^\dagger)_*$ . It is easy to see that  $r_*$  acts as  $(r^\dagger)^*$ :

$$\begin{aligned} \text{cp} \circ (\text{id} \otimes r_*) &\stackrel{(3)}{=} j \circ \overline{\text{cp}} \circ \tau^{-1} \circ (\overline{\text{id}} \otimes r_*) \\ &= j \circ \overline{\text{cp}} \circ \overline{(\overline{r^*} \otimes \text{id})} \circ \tau^{-1} \\ &= j \circ \overline{r} \circ \tau^{-1} \\ &= r^\dagger, \quad \text{as just shown.} \quad \square \end{aligned}$$

We summarise the situation.

**Proposition 3.5.** *A comparison cluster  $\text{cp} = (\overline{F(X)} \otimes F(X) \xrightarrow{\text{cp}_X} \Omega)_X$  in a category  $\mathbf{A}$  gives rise a category  $\text{TRel}(\mathbf{A}, \text{cp})$  of tame relations; it has indices  $X$  as objects, and its morphisms  $X \rightarrow Y$  are tame relations  $\overline{F(X)} \otimes F(Y) \rightarrow \Omega$ . Comparison relations  $\text{cp}_X$  form identity maps on  $X$ , and composition is given by  $\bullet$ , as in Definition 3.2.*

*In case the comparison cluster is symmetric,  $\text{TRel}(\mathbf{A}, \text{cp})$  is a dagger category, with dagger  $(-)^{\dagger}$  as in Lemma 3.4.*

**Proof** We briefly check the basic properties, using Lemma 3.3 and 3.4.

$$\begin{aligned} \text{cp} \bullet r &= \text{cp} \circ (\overline{r^*} \otimes \text{id}) & \text{cp}^\dagger &= \text{cp} \circ (\overline{\text{cp}^*} \otimes \text{id}) \\ &= r & &= \text{cp} \circ (\overline{\text{id}} \otimes \text{id}) \\ s \bullet \text{cp} &= s \circ (\overline{\text{cp}_*} \otimes \text{id}) & &= \text{cp} \\ &= s \circ (\overline{\text{id}} \otimes \text{id}) & (s \bullet r)^\dagger &= \text{cp} \circ (\overline{(s \bullet r)^*} \otimes \text{id}) \\ &= s & &= \text{cp} \circ (\overline{r^*} \otimes \text{id}) \circ (\overline{s^*} \otimes \text{id}) \\ t \bullet (s \bullet r) &= t \circ (\overline{(s \bullet r)_*} \otimes \text{id}) & &= r^\dagger \circ (\overline{(s^\dagger)_*} \otimes \text{id}) \\ &= t \circ (\overline{(s_* \circ r_*)} \otimes \text{id}) & &= r^\dagger \bullet s^\dagger \\ &= t \circ (\overline{s_*} \otimes \text{id}) \circ (\overline{r_*} \otimes \text{id}) & r^{\dagger\dagger} &= \text{cp} \circ (\overline{(r^\dagger)^*} \otimes \text{id}) \\ &= (t \bullet s) \circ (\overline{r_*} \otimes \text{id}) & &= \text{cp} \circ (\overline{r^*} \otimes \text{id}) \\ &= (t \bullet s) \bullet r & &= r. \quad \square \end{aligned}$$

In the sequel we focus on symmetric comparison clusters. We end this section with some easy but useful observation.

**Lemma 3.6.** *For a map  $r: X \rightarrow Y$  in the dagger category  $\text{TRel}(\mathbf{A}, \text{cp})$  of a symmetric comparison cluster one has:*

$$\begin{aligned} r \text{ is a dagger mono, i.e. } r^\dagger \bullet r &= \text{id} \iff r^* \circ r_* = \text{id} \\ r \text{ is a dagger epi, i.e. } r \bullet r^\dagger &= \text{id} \iff r_* \circ r^* = \text{id}. \end{aligned}$$

*As a result we can characterise dagger isomorphisms (or: unitary maps) as:*

$$r \text{ is a dagger iso} \stackrel{\text{def}}{\iff} r^\dagger = r^{-1} \iff \begin{cases} r_* \circ r^* = \text{id} \\ r^* \circ r_* = \text{id} \end{cases} \iff (r^\dagger)_* = (r_*)^{-1}.$$

**Proof** Assume  $r^\dagger \bullet r = \text{id}$ . Then, using Lemma 3.3 and 3.4,  $r^* \circ r_* = r^* \circ (r^\dagger)^* = (r^\dagger \bullet r)^* = \text{cp}^* = \text{id}$ . Conversely, if  $r^* \circ r_* = \text{id}$ , then  $r^\dagger \bullet r = r^\dagger \circ (r_* \otimes \text{id}) = \text{cp} \circ (r^* \otimes \text{id}) \circ (r_* \otimes \text{id}) = \text{cp} = \text{id}$ . The dagger epi case is handled similarly, and the result for dagger isos follows by combining these two cases.  $\square$

Later on, in Section 5, we shall see examples of dagger monos in a category of tame relation (see especially Lemma 5.3).

**Lemma 3.7.** *In the same context as the previous lemma, an endomap  $r: X \rightarrow X$  is self-adjoint (i.e.  $r^\dagger = r$ ) iff  $r_* = r^*$ .*

*It is a projection (i.e.  $r \bullet r = r = r^\dagger$ ) iff  $r_* = r^*$  and  $r_* \circ r_* = r_*$ .*

**Proof** If  $r = r^\dagger$  then  $r_* = (r^\dagger)_* = r^*$ . Conversely, if  $r_* = r^*$ , then  $r^\dagger = \text{cp} \circ (\overline{r^*} \otimes \text{id}) = \text{cp} \circ (\overline{r_*} \otimes \text{id}) = r$ .

If  $r$  is a projection, then it is a self-adjoint and so  $r_* = r^*$ . Further,  $\text{cp} \circ (\overline{r_*} \otimes \text{id}) = r = r \bullet r = r \circ (\overline{r_*} \otimes \text{id}) = \text{cp} \circ (\overline{r_*} \otimes \text{id}) \circ (\overline{r_*} \otimes \text{id}) = \text{cp} \circ (\overline{r_* \circ r_*} \otimes \text{id})$ . Hence  $\overline{r_*} = \overline{r_* \circ r_*}$ , by the mono-requirement for  $\text{cp}$ , and thus  $r_* = r_* \circ r_*$ . The converse is obvious.  $\square$

## 4. Examples of categories of tame relations

All the illustrations of comparison clusters in this section will be symmetric—resulting in dagger categories of tame relations. In many of the examples below we have closed structure—with an exponent  $\dashv$  for  $\otimes$ . Thus we can equivalently describe such relations  $\overline{F(X)} \otimes F(Y) \rightarrow \Omega$  as maps  $F(Y) \rightarrow (\overline{F(X)} \dashv \Omega)$ . This is often more convenient, since it avoids tensors.

### 4.1. Orthomodular lattices and Galois connections

In Subsection 2.1 we have seen that the order on a poset  $X$  forms a non-symmetric comparison relation  $\leq: \overline{X} \times X \rightarrow 2$  in  $\mathbf{PoSets}$ , where  $\overline{(-)}$  is order-reversal. Now assume that  $X$  is an orthomodular lattice (see [18] for details), with orthocomplement  $(-)^{\perp}: X \rightarrow \overline{X}$ . It satisfies, among other things,  $x^{\perp\perp} = x$  and:  $x^{\perp} \leq y$  iff  $y^{\perp} \leq x$ . When  $x \leq y^{\perp}$  one calls  $x, y$  orthogonal, which is also written as  $x \perp y$ . We obtain a comparison relation  $\text{cp}_{\perp}: X \times X \rightarrow 2$  in  $\mathbf{PoSets}$  (with identity involution), via  $\text{cp}_{\perp}(x, y) = 1$  iff  $x^{\perp} \leq y$ . By using orthocomplement in the first coordinate the contravariance disappears. This relation is the same as  $(x, y) \mapsto x^{\perp} \perp y^{\perp}$ , that is, as orthogonality of orthocomplements. It forms a symmetric comparison relation, since orthogonality is symmetric. The resulting category of tame relations is known from [6, 13].

**Proposition 4.1.** *The category of tame relations  $T\text{Rel}(\mathbf{PoSets}, \text{cp}_{\perp})$  for the symmetric comparison cluster  $(X \times X \xrightarrow{\text{cp}_{\perp}} 2)_{X \in \mathbf{OrthMod}}$  given by orthogonality of orthocomplements, is the category  $\mathbf{OMLatGal}$  of orthomodular lattices and (antitone) Galois connections between them.*

**Proof** A tame relations  $r: X \rightarrow Y$ , for  $X, Y$  orthomodular lattices is determined by monotone functions  $r_*: X \rightarrow Y$  and  $r^*: Y \rightarrow X$  satisfying:

$$\begin{aligned} r_*(x)^\perp \leq y &\iff \mathbf{cp}_\perp(r_*(x), y) = 1 \iff r(x, y) = 1 \\ &\iff \mathbf{cp}_\perp(x, r^*(y)) \iff x^\perp \leq r^*(y). \end{aligned}$$

These  $r_*$  and  $r^*$  are completely determined by monotone functions  $r_\# = r_* \circ (-)^\perp: \overline{X} \rightarrow Y$  and  $r^\# = r^* \circ (-)^\perp: Y \rightarrow \overline{X}$  satisfying:

$$x = x^{\perp\perp} \leq r^\#(y) = r^*(y^\perp) \iff r_*(x^\perp)^\perp \leq y^\perp \iff y \leq r_*(x^\perp) = r_\#(x).$$

This precisely says that  $r_\#, r^\#$  form an antitone Galois connection—or an adjunction  $r^\# \dashv r_\#$ .  $\square$

In [13] it is shown that **OMLatGal** is a dagger kernel category with (dagger) biproducts, and that every dagger kernel category maps into it.

## 4.2. Locally bifinite relations and partial injections

We have already seen the finite powerset  $\mathcal{P}_{fin}(X) = \{U \subseteq X \mid U \text{ is finite}\}$  as free functor  $\mathcal{P}_{fin}: \mathbf{Sets} \rightarrow \mathbf{JSL}$ , left adjoint to the forgetful functor from the category of join semi-lattices (finite joins only). This category **JSL** is in fact the category of (Eilenberg-Moore) algebras of the commutative (symmetric monoidal) monad  $\mathcal{P}_{fin}$ . Hence **JSL** is symmetric monoidal closed, following the constructions in [22, 21], where  $\mathcal{P}_{fin}$  preserves the monoidal structure:  $\mathcal{P}_{fin}(1) = 2$  is tensor unit and  $\mathcal{P}_{fin}(X \times Y) \cong \mathcal{P}_{fin}(X) \otimes \mathcal{P}_{fin}(Y)$ . We first review the comparison structure on free semilattices  $\mathcal{P}_{fin}(X)$ , with respect to the trivial (identity) involution on **JSL**.

