# On affine variety codes from the Klein quartic 

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#### Abstract

We study a family of primary affine variety codes defined from the Klein quartic. The duals of these codes have previously been treated in [12, Ex. 3.2]. Among the codes that we construct almost all have parameters as good as the best known codes according to 9] and in the remaining few cases the parameters are almost as good. To establish the code parameters we apply the footprint bound [10|7] from Gröbner basis theory and for this purpose we develop a new method where we inspired by Buchbergers algorithm perform a series of symbolic computations.


## 1 Introduction

Affine variety codes [5] are codes defined by evaluating multivariate polynomials at the points of an affine variety. Despite having a simple description such codes constitute the entire class of linear codes [5, Pro. 1]. Given a description of a code as an affine variety code it is easy to determine the length $n$ and dimension $k$, but no simple general method is known which easily estimates the minimum distance $d$. Of course such methods exists for particular classes of affine variety codes. For instance the Goppa bound for one-point algebraic geometric codes extends to an improved bound on the more general class of order domain codes [11|6], and in larger generality the Feng-Rao bounds and their variants can be successfully applied to many different types of codes [2|3|4]12|13|6|8]. In this paper we consider a particular family of primary affine variety codes for which none of the above mentioned bounds provide accurate information. More precisely we consider primary codes defined from the Klein quartic using the same weighted degree lexicographic ordering as in [12, Ex. 3.2] where they studied the corresponding dual codes. A common property of the Feng-Rao bound for primary codes and its variants are that they can be viewed [6]8] as consequences of the footprint bound [10|7] from Gröbner basis theory. To establish more accurate information for the codes under
consideration it is therefore natural to try to apply the footprint bound in a more direct way, which is exactly what we do in the present paper using ingredients from Buchberger's algorithm and by considering an exhaustive number of special cases. Our analysis reveals that the codes under consideration are in most cases as good as the best known codes according to $[9]$ and for the remaining few cases the minimum distance is only one less than the best known codes of the same dimension.

The paper is organized as follows. In Section 2 we introduce the footprint of an ideal and define affine variety codes. We then describe how the footprint bound can be applied to determine the Hamming weight of a code word. Then in Section 3 we apply symbolic computations leading to estimates on the minimum distance on each of the considered codes the information of which we collect in Section 4.

## 2 Affine variety codes and the footprint bound

The footprint (also called the delta-set) is defined as follows:
Definition 1. Given a field $k$, a monomial ordering $\prec$ and an ideal $J \subseteq$ $k\left[X_{1}, \ldots, X_{m}\right]$ the footprint of $J$ is
$\Delta_{\prec}(J)=\{M \mid M$ is a monomial which is not leading monomial of any polynomial in J\}

From [1, Prop. 7, Sec. 5.3] we have the following well-known result.
Theorem 1. Let the notation be as in the above definition. The set

$$
\left\{M+J \mid M \in \Delta_{\prec}(J)\right\}
$$

is a basis for $k\left[X_{1}, \ldots, X_{m}\right] / J$ as a vector space over $k$.
Recall that by definition a Gröbner basis is a finite basis for the ideal $J$ from which one can easily determine the footprint. Concretely a monomial is a leading monomial of some polynomial in the ideal if and only if it is divisible by a leading monomial of some polynomial in the Gröbner basis. The following corollary is an instance of the more general footprint bound [10.
Corollary 1. Let $I \subseteq \mathbb{F}_{q}\left[X_{1}, \ldots, X_{m}\right]$ be an ideal and $I_{q}=I+\left\langle X_{1}^{q}-\right.$ $\left.X_{1}, \ldots, X_{m}^{q}-X_{m}\right\rangle$. The variety of $I_{q}$ is of size $\# \Delta_{\prec}\left(I_{q}\right)$ for any monomial ordering $\prec$.

Proof. Let the variety of $I_{q}$ be $\left\{P_{1}, \ldots, P_{n}\right\}$ with $P_{i} \neq P_{j}$ for $i \neq j$. The field $\mathbb{F}_{q}$ being perfect, the ideal $I_{q}$ is radical because it contains a univariate square-free polynomial in each variable and by the ideal-variety correspondence therefore $I_{q}$ is in fact the vanishing ideal of $\left\{P_{1}, \ldots, P_{n}\right\}$. Therefore the evaluation map ev : $\mathbb{F}_{q}\left[X_{1}, \ldots, X_{m}\right] / I_{q} \rightarrow \mathbb{F}_{q}^{n}$ given by $\operatorname{ev}\left(F+I_{q}\right)=\left(F\left(P_{1}\right), \ldots, F\left(P_{n}\right)\right)$ is injective. On the other hand the evaluation map is also surjective which is seen by applying Lagrange interpolation. We have demonstrated that ev is a bijection and the corollary follows from Theorem [1.

We are now ready to define primary affine variety codes formally.
Definition 2. Let the notation be as in the proof of Corollary 1. Given an ideal $I \subseteq \mathbb{F}_{q}\left[X_{1}, \ldots, X_{m}\right]$ and a monomial ordering $\prec$ choose $L \subseteq \Delta_{\prec}\left(I_{q}\right)$. Then

$$
C(I, L)=\operatorname{Span}_{\mathbb{F}_{q}}\left\{\operatorname{ev}\left(M+I_{q}\right) \mid M \in L\right\}
$$

is called a primary affine variety code.
From the above discussion it is clear that $C(I, L)$ is a code of length $n=\# \Delta_{\prec}\left(I_{q}\right)$ and dimension $k=\# L$. Given a code word $\boldsymbol{c}=\operatorname{ev}\left(F+I_{q}\right)$ then by Corollary 1 we have

$$
w_{H}(\boldsymbol{c})=n-\# \Delta_{\prec_{w}}\left(\langle F\rangle+I_{q}\right)=\# \Delta_{\prec_{w}}\left(I_{q}\right) \cap \operatorname{lm}\left(\langle F\rangle+I_{q}\right)=\# \square_{\prec_{w}}(F),
$$

where $\square_{\prec_{w}}(F):=\Delta_{\prec_{w}}\left(I_{q}\right) \cap \operatorname{lm}\left(\langle F\rangle+I_{q}\right)$. Reducing a polynomial modulo a Gröbner basis for $I_{q}$ one obtains a (unique) polynomial which has support in the footprint $\Delta\left(I_{q}\right)$ (this is the result behind Theorem (1). Hence we shall always assume that $F$ is of this form. In the rest of the paper we concentrate on estimating $\# \square_{\prec}(F)$ using only information on the leading monomial. We do this for a concrete class of codes defined from the Klein quartic, but the method that we describe can be applied to any affine variety code of moderate dimension. In particular it can be applied whenever the length of the codes are moderate.

## 3 Code words from the Klein curve

In the remaining part of the paper $I$ will always be the ideal

$$
I=\left\langle Y^{3}+X^{3} Y+X\right\rangle \subseteq \mathbb{F}_{8}[X, Y]
$$

and consequently $I_{8}=\left\langle Y^{3}+X^{3} Y+X, X^{8}+X, Y^{8}+Y\right\rangle$. The corresponding variety $\sqrt[3]{ }$ is of size 22 , hence we write it as $\left\{P_{1}, \ldots, P_{22}\right\}$. The evaluation map then becomes $\operatorname{ev}\left(F+I_{8}\right)=\left(F\left(P_{1}\right), \ldots, F\left(P_{22}\right)\right)$.

As monomial ordering we choose the same ordering as in [12, Ex. 3.2], namely the weighted degree lexicographic ordering $\prec_{w}$ given by the rule that $X^{\alpha} Y^{\beta} \prec_{w} X^{\gamma} Y^{\delta}$ if either (i) or (ii) below holds

$$
\text { (i) } 2 \alpha+3 \beta<2 \gamma+3 \delta, \quad \text { (ii) } 2 \alpha+3 \beta=2 \gamma+3 \delta \text { but } \beta<\delta \text {. }
$$

By inspection $\left\{Y^{3}+X^{3} Y+X, X^{8}-X, X^{7} Y+Y\right\}$ is a Gröbner basis for $I_{8}$ with respect to $\prec_{w}$. Hence, the footprint $\Delta_{\prec_{w}}\left(I_{8}\right)$ and the corresponding weights are as in Figure [1 We remind the reader that for $L \subseteq \Delta_{\prec_{w}}\left(I_{8}\right)$ the code $C(I, L)$ equals ev $\left(\operatorname{Span}_{\mathbb{F}_{8}}(L)+I_{8}\right)$ which is of length $n=22$ and dimension $k=\# L$.

| $Y^{2}$ | $X Y^{2}$ | $X^{2} Y^{2}$ | $X^{3} Y^{2}$ | $X^{4} Y^{2}$ | $X^{5} Y^{2}$ | $X^{6} Y^{2}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $Y$ | $X Y$ | $X^{2} Y$ | $X^{3} Y$ | $X^{4} Y$ | $X^{5} Y$ | $X^{6} Y$ |  |
| 1 | $X$ | $X^{2}$ | $X^{3}$ | $X^{4}$ | $X^{5}$ | $X^{6}$ | $X^{7}$ |
| 6 | 8 |  |  | 12 |  | 14 | 16 |
|  | 18 |  |  |  |  |  |  |
| 3 | 5 | 7 | 9 | 11 | 13 | 15 |  |
| 0 | 2 | 4 | 6 | 8 | 10 | 12 | 14 |

Fig. 1. The footprint $\Delta_{\prec_{w}}\left(I_{8}\right)$ with corresponding weights.