As  $\Omega \in \mathbf{JSL}$  we take the two-element lattice  $2 = \mathcal{P}_{fin}(1)$ . Then we have correspondences between ‘abstract’ relations and ordinary relations, in:

$$\begin{array}{ccc} X \times Y \longrightarrow 2 & \text{in } \mathbf{Sets} \\ \hline \mathcal{P}_{fin}(X \times Y) \longrightarrow 2 & \text{in } \mathbf{JSL} \\ \hline \mathcal{P}_{fin}(X) \otimes \mathcal{P}_{fin}(Y) \longrightarrow 2 & \text{in } \mathbf{JSL} \\ \hline \mathcal{P}_{fin}(Y) \longrightarrow (\mathcal{P}_{fin}(X) \multimap 2) & \text{in } \mathbf{JSL} \end{array} \quad (5)$$

Starting from the equality relation  $=: X \times X \rightarrow 2$  in **Sets** this correspondence yields a comparison relation  $\mathbf{cp}_\mathcal{P}: \mathcal{P}_{fin}(X) \rightarrow (\mathcal{P}_{fin}(X) \multimap 2)$  given by:

$$\mathbf{cp}_\mathcal{P}(U)(U') = \bigvee_{(x, x') \in U \times U'} (x = x') = \begin{cases} 1 & \text{if } U \cap U' \neq \emptyset \\ 0 & \text{otherwise.} \end{cases} \quad (6)$$

Clearly, this relation  $\mathbf{cp}_\mathcal{P}$  is symmetric; it is also monic: if  $\mathbf{cp}_\mathcal{P}(U) = \mathbf{cp}_\mathcal{P}(V)$ , then:

$$x \in U \iff \mathbf{cp}_\mathcal{P}(U)(\{x\}) = 1 \iff \mathbf{cp}_\mathcal{P}(V)(\{x\}) = 1 \iff x \in V.$$

Hence  $U = V$ .

**Proposition 4.2.** *The dagger category  $T\text{Rel}(\mathbf{JSL}, \text{cp}_{\mathcal{P}})$  of tame relations for the symmetric comparison cluster  $\text{cp}_{\mathcal{P}}: \mathcal{P}_{\text{fin}}(X) \otimes \mathcal{P}_{\text{fin}}(X) \rightarrow 2$  determined by (6) is the category of sets with bifinite relations between them, i.e. with those relations  $r \subseteq X \times Y$  where for each  $x \in X$  and  $y \in Y$  both the sets*

$$\{z \in Y \mid r(x, z)\} \quad \text{and} \quad \{w \in X \mid r(w, y)\}$$

*are finite. Such a relation factors in two directions as  $X \rightarrow \mathcal{P}_{\text{fin}}(Y)$  and as  $Y \rightarrow \mathcal{P}_{\text{fin}}(X)$ . Thus we also write  $\mathbf{BifRel} = T\text{Rel}(\mathbf{JSL}, \text{cp}_{\mathcal{P}})$  for this category of sets and bifinite relations.*

**Proof** Assume  $r \subseteq X \times Y$ , which corresponds to  $\hat{r}: \mathcal{P}_{\text{fin}}(Y) \rightarrow (\mathcal{P}_{\text{fin}}(X) \multimap 2)$  in  $\mathbf{JSL}$  like in (5), given by  $\hat{r}(V)(U) = 1$  iff  $r(x, y)$  holds for some  $x \in U$  and  $y \in V$ . We shall prove the equivalence of:

- (a) for each  $y \in Y$ , the set  $\{x \mid R(x, y)\} \subseteq X$  is finite;
- (b) there is a necessarily unique map  $r^*: \mathcal{P}_{\text{fin}}(Y) \rightarrow \mathcal{P}_{\text{fin}}(X)$  in  $\mathbf{JSL}$  in the diagram:

$$\begin{array}{ccc} \mathcal{P}_{\text{fin}}(Y) & \xrightarrow{\hat{r}} & \mathcal{P}_{\text{fin}}(X) \multimap 2 \\ & \searrow r^* & \nearrow \text{cp}_{\mathcal{P}} \\ & \mathcal{P}_{\text{fin}}(X) & \end{array}$$

This diagram corresponds to the triangle on the right in (4). The analogous statement for  $r_*$  is left to the reader.

So assume (a) holds. Then we can define  $r^*(V) \in \mathcal{P}_{\text{fin}}(X)$ , for  $V \in \mathcal{P}_{\text{fin}}(Y)$ , as finite union of finite sets, namely as  $r^*(V) = \bigcup_{y \in V} \{x \mid r(x, y)\}$ . It makes the triangle in (b) commute:

$$\begin{aligned} (\text{cp}_{\mathcal{P}} \circ r^*)(V)(U) = 1 &\iff \text{cp}_{\mathcal{P}}(r^*(V))(U) = 1 \\ &\iff U \cap r^*(V) \neq \emptyset \\ &\iff \exists x \in U. \exists y \in V. r(x, y) \\ &\iff \hat{r}(V)(U) = 1. \end{aligned}$$

Conversely, assume (b) holds, so that we have a map  $r^*: \mathcal{P}_{\text{fin}}(Y) \rightarrow \mathcal{P}_{\text{fin}}(X)$  in  $\mathbf{JSL}$  in the above triangle. Then:

$$\begin{aligned} r(x, y) &\iff \hat{r}(\{y\})(\{x\}) = 1 \\ &\iff \text{cp}_{\mathcal{P}}(r^*(\{y\}))(\{x\}) = 1 \\ &\iff \{x\} \cap r^*(\{y\}) \neq \emptyset \\ &\iff x \in r^*(\{y\}). \end{aligned}$$

Since  $r^*(\{y\}) \in \mathcal{P}_{\text{fin}}(X)$  there are at most finitely many  $x$  that satisfy  $R(x, y)$ .

Finally, it is easy to see that composition in the category  $\mathbf{BifRel} = T\text{Rel}(\mathbf{JSL}, \text{cp}_{\mathcal{P}})$  is just relational composition, and that the dagger is relational converse.  $\square$

For a map  $r: X \times Y \rightarrow 2$ , as morphism in  $\mathbf{BifRel} = T\mathbf{Rel}(\mathbf{JSL}, \mathbf{cp}_{\mathcal{P}})$ , the ‘adjointness’ correspondence (4) takes the form:

$$r_*(U) \cap V \neq \emptyset \iff \exists x \in U. \exists y \in V. r(x, y) \iff U \cap r^*(V) \neq \emptyset,$$

for  $U \in \mathcal{P}_{fin}(X)$  and  $V \in \mathcal{P}_{fin}(Y)$ . Moreover, such a map  $r: X \rightarrow Y$  is unitary if and only if it is given by an isomorphism of sets  $X \cong Y$ .

Our next example is fairly similar to the previous one. Below in Subsection 4.3 we shall capture this similarity in terms of certain monads. But we prefer to describe this second example concretely, because it leads to a well-known category, namely the category  $\mathbf{PInj}$  of sets and partial injections between them (see *e.g.* [10, 11]). We start with the category  $\mathbf{Sets}_\bullet$  of pointed sets. Objects are sets  $X$  containing a distinguished base point  $\bullet \in X$ . Morphisms are ordinary functions that preserve this base point. This category  $\mathbf{Sets}_\bullet$  is equivalent to the category  $\mathbf{Pfn}$  of sets and partial functions between them.

There is a “lift” functor  $\mathcal{L} = 1 + (-): \mathbf{Sets} \rightarrow \mathbf{Sets}_\bullet$  that adds such a base point to set; it is left adjoint to the forgetful functor  $\mathbf{Sets}_\bullet \rightarrow \mathbf{Sets}$ . An element  $z \in \mathcal{L}(X) = 1 + X$  is either of the form  $z = \bullet \in 1$  or  $z = x \in X$ , for a unique  $x \in X$ . Thus one can see  $z \in \mathcal{L}(X)$  as a subset of  $X$  with at most one element (a ‘subsingleton’). This category  $\mathbf{Sets}_\bullet$  is the category of algebras of  $\mathcal{L}$ , as monad on  $\mathbf{Sets}$ ; thus,  $\mathbf{Sets}_\bullet$  is symmetric monoidal closed, following the constructions in [22, 21]. If we take  $\Omega = 2 = \mathcal{L}(1) \in \mathbf{Sets}_\bullet$ , then we have a bijective correspondence between abstract relations  $\mathcal{L}(X) \otimes \mathcal{L}(Y) \rightarrow 2$  and ordinary relations  $X \times Y \rightarrow 2$  in  $\mathbf{Sets}$ , like in (5).

The comparison relation  $\mathbf{cp}_{\mathcal{L}}$  we use here for  $\mathcal{L}$  is the same as before— for  $\mathcal{P}_{fin}$  in (6), when considered as relation  $=: X \times X \rightarrow 2$ . But when we translate it into a map  $\mathbf{cp}_{\mathcal{L}}: \mathcal{L}(X) \rightarrow (\mathcal{L}(X) \multimap 2)$  it becomes:

$$\mathbf{cp}_{\mathcal{L}}(z)(z') = \begin{cases} 1 & \text{if } z = x = z' \text{ for some (necessarily unique) } x \in X \\ 0 & \text{otherwise.} \end{cases} \quad (7)$$

Again this relation is symmetric, and satisfies the mono requirement from Definition 2.2: if  $\mathbf{cp}_{\mathcal{L}}(z) = \mathbf{cp}_{\mathcal{L}}(w)$ , then for each  $x \in X$ ,

$$z = x \iff \mathbf{cp}_{\mathcal{L}}(z)(x) = 1 \iff \mathbf{cp}_{\mathcal{L}}(w)(x) = 1 \iff w = x.$$

Hence  $z = w$ .

**Proposition 4.3.** *The dagger category  $T\mathbf{Rel}(\mathbf{Sets}_\bullet, \mathbf{cp}_{\mathcal{L}})$  of tame relations for the comparison relations (7) is the category  $\mathbf{PInj}$  of sets with partial injections between them: relations  $r \subseteq X \times Y$  satisfying both:*

$$r(x, y) \text{ and } r(x, y') \implies y = y' \quad r(x, y) \text{ and } r(x', y) \implies x = x'.$$

*That is:  $r$  factors both as  $X \rightarrow \mathcal{L}(Y)$  and as  $Y \rightarrow \mathcal{L}(Y)$ .*

**Proof** We prove the equivalence of:

$$(a) \ r(x, y) \text{ and } r(x', y) \implies x = x';$$

- (b) there is a necessarily unique map  $r^*: \mathcal{L}(Y) \rightarrow \mathcal{L}(X)$  in **Sets $_{\bullet}$**  in the diagram:

$$\begin{array}{ccc}
 \mathcal{L}(Y) & \xrightarrow{\widehat{r}} & \mathcal{L}(X) \multimap 2 \\
 & \searrow^{r^*} & \nearrow^{\text{cp}_{\mathcal{L}}} \\
 & & \mathcal{L}(X)
 \end{array}$$

where  $\widehat{r}(z)(w) = 1$  iff  $w = x \in X$  and  $z = y \in Y$  and  $r(x, y)$ .

Assuming (a) we define:

$$r^*(z) = \begin{cases} x & \text{if } z = y \in Y \text{ and there is a (necessarily unique) } x \text{ with } r(x, y) \\ \bullet & \text{otherwise.} \end{cases}$$

Then:

$$\begin{aligned}
 (\text{cp}_{\mathcal{L}} \circ r^*)(z)(w) = 1 &\iff \text{cp}_{\mathcal{L}}(r^*(z))(w) = 1 \\
 &\iff r^*(z) = x = w \in X \\
 &\iff z = y \in Y \text{ and } w = x \in X \text{ and } r(x, y) \\
 &\iff \widehat{r}(z)(w).
 \end{aligned}$$

Conversely, assume  $r^*: \mathcal{L}(Y) \rightarrow \mathcal{L}(X)$  as in (b). Then:

$$\begin{aligned}
 r(x, y) &\iff \widehat{r}(y)(x) = 1 \\
 &\iff \text{cp}_{\mathcal{L}}(r^*(y))(x) = 1 \\
 &\iff r^*(y) = x.
 \end{aligned}$$

There is thus at most one such  $x$ . □

In the end we note that there is an obvious inclusion of categories:

$$\mathbf{PInj} = T\text{Rel}(\mathbf{Sets}_{\bullet}, \text{cp}_{\mathcal{L}}) \longrightarrow T\text{Rel}(\mathbf{JSL}, \text{cp}_{\mathcal{P}}) = \mathbf{BifRel}$$

since a relation  $X \times Y \rightarrow 2$  that is ‘bi-singlevalued’ is also ‘bifinite’.