Our method to estimate $\# \square_{\swarrow_{w}}(F)$ (which corresponds to estimating the Hamming weight of the corresponding code word) consists in two parts. First we observe that all monomials in $\Delta_{\prec w}\left(I_{8}\right)$ divisible by the leading monomial of $F$ are in $\square_{\prec_{w}}(F)$. In the second part we then for a number of exhaustive special cases find more monomials in $\square_{\prec_{w}}(F)$ by establishing clever combinations of polynomials that we already know are in $\langle F\rangle+I_{q}$. To describe how such combinations are derived we will need the following notation. Consider polynomials $S(X, Y), D(X, Y)$ and

[^0]$R(X, Y)$. By
\[

$$
\begin{equation*}
S(X, Y) \xrightarrow{D(X, Y)} R(X, Y) \tag{1}
\end{equation*}
$$

\]

we shall indicate that $R(X, Y)=S(X, Y)-Q(X, Y) D(X, Y)$ for some polynomial $Q(X, Y)$. The important fact - which we shall use frequently throughout the paper - is that $R(X, Y) \in\langle S(X, Y), D(X, Y)\rangle$. Observe that although we will always use the above "operation" to decrease the leading monomial (meaning that $\operatorname{lm}(R) \prec \operatorname{lm}(S)$ ), we may still have monomials left in the support of $R(X, Y)$ which are divisible by the leading monomial of $D(X, Y)$. Hence, (11) does not necessarily correspond to the usual (full) division as described in [1, Sec. 2.3].

Remark 1. The Feng-Rao bound can be applied to any affine variety code; but it works most efficiently when the ideal $I$ and the monomial ordering $\prec$ under consideration satisfy the order domain conditions [6, Sec. 7]. That is,

1. The ordering $\prec$ must be a weighted degree lexicographic ordering (or in larger generality a generalized weighted degree ordering [6, Def. 8]).
2. A Gröbner basis for $I$ must exist with the property that any polynomial in it contains in its support (exactly) two monomials of the highest weight.
3. No two different monomials in $\Delta_{\prec}(I)$ are of the same weight.

In such cases the method often establishes many more monomials in $\square_{\prec}(F)$ than those divisible by the leading monomial of $F$. In 8] an improved Feng-Rao bound was presented which treats in addition efficiently certain families of cases where the conditions 1. and 2. are satisfied, but 3 . is not. Even though the ideal and monomial ordering studied in the present section exactly satisfy conditions 1 . and 2 ., but not 3 , the improved Feng-Rao bound produces the same information as the Feng-Rao bound in this case. By inspection both methods only "detect" monomials divisible by the leading monomial of $F$ as being members of $\square_{\prec_{w}}(F)$.

Below we treat the 22 different possible leading monomials - corresponding to the different members of $\Delta_{\prec_{w}}\left(I_{8}\right)$ - one by one. For simplicity, we shall in our calculations always assume that the leading coefficient of $F$ is 1 which is not really a restriction as our goal is to estimate Hamming weights.

### 3.1 Leading monomial equal to $Y$

Consider $\boldsymbol{c}=\operatorname{ev}\left(F+I_{8}\right)$ where $F(X, Y)=Y+a_{1} X+a_{2}$. Clearly

$$
\left\{Y, Y^{2}, X Y, X Y^{2}, \ldots, X^{6} Y, X^{6} Y^{2}\right\} \subset \square_{\prec_{w}}(F)
$$

We next establish more monomials in $\square_{\prec_{w}}(F)$ under different conditions on the coefficients $a_{1}, a_{2}$. Consider

$$
\begin{array}{ll} 
& Y^{2} F(X, Y) \\
\xrightarrow{Y^{3}+X^{3} Y}+X & X^{3} Y+a_{1} X Y^{2}+a_{2} Y^{2}+X \\
F(X, Y) & a_{1} X^{4}+\left(a_{1}^{3}+a_{2}\right) X^{3}+a_{1}^{2} a_{2} X^{2}+\left(a_{1} a_{2}^{2}+1\right) X+a_{2}^{3} .
\end{array}
$$

If $a_{1} \neq 0$ then we have

$$
\left\{X^{4}, X^{5}, X^{6}, X^{7}\right\} \subset \square_{\prec_{w}}(F)
$$

Next assume $a_{1}=0$. If $a_{2} \neq 0$ then we obtain

$$
\left\{X^{3}, X^{4}, X^{5}, X^{6}, X^{7}\right\} \subset \square_{\prec}(F)
$$

Finally, assume $a_{1}=a_{2}=0$ in which case we have

$$
\left\{X, X^{2}, X^{3} X^{4}, X^{5}, X^{6}, X^{7}\right\} \subset \square_{\prec}(F)
$$

In conclusion we have shown that $\square_{\prec_{w}}(F)$ contains at least $14+4=18$ elements which implies $w_{H}(\boldsymbol{c}) \geq 18$.

### 3.2 Leading monomial equal to $\boldsymbol{Y}^{2}$

Consider a codeword $\boldsymbol{c}=\operatorname{ev}\left(F+I_{8}\right)$ where

$$
F(X, Y)=Y^{2}+a_{1} X^{3}+a_{2} X Y+a_{3} X^{2}+a_{4} Y+a_{5} X+a_{6}
$$

Independently of the coefficients $a_{1}, \ldots, a_{6}$ we see that

$$
\begin{equation*}
\left\{Y^{2}, X Y^{2}, \ldots, X^{6} Y^{2}\right\} \subset \square_{\prec_{w}}(F) \tag{2}
\end{equation*}
$$

We next consider an exhaustive series of conditions under which we establish more monomials in $\square_{\prec_{w}}(F)$. We have

$$
\begin{gather*}
Y F(X, Y) \\
Y^{3}+X^{3} Y+X \\
\left(a_{1}+1\right) X^{3} Y+a_{2} X Y^{2}+a_{3} X^{2} Y  \tag{3}\\
+a_{4} Y^{2}+a_{5} X Y+a_{6} Y+X
\end{gather*}
$$

If $a_{1} \neq 1$ then the leading monomial of the last polynomial becomes $X^{3} Y$ and consequently

$$
\begin{equation*}
\left\{X^{3} Y, X^{4} Y, X^{5} Y, X^{6} Y\right\} \in \square_{\prec w}(F) . \tag{4}
\end{equation*}
$$

Continuing the calculations for this case we obtain:

$$
\begin{aligned}
& Y\left(\left(a_{1}+1\right) X^{3} Y+a_{2} X Y^{2}+a_{3} X^{2} Y+a_{4} Y^{2}+a_{5} X Y+a_{6} Y+X\right) \\
& F(X, Y) \\
&\left(a_{1}+1\right)\left(a_{1} X^{6}+a_{2} X^{4} Y+a_{3} X^{5}+a_{4} X^{3} Y+a_{5} X^{4}+a_{6} X^{3}\right) \\
&+a_{2} X Y^{3}+a_{3} X^{2} Y^{2}+a_{4} Y^{3}+a_{5} X Y^{2}+a_{6} Y^{2}+X Y .
\end{aligned}
$$