### 4.3. Monad-based examples

The previous two examples of functors with equality arise from certain monads, namely finite powerset  $\mathcal{P}_{fin}$  and lift  $\mathcal{L}$ . The constructions involved will be generalised next. Subsequently, in the next subsection, the multiset monad  $\mathcal{M}$  and the distribution monad  $\mathcal{D}$  will be used as additional examples.

So let  $\mathbf{B}$  be an involutive symmetric monoidal category (SMC) carrying an involutive monad  $T = (T, \eta, \mu, \sigma)$  which is symmetric monoidal (or ‘commutative’), via maps  $I \rightarrow T(I)$  and  $T(X) \otimes T(Y) \rightarrow T(X \otimes Y)$ , and with its involution described via a distributive law  $\nu_X: T(\overline{X}) \Rightarrow \overline{T(X)}$ , commuting appropriately with these two maps and with  $\eta$  and  $\mu$ , and satisfying  $\overline{\nu} \circ \nu \circ T(\iota) = \iota$ . These requirements imply that  $\nu$  is an isomorphism, see [16] for further details.

In case the category  $\text{Alg}(T)$  of (Eilenberg-Moore) algebras has enough coequalisers, it is also involutive symmetric monoidal, and the free functor  $F: \mathbf{B} \rightarrow \text{Alg}(T)$  is strong monoidal. The monoidal construction is described

in [22, 21] and the involution structure in [16]. Additionally, exponents  $\dashv$  in  $\text{Alg}(T)$  can be obtained from exponents in the underlying category  $\mathbf{B}$ , via equalisers.

This situation applies to (involutive) commutative monads  $T$  on  $\mathbf{Sets}$ . The resulting category of algebras  $\text{Alg}(T)$  is always monoidal closed. The finite powerset  $\mathcal{P}_{fin}$  and the lift monad  $\mathcal{L}$  are instances, with identity involutions; the multiset and distribution monad form other examples below. In the rest of this subsection we restrict to  $\mathbf{Sets}$  as base category.

The candidate comparison relations are defined on free objects, given by the free functor  $F: \mathbf{Sets} \rightarrow \text{Alg}(T)$ . We assume an object  $\Theta \in \mathbf{Sets}$  for which the free algebra  $\Omega = T(\Theta) \in \text{Alg}(T)$  contains two different objects  $0, 1 \in \Omega$ . In our examples it is usually obvious what these elements  $0, 1$  are, for instance, for  $\Omega = 2 = \{0, 1\}$ , or for  $\Omega = [0, 1]$ , or for  $\Omega$  a semiring  $S$ , with  $0$  as additive unit, and  $1$  as multiplicative unit.

Since we use the identity involution on  $\mathbf{Sets}$  there is a map

$$\bar{\Omega} = \overline{T(\Theta)} \xrightarrow[\cong]{\nu_{\Theta}^{-1}} T(\bar{\Theta}) = T(\Theta) = \Omega$$

that makes this  $\Omega$ , like any free algebra, into a self-conjugate object in  $\text{Alg}(T)$ .

In this situation we can define an equality function:

$$\text{eq}_X: \bar{X} \times X \longrightarrow \Omega \quad \text{by} \quad (x, x') \longmapsto \begin{cases} 1 & \text{if } x = x' \\ 0 & \text{otherwise,} \end{cases} \quad (8)$$

where  $\bar{X} = X$  is the trivial involution on  $\mathbf{Sets}$  (written only for formal reasons). This equality map in  $\mathbf{Sets}$  gives rise to a comparison relation  $\text{cp}$  in the category  $\text{Alg}(T)$  on free algebras, via:

$$\text{cp} \stackrel{\text{def}}{=} \left( \overline{F(X)} \otimes F(X) \xrightarrow[\cong]{\nu^{-1} \otimes \text{id}} F(\bar{X}) \otimes F(X) \xrightarrow[\cong]{\xi} F(\bar{X} \times X) \xrightarrow{F(\text{eq})} F(\Omega) \xrightarrow{\mu} \Omega \right)$$

where  $\mu$  is the monad's multiplication  $T^2(\Theta) \rightarrow T(\Theta) = \Omega$ . It is not hard to see that this  $\text{cp}$  is automatically symmetric. The mono-requirements from Definition 2.2 have to be checked explicitly in specific situations.

This general form of comparison, obtained by lifting equality (8) to a category of algebras, turns out to be appropriate in many situations of interest. For instance, for the finite powerset monad  $\mathcal{P}_{fin}$ , with  $\nu = \text{id}$ , we get the earlier comparison relation (6), since for  $U, U' \in \mathcal{P}_{fin}(X)$  this description yields:

$$\begin{aligned} \text{cp}(U, U') &= \bigvee \mathcal{P}_{fin}(\text{eq})(\xi(U, U')) = \bigvee \mathcal{P}_{fin}(\text{eq})(U \times U') \\ &= \bigvee \{\text{eq}(x, x') \mid x \in U, x' \in U'\} \\ &= \begin{cases} 1 & \text{if } \exists x \in U. \exists x' \in U'. x = x' \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 1 & \text{if } U \cap U' \neq \emptyset \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

The comparison relations (7) for the lift monad  $\mathcal{L}$  are also of this kind. We shall see more examples in Subsections 4.4 and 4.6 below.

In our set-theoretic examples we often take  $\Theta = 1$ —but not always, see the distribution monad example below. There are now several ways to describe ‘relations’:

$$\begin{array}{ll} X \times Y \longrightarrow \Omega = T(\Theta) & \text{in } \mathbf{Sets} \\ \overline{F(X)} \otimes F(Y) \cong \overline{F(X \times Y)} \longrightarrow \Omega & \text{in } \mathit{Alg}(T), \text{ by freeness} \\ F(Y) \longrightarrow \overline{\overline{F(X)} \multimap \Omega} & \text{in } \mathit{Alg}(T), \text{ with exponents} \end{array}$$

We often use such correspondences implicitly and freely switch between different (Curry-ied or non-Curry-ied) notations for comparison.

#### 4.4. Multiset and distribution monads

We sketch two more applications of the monad-based construction described above, involving the multiset monad  $\mathcal{M}_S$  and the distribution monad  $\mathcal{D}$ . We shall use the multiset monad in full generality, over a (commutative) involutive semiring  $S$ , like the complex numbers  $\mathbb{C}$ . Such a semiring consists of a commutative additive monoid  $(S, +, 0)$  and a (commutative) multiplicative monoid  $(S, \cdot, 1)$ , where multiplication distributes over addition, together with an involution  $\overline{(-)}: S \rightarrow S$  satisfying  $\overline{\overline{s}} = s$ , and forming a map of semirings. One can define a “multiset” functor  $\mathcal{M}_S: \mathbf{Sets} \rightarrow \mathbf{Sets}$  by:

$$\mathcal{M}_S(X) = \{\varphi: X \rightarrow S \mid \text{supp}(\varphi) \text{ is finite}\},$$

where  $\text{supp}(\varphi) = \{x \in X \mid \varphi(x) \neq 0\}$  is the support of  $\varphi$ . For a function  $f: X \rightarrow Y$  one defines  $\mathcal{M}_S(f): \mathcal{M}_S(X) \rightarrow \mathcal{M}_S(Y)$  by:

$$\mathcal{M}_S(f)(\varphi)(y) = \sum_{x \in f^{-1}(y)} \varphi(x).$$

Such a multiset  $\varphi \in \mathcal{M}_S(X)$  may be written as formal sum  $s_1x_1 + \dots + s_kx_k$  where  $\text{supp}(\varphi) = \{x_1, \dots, x_k\}$  and  $s_i = \varphi(x_i) \in S$  describes the “multiplicity” of the element  $x_i$ . This formal sum notation might suggest an order  $1, 2, \dots, k$  among the summands, but this sum is considered, up-to-permutation of the summands. Also, the same element  $x \in X$  may be counted multiple times, but  $s_1x + s_2x$  is considered to be the same as  $(s_1 + s_2)x$  within such expressions. With this formal sum notation one can write the application of  $\mathcal{M}_S$  on a map  $f$  as  $\mathcal{M}_S(f)(\sum_i s_i x_i) = \sum_i s_i f(x_i)$ .

This multiset functor is a monad, whose unit  $\eta: X \rightarrow \mathcal{M}_S(X)$  is  $\eta(x) = 1x$ , and multiplication  $\mu: \mathcal{M}_S(\mathcal{M}_S(X)) \rightarrow \mathcal{M}_S(X)$  is  $\mu(\sum_i s_i \varphi_i)(x) = \sum_i s_i \cdot \varphi_i(x)$ . There is also an involution  $\nu: \mathcal{M}_S(X) \rightarrow \mathcal{M}_S(X)$  given by  $\nu(\sum_i s_i x_i) = \sum_i \overline{s_i} x_i$ .

For the semiring  $S = \mathbb{N}$  one gets the free commutative monoid  $\mathcal{M}_{\mathbb{N}}(X)$  on a set  $X$ . The monad  $\mathcal{M}_{\mathbb{N}}$  is also known as the ‘bag’ monad, containing ordinary ( $\mathbb{N}$ -valued) multisets. If  $S = \mathbb{Z}$  one obtains the free Abelian group  $\mathcal{M}_{\mathbb{Z}}(X)$  on  $X$ . The Boolean semiring  $2 = \{0, 1\}$  yields the finite powerset monad  $\mathcal{P}_{\text{fin}} = \mathcal{M}_2$ . By taking the complex numbers  $\mathbb{C}$  as semiring one obtains the free vector space  $\mathcal{M}_{\mathbb{C}}(X)$  on  $X$  over  $\mathbb{C}$ .

An (Eilenberg-Moore) algebra  $a: \mathcal{M}_S(X) \rightarrow X$  for the multiset monad corresponds to a monoid structure on  $X$ —given by  $x + y = a(1x + 1y)$ —together with a scalar multiplication  $\bullet: S \times X \rightarrow X$  given by  $s \bullet x = a(sx)$ . It preserves the additive structure (of  $S$  and of  $X$ ) in each coordinate separately. This makes  $X$  a module, over the semiring  $S$ . Conversely, such an  $S$ -module structure on a commutative monoid  $M$  yields an algebra  $\mathcal{M}_S(M) \rightarrow M$  by  $\sum_i s_i x_i \mapsto \sum_i s_i \bullet x_i$ . Thus the category of algebras  $\text{Alg}(\mathcal{M}_S)$  is isomorphic to the category  $\mathbf{Mod}_S$  of  $S$ -modules. When  $S$  happens to be a field, this category  $\mathbf{Mod}_S$  is the category of vector spaces  $\mathbf{Vect}_S$  over  $S$ . It carries an involution in case  $S$  is involutive, see [16].