If $a_{1} \neq 0$ then we also have

$$
\left\{X^{6}, X^{7}\right\} \subset \square_{\prec_{w}}(F) .
$$

Assuming next that $a_{1}=0$ the above expression becomes

$$
\begin{aligned}
& a_{2} X^{4} Y+a_{3} X^{5}+a_{4} X^{3} Y+a_{5} X^{4}+a_{6} X^{3}+a_{2} X Y^{3} \\
& +a_{3} X^{2} Y^{2}+a_{4} Y^{3}+a_{5} X Y^{2}+a_{6} Y^{2}+X Y \\
Y^{3}+X^{3} Y+X & a_{3} X^{5}+a_{4} X^{3} Y+a_{5} X^{4}+a_{6} X^{3}+a_{3} X^{2} Y^{2}+a_{4} Y^{3} \\
& +a_{5} X Y^{2}+a_{6} Y^{2}+X Y+a_{2} X^{2} \\
\xrightarrow{F(X, Y)} & a_{3} X^{5}+a_{5} X^{4}+a_{6} X^{3}+a_{3} a_{2} X^{3} Y+a_{3}^{2} X^{4}+a_{3} a_{4} X^{2} Y \\
& +a_{3} a_{5} X^{3}+a_{3} a_{6} X^{2}+a_{5} X Y^{2}+a_{6} Y^{2}+X Y+a_{2} X^{2} .
\end{aligned}
$$

If $a_{3} \neq 0$ then

$$
\left\{X^{5}, X^{6}, X^{7}\right\} \subset \square_{\prec_{w}}(F) .
$$

Hence, continuing under the assumption $a_{3}=0$ we are left with

$$
\begin{aligned}
& a_{5} X^{4}+a_{6} X^{3}+a_{5} X Y^{2}+a_{6} Y^{2}+X Y+a_{2} X^{2} \\
\stackrel{F(X, Y)}{ } & a_{5} X^{4}+a_{6} X^{3}+a_{5} a_{2} X^{2} Y+a_{5} a_{4} X Y+a_{5}^{2} X^{2}+a_{5} a_{6} X+a_{6} Y^{2} \\
& +X Y+a_{2} X^{2} .
\end{aligned}
$$

if $a_{5} \neq 0$ then

$$
\left\{X^{4}, X^{5}, X^{6}, X^{7}\right\} \subset \square_{\prec_{w}}(F) .
$$

Hence, assume $a_{5}=0$ and we are left with

$$
\begin{aligned}
& a_{6} X^{3}+a_{6} Y^{2}+X Y+a_{2} X^{2} \\
\stackrel{F(X, Y)}{\longrightarrow} & a_{6} X^{3}+a_{6} a_{2} X Y+a_{6} a_{4} Y+a_{6}^{2}+X Y+a_{2} X^{2} .
\end{aligned}
$$

If $a_{6} \neq 0$ then

$$
\left\{X^{3}, X^{4}, X^{5}, X^{6}, X^{7}\right\} \subset \square_{\prec_{w}}(F)
$$

If on the other hand $a_{6}=0$ then we are left with $X Y+a_{2} X^{2}$ in which case we obtain

$$
\left\{X Y, X^{2} Y\right\} \subset \square_{\prec w}(F) .
$$

In conclusion, for the case $a_{1} \neq 1$ we obtained in addition to the elements in (2) the elements in (4) and at least 2 more. That is, in addition to the elements in (2) at least 6 more.
Assume in the following that $a_{1}=1$ and continue the reduction from (3)

$$
\begin{align*}
& \xrightarrow{F(X, Y)} a_{2} X^{4}+\left(a_{3}+a_{2}^{2}\right) X^{2} Y+\left(a_{2} a_{3}+a_{4}\right) X^{3}+a_{5} X Y \\
& \quad+\left(a_{2} a_{5}+a_{3} a_{4}\right) X^{2}+\left(a_{6}+a_{4}^{2}\right) Y+\left(1+a_{4} a_{5}\right) X+a_{4} a_{6} . \tag{5}
\end{align*}
$$

If $a_{2} \neq 0$ then

$$
\left\{X^{4}, X^{5}, X^{7}, X^{4} Y, X^{5} Y, X^{6} Y\right\} \subset \square_{\prec_{w}}(F) .
$$

Next assume $a_{2}=0$. If $a_{3} \neq 0$ then

$$
\begin{equation*}
\left\{X^{2} Y, X^{3} Y, X^{4} Y, X^{5} Y, X^{6} Y\right\} \subset \square_{\prec w}(F) . \tag{6}
\end{equation*}
$$

Continuing the reduction under the assumption $a_{3} \neq 0$ we multiply (5) by $Y$ and continue the reduction:

$$
\begin{aligned}
& a_{3} X^{2} Y^{2}+a_{4} X^{3} Y+a_{5} X Y^{2}+a_{3} a_{4} X^{2} Y+\left(a_{6}+a_{4}^{2}\right) Y \\
&+\left(1+a_{4} a_{5}\right) X+a_{4} a_{6} \\
& F(X, Y) \\
& F a_{3} X^{5}+a_{3}^{2} X^{4}+a_{3} a_{4} X^{2} Y+a_{3} a_{5} X^{3}+a_{3} a_{6} X^{2}+a_{4} X^{3} Y+a_{5} X Y^{2} \\
&+a_{3} a_{4} X^{2} Y+\left(a_{6}+a_{4}^{2}\right) Y+\left(1+a_{4} a_{5}\right) X+a_{4} a_{6} .
\end{aligned}
$$

As $a_{3} \neq 0$ we obtain in addition to (22) and (6) that

$$
\left\{X^{5}, X^{6}, X^{7}\right\} \subset \square_{\prec_{w}}(F) .
$$

That is, in addition to (2) we found in total 8 more elements in $\square_{\prec_{w}}(F)$.
Next assume $a_{3}=0$ and continue from (5). If $a_{4} \neq 0$ then

$$
\left\{X^{3}, X^{4}, X^{5}, X^{6}, X^{7}, X^{3} Y, X^{4} Y, X^{5} Y, X^{6} Y\right\} \subset \square_{\prec_{w}}(F) .
$$

Next assume $a_{4}=0$. if $a_{5} \neq 0$ then

$$
\left\{X Y, X^{2} Y, X^{3} Y, X^{4} Y, X^{5} Y, X^{6} Y\right\} \subset \square_{\prec_{w}}(F)
$$

Hence, assume $a_{5}=0$. If $a_{6} \neq 0$ then

$$
\left\{Y, X Y, X^{2} Y, X^{3} Y, X^{4} Y, X^{5} Y, X^{6} Y\right\} \subset \square_{\prec w}(F) .
$$

Finally, assume $a_{6}=0$. But then

$$
\left\{X, X^{2}, X^{3}, X^{4}, X^{5}, X^{6}, X^{7}, X Y, X^{2} Y, X^{3} Y, X^{4} Y, X^{5} Y, X^{6} Y\right\} \subset \square_{\prec w}(F)
$$

In conclusion, we have at least $7+\min \{6,6,8,9,6,7,13\}=13$ monomials in $\square_{\prec}(F)$ and therefore $w_{H}(\boldsymbol{c}) \geq 13$.

### 3.3 Leading monomial equal to $\boldsymbol{X Y}$

Consider $\boldsymbol{c}=\operatorname{ev}\left(F+I_{8}\right)$ where

$$
F(X, Y)=X Y+a_{1} X^{2}+a_{2} Y+a_{3} X+a_{4} .
$$

For sure

$$
\begin{align*}
& \left\{X Y, X^{2} Y, X^{3} Y, X^{4} Y, X^{5} Y, X^{6} Y,\right. \\
& \left.\quad X Y^{2}, X^{2} Y^{2}, X^{3} Y^{2}, X^{4} Y^{2}, X^{5} Y^{2}, X^{6} Y^{2}\right\} \subset \square_{\prec_{w}}(F) . \tag{7}
\end{align*}
$$

We next consider an exhaustive series of conditions under which we establish more monomials in $\square_{\text {prec }}(F)$. We have

$$
\begin{aligned}
& Y^{2} F(X, Y) \\
Y^{3}+X^{3} Y+X & a_{1} X^{5}+a_{3} X^{4}+a_{4} X^{3} \\
& +a_{1}\left(a_{1}^{2} X^{4}+a_{2}^{2} Y^{2}+a_{3}^{2} X^{2}+a_{4}^{2}\right) \\
& +a_{3} X Y^{2}+a_{4} Y^{2}+X^{2}+a_{2} X .
\end{aligned}
$$

If $a_{1} \neq 0$ then

$$
\left\{X^{5}, X^{6}, X^{7}\right\} \subset \square_{\prec_{w}}(F) .
$$

Hence, assume $a_{1}=0$ and continue the reduction:

$$
\xrightarrow{F(X, Y)} a_{3} a_{2} Y^{2}+a_{3}^{2} X Y+a_{3} a_{4} Y+a_{3} X^{4}+a_{4} X^{3}+a_{4} Y^{2}+X^{2}+a_{2} X .
$$

If $a_{3} \neq 0$ then

$$
\left\{X^{4}, X^{5}, X^{6}, X^{7}\right\} \subset \square_{\prec_{w}}(F)
$$

Hence, assume $a_{3}=0$ in which case the above becomes

$$
a_{4} Y^{2}+a_{4} X^{3}+X^{2}+a_{2} X
$$

If $a_{4}=0$ then

$$
\left\{X^{2}, X^{3}, X^{4}, X^{5}, X^{6}, X^{7}\right\} \subset \square_{\prec w}(F) .
$$

Hence, assume $a_{4} \neq 0$, in which case we have

$$
\left\{Y^{2}\right\} \subset \square_{\prec w}(F) .
$$

We continue the calculations to add more elements. We have:

$$
X^{2}\left(a_{4} Y^{2}+a_{4} X^{3}+X^{2}+a_{2} X\right) \xrightarrow{F(X, Y)} a_{4} X^{5}+a_{2}^{2} Y^{2}+a_{2} X+a_{4}^{2} .
$$

But then

$$
\left\{X^{5}, X^{6}, X^{7}\right\} \subset \square_{\prec_{w}}(F) .
$$

That is, for the case $a_{4} \neq 0 \square_{\prec_{w}}(F)$ contains in addition to (7) at least $1+3=4$ more monomials.