We show that free modules  $\mathcal{M}_S(X)$  carry a comparison relation. We take  $\Theta = 1 \in \mathbf{Sets}$ , so that  $\Omega = \mathcal{M}_S(1) = S \in \mathbf{Mod}_S$ . We shall call maps  $X \times Y \rightarrow S$  multirelations, in analogy with multisets; they may be seen as fuzzy relations, assigning a possibly more general value than 0,1 to a pair of elements. Such multirelations can thus also be described as module maps  $\mathcal{M}_S(Y) \rightarrow (\overline{\mathcal{M}_S(X)} \multimap S)$ , like in (5). The comparison relation, as a map  $\text{cp}: \mathcal{M}_S(X) \rightarrow (\overline{\mathcal{M}_S(X)} \multimap S)$  is given by (finite) sums:

$$\text{cp}_{\mathcal{M}}(\varphi)(\varphi') = \sum_x \overline{\varphi'(x)} \cdot \varphi(x). \quad (9)$$

This comparison captures the usual inner product (or ‘dot’ product) for vectors wrt. a basis. Symmetry amounts to  $\overline{\text{cp}_{\mathcal{M}}(\varphi)(\varphi')} = \text{cp}_{\mathcal{M}}(\varphi')(\varphi)$ , and thus clearly holds. In order to see that  $\text{cp}_{\mathcal{M}}$  is injective, assume  $\text{cp}_{\mathcal{M}}(\varphi) = \text{cp}_{\mathcal{M}}(\psi)$ . Then, for each  $x \in X$ ,

$$\varphi(x) = \text{cp}_{\mathcal{M}}(\varphi)(1x) = \text{cp}_{\mathcal{M}}(\psi)(1x) = \psi(x).$$

Hence  $\varphi = \psi$ , as functions  $X \rightarrow S$ .

The following result is no surprise anymore. The proof proceeds along the lines of Propositions 4.2 and 4.3; details are left to the interested reader.

**Proposition 4.4.** *Let  $S$  be an involutive commutative semiring. The dagger category  $T\text{Rel}(\mathbf{Mod}_S, \text{cp}_{\mathcal{M}})$  of tame relations for the comparison relation  $\text{cp}_{\mathcal{M}}: \overline{\mathcal{M}_S(X)} \otimes \mathcal{M}_S(X) \rightarrow S$  corresponding to (9) contains sets as objects and morphisms  $X \rightarrow Y$  are ‘multirelations’  $r: X \times Y \rightarrow S$  for which the two obvious maps obtained by abstraction (or Curry-ing) satisfy:*

$$\begin{cases} \Lambda(r): Y \longrightarrow S^X \text{ factors through } Y \longrightarrow \mathcal{M}_S(X) \\ \Lambda(r^\dagger): X \longrightarrow S^Y \text{ factors through } X \longrightarrow \mathcal{M}_S(Y). \end{cases}$$

*More concretely, this means that for each  $x \in X$  there are only finitely many  $y \in Y$  with  $r(x, y) \neq 0$ , and vice-versa.*

*We shall also write  $\mathbf{BifMRel}_S = T\text{Rel}(\mathbf{Mod}_S, \text{cp}_{\mathcal{M}})$  for this category of sets and bifinite multirelations.*

A bifinite multirelation  $r: X \times Y \rightarrow S$ , as morphism  $X \rightarrow Y$  in the category  $\mathbf{BifMRel}_S = T\text{Rel}(\mathbf{Mod}_S, \text{cp}_{\mathcal{M}})$ , satisfies the ‘adjointness’ correspondence (from (4)):

$$\text{cp}_{\mathcal{M}}(r_*(\varphi), \psi) = \sum_{x,y} \overline{\varphi(x)} \cdot r(x, y) \cdot \psi(y) = \text{cp}_{\mathcal{M}}(\varphi, r^*(\psi)),$$

for  $\varphi \in \mathcal{M}_S(X)$  and  $\psi \in \mathcal{M}_S(Y)$ . This category of bifinite multirelations will be investigated more closely in Section 5. Here we only mention that there is an inclusion of categories:

$$\mathbf{BifRel} = TRel(\mathbf{JSL}, \mathbf{cp}_{\mathcal{P}}) \longrightarrow TRel(\mathbf{Sets}_{\bullet}, \mathbf{cp}_{\mathcal{L}}) = \mathbf{BifMRel}$$

since we can turn a bifinite relation  $X \times Y \rightarrow 2$  into a bifinite multirelation  $X \times Y \rightarrow S$  via the inclusion  $\{0, 1\} \hookrightarrow S$ .

Analogously to the multiset monad the distribution monad  $\mathcal{D}: \mathbf{Sets} \rightarrow \mathbf{Sets}$  is defined as:

$$\mathcal{D}(X) = \{\varphi: X \rightarrow [0, 1] \mid \text{supp}(\varphi) \text{ is finite and } \sum_{x \in X} \varphi(x) = 1\}. \quad (10)$$

Elements of  $\mathcal{D}(X)$  are convex combinations  $s_1x_1 + \dots + s_kx_k$ , where the probabilities  $s_i \in [0, 1]$  satisfy  $\sum_i s_i = 1$ . Unit and multiplication making  $\mathcal{D}$  a monad can be defined as for  $\mathcal{M}_S$ . The distribution monad  $\mathcal{D}$  is always symmetric monoidal (commutative) and its category of algebras is the category  $\mathbf{Conv}$  of convex sets with affine maps between them, see also [19, 8, 12].

The functor  $\mathcal{D}: \mathbf{Sets} \rightarrow \mathbf{Conv}$  also comes with equality. We now choose  $\Theta = 2 \in \mathbf{Sets}$ , so that  $\Omega = \mathcal{D}(2) = [0, 1] \in \mathbf{Conv}$ . Comparison  $\mathbf{cp}: \mathcal{D}(Y) \rightarrow (\mathcal{D}(X) \multimap [0, 1])$  can be defined as in (9) (but without conjugation).

**Proposition 4.5.** *The dagger category of tame relations  $TRel(\mathbf{Conv}, \mathbf{cp})$  has morphisms  $X \rightarrow Y$  given by discrete ‘bistochastic’ relations  $X \times Y \rightarrow [0, 1]$  satisfying both:*

$$\begin{cases} X \rightarrow [0, 1]^Y \text{ factors through } X \rightarrow \mathcal{D}(Y) \\ Y \rightarrow [0, 1]^X \text{ factors through } Y \rightarrow \mathcal{D}(X). \end{cases}$$

We also write  $\mathbf{dBisRel} = TRel(\mathbf{Conv}, \mathbf{cp})$  for this category of discrete bistochastic relations.

These bistochastic relations are reversible by definition. Reversibility of arbitrary stochastic relations is studied for instance in [7].

#### 4.5. Hilbert spaces

The so-called  $\ell^2$ -construction can be seen as an infinite version of the multiset monad  $\mathcal{M}_{\mathbb{C}}$ . For a set  $X$  one takes the square-summable sequences indexed by  $X$ , as in:

$$\ell^2(X) = \{\varphi: X \rightarrow \mathbb{C} \mid \sum_{x \in X} \|\varphi(x)\|^2 < \infty\},$$

where  $\|\varphi(x)\|^2 = \varphi(x) \cdot \overline{\varphi(x)}$ . As is well-known, the  $\ell^2$ -construction forms a functor  $\ell^2: \mathbf{PInj} \rightarrow \mathbf{Hilb}$ , but not a functor  $\mathbf{Sets} \rightarrow \mathbf{Hilb}$ , see e.g. [1, 10]. However, in the present setting we do not need functoriality for the indices of comparison relations. Thus we have the (standard) inner products

$$\overline{\ell^2(X)} \otimes \ell^2(X) \xrightarrow{\langle - | - \rangle} \mathbb{C} \quad \text{given by} \quad \langle \varphi | \varphi' \rangle = \sum_x \overline{\varphi(x)} \cdot \varphi'(x)$$

forming a symmetric cluster of comparison relations, much like in (9) for multisets. As we show below, it does not matter if we consider these inner

products as morphisms in  $\mathbf{Vect}_{\mathbb{C}}$  or in  $\mathbf{Hilb}$ . The resulting category of tame relations has sets as objects and continuous linear functions  $\ell^2(X) \rightarrow \ell^2(Y)$  as morphisms  $X \rightarrow Y$ . This follows from the lemma below.

Given an arbitrary Hilbert space  $H$ , we can consider its inner product  $\overline{H} \otimes H \xrightarrow{\langle - | - \rangle} \mathbb{C}$  as a comparison relation in the category  $\mathbf{Vect}_{\mathbb{C}}$ , as already mentioned in Section 3. It is well-known that a linear map between Hilbert spaces is continuous if and only if it is bounded. Jorik Mandemaker suggested the next result (and proof), which shows that boundedness/continuity can be captured in terms of tameness.

**Lemma 4.6.** *Consider two Hilbert spaces  $H_1, H_2$ , with their inner products  $\overline{H}_i \otimes H_i \xrightarrow{\langle - | - \rangle} \mathbb{C}$  as comparisons in  $\mathbf{Vect}_{\mathbb{C}}$ . There is a bijective correspondence between:*

$$\frac{\text{tame } \overline{H}_1 \otimes H_2 \xrightarrow{r} \mathbb{C}}{\text{bounded } H_1 \xrightarrow{f} H_2}$$

As a result,  $\mathbf{Hilb} = \text{TRel}(\mathbf{Vect}_{\mathbb{C}}, \langle - | - \rangle)$ , where  $\langle - | - \rangle$  is the comparison cluster  $(\overline{H} \otimes H \xrightarrow{\langle - | - \rangle_H} \mathbb{C})_{H \in \mathbf{Hilb}}$  indexed by Hilbert spaces.

Thus, morphisms between Hilbert spaces can also be understood as (tame) relations, like morphisms in many other categories of interest in quantum foundations.

**Proof** If a linear map  $f: H_1 \rightarrow H_2$  is bounded, then it has a dagger  $f^\dagger: H_2 \rightarrow H_1$  satisfying  $\langle f(x) | y \rangle = \langle x | f^\dagger(y) \rangle$ , for all  $x \in H_1$  and  $y \in H_2$ . Thus, by construction, the relation  $r(x, y) = \langle f(x) | y \rangle = \langle x | f^\dagger(y) \rangle$  is tame, with  $r_* = f$  and  $r^* = f^\dagger$ .

Conversely, given a tame relation  $r: \overline{H}_1 \otimes H_2 \rightarrow \mathbb{C}$  we use the Closed Graph Theorem in order to show that  $r_*: H_1 \rightarrow H_2$  is continuous. Assume we have a Cauchy sequence  $(x_n)_{n \in \mathbb{N}}$  in  $H_1$  with limit  $x$ , and let the sequence  $(r_*(x_n))_{n \in \mathbb{N}}$  in  $H_2$  have limit  $z$ . It suffices to show  $r_*(x) = z$ . We use that the inner product is continuous (which follows from Cauchy-Schwarz), in:

$$\begin{aligned} \langle r_*(x) | y \rangle &= \langle r_*(\lim_n x_n) | y \rangle = \langle \lim_n x_n | r^*(y) \rangle \\ &= \lim_n \langle x_n | r^*(y) \rangle \\ &= \lim_n \langle r_*(x_n) | y \rangle \\ &= \langle \lim_n r_*(x_n) | y \rangle = \langle z | y \rangle. \end{aligned}$$

Since this holds for each  $y$ , we get  $r_*(x) = z$  by the mono-property of comparisons (or inner products).  $\square$

#### 4.6. Formal distributions

We now use the present framework of comparisons for re-describing the dagger category of formal distributions introduced in [4]. First we show how to capture polynomials via multiset monads (from Subsection 4.4). Laurent polynomials, with negative powers  $x^{-1}$ , are used in [4], but here we stick to ordinary polynomials.

As described in the previous subsection, a multiset  $\varphi \in \mathcal{M}_{\mathbb{N}}(X)$  can be described as a formal sum  $n_1x_1 + n_2x_2 + \cdots + n_kx_k$ , with  $n_i \in \mathbb{N}$ . We might as well write  $\varphi$  multiplicatively, as in  $x_1^{n_1}x_2^{n_2} \cdots x_k^{n_k}$ . This is convenient, because we can now describe a (multivariate) polynomial as a ‘multiset of multisets’  $p \in \mathcal{M}_S(\mathcal{M}_{\mathbb{N}}(X))$ . If we use additive notation for the outer multiset  $\mathcal{M}_S$  we can write  $p$  as formal sum:

$$p = \sum_i s_i \varphi_i \quad \text{where} \quad \varphi_i = x_{i1}^{n_{i1}} \cdots x_{ik_i}^{n_{ik_i}} \in \mathcal{M}_{\mathbb{N}}(X).$$

The univariate polynomials, with only one variable, appear by taking  $X = 1$ , namely as  $p \in \mathcal{M}_S(\mathcal{M}_{\mathbb{N}}(1)) = \mathcal{M}_S(\mathbb{N})$ . Such a  $p$  can be written as  $\sum_i s_i n_i$ , or as polynomial  $\sum_i s_i x^{n_i}$  for some variable  $x$ .

We write  $S[X] = \mathcal{M}_S \mathcal{M}_{\mathbb{N}}(X)$  for the set of (multivariate) polynomials with variables from an arbitrary set  $X$  and coefficients from the commutative semiring  $S$ . These polynomials are finite, by construction. Possibly infinite polynomials—also known as power series or as formal distributions—are obtained via the function space  $S[[X]] = S^{\mathcal{M}_{\mathbb{N}}(X)}$ . Thus  $q \in S[[X]]$  can be written as (possibly infinite) formal sum  $\sum_{\varphi \in \mathcal{M}_{\mathbb{N}}(X)} q(\varphi)\varphi$ , where  $q(\varphi) \in S$  gives the coefficient. There is an obvious inclusion  $S[X] \hookrightarrow S[[X]]$  that will play the role of comparison below. But first we need to relate finite and infinite polynomials more closely.

**Lemma 4.7.** *For a commutative semiring  $S$  and set  $X$  of ‘variables’, modules  $S[X], S[[X]] \in \mathbf{Mod}_S$  of finite and infinite multivariate polynomials, with variables from  $X$ , are defined as:*

$$S[X] = \mathcal{M}_S \mathcal{M}_{\mathbb{N}}(X) \quad \text{and} \quad S[[X]] = S^{\mathcal{M}_{\mathbb{N}}(X)} = \mathbf{Hom}_{\mathbf{Sets}}(\mathcal{M}_{\mathbb{N}}(X), S).$$

Then:  $S[[X]] \cong (S[X] \multimap S)$ , where  $\multimap$  is exponent in  $\mathbf{Mod}_S$ .

**Proof** We use the following chain of isomorphisms, which exploits that multiset  $\mathcal{M}_S: \mathbf{Sets} \rightarrow \mathbf{Mod}_S$  is the free functor and that the category  $\mathbf{Mod}_S$  of modules is monoidal closed.

$$\begin{aligned} S[[X]] &= S^{\mathcal{M}_{\mathbb{N}}(X)} = \mathbf{Hom}_{\mathbf{Sets}}(\mathcal{M}_{\mathbb{N}}(X), S) \\ &\cong \mathbf{Hom}_{\mathbf{Mod}_S}(\mathcal{M}_S \mathcal{M}_{\mathbb{N}}(X), S) \\ &\cong \mathbf{Hom}_{\mathbf{Mod}_S}(S[X] \otimes \mathcal{M}_S(1), S) \\ &\quad \text{since } \mathcal{M}_S(1) \cong S \text{ is the tensor unit} \\ &\cong \mathbf{Hom}_{\mathbf{Mod}_S}(\mathcal{M}_S(1), S[X] \multimap S) \\ &\cong \mathbf{Hom}_{\mathbf{Sets}}(1, S[X] \multimap S) \\ &\cong (S[X] \multimap S). \quad \square \end{aligned}$$

We shall introduce comparisons  $\text{cp}: S[X] \otimes S[X] \rightarrow S$  in the category  $\mathbf{Mod}_S$  of  $S$ -modules, following the recipe from Subsection 4.3. We start from the equality relation  $\text{eq}: \mathcal{M}_{\mathbb{N}}(X) \times \mathcal{M}_{\mathbb{N}}(X) \rightarrow S$ , following (8), which gives

rise to  $\text{cp}$  as composite:

$$\begin{aligned}
 S[X] \otimes S[X] &= \mathcal{M}_S(\mathcal{M}_{\mathbb{N}}(X)) \otimes \mathcal{M}_S(\mathcal{M}_{\mathbb{N}}(X)) \\
 &\quad \wr \parallel \\
 &\mathcal{M}_S(\mathcal{M}_{\mathbb{N}}(X) \times \mathcal{M}_{\mathbb{N}}(X)) \xrightarrow{\mathcal{M}_S(\text{eq})} \mathcal{M}_S(S) \xrightarrow{\mu} S
 \end{aligned} \tag{11}$$

Concretely,  $\text{cp}(p, p') = \sum_{\varphi \in \mathcal{M}_{\mathbb{N}}(X)} p(\varphi) \cdot p'(\varphi)$ , like in (9).

More generally, relations in this setting will be module maps of the form  $S[X] \otimes S[Y] \rightarrow S$ . Obviously, by Curry-ing they can also be described as maps  $S[Y] \rightarrow (S[X] \multimap S) \cong S[[X]]$ . It is not hard to see that the map  $S[X] \rightarrow S[[X]]$  corresponding to comparison  $\text{cp}$  in (11) is inclusion. In particular, this shows that the mono-requirement from Definition 2.2 is satisfied.

There is one further observation that we need to make.

**Lemma 4.8.** *Each multiset monad  $\mathcal{M}_S$  is an ‘additive’ monad [5]: it maps finite coproducts to products, in a canonical way:  $\mathcal{M}_S(0) \cong 1$  and  $\mathcal{M}_S(X + Y) \cong \mathcal{M}_S(X) \times \mathcal{M}_S(Y)$ . The latter isomorphism will be written explicitly as:*

$$\begin{array}{ccc}
 & \chi \mapsto \langle \chi(\kappa_1-), \chi(\kappa_2-) \rangle & \\
 \mathcal{M}_S(X + Y) & \xrightarrow{\quad \cong \quad} & \mathcal{M}_S(X) \times \mathcal{M}_S(Y) \\
 & \varphi \star \psi \longleftarrow (\varphi, \psi) & 
 \end{array}$$

where the operation  $\star$  multiplies  $\varphi, \psi$ , after appropriate relabeling has put them in the same set of multisets:

$$\varphi \star \psi = \mathcal{M}_S(\kappa_1)(\varphi) \cdot \mathcal{M}_S(\kappa_2)(\psi).$$

(We use multiplicative notation  $\cdot$  in the definition of  $\star$  for multiset addition because later on we use  $\star$  when we read multisets multiplicatively; the  $\kappa_i$  are the coprojections associated with the coproduct.)  $\square$

Using this additivity of the multiset monad we show that relations can be described in another way as formal distributions.

**Proposition 4.9.** *In the setting described above, there is an isomorphism of modules between formal distributions in the coproduct  $X + Y$  and relations on  $X$  and  $Y$ , as in:*

$$S[[X + Y]] \cong (S[X] \otimes S[Y] \multimap S).$$

The formal distribution in  $S[[X + X]]$  corresponding to the comparison relation  $\text{cp}: S[X] \otimes S[X] \rightarrow S$  is the function:

$$\lambda\varphi \in \mathcal{M}_{\mathbb{N}}(X + X). \begin{cases} 1 & \text{if } \varphi(\kappa_1-) = \varphi(\kappa_2-) \text{ in } \mathcal{M}_S(X) \\ 0 & \text{otherwise.} \end{cases}$$

**Proof** Because multiset monads are additive and free functors we have:

$$\begin{aligned}
 S[X + Y] &= \mathcal{M}_S(\mathcal{M}_{\mathbb{N}}(X + Y)) \\
 &\cong \mathcal{M}_S(\mathcal{M}_{\mathbb{N}}(X) \times \mathcal{M}_{\mathbb{N}}(Y)) \\
 &\cong \mathcal{M}_S(\mathcal{M}_{\mathbb{N}}(X)) \otimes \mathcal{M}_S(\mathcal{M}_{\mathbb{N}}(Y)) \\
 &= S[X] \otimes S[Y].
 \end{aligned}$$

Hence Lemma 4.7 gives:

$$\begin{aligned}
 S[[X + Y]] &\cong (S[X + Y] \multimap S) \\
 &\cong (S[X] \otimes S[Y] \multimap S) \\
 &\cong (S[X] \multimap (S[Y] \multimap S)).
 \end{aligned}$$

The formal power series in  $S[[X + X]]$  can be obtained by following these isomorphisms backwards.  $\square$

In [4] a category of formal distributions is defined with (finite) sets as objects and morphisms  $X \rightarrow Y$  given by “tame” formal distributions  $p \in S[[X + Y]]$ . Here we re-describe them in the current framework, namely as category  $TRel(\mathbf{Mod}_S, \mathbf{cp})$  for the comparison cluster (11). Indeed, for a morphism  $r: X \rightarrow Y$  in this category, considered as a map of modules  $r: S[Y] \rightarrow (S[X] \multimap S)$ , tameness means the existence of a map  $r^*: S[Y] \rightarrow S[X]$  in  $\mathbf{Mod}_S$ , as indicated:

$$\begin{array}{ccc}
 S[Y] & \xrightarrow{r} & (S[X] \multimap S) = S[[X]] \\
 \searrow & & \nearrow \text{cp} \\
 & r^* & S[X]
 \end{array}$$

(And similarly for  $r_*$ .) We can translate this condition to formal distributions as morphisms, using Lemma 4.9. Indeed, a formal distribution  $p \in S[[X + Y]]$  gives rise to a map  $\widehat{p}: S[Y] \rightarrow S[[X]]$ , namely:

$$\widehat{p}(q) = \lambda\varphi \in \mathcal{M}_{\mathbb{N}}(X). \quad \sum_{\psi \in \mathcal{M}_{\mathbb{N}}(Y)} p(\varphi \star \psi)$$

where  $\star$  is the operation for additivity from Lemma 4.8. Tameness says that  $\widehat{p}(q)$  is a finite polynomial, for each  $q \in S[Y]$ ; it means that  $\widehat{p}$  factors as  $S[Y] \rightarrow S[X]$  (and vice-versa).

For completeness we include formulations of composition and dagger for formal distributions. Given  $p \in S[[X + Y]]$  and  $q \in S[[Y + Z]]$  we have:

$$\begin{aligned}
 p \bullet q &= \lambda\chi \in \mathcal{M}_{\mathbb{N}}(X + Z). \quad \sum_{\psi \in \mathcal{M}_{\mathbb{N}}(Y)} p(\chi(\kappa_1-) \star \psi) \cdot q(\psi \star \chi(\kappa_2-)) \\
 (p)^\dagger &= \lambda\chi \in \mathcal{M}_{\mathbb{N}}(Y + X). \quad p(\chi(\kappa_2-) \star \chi(\kappa_1-)).
 \end{aligned}$$

The tameness requirement ensures that these sums  $\sum$  exist. It is not hard to see that the formal distribution described at the end of Proposition 4.9 is the identity map.

In the end we see that this formal distribution example fits in the general recipe for monads  $T$  from Subsection 4.3, except that we start with an (additional) additive monad  $R$ . Equality is used on  $R$ , in the form of maps  $\text{eq}: R(X) \times R(X) \rightarrow T(\Theta) = \Omega$ , and is lifted to comparisons  $\text{cp}: TR(X) \otimes TR(X) \rightarrow \Omega$ . Additivity of  $R$  allows us to translate between coproducts and products to make the machinery work (via the  $\star$ 's above). Hence one may construct other examples of dagger categories of this kind.