In conclusion $w_{H}(\boldsymbol{c}) \geq 12+\min \{3,4,6,4\}=15$, and if $a_{1}=0$ then $w_{H}(\boldsymbol{c}) \geq 16$.

### 3.4 Leading monomial equal to $X^{2} Y$

Consider $\boldsymbol{c}=\operatorname{ev}\left(F+I_{8}\right)$ where

$$
F(X, Y)=X^{2} Y+a_{1} Y^{2}+a_{2} X^{3}+a_{3} X Y+a_{4} X^{2}+a_{5} Y+a_{6} X+a_{7}
$$

For sure

$$
\begin{array}{r}
\left\{X^{2} Y, X^{3} Y, X^{4} Y, X^{5} Y, X^{6} Y, X^{2} Y^{2}, X^{3} Y^{2}, X^{4} Y^{2}, X^{5} Y^{2}, X^{6} Y^{2}\right\} \\
\subset \square_{\prec w}(F) . \tag{8}
\end{array}
$$

We next consider an exhaustive series of conditions under which we establish more monomials in $\square_{\prec_{w}}(F)$. We have

$$
\begin{aligned}
& Y^{2} F(X, Y) \\
=\quad & X^{2} Y^{3}+a_{1} Y^{4}+a_{2} X^{3} Y^{2}+a_{3} X Y^{3}+a_{4} X^{2} Y^{2}+a_{5} Y^{3} \\
& +a_{6} X Y^{2}+a_{7} Y^{2} \\
Y^{3}+\xrightarrow{X^{3} Y+X} & X^{5} Y+a_{1} X^{3} Y^{2}+a_{2} X^{3} Y^{2}+a_{3} X^{4} Y+a_{4} X^{2} Y^{2}+a_{5} X^{3} Y \\
& +a_{6} X Y^{2}+a_{7} Y^{2}+X^{3}+a_{1} X Y+a_{3} X^{2}+a_{5} X \\
\xrightarrow{F(X, Y)} & a_{2} X^{6}+a_{4} X^{5}+a_{6} X^{4}+a_{7} X^{3}+a_{2} X^{3} Y^{2}+a_{4} X^{2} Y^{2} \\
& +a_{6} X Y^{2}+a_{7} Y^{2}+X^{3}+a_{1} X Y+a_{3} X^{2}+a_{5} X .
\end{aligned}
$$

If $a_{2} \neq 0$ then

$$
\left\{X^{6}, X^{7}\right\} \subset \square_{\prec w}(F)
$$

Hence, assume $a_{2}=0$, in which case we have

$$
\begin{aligned}
& a_{4} X^{5}+a_{6} X^{4}+a_{7} X^{3}+a_{4} X^{2} Y^{2}+a_{6} X Y^{2}+a_{7} Y^{2}+X^{3} \\
& +a_{1} X Y+a_{3} X^{2}+a_{5} X \\
\stackrel{F(X, Y)}{\longrightarrow} & a_{4} X^{5}+a_{6} X^{4}+a_{7} X^{3}+a_{4} Y\left(a_{1} Y^{2}+a_{3} X Y+a_{4} X^{2}\right. \\
& \left.+a_{5} Y+a_{6} X+a_{7}\right)+a_{6} X Y^{2}+a_{7} Y^{2}+X^{3}+a_{1} X Y+a_{3} X^{2}+a_{5} X .
\end{aligned}
$$

If $a_{4} \neq 0$ then

$$
\left\{X^{5}, X^{6}, X^{7}\right\} \subset \square_{\prec w}(F)
$$

Hence, assume $a_{4}=0$ and continuer

$$
\begin{aligned}
& X\left(a_{6} X^{4}+a_{7} X^{3}+a_{6} X Y^{2}+a_{7} Y^{2}+X^{3}+a_{1} X Y+a_{3} X^{2}+a_{5} X\right) \\
F(X, Y) & a_{6} X^{5}+a_{7} X^{4}+a_{6} Y\left(a_{1} Y^{2}+a_{3} X Y+a_{5} Y+a_{6} X+a_{7}\right) \\
& +a_{7} X Y^{2}+X^{4}+a_{1} X^{2} Y+a_{3} X^{3}+a_{5} X^{2} .
\end{aligned}
$$

If $a_{6} \neq 0$ then

$$
\left\{X^{5}, X^{6}, X^{7}\right\} \subset \square_{\prec_{w}}(F) .
$$

Hence, assume $a_{6}=0$, in which case we have

$$
\begin{aligned}
& X\left(a_{7} X^{4}+a_{7} X Y^{2}+X^{4}+a_{1} X^{2} Y+a_{3} X^{3}+a_{5} X^{2}\right) \\
F(X, Y) & \left(a_{7}+1\right) X^{5}+a_{7} Y\left(a_{1} Y^{2}+a_{3} X Y+a_{5} Y+a_{7}\right) \\
& +a_{1} X^{3} Y+a_{3} X^{4}+a_{5} X^{3} .
\end{aligned}
$$

If $a_{7} \neq 1$ then

$$
\left\{X^{5}, X^{6}, X^{7}\right\} \subset \square_{\prec_{w}}(F) .
$$

Hence, assume $a_{7}=1$ and continue the reduction

$$
\xrightarrow{Y^{3}+X^{3} Y+X} \begin{aligned}
& a_{1} Y^{3}+a_{3} X Y^{2}+a_{5} Y^{2}+Y+a_{1} X^{3} Y+a_{3} X^{4}+a_{5} X^{3} \\
& a_{3} X Y^{2}+a_{5} Y^{2}+Y+a_{3} X^{4}+a_{5} X^{3}+a_{1} X
\end{aligned}
$$

which we multiply by $X$ before continuing reduction

$$
\begin{aligned}
& a_{3} X^{2} Y^{2}+a_{5} X Y^{2}+X Y+a_{3} X^{5}+a_{5} X^{4}+a_{1} X^{2} \\
\stackrel{F(X, Y)}{ } & a_{3} Y\left(a_{1} Y^{2}+a_{3} X Y+a_{5} Y+1\right) \\
& +a_{5} X Y^{2}+X Y+a_{3} X^{5}+a_{5} X^{4}+a_{1} X^{2} .
\end{aligned}
$$

If $a_{3} \neq 0$ then

$$
\left\{X^{5}, X^{6}, X^{7}\right\} \subset \square_{\prec_{w}}(F) .
$$

Hence, assume $a_{3}=0$ and continue

$$
\begin{aligned}
& X\left(a_{5} X Y^{2}+X Y+a_{5} X^{4}+a_{1} X^{2}\right) \\
\stackrel{F(X, Y)}{\longrightarrow} & a_{5}\left(a_{1} Y^{3}+a_{5} Y^{2}+Y\right)+a_{1} Y^{2}+a_{5} Y+1+a_{5} X^{5}+a_{1} X^{3} .
\end{aligned}
$$

If $a_{5} \neq 0$ then

$$
\left\{X^{5}, X^{6}, X^{7}\right\} \subset \square_{\prec_{w}}(F) .
$$

Hence, assume $a_{5}=0$ and multiply the resulting expression by $Y$

$$
\begin{aligned}
& Y\left(a_{1} Y^{2}+a_{1} X^{3}+1\right) \\
& Y^{3}+\xrightarrow{X^{3} Y+X} Y+a_{1} X
\end{aligned}
$$

and we conclude

$$
\left\{Y, Y^{2}, X Y, X Y^{2}\right\} \subset \square_{\prec w}(F) .
$$

In conclusion $w_{H}(\boldsymbol{c}) \geq 10+\min \{2,3,3,3,3,3,4\}=12$, and if $a_{2}=0$ then $w_{H}(\boldsymbol{c}) \geq 13$.

### 3.5 Leading monomial equal to $X Y^{2}$

Consider $\boldsymbol{c}=\operatorname{ev}\left(F+I_{8}\right)$ where

$$
\begin{aligned}
F(X, Y)= & X Y^{2}+a_{1} X^{4}+a_{2} X^{2} Y+a_{3} Y^{2}+a_{4} X^{3} \\
& +a_{5} X Y+a_{6} X^{2}+a_{7} Y+a_{8} X+a_{9} .
\end{aligned}
$$

For sure

$$
\begin{equation*}
\left\{X Y^{2}, X^{2} Y^{2}, X^{3} Y^{2}, X^{4} Y^{2}, X^{5} Y^{2}, X^{6} Y^{2}\right\} \subset \square_{\prec w}(F) . \tag{9}
\end{equation*}
$$

We next consider an exhaustive series of conditions under which we establish more monomials in$\prec_{w}(F)$. We have

$$
\begin{align*}
& Y F(X, Y) \\
& Y^{3}+X^{3} Y+X \\
& \left(a_{1}+1\right) X^{4} Y+a_{2} X^{2} Y^{2}+a_{3} Y^{3}+a_{4} X^{3} Y+a_{5} X Y^{2}  \tag{10}\\
& +a_{6} X^{2} Y+a_{7} Y^{2}+a_{8} X Y+X^{2}+a_{9} Y .
\end{align*}
$$

If $a_{1} \neq 1$ then

$$
\begin{equation*}
\left\{X^{4} Y, X^{5} Y, X^{6} Y\right\} \subset \square_{\prec w}(F) . \tag{11}
\end{equation*}
$$

Continuing the calculations for this case we obtain

$$
\begin{aligned}
& Y\left(\left(a_{1}+1\right) X^{4} Y+a_{2} X^{2} Y^{2}+a_{3} Y^{3}+a_{4} X^{3} Y+a_{5} X Y^{2}+a_{6} X^{2} Y\right. \\
& \left.+a_{7} Y^{2}+a_{8} X Y+X^{2}+a_{9} Y\right) \\
\stackrel{F(X, Y)}{\longrightarrow} & (a+1)\left(a_{1} X^{7}+a_{2} X^{5} Y+a_{3} X^{3} Y^{2}+a_{4} X^{6}+a_{5} X^{4} Y+a_{6} X^{5}\right. \\
& \left.+a_{7} X^{3} Y+a_{8} X^{4}+a_{9} X^{3}\right)+a_{2} X^{2} Y^{3}+a_{3} Y^{4}+a_{4} X^{3} Y^{2}+a_{5} X Y^{3} \\
& +a_{6} X^{2} Y^{2}+a_{7} Y^{3}+a_{8} X Y^{2}+X^{2} Y+a_{9} Y^{2} .
\end{aligned}
$$

If $a_{1} \neq 0$ then we also have

$$
\left\{X^{7}\right\} \subset \square_{\prec_{w}}(F) .
$$

Assuming next that $a_{1}=0$ the above expression becomes

$$
\begin{aligned}
& a_{2} X^{5} Y+a_{3} X^{3} Y^{2}+a_{4} X^{6}+a_{5} X^{4} Y+a_{6} X^{5}+a_{7} X^{3} Y+a_{8} X^{4} \\
& +a_{9} X^{3}+a_{2} X^{2} Y^{3}+a_{3} Y^{4}+a_{4} X^{3} Y^{2} \\
& +a_{5} X Y^{3}+a_{6} X^{2} Y^{2}+a_{7} Y^{3}+a_{8} X Y^{2}+X^{2} Y+a_{9} Y^{2} \\
Y^{3}+X^{3} Y+X & a_{4} X^{6}+a_{6} X^{5}+a_{8} X^{4}+a_{9} X^{3}+a_{3} X Y+a_{4} X^{3} Y^{2} \\
\xrightarrow{F(X, Y)} & +a_{6} X^{2} Y^{2}+a_{8} X Y^{2}+X^{2} Y+a_{9} Y^{2}+a_{2} X^{3}+a_{5} X^{2}+a_{7} X \\
& a_{4} X^{6}+a_{6} X^{5}+a_{8} X^{4}+a_{9} X^{3}+a_{3} X Y \\
& +\left(a_{4} X^{2}+a_{6} X+a_{8}\right)\left(a_{2} X^{2} Y+a_{3} Y^{2}+a_{4} X^{3}+a_{5} X Y+a_{6} X^{2}\right. \\
& \left.+a_{7} Y+a_{8} X+a_{9}\right)+X^{2} Y+a_{9} Y^{2}+a_{2} X^{3}+a_{5} X^{2}+a_{7} X .
\end{aligned}
$$

If $a_{4} \neq 0$ then

$$
\left\{X^{6}, X^{7}\right\} \subset \square_{\prec_{w}}(F) .
$$

Hence, we assume $a_{4}=0$. From the above expression we see that if next $a_{6} \neq 0$ then

$$
\left\{X^{5}, X^{6}, X^{7}\right\} \subset \square_{\prec_{w}}(F)
$$

Hence, assume $a_{6}=0$. Investigating again the above expression we now see that for $a_{8} \neq 0$ it holds that

$$
\left\{X^{4}, X^{5}, X^{6}, X^{7}\right\} \subset \square_{\prec w} .
$$

Continuing from the same expression, but now under the assumption that $a_{8}=0$ we see that

$$
\left\{X^{2} Y, X^{3} Y\right\} \subset \square_{\prec w}(F) .
$$

In conclusion, for the case $a_{1} \neq 1$ we have in addition to (9) and (11) established at least one more element in $\square_{\prec_{w}}(F)$. That is, in addition to
(9) we have at least 4 elements in $\square_{\prec_{w}}(F)$. Furthermore, if $a_{1} \neq 1$ and $a_{1} \neq 0$ then we have at least one more element in addition in this set.

In the following we assume $a_{1}=1$ and continue the calculations from (10) as follows

$$
\begin{aligned}
Y^{3}+X^{3} Y+X & a_{2} X^{2} Y^{2}+\left(a_{3}+a_{4}\right) X^{3} Y+a_{5} X Y^{2}+a_{6} X^{2} Y \\
& +a_{7} Y^{2}+a_{8} X Y+X^{2}+a_{9} Y+a_{3} X . \\
\xrightarrow{F(X, Y)} & a_{2} X^{5}+\left(a_{2}^{2}+a_{3}+a_{4}\right) X^{3} Y+\left(a_{2} a_{3}+a_{5}\right) X Y^{2}+a_{2} a_{4} X^{4} \\
& +\left(a_{2} a_{5}+a_{6}\right) X^{2} Y+a_{7} Y^{2}+a_{2} a_{6} X^{3}+\left(a_{2} a_{7}+a_{8}\right) X Y \\
& +\left(a_{2} a_{8}+1\right) X^{2}+a_{9} Y+\left(a_{2} a_{9}+a_{3}\right) X .
\end{aligned}
$$

If $a_{2} \neq 0$ then

$$
\left\{X^{5}, X^{5} Y, X^{6}, X^{6} Y, X^{7}\right\} \subset \square_{\prec w}(F) .
$$

Hence, assume $a_{2}=0$. But then if $a_{3} \neq a_{4}$ we get

$$
\begin{equation*}
\left\{X^{3} Y, X^{4} Y, X^{5} Y, X^{6} Y\right\} \subset \square_{\prec w}(F) . \tag{12}
\end{equation*}
$$

Multiplying the above polynomial by $Y$ and continuing the reduction we obtain:

$$
\begin{aligned}
& \left(a_{3}+a_{4}\right) X^{3} Y^{2}+a_{5} X Y^{3}+a_{6} X^{2} Y^{2}+a_{7} Y^{3} \\
& a_{8} X Y^{2}+X^{2} Y+a_{9} Y^{2}+a_{3} X Y \\
\stackrel{F(X, Y)}{ } & \left(\left(a_{3}+a_{4}\right) X^{2}+a_{5} Y+a_{6} X+a_{8}\right) \\
& \left(X^{4} a_{3} Y^{2}+a_{4} X^{3}+a_{5} X Y+a_{6} X^{2}+a_{7} Y+a_{8} X+a_{9}\right) \\
& +a_{7} Y^{3}+a_{8} X Y^{2}+X^{2} Y+a_{9} Y^{2}+a_{3} X Y
\end{aligned}
$$

implying that

$$
\left\{X^{6}, X^{7}\right\} \subset \square_{\prec_{w}}(F)
$$

Hence, for the case $a_{1}=1, a_{2}=0, a_{3} \neq a_{4}$ in addition to (9) we found 6 more elements in $\square_{\prec_{w}}(F)$. Namely, the above 2 and the 4 in (12).