## 5. The category of bifinite multirelations

Subsection 4.4 introduced the category  $\mathbf{BifMRel}_S = T\text{Rel}(\mathbf{Mod}_S, \text{cp}_{\mathcal{M}})$  of sets and bifinite multirelations, with values in an involutive semiring  $S$  (such as  $\mathbb{C}$ ). Here we shall investigate its categorical structure in more detail.

(Describing the categorical structure of categories  $T\text{Rel}(\mathbf{A}, \text{cp})$  in full generality turns out to be rather involved. In contrast, for several examples, this structure is essentially straightforward. That is why we prefer this more concrete approach.)

There is a special reason why we concentrate on  $\mathbf{BifMRel}_S$ —and not on other categories of tame relations. The category  $\mathbf{BifMRel}_S$  may be seen a universe for ‘discrete’ quantum computation (like in [14]), just like the category of Hilbert spaces may be used for ‘continuous’ computation. We shall illustrate this in a moment, but first we describe the category  $\mathbf{BifMRel}_S$  concretely, and state an elementary result.

Objects in the category  $\mathbf{BifMRel}_S$  are sets; it is important that infinite sets are allowed as objects, so that computations with infinitely many (orthogonal) states can be covered—unlike in finite-dimensional vector (or Hilbert) spaces. A morphism  $r: X \rightarrow Y$  in  $\mathbf{BifMRel}_S$  is a multirelation  $r: X \times Y \rightarrow S$  such that for each  $x \in X$  the subset  $\{y \mid r(x, y) \neq 0\} = \text{supp}(r(x, -))$  is finite, and similarly, for each  $y \in Y$  the set  $\{x \mid r(x, y) \neq 0\} = \text{supp}(r(-, y))$  is finite. Composition of  $r: X \rightarrow Y$  with  $s: Y \rightarrow Z$  can be described as matrix composition:  $(s \bullet r)(x, z) = \sum_y r(x, y) \cdot s(y, z)$ . The dagger  $r^\dagger: Y \rightarrow X$  is given by the adjoint matrix:  $r^\dagger(y, x) = \overline{r(x, y)}$ , obtained by mirroring and conjugation in  $S$ . Notice that the special case  $S = 2 = \{0, 1\}$  covers the category  $\mathbf{BifRel} = \mathbf{BifMRel}_2$  of bifinite relations.

We show how unitary maps give rise to bistochastic relations (for the standard semiring examples in this context).

**Lemma 5.1.** *Assume an involutive semiring  $S$  like  $2, \mathbb{R}, \mathbb{R}_{\geq 0}$  or  $\mathbb{C}$ , for which the mapping  $a \mapsto a \cdot \bar{a}$  yields a function  $S \rightarrow \mathbb{R}_{\geq 0}$ , which we write as squared norm  $\| - \|^2$ . A unitary map  $r: X \rightarrow Y$  in  $\mathbf{BifMRel}_S$  then yields a discrete bistochastic relation,  $\|r\|^2: X \rightarrow Y$ , i.e. a morphism in the category  $\mathbf{dBisRel}$  from Proposition 4.5, given by  $\|r\|^2(x, y) = \|r(x, y)\|^2$ .*

**Proof** Suppose  $r: X \rightarrow Y$  in  $\mathbf{BifMRel}$  is unitary, i.e.  $r^\dagger$  is  $r$ 's inverse. Then, for each  $x \in X$ ,

$$1 = \text{id}_X(x, x) = (r^\dagger \bullet r)(x, x) = \sum_y r(x, y) \cdot \overline{r(x, y)} = \sum_y \|r(x, y)\|^2.$$

And similarly for  $y \in Y$ . Hence, post-composition with the squared norm  $\|-\|^2: S \rightarrow \mathbb{R}_{\geq 0}$  turns the unitary bifinite multirelation  $r: X \times Y \rightarrow S$  into a bistochastic relation  $X \times Y \rightarrow [0, 1]$ .  $\square$

Notice that an arbitrary morphism  $q: 1 \rightarrow 2$  corresponds to a map  $q: 1 \times 2 \rightarrow S$ , and thus to two scalars  $a = q(*, 0) \in S$  and  $b = q(*, 1) \in S$ , where we use  $1 = \{*\}$  and  $2 = \{0, 1\}$ . One can call such a  $q$  a *unit* if  $\|q\|^2 = 1$ , i.e. if  $(q^\dagger \bullet q)(*, *) = \|a\|^2 + \|b\|^2 = 1$  in  $\mathbb{R}_{\geq 0}$ . Such a unit is a *quantum bit* for  $S = \mathbb{C}$  and a *classical bit* for  $S = 2$ .

We briefly illustrate the use of the category  $\mathbf{BifMRel}_{\mathbb{C}}$  to model discrete quantum computations (on an infinite state space). In [14] quantum walks (see also [20, 26]) are investigated in relation to possibilistic and probabilistic walks. Such walks involves discrete steps on an infinite line, given by the integers  $\mathbb{Z}$ . In a single move, left or right steps can be made, described as  $-1$  decrements or  $+1$  increments. The walks are steered by Hadamard’s matrix acting on a qubit. They can be described via a function  $\mathbb{C}^2 \otimes \mathcal{M}_{\mathbb{C}}(\mathbb{Z}) \rightarrow \mathbb{C}^2 \otimes \mathcal{M}_{\mathbb{C}}(\mathbb{Z})$ , where  $\mathbb{C}^2$  represents the qubit, see [14]. Alternatively, they can be described via a bifinite multirelation on  $\mathbb{Z} + \mathbb{Z}$ . We write  $\kappa_1$  and  $\kappa_2$  as left and right coprojection for this coproduct, corresponding to the up and down orientations of the qubit that steers the movement. This kind of quantum walk can now be given as an endomap  $q: \mathbb{Z} + \mathbb{Z} \rightarrow \mathbb{Z} + \mathbb{Z}$  in  $\mathbf{BifMRel}_{\mathbb{C}}$ , which we describe by listing only the non-zero values of  $q$ , as multirelation:

$$(\mathbb{Z} + \mathbb{Z}) \times (\mathbb{Z} + \mathbb{Z}) \xrightarrow{q} \mathbb{C} \quad \text{where} \quad \begin{cases} q(\kappa_1 n, \kappa_1(n-1)) = \frac{1}{\sqrt{2}} \\ q(\kappa_1 n, \kappa_2(n+1)) = \frac{1}{\sqrt{2}} \\ q(\kappa_2 n, \kappa_1(n-1)) = \frac{1}{\sqrt{2}} \\ q(\kappa_2 n, \kappa_2(n+1)) = -\frac{1}{\sqrt{2}} \end{cases}$$

The  $n \in \mathbb{Z}$  in the first argument of  $q$  represents the current position; the second argument describes the successor position, which is either a step left or right. The labels  $\kappa_i$  capture orientations.

It is not hard to see that this map  $q$  is unitary. By iterating the map in  $\mathbf{BifMRel}$ , like in  $q^2 = q \bullet q, q^3 = q \bullet q \bullet q, \dots$ , and subsequently taking the resulting bistochastic relation (see Lemma 5.1), one can compute the iterated distributions of the original quantum walk (and the stationary distribution as suitable limit).

In the remainder of this section we investigate some of the categorical structure of the category of bifinite multirelations. It will clarify, for instance, that the above “walks” map  $q$  is an endomap  $\mathbb{Z} \oplus \mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z}$ , where  $\oplus$  is a biproduct.

**Proposition 5.2.** *For an involutive commutative semiring  $S$ , the category  $\mathbf{BifMRel}_S$  of sets and bifinite  $S$ -valued multirelations has (symmetric) dagger tensors  $(\times, 1)$  and dagger biproducts  $(+, 0)$ , where tensors distribute over biproducts.*

The set of scalars in  $\mathbf{BifMRel}_S$  (endomaps of the tensor unit  $1$ ) is  $S$ . The induced additive structure on homsets is obtained pointwise from  $S$ . The homsets are (Abelian) groups iff  $S$  is a ring.

The objects  $X$  in  $\mathbf{BifMRel}_S$  that are finite (as a set) are  $S$ -modules of the form  $S^n$  that carry a compact structure. The induced monoidal trace operation  $tr(s): X \rightarrow Y$ , for  $s: X \times A \rightarrow Y \times A$  is given by the sum of the ‘diagonal’ elements:

$$tr(s)(x, y) = \sum_{a \in A} \frac{s(x, a, y, a)}{\#A \cdot 1}.$$

Here we assume that the number of elements  $\#A \in \mathbb{N}$  is not zero; otherwise, trivially,  $A = 0$  is the zero-object and  $s$  is the zero-map.

In case  $S$  is a field like  $\mathbb{R}$  or  $\mathbb{C}$ , the latter category of modules  $S^n$  is of course the category of finite-dimensional vector (or Hilbert) spaces.

**Proof** The tensor is given on objects by Cartesian product:  $X_1 \otimes X_2 = X_1 \times X_2$ . And if we have  $r_i: X_i \rightarrow Y_i$ , then  $r_1 \otimes r_2: X_1 \otimes X_2 \rightarrow Y_1 \otimes Y_2$  is given by the function:

$$\begin{aligned} (X_1 \times Y_1) \times (X_2 \times Y_2) &\xrightarrow{r_1 \otimes r_2} S \\ \langle (x_1, y_1), (x_2, y_2) \rangle &\longmapsto r_1(x_1, y_1) \cdot r_2(x_2, y_2). \end{aligned}$$

The singleton set  $1$ , say  $1 = \{*\}$ , is tensor unit. The monoidal (dagger) isomorphisms are given by equalities, such as:

$$\begin{aligned} (1 \times X) \times X &\xrightarrow{\lambda} S & (X \times Y) \times (Y \times X) &\xrightarrow{\gamma} S \\ \langle (*, x), x' \rangle &\longmapsto \begin{cases} 1 & \text{if } x = x' \\ 0 & \text{otherwise} \end{cases} & \langle (x, y), (y', x') \rangle &\longmapsto \begin{cases} 1 & \text{if } x = x', y = y' \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The endomaps on the tensor unit  $1$  are maps  $1 \times 1 \rightarrow S$ , corresponding to elements of the semiring  $S$ .

The category  $\mathbf{BifMRel}_S$  also has biproducts, given on objects by finite coproducts on sets (whose coprojections we write as  $\kappa_i$ , like above). The empty set  $0$  is zero object in  $\mathbf{BifMRel}_S$ , with empty multirelations  $X \rightarrow 0$  and  $0 \rightarrow Y$ . The resulting zero map  $0: X \rightarrow Y$  is the relation  $0: X \times Y \rightarrow S$  that is always  $0$ . The coprojections and projections  $X_i \xrightarrow{\kappa_i} X_1 \oplus X_2 \xrightarrow{\pi_i} X_i$  in  $\mathbf{BifMRel}_S$  are given by:

$$\begin{aligned} X_i \times (X_1 + X_2) &\xrightarrow{\kappa_i} S & (X_1 + X_2) \times X_i &\xrightarrow{\pi_i} S \\ \langle x, u \rangle &\longmapsto \begin{cases} 1 & \text{if } u = \kappa_i x \\ 0 & \text{otherwise} \end{cases} & \langle u, x \rangle &\longmapsto \begin{cases} 1 & \text{if } u = \kappa_i x \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

(Notice that two different coprojections  $\kappa$  occur: in  $\mathbf{BifMRel}_S$  and in  $\mathbf{Sets}$ .)

We have  $\pi_i = (\kappa_i)^\dagger$  in  $\mathbf{BifMRel}_S$ . Triples and cotriples, for  $r_i: Z \rightarrow X_i$  and  $t_i: X_i \rightarrow Z$  are given by:

$$\begin{array}{ccc} Z \times (X_1 + X_2) & \xrightarrow{\langle r_1, r_2 \rangle} & S & & (X_1 + X_2) \times Y & \xrightarrow{[t_1, t_2]} & S \\ \langle z, u \rangle & \longmapsto & r_i(z, x), \text{ for } u = \kappa_i x & & \langle u, z \rangle & \longmapsto & t_i(x, z), \text{ for } u = \kappa_i x \end{array}$$

It is not hard to see that  $\langle r_1, r_2 \rangle^\dagger = [(r_1)^\dagger, (r_2)^\dagger]$ .

There are distributivity (dagger) isomorphisms  $X \otimes (Y_1 \oplus Y_2) \xrightarrow{\cong} (X \otimes Y_1) \oplus (X \otimes Y_2)$  given by:

$$\begin{array}{ccc} \left( X \times (Y_1 + Y_2) \right) \times \left( (X \times Y_1) + (X \times Y_2) \right) & \xrightarrow{\quad} & S \\ \langle \langle x, u \rangle, v \rangle & \longmapsto & \begin{cases} 1 & \text{if } u = \kappa_i y \text{ and } v = \kappa_i(x, y) \\ 0 & \text{otherwise.} \end{cases} \end{array}$$

Finally we note that if  $X$  is a finite set, say with  $n = |X|$  elements, then  $\mathcal{M}_S(X) \cong S^n$ . Further, each multirelation  $r: X \times Y \rightarrow S$  is automatically bifinite, if  $X, Y$  are finite. Such a morphism is thus determined by the associated map  $X \rightarrow \mathcal{M}_S(Y) \cong S^{|Y|}$ . The latter corresponds to a linear map  $S^{|X|} \rightarrow S^{|Y|}$ . The compact structure on a finite set  $X$  has  $X^* = X$  with unit  $\eta: 1 \rightarrow X^* \otimes X$  and counit  $\varepsilon: X \otimes X^* \rightarrow 1$  given by:

$$\begin{array}{ccc} 1 \times (X \times X) & \xrightarrow{\eta} & S & & (X \times X) \times 1 & \xrightarrow{\varepsilon} & S \\ \langle *, (x, x') \rangle & \longmapsto & \begin{cases} 1 & \text{if } x = x' \\ 0 & \text{otherwise} \end{cases} & & \langle (x, x'), * \rangle & \longmapsto & \begin{cases} 1 & \text{if } x = x' \\ 0 & \text{otherwise.} \end{cases} \end{array}$$

These multirelations are bifinite because  $X$  is finite. The formula for traces is obtained in the standard manner from compact structure  $\eta, \varepsilon$ , see [17].  $\square$

The logical structure of the category  $\mathbf{BifMRel}_S$  will be described in terms of its dagger kernels, following [11]. We first borrow some more terminology from linear algebra. Two multisets  $\varphi, \psi \in \mathcal{M}_S(X)$  will be called orthogonal, written as  $\varphi \perp \psi$ , if  $\mathbf{cp}_{\mathcal{M}}(\varphi, \psi) = 0$ . Recall that this corresponds to the usual condition  $\sum_x \overline{\varphi(x)} \cdot \psi(x) = 0$ , for the “dot” inner product. A subset  $V \subseteq \mathcal{M}_S(X)$  will be called orthogonal if all its pairs of (different) elements are orthogonal; it will be called orthonormal if additionally each  $\varphi \in V$  satisfies  $\|\varphi\|^2 \stackrel{\text{def}}{=} \mathbf{cp}_{\mathcal{M}}(\varphi, \varphi) = 1$ .

Since dagger kernels are both kernels and dagger monos, the following characterisation sheds light on the situation.

**Lemma 5.3.** *A morphism  $r: X \rightarrow Y$  in  $\mathbf{BifMRel}_S$  is a dagger mono iff the set of multisets  $\{r(x, -) \mid x \in X\} \subseteq \mathcal{M}_S(Y)$  is orthonormal.*

**Proof** The crucial point is:

$$(r^\dagger \bullet r)(x, x') = \sum_y r(x, y) \cdot \overline{r(x', y)} = \mathbf{cp}_{\mathcal{M}}(r(x', -), r(x, -)).$$

Thus:

$$\begin{aligned}
r \text{ is dagger mono} &\iff r^\dagger \bullet r = \text{id} \\
&\iff \forall x, x'. (r^\dagger \bullet r)(x, x') = \begin{cases} 1 & \text{if } x = x' \\ 0 & \text{otherwise} \end{cases} \\
&\iff \forall x, x'. \text{cp}_{\mathcal{M}}(r(x', -), r(x, -)) = \begin{cases} 1 & \text{if } x = x' \\ 0 & \text{otherwise} \end{cases} \\
&\iff \{r(x, -) \mid x \in X\} \text{ is orthonormal.} \quad \square
\end{aligned}$$

Before giving the general construction of dagger kernels, it may be helpful to see an illustration first.

*Example 5.4.* We use  $S = \mathbb{R}$  as semiring (actually as field) and start from a morphism  $r: \mathbb{N} \rightarrow \mathbb{N}$  in  $\mathbf{BifMRel}_S$ , described as the following multirelation  $r: \mathbb{N} \times \mathbb{N} \rightarrow S$ .

$$r(x, y) = \begin{cases} 1 & \text{if } x = 2y \\ -1 & \text{if } x = 2y + 1 \\ 0 & \text{otherwise.} \end{cases}$$

This  $r$  involves an infinite number of multisets:

$$r(-, 0) = 1 \cdot 0 + (-1) \cdot 1, \quad r(-, 1) = 1 \cdot 2 + (-1) \cdot 3, \quad r(-, 2) = \dots \quad \textit{etc.}$$

We illustrate how to interpret them as infinitely many linear equations:

$$v_0 = v_1 \quad v_2 = v_3 \quad v_4 = v_5 \quad \textit{etc.}$$

Assume we have a map  $f: 1 \rightarrow \mathbb{N}$  with  $r \bullet f = 0$ . Then, for each  $y \in \mathbb{N}$ ,

$$0 = (r \bullet f)(*, y) = \sum_x f(x) \cdot r(x, y) = f(2y) - f(2y + 1).$$

Thus, this  $f$ , as function  $f: \mathbb{N} \rightarrow S$  with finite support, satisfies  $f(2y) = f(2y + 1)$ . It thus provides a “solution”  $f(0) = f(1), f(2) = f(3), \dots$  to the “equations”  $r(-, y) = 0$ .

We wish to describe the dagger kernel of  $r$  as the solution space for these equations  $r(-, y) = 0$ . Lemma 5.3 tells that we have to look for an orthonormal basis for this space. An obvious choice for such a basis is the infinite set of multisets:

$$B = \{\varphi_i \mid i \in \mathbb{N}\} \quad \text{where} \quad \varphi_i = \frac{1}{\sqrt{2}} \cdot (2i) + \frac{1}{\sqrt{2}} \cdot (2i + 1) \in \mathcal{M}_{\mathbb{R}}(X).$$

We take as kernel the map  $\ker(r): B \rightarrow \mathbb{N}$ , given as function  $\ker(r): B \times \mathbb{N} \rightarrow S$  simply by:

$$k(\varphi_i, x) = \varphi_i(x).$$

Clearly, this is well-defined, in the sense that  $\ker(r)$  is bifinite, as multirelation. Further,  $\ker(r)$  satisfies the appropriate properties:

$$\begin{aligned}
(r \bullet \ker(r))(\varphi_i, y) &= \sum_x \ker(r)(\varphi_i, x) \cdot r(x, y) \\
&= \varphi_i(2y) \cdot 1 + \varphi_i(2y + 1) \cdot -1 \\
&= \begin{cases} \frac{1}{\sqrt{2}} + -\frac{1}{\sqrt{2}} & \text{if } i = y \\ 0 & \text{otherwise} \end{cases} \\
&= 0 \\
(\ker(r)^\dagger \circ \ker(r))(\varphi_i, \varphi_j) &= \sum_x \ker(r)(\varphi_i, x) \cdot \overline{\ker(r)(\varphi_j, x)} \\
&= \begin{cases} (\frac{1}{\sqrt{2}})^2 + (\frac{1}{\sqrt{2}})^2 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} 1 & \text{if } \varphi_i = \varphi_j \\ 0 & \text{otherwise} \end{cases} \\
&= \text{id}(\varphi_i, \varphi_j).
\end{aligned}$$

Next assume we have a map  $t: Z \rightarrow \mathbb{N}$  in  $\mathbf{BifMRel}_S$  satisfying  $r \bullet t = 0$ . We have to show that  $t$  factors through the kernel  $\ker(r)$ . For each  $z \in Z$  and  $y \in \mathbb{N}$  we have  $0 = (r \bullet t)(z, y) = \sum_x t(z, x) \cdot r(x, y) = t(z, 2y) \cdot 1 + t(z, 2y + 1) \cdot -1$ . Hence  $t(z, 2y) = t(z, 2y + 1)$ , so that  $t$  solves the equations  $r(-, y) = 0$ . Since  $t$  is bifinite, there are for a fixed  $z \in Z$ , only finitely many  $y$  with  $t(z, y) \neq 0$ . Hence we can express  $t(z, -) \in \mathcal{M}_{\mathbb{R}}(\mathbb{N})$  in terms of the base vectors in  $B$ , say as:

$$t(z, -) = a_1 \cdot \varphi_{y_1} + \cdots + a_n \cdot \varphi_{y_n}, \quad \text{where} \quad a_i = t(z, 2y_i) \cdot \sqrt{2} \in \mathbb{R},$$

for certain  $y_1, \dots, y_n \in \mathbb{N}$  (depending on  $z$ ). We thus define the required map  $t': Z \rightarrow B$  by  $t'(z, \varphi_{y_i}) = a_i$ , for these  $y_1, \dots, y_n$  (and 0 elsewhere). Then:

$$\begin{aligned}
(\ker(r) \bullet t')(z, x) &= \sum_i t'(z, \varphi_i) \cdot \ker(r)(\varphi_i, x) \\
&= \sum_i a_i \cdot \varphi_{y_i}(x) \\
&= t(z, x).
\end{aligned}$$

**Proposition 5.5 (AC).** *The category  $\mathbf{BifMRel}_S$ , restricted to countable objects, has dagger kernels, assuming  $S = \mathbb{R}$  or  $S = \mathbb{C}$ .*

**Proof** For an arbitrary map  $r: X \rightarrow Y$  we consider, like in Example 5.4, the multisets  $r(-, y) \in \mathcal{M}_S(X)$  as equations, whose solutions, also in  $\mathcal{M}_S(X)$ , give rise to kernels. The support  $\text{supp}(r(-, y)) = \{x \mid r(x, y) \neq 0\}$  of such an equation captures the variables that occur. We first collect all such variables in a subset  $X_r \subseteq X$ , and then describe the set of solutions in terms of multisets over these variables.

$$\begin{aligned}
X_r &= \bigcup_{y \in Y} \text{supp}(r(-, y)) \\
\text{Sol}_r &= \{\varphi \in \mathcal{M}_S(X_r) \mid \forall y \in Y. \sum_x \varphi(x) \cdot r(x, y) = 0\}.
\end{aligned}$$

Clearly,  $\text{Sol}_r \subseteq \mathcal{M}_S(X)$  is a linear subspace. Hence, using the Axiom of Choice, we can choose a basis  $B_r \subseteq \text{Sol}_r$ , of linearly independent, with norm 1. Since the set  $\{1x \mid x \in X\}$  is a countable basis for  $\mathcal{M}_S(X)$ ,  $B_r$  has at most countably many elements.