In the following we assume $a_{3}=a_{4}$. Continuing the reduction we obtain

$$
\begin{align*}
\stackrel{F(X, Y)}{\longrightarrow} & a_{5} X^{4}+a_{6} X^{2} Y+\left(a_{4} a_{5}+a_{7}\right) Y^{2}+a_{4} a_{5} X^{3}+\left(a_{5}^{2}+a_{8}\right) X Y \\
& +\left(a_{5} a_{6}+1\right) X^{2}+\left(a_{5} a_{7}+a_{9}\right) Y+\left(a_{5} a_{8}+a_{4}\right) X+a_{5} a_{9} . \tag{13}
\end{align*}
$$

If $a_{5} \neq 0$ then

$$
\left\{X^{4}, X^{4} Y, X^{5}, X^{5} Y, X^{6}, X^{6} Y, X^{7}\right\} \subset \square_{\prec w}(F) .
$$

Hence, we next assume $a_{5}=0$. If then $a_{6} \neq 0$ we obtain

$$
\left\{X^{2} Y, X^{3} Y, X^{4} Y, X^{5} Y, X^{6} Y\right\} \subset \square_{\prec_{w}}(F),
$$

and we therefore now assume $a_{6}=0$. We next multiply the considered polynomial by $X$ and continue the reduction

$$
\begin{aligned}
& a_{7} X Y^{2}+a_{8} X^{2} Y+X^{3}+a_{9} X Y+a_{4} X^{2} \\
& F(X, Y) \\
& a_{7} X^{4}+a_{8} X^{2} Y+a_{4} a_{7} Y^{2}+\left(a_{4} a_{7}+1\right) X^{3} \\
&+a_{9} X Y+a_{4} X^{2}+a_{7}^{2} Y+a_{7} a_{8} X+a_{7} a_{9} .
\end{aligned}
$$

If $a_{7} \neq 0$ then

$$
\left\{Y^{2}, X^{4}, X^{4} Y, X^{5}, X^{5} Y, X^{6}, X^{6} Y, X^{7}\right\} \subset \square_{\prec_{w}}(F) .
$$

Here - although it has no implication for what we want to prove - we used (13) to demonstrate that $Y^{2}$ is also in the set. Hence, assume now that $a_{7}=0$. Then if $a_{8} \neq 0$ we obtain

$$
\left\{X^{2} Y, X^{3} Y, X^{4} Y, X^{5} Y, X^{6} Y\right\} \subset \square_{\prec w}(F)
$$

Finally, if $a_{8}=0$ the leading monomial becomes $X^{3}$ and we therefore have

$$
\left\{X^{3}, X^{3} Y, X^{4}, X^{4} Y, X^{5}, X^{5} Y, X^{6}, X^{6} Y, X^{7}\right\} \subset \square_{\prec_{w}}(F) .
$$

In conclusion we have established the existence of at least $6+\min \{4,5,6,7,5,8,5\}=$ 10 elements in $\square_{\prec_{w}}(F)$, and therefore $w_{H}(\boldsymbol{c}) \geq 10$. Moreover, by inspection of the results in the present section we see that $w_{H}(\boldsymbol{c}) \geq 6+5=11$ holds when $a_{1} \in\{0,1\}$.

### 3.6 Leading monomial equal to $X^{3} Y$

Consider $\boldsymbol{c}=\operatorname{ev}\left(F+I_{8}\right)$ where

$$
\begin{aligned}
F(X, Y)= & X^{3} Y+a_{1} X Y^{2}+a_{2} X^{4}+a_{3} X^{2} Y+a_{4} Y^{2} \\
& +a_{5} X^{3}+a_{6} X Y+a_{7} X^{2}+a_{8} Y+a_{9} X+a_{10} .
\end{aligned}
$$

For sure

$$
\left\{X^{3} Y, X^{4} Y, X^{5} Y, X^{6} Y, X^{3} Y^{2}, X^{4} Y^{2}, X^{5} Y^{2}, X^{6} Y^{2}\right\} \subset \square_{\prec_{w}}(F)
$$

We next consider an exhaustive series of conditions under which we establish more monomials in $\square_{\prec w}(F)$. The strategy in this subsection is different from other sections in that we here do not reduce modulo $F(X, Y)$ but instead in addition to reducing modulo $Y^{3}+X^{3} Y+X$ also reduce modulo the polynomials $X^{8}+X, X^{7} Y+Y \in\left\langle Y^{3}+X^{3} Y+X, Y^{8}+Y, X^{8}+X\right\rangle$.

We start by multiplying $F(X, Y)$ by $X^{7}$ to obtain

$$
\begin{aligned}
& X^{10} Y+a_{1} X^{8} Y^{2}+a_{2} X^{11}+a_{3} X^{9} Y+a_{4} X^{7} Y^{2}+a_{5} X^{10}+a_{6} X^{8} Y \\
& +a_{7} X^{9}+a_{8} X^{7} Y+a_{9} X^{8}+a_{10} X^{7} \\
\xrightarrow{X^{8}+X} & X^{3} Y+a_{1} X Y^{2}+a_{2} X^{4}+a_{3} X^{2} Y+a_{4} X^{7} Y^{2} \\
& +a_{5} X^{3}+a_{6} X Y+a_{7} X^{2}+a_{8} X^{7} Y+a_{9} X+a_{10} X^{7} \\
\text { X}^{7} Y+Y & X^{3} Y+a_{1} X Y^{2}+a_{2} X^{4}+a_{3} X^{2} Y+a_{4} Y^{2} \\
& +a_{5} X^{3}+a_{6} X Y+a_{7} X^{2}+a_{8} Y+a_{5} X+a_{10} X^{7} .
\end{aligned}
$$

If $a_{10} \neq 0$ then

$$
\left\{X^{7}\right\} \subset \square_{\prec_{w}}(F) .
$$

Hence, assume $a_{10}=0$ and multiply the resulting expression by $Y^{2}$ to obtain

$$
\begin{aligned}
& X^{3} Y^{3}+a_{1} X Y^{4}+a_{2} X^{4} Y^{2}+a_{3} X^{2} Y^{3}+a_{4} Y^{4} \\
& +a_{5} X^{3} Y^{2}+a_{6} X Y^{3}+a_{7} X^{2} Y^{2}+a_{8} Y^{3}+a_{9} X Y^{2} \\
Y^{3}+X^{3} Y+X & X^{6} Y+X^{4}+a_{1} X^{4} Y^{2}+a_{1} X^{2} Y+a_{2} X^{4} Y^{2}+a_{3} X^{5} Y \\
& +a_{3} X^{3}+a_{4} X^{3} Y^{2}+a_{4} X Y+a_{5} X^{3} Y^{2}+a_{6} X^{4} Y+a_{6} X^{2} \\
& +a_{7} X^{2} Y^{2}+a_{8} X^{3} Y+a_{8} X+a_{9} X Y^{2}
\end{aligned}
$$

which we multiply by $X^{6}$ to obtain

$$
\begin{aligned}
& X^{12} Y+X^{10}+a_{1} X^{10} Y^{2}+a_{1} X^{8} Y+a_{2} X^{10} Y^{2}+a_{3} X^{11} Y+a_{3} X^{9} \\
& +a_{4} X^{9} Y^{2}+a_{4} X^{7} Y+a_{5} X^{9} Y^{2}+a_{6} X^{10} Y+a_{6} X^{8}+a_{7} X^{8} Y^{2} \\
& +a_{8} X^{9} Y+a_{8} X^{7}+a_{9} X^{7} Y^{2} \\
X^{7} Y+Y & \cdots \\
\xrightarrow{X^{8}+X} & X^{5} Y+X^{3}+a_{1} X^{3} Y^{2}+a_{1} X Y+a_{2} X^{3} Y^{2}+a_{3} X^{4} Y+a_{3} X^{2} \\
& +a_{4} X^{2} Y^{2}+a_{4} Y+a_{5} X^{2} Y^{2}+a_{6} X^{3} Y+a_{6} X+a_{7} X Y^{2}+a_{8} X^{2} Y \\
& +a_{8} X^{7}+a_{9} Y^{2} .
\end{aligned}
$$

If $a_{8} \neq 0$ then

$$
\left\{X^{7}\right\} \subset \square_{\prec_{w}}(F) .
$$

Hence, assume $a_{8}=0$. We next multiply $F(X, Y)$ by $Y$ and obtain

$$
\begin{aligned}
& X^{3} Y^{2}+a_{1} X Y^{3}+a_{2} X^{4} Y+a_{3} X^{2} Y^{2}+a_{4} Y^{3}+a_{5} X^{3} Y \\
& +a_{6} X Y^{2}+a_{7} X^{2} Y+a_{9} X Y \\
Y^{3}+X^{3} Y+X & X^{3} Y^{2}+a_{1} X^{4} Y+a_{1} X^{2}+a_{2} X^{4} Y+a_{3} X^{2} Y^{2} \\
& +a_{4} X^{3} Y+a_{4} X+a_{5} X^{3} Y+a_{6} X Y^{2}+a_{7} X^{2} Y+a_{9} X Y
\end{aligned}
$$

which we multiply by $X^{6}$ to obtain

$$
\begin{aligned}
& X^{9} Y^{2}+a_{1} X^{10} Y+a_{1} X^{8}+a_{2} X^{10} Y+a_{3} X^{8} Y^{2} \\
& +a_{4} X^{9} Y+a_{4} X^{7}+a_{5} X^{9} Y+a_{6} X^{7} Y^{2}+a_{7} X^{8} Y+a_{9} X^{7} Y \\
X^{7} Y Y Y & X^{2} Y^{2}+a_{1} X^{3} Y+a_{1} X+a_{2} X^{3} Y+a_{3} X Y^{2} \\
& +a_{4} X^{2} Y+a_{4} X^{7}+a_{5} X^{2} Y+a_{6} Y^{2}+a_{7} X Y+a_{9} Y .
\end{aligned}
$$

If $a_{4} \neq 0$ then

$$
\left\{X^{7}\right\} \subset \square_{\prec_{w}}(F) .
$$

Hence, assume $a_{4}=0$. We next multiply $F(X, Y)$ by $X^{6}$ to obtain

$$
\begin{aligned}
& X^{9} Y+a_{1} X^{7} Y^{2}+a_{2} X^{10}+a_{3} X^{8} Y+a_{5} X^{9} \\
& +a_{6} X^{7} Y+a_{7} X^{8}+a_{9} X^{7} \\
X^{7} Y+Y & \ldots \\
\xrightarrow{X^{8}+X} & X^{2} Y+a_{1} Y^{2}+a_{2} X^{3}+a_{3} X Y+a_{5} X^{2}+a_{6} Y+a_{7} X+a_{9} X^{7} .
\end{aligned}
$$

If $a_{9} \neq 0$ then

$$
\left\{X^{7}\right\} \subset \square_{\prec_{w}}(F) .
$$

Hence, assume $a_{9}=0$ and multiply by $Y^{2}$

$$
\begin{aligned}
& X^{2} Y^{3}+a_{1} Y^{4}+a_{2} X^{3} Y^{2}+a_{3} X Y^{3}+a_{5} X^{2} Y^{2}+a_{6} Y^{3}+a_{7} X Y^{2} \\
& Y^{3}+X^{3} Y+X X^{5} Y+X^{3}+a_{1} X^{3} Y^{2}+a_{1} X Y+a_{2} X^{3} Y^{2}+a_{3} X^{4} Y \\
& +a_{3} X^{2}+a_{5} X^{2} Y^{2}+a_{6} X^{3} Y+a_{6} X+a_{7} X Y^{2}
\end{aligned}
$$

which we then multiply by $X^{6}$ to obtain

$$
\begin{aligned}
& X^{11} Y+X^{9}+a_{1} X^{9} Y^{2}+a_{1} X^{7} Y+a_{2} X^{9} Y^{2}+a_{3} X^{10} Y+a_{3} X^{8} \\
& +a_{5} X^{8} Y^{2}+a_{6} X^{9} Y+a_{6} X^{7}+a_{7} X^{7} Y^{2} \\
\xrightarrow{X^{7} Y+Y} & \cdots \\
\xrightarrow{X^{8}+X} & X^{4} Y+X^{2}+a_{1} X^{2} Y^{2}+a_{1} Y+a_{2} X^{2} Y^{2}+a_{3} X^{3} Y+a_{3} X \\
& +a_{5} X Y^{2}+a_{6} X^{2} Y+a_{6} X^{7}+a_{7} Y^{2} .
\end{aligned}
$$

If $a_{6} \neq 0$ then

$$
\left\{X^{7}\right\} \subset \square_{\prec w}(F) .
$$

Hence, assume $a_{6}=0$. We next multiply $F(X, Y)$ by $Y$ and continue the reductions:

$$
\begin{aligned}
& X^{3} Y^{2}+a_{1} X Y^{3}+a_{2} X^{4} Y+a_{3} X^{2} Y^{2}+a_{5} X^{3} Y+a_{7} X^{2} Y \\
& Y^{3}+X^{3} Y+X X^{3} Y^{2}+a_{1} X^{4} Y+a_{1} X^{2}+a_{2} X^{4} Y+a_{3} X^{2} Y^{2}+a_{5} X^{3} Y+a_{7} X^{2} Y
\end{aligned}
$$

which we multiply by $X^{5}$

$$
\begin{aligned}
& X^{8} Y^{2}+a_{1} X^{9} Y+a_{1} X^{7}+a_{2} X^{9} Y+a_{3} X^{7} Y^{2}+a_{5} X^{8} Y+a_{7} X^{7} Y \\
& \xrightarrow{X^{7} Y+Y} X Y^{2}+a_{1} X^{2} Y+a_{1} X^{7}+a_{2} X^{2} Y+a_{3} Y^{2}+a_{5} X Y+a_{7} Y .
\end{aligned}
$$

If $a_{1} \neq 0$ then

$$
\left\{X^{7}\right\} \subset \square_{\prec w}(F) .
$$

Hence, assume $a_{1}=0$. We next multiply $F(X, Y)$ by $X^{5}$

$$
\begin{aligned}
& X^{8} Y+a_{2} X^{9}+a_{3} X^{7} Y+a_{5} X^{8}+a_{7} X^{7} \\
\xrightarrow{X^{7} Y+Y} & \\
\xrightarrow{X^{8}+X} & X Y+a_{2} X^{2}+a_{3} Y+a_{5} X+a_{7} X^{7} .
\end{aligned}
$$

If $a_{7} \neq 0$ then

$$
\left\{X^{7}\right\} \subset \square_{\prec_{w}}(F) .
$$

Hence, assume $a_{7}=0$. Next we multiply $F(X, Y)$ by $Y^{2}$ and obtain

$$
\begin{aligned}
& X^{3} Y^{3}+a_{2} X^{4} Y^{2}+a_{3} X^{2} Y^{3}+a_{5} X^{3} Y^{2} \\
& \xrightarrow{Y^{3}+X^{3} Y}{ }^{X} X^{6} Y+X^{4}+a_{2} X^{4} Y^{2}+a_{3} X^{5} Y+a_{3} X^{3}+a_{5} X^{3} Y^{2}
\end{aligned}
$$

which we multiply by $X^{4}$

$$
\begin{aligned}
& X^{10} Y+X^{8}+a_{2} X^{8} Y^{2}+a_{3} X^{9} Y+a_{3} X^{7}+a_{5} X^{7} Y^{2} \\
\xrightarrow{X^{7} Y+Y} & \\
\xrightarrow{X^{8}+X} & X^{3} Y+X+a_{2} X Y^{2}+a_{3} X^{2} Y+a_{3} X^{7}+a_{5} Y^{2} .
\end{aligned}
$$

If $a_{3} \neq 0$ then

$$
\left\{X^{7}\right\} \subset \square_{\prec w}(F)
$$

Hence, assume $a_{3}=0$. We now multiply $F(X, Y)$ by $X^{4}$

$$
\begin{aligned}
& \quad X^{7} Y+a_{2} X^{8}+a_{5} X^{7} \\
& \xrightarrow{X^{7} Y+Y} \ldots \\
& \xrightarrow{X^{8}+x} Y+a_{2} X+a_{5} X^{7} .
\end{aligned}
$$

If $a_{5} \neq 0$ then

$$
\left\{X^{7}\right\} \subset \square_{\prec_{w}} .
$$

Hence, assume finally that $a_{5} \neq 0$ and multiply $F(X, Y)$ by $Y^{2}$ to obtain

$$
\begin{gathered}
X^{3} Y^{3}+a_{2} X^{4} Y^{2} \\
Y^{3}+\xrightarrow{X^{3} Y}+X X^{6} Y+X^{4}+a_{2} X^{4} Y^{2} .
\end{gathered}
$$

This expression is then multiplied by $X^{3}$

$$
\begin{aligned}
& X^{9} Y+X^{7}+a_{2} X^{7} Y^{2} \\
X^{7} Y+Y & X^{2} Y+X^{7}+a_{2} Y^{2}
\end{aligned}
$$

and

$$
\left\{X^{7}\right\} \subset \square_{\prec_{w}} \subset \square_{\prec_{w}}(F) .
$$

In conclusion $w_{H}(\boldsymbol{c}) \geq 8+1=9$.