We claim: for each  $z \in X$ , the set  $\{\varphi \in B_r \mid \varphi(z) \neq 0\}$  is finite. Suppose not, *i.e.* suppose there are infinitely many  $\varphi_i \in B_r$  with  $\varphi_i(z) \neq 0$ . Since  $\varphi_i \in \mathcal{M}_S(X_r)$  and  $z \in \text{supp}(\varphi_i) \subseteq X_r$ , there must be an  $y_i \in Y$  with  $r(z, y_i) \neq 0$ . Because  $r$  is bifinite there can only be finitely many such  $y_i$ , say  $y_1, \dots, y_n$ . Since the  $\varphi_i$  are in  $B_r \subseteq \text{Sol}_r$ , we have  $\sum_x \varphi_i(x) \cdot r(x, y_j) = 0$  for each  $i$  and  $j \leq n$ . The solution space of these  $n$  equations  $r(-, y_j)$  has finite dimension. Hence it cannot contain infinitely many linearly independent  $\varphi_i$ .

We now define a kernel object  $\text{Ker}(r) = (X - X_r) \cup B_r$ , with kernel map  $\text{ker}(r): \text{Ker}(r) \rightarrow X$  given by:

$$\text{ker}(r)(x, x') = \begin{cases} 1 & \text{if } x \in X - X_r \text{ and } x = x' \\ 0 & \text{if } x \in X - X_r \text{ and } x \neq x' \end{cases} \quad \text{ker}(\varphi, x) = \varphi(x).$$

This gives a bifinite multirelation by the claim above. We check that this  $\text{ker}(r)$  is a dagger kernel in three steps.

- In order to obtain that  $\text{ker}(r)$  is a dagger mono by applying Lemma 5.3 we need to transform the set of base vectors  $B_r \subseteq \text{Sol}_r$  into an orthonormal basis. This can be done in a standard way, via Gram-Schmidt, using the inner product  $\text{cp}_{\mathcal{M}}$  from (9). Because  $B_r$  is countable, we can write  $B_r = \{\varphi_n \mid n \in \mathbb{N}\}$  and replace each  $\varphi_n$  by  $\varphi'_n$  obtained as:  $\varphi'_0 = \varphi_0$  and:

$$\varphi'_{n+1} = \varphi_{n+1} - \text{cp}_{\mathcal{M}}(\varphi'_0, \varphi_{n+1})\varphi'_0 - \dots - \text{cp}_{\mathcal{M}}(\varphi'_n, \varphi_{n+1})\varphi'_n.$$

By construction,  $\text{cp}_{\mathcal{M}}(\varphi'_k, \varphi'_n) = 0$ , for  $k < n$ . Hence we may assume that  $B_r$  is orthonormal.

- We have  $r \bullet \text{ker}(r) = 0$ , since for  $x \in X - X_r$  and  $\varphi \in B_r$ ,

$$\begin{aligned} (r \bullet \text{ker}(r))(x, y) &= \sum_{x' \in X} \text{ker}(r)(x, x') \cdot r(x', y) \\ &= r(x, y) \\ &= 0 \quad \text{since } x \notin X_r \\ (r \bullet \text{ker}(r))(\varphi, y) &= \sum_{x' \in X} \text{ker}(r)(\varphi, x') \cdot r(x', y) \\ &= \sum_{x' \in X} \varphi(x') \cdot r(x', y) \\ &= 0, \quad \text{since } \varphi \in B_r \subseteq \text{Sol}_r. \end{aligned}$$

- We also check the universal property of  $\text{ker}(r)$ . Let  $t: Z \rightarrow X$  satisfy  $r \bullet t = 0$ . We split each multiset  $t(z, -) \in \mathcal{M}_S(X)$  in two parts:

$$t(z, -) = t_1(z, -) + t_2(z, -) \quad \text{where} \quad \begin{cases} \text{supp}(t_1(z, -)) \subseteq X_r \\ \text{supp}(t_2(z, -)) \cap X_r = \emptyset. \end{cases}$$

Then  $t_1(z, -) \in \text{Sol}_r$ , since for each  $y \in Y$ ,

$$\begin{aligned} 0 &= (r \bullet t)(z, y) = \sum_x t(z, x) \cdot r(x, y) \\ &= \sum_x t_1(z, x) \cdot r(x, y) + t_2(z, x) \cdot r(x, y) \\ &= \sum_x t_1(z, x) \cdot r(x, y). \end{aligned}$$

Since  $t$  is bifinite there are only finitely many  $y$  with  $t_1(z, y) \neq 0$ . Hence each  $t_1(z, -) \in \mathcal{M}_S(X_r)$  can be expressed in terms of finitely many base vectors from  $B_r$ , say as:

$$t_1(z, -) = a_1^z \cdot \varphi_1^z + \cdots + a_{n_z}^z \cdot \varphi_{n_z}^z.$$

We then define the required mediating map  $t': Z \rightarrow \text{Ker}(r)$  as function  $t': Z \times ((X - X_r) \cup B_r) \rightarrow S$ , given on  $x \in X - X_r$  and  $\varphi \in B_r$  by:

$$t'(z, x) = t(z, x) = t_2(z, x) \quad \text{and} \quad t'(z, \varphi) = \begin{cases} a_i^z & \text{if } \varphi = \varphi_i^z \\ 0 & \text{otherwise.} \end{cases}$$

This  $t'$  is bifinite, and is the right map, since:

$$\begin{aligned} &(\text{ker}(r) \bullet t')(z, x) \\ &= \sum_{k \in \text{Ker}(r)} t'(z, k) \cdot \text{ker}(r)(k, x) \\ &= \left( \sum_{x' \in X - X_r} t'(z, x') \cdot \text{ker}(r)(x', x) \right) + \left( \sum_{\varphi \in B_r} t'(z, \varphi) \cdot \text{ker}(r)(\varphi, x) \right) \\ &= \begin{cases} t'(z, x) & \text{if } x \notin X_r \\ \sum_i a_i^z \cdot \varphi_i^z(x) & \text{if } x \in X_r \end{cases} \\ &= \begin{cases} t_2(z, x) & \text{if } x \notin X_r \\ t_1(z, x) & \text{if } x \in X_r \end{cases} \\ &= t(z, x). \quad \square \end{aligned}$$

The category  $\mathbf{BifMRel}_S$  is thus a dagger kernel category. The kernel subobjects of an object then form an orthomodular lattice, see [11]. Further investigation is needed to see if  $\mathbf{BifMRel}_S$  can really be seen as a “light” version of the category of Hilbert spaces, suitable for discrete quantum computations. Further steps in more logic-oriented investigations, including measurement, can be found in [15].

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## References

- [1] M. Barr. Algebraically compact functors. *Journ. of Pure & Appl. Algebra*, 82:211–231, 1992.
- [2] E.J. Beggs and S. Majid. Bar categories and star operations. *Algebras and Representation Theory*, 12:103–152, 2009.

- [3] J. Bénabou. Les distributeurs. Rapport 33. Seminaires de Mathématiques Pure, Inst. de Mathématiques, Univ. Cath. Louvain, 1973.
- [4] R. Blute and P. Panangaden. Dagger categories and formal distributions. In B. Coecke, editor, *New Structures in Physics*, number 813 in Lect. Notes Physics, pages 421–436. Springer, Berlin, 2011.
- [5] D. Coumans and B. Jacobs. Scalars, monads and categories. In C. Heunen and M. Sadrzadeh, editors, *Compositional methods in Physics and Linguistics*. Oxford Univ. Press, 2012. See [arxiv.org/abs/1003.0585](http://arxiv.org/abs/1003.0585).
- [6] G.D. Crown. On some orthomodular posets of vector bundles. *Journ. of Natural Sci. and Math.*, 15(1-2):11–25, 1975.
- [7] E.-E. Doberkat. The converse of a stochastic relation. In A.D. Gordon, editor, *Foundations of Software Science and Computation Structures*, number 2620 in Lect. Notes Comp. Sci., pages 233–249. Springer, Berlin, 2003.
- [8] E.-E. Doberkat. Eilenberg-Moore algebras for stochastic relations. *Inf. & Comp.*, 204(12):1756–1781, 2006. Erratum and addendum in: 206(12):1476–1484, 2008.
- [9] J.M. Egger. On involutive monoidal categories. *Theory and Applications of Categories*, 25(14):368–393, 2011.
- [10] E. Haghverdi and Ph. Scott. A categorical model for the geometry of interaction. *Theor. Comp. Sci.*, 350:252–274, 2006.
- [11] C. Heunen and B. Jacobs. Quantum logic in dagger kernel categories. *Order*, 27(2):177–212, 2010.
- [12] B. Jacobs. Convexity, duality, and effects. In C.S. Calude and V. Sassone, editors, *IFIP Theoretical Computer Science 2010*, number 82(1) in IFIP Adv. in Inf. and Comm. Techn., pages 1–19. Springer, Boston, 2010.
- [13] B. Jacobs. Orthomodular lattices, Foulis semigroups and dagger kernel categories. *Logical Methods in Comp. Sci.*, 6(2), 2010. See [arxiv.org/pdf/0912.0931](http://arxiv.org/pdf/0912.0931).
- [14] B. Jacobs. Coalgebraic walks, in quantum and Turing computation. In M. Hofmann, editor, *Foundations of Software Science and Computation Structures*, number 6604 in Lect. Notes Comp. Sci., pages 12–26. Springer, Berlin, 2011.
- [15] B. Jacobs. New directions in categorical logic, for classical, probabilistic and quantum logic. See [arxiv.org/abs/1205.3940](http://arxiv.org/abs/1205.3940), 2012.
- [16] B. Jacobs. Involutive categories and monoids, with a GNS-correspondence. *Found. of Physics*, 8 Sept. 2011.
- [17] A. Joyal, R. Street, and D. Verity. Traced monoidal categories. *Math. Proc. Cambridge Phil. Soc.*, 119(3):425–446, 1996.
- [18] G. Kalmbach. *Orthomodular Lattices*. Academic Press, London, 1983.
- [19] K. Keimel. The monad of probability measures over compact ordered spaces and its Eilenberg-Moore algebras. *Topology and its Applications*, 156:227–239, 2008.
- [20] J. Kempe. Quantum random walks – an introductory overview. *Contemporary Physics*, 44:307–327, 2003.
- [21] A. Kock. Bilinearity and cartesian closed monads. *Math. Scand.*, 29:161–174, 1971.
- [22] A. Kock. Closed categories generated by commutative monads. *Journ. Austr. Math. Soc.*, XII:405–424, 1971.

- [23] F.W. Lawvere. Metric spaces, generalized logic, and closed categories. *Seminario Matematico e Fisico. Rendiconti di Milano*, 43:135–166, 1973.
- [24] N.D. Mermin. *Quantum Computer Science: An Introduction*. Cambridge Univ. Press, 2007.
- [25] M.A. Nielsen and I.L. Chuang. *Quantum Computation and Quantum Information*. Cambridge Univ. Press, 2000.
- [26] S. Venegas-Andraca. *Quantum Walks for Computer Scientists*. Morgan & Claypool, 2008.

Bart Jacobs

Institute for Computing and Information Sciences (iCIS),

Radboud University Nijmegen, The Netherlands.

Webaddress: [www.cs.ru.nl/B.Jacobs](http://www.cs.ru.nl/B.Jacobs)

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