### 3.7 Leading monomial equal to $X^{2} Y^{2}$

Consider $\boldsymbol{c}=\operatorname{ev}\left(F+I_{8}\right)$ where

$$
\begin{aligned}
F(X, Y)= & X^{2} Y^{2}+a_{1} X^{5}+a_{2} X^{3} Y+a_{3} X Y^{2}+a_{4} X^{4}+a_{5} X^{2} Y \\
& +a_{6} Y^{2}+a_{7} X^{3}+a_{8} X Y+a_{9} X^{2}+a_{10} Y+a_{11} X+a_{12} .
\end{aligned}
$$

For sure

$$
\left\{X^{2} Y^{2}, X^{3} Y^{2}, X^{4} Y^{2}, X^{5} Y^{2}, X^{6} Y^{2}\right\} \subset \square_{\prec w}(F)
$$

We next consider an exhaustive series of conditions under which we establish more monomials in $\square_{\prec_{w}}(F)$. We have

$$
\begin{array}{rl}
Y^{3}+X^{3} Y+X & Y F(X, Y) \\
& \left(1+a_{1}\right) X^{5} Y+a_{2} X^{3} Y^{2}+a_{3} X Y^{3}+a_{4} X^{4} Y+a_{5} X^{2} Y^{2} \\
& +a_{6} Y^{3}+a_{7} X^{3} Y+a_{8} X Y^{2}+a_{9} X^{2} Y+a_{10} Y^{2}+X^{3} \\
& +a_{11} X Y+a_{12} Y .
\end{array}
$$

If $a_{1} \neq 1$ then

$$
\left\{X^{5} Y, X^{6} Y\right\} \subset \square_{\prec_{w}}(F)
$$

Hence, assume $a_{1}=1$ and continue the reduction.

$$
\begin{aligned}
Y^{3}+X^{3} Y+X & a_{2} X^{3} Y^{2}+\left(a_{3}+a_{4}\right) X^{4} Y+a_{5} X^{2} Y^{2}+\left(a_{6}+a_{7}\right) X^{3} Y+a_{8} X Y^{2} \\
& +a_{9} X^{2} Y+a_{10} Y^{2}+X^{3}+a_{11} X Y+a_{3} X^{2}+a_{12} Y+a_{6} X \\
\xrightarrow{F(X, Y)} & a_{2} X^{6}+\left(a_{3}+a_{4}+a_{2}^{2}\right) X^{4} Y+\left(a_{2} a_{3}+a_{5}\right) X^{2} Y^{2}+a_{2} a_{4} X^{5} \\
& +\left(a_{2} a_{5}+a_{6}+a_{7}\right) X^{3} Y+\left(a_{2} a_{6}+a_{8}\right) X Y^{2}+a_{2} a_{7} X^{4} \\
& +\left(a_{2} a_{8}+a_{9}\right) X^{2} Y+a_{10} Y^{2}+\left(a_{2} a_{9}+1\right) X^{3}+\left(a_{2} a_{10}+a_{11}\right) X Y \\
& +\left(a_{2} a_{11}+a_{3}\right) X^{2}+a_{12} Y+\left(a_{2} a_{12}+a_{6}\right) X .
\end{aligned}
$$

If $a_{2} \neq 0$ then

$$
\left\{X^{6}, X^{6} Y, X^{7}\right\} \subset \square_{\prec}(F) .
$$

Hence, assume $a_{2}=0$. If $a_{3} \neq a_{4}$ then we have

$$
\left\{X^{4} Y, X^{5} Y, X^{6} Y\right\} \subset \square_{\prec_{w}}(F) .
$$

Assuming $a_{3}=a_{4}$ we continue the reduction as follows

$$
\begin{aligned}
\stackrel{F(X, Y)}{\longrightarrow} & a_{5} X^{5}+\left(a_{6}+a_{7}\right) X^{3} Y+\left(a_{4} a_{5}+a_{8}\right) X Y^{2}+a_{4} a_{5} X^{4}+\left(a_{5}^{2}+a_{9}\right) X^{2} Y \\
& +\left(a_{5} a_{6}+a_{10}\right) Y^{2}+\left(a_{5} a_{7}+1\right) X^{3}+\left(a_{5} a_{8}+a_{11}\right) X Y+\left(a_{5} a_{9}+a_{4}\right) X^{2} \\
& +\left(a_{5} a_{10}+a_{12}\right) Y+\left(a_{5} a_{11}+a_{6}\right) X+a_{5} a_{12} .
\end{aligned}
$$

If $a_{5} \neq 0$ then

$$
\left\{X^{5}, X^{5} Y, X^{6}, X^{6} Y, X^{7}\right\} \subset \square_{\prec w}(F) .
$$

Hence, assume $a_{5}=0$. But then if $a_{6} \neq a_{7}$

$$
\left\{X^{3} Y, X^{4} Y, X^{5} Y, X^{6} Y\right\} \subset \square_{\prec_{w}}(F),
$$

and we therefore next assume $a_{6}=a_{7}$. We now multiply the above polynomial by $X$ and continue the reduction

$$
\begin{aligned}
& X\left(a_{8} X Y^{2}+a_{9} X^{2} Y+a_{10} Y^{2}+X^{3}+a_{11} X Y+a_{4} X^{2}+a_{12} Y+a_{6} X\right) \\
F(X, Y) & a_{8} X^{5}+a_{9} X^{3} Y+\left(a_{4} a_{8}+a_{10}\right) X Y^{2}+\left(a_{4} a_{8}+1\right) X^{4}+a_{11} X^{2} Y \\
& +a_{7} a_{8} Y^{2}+\left(a_{4}+a_{7} a_{8}\right) X^{3}+\left(a_{8}^{2}+a_{12}\right) X Y+\left(a_{6}+a_{8} a_{9}\right) X^{2} \\
& +a_{8} a_{10} Y+a_{8} a_{11} X+a_{8} a_{12} .
\end{aligned}
$$

If $a_{8} \neq 8$ then

$$
\left\{X^{5}, X^{5} Y, X^{6}, X^{6} Y, X^{7}\right\} \subset \square_{\prec w}(F),
$$

and if $a_{8}=0$ but $a_{9} \neq 0$ then

$$
\left\{X^{3} Y, X^{4} Y, X^{5} Y, X^{6} Y\right\} \subset \square_{\prec_{w}}(F) .
$$

Hence, assume $a_{8}=a_{9}=0$ and multiply the resulting polynomial by $X$ after which we continue the reduction.

$$
\begin{aligned}
& X\left(a_{10} X Y^{2}+X^{4}+a_{11} X^{2} Y+a_{4} X^{3}+a_{12} X Y+a_{6} X^{2}\right) \\
& F(X, Y) \\
& \left(a_{10}+1\right) X^{5}+a_{11} X^{3} Y+a_{4} a_{10} X Y^{2}+\left(a_{4}+a_{4} a_{10}\right) X^{4}+a_{12} X^{2} Y \\
& +a_{7} a_{10} Y^{2}+\left(a_{7} a_{10}+a_{7}\right) X^{2}+a_{10}^{2} Y+a_{10} a_{11} X+a_{10} a_{12} .
\end{aligned}
$$

If $a_{10} \neq 1$ then

$$
\left\{X^{5}, X^{5} Y, X^{6}, X^{6} Y, X^{7}\right\} \subset \square_{\prec w}(F)
$$

Hence, assume $a_{10}=1$. If $a_{11} \neq 0$ then

$$
\left\{X^{3} Y, X^{4} Y, X^{5} Y, X^{6} Y\right\} \subset \square_{\prec w}(F) .
$$

Hence assume $a_{11}=0$ and multiply the resulting polynomial by $X$ and continue the reduction

$$
\xrightarrow{F(X, Y)} \begin{aligned}
& X\left(a_{4} X Y^{2}+a_{12} X^{2} Y+a_{7} Y^{2}+Y+a_{12}\right) \\
& a_{4} X^{5}+a_{12} X^{3} Y+\left(a_{4}^{2}+a_{7}\right) X Y^{2}+a_{4}^{2} X^{4}+a_{4} a_{7} Y^{2} \\
& +a_{4} a_{7} X^{3}+X Y+a_{4} Y+a_{12} X+a_{4} a_{12} .
\end{aligned}
$$

If $a_{4} \neq 0$ then

$$
\left\{X^{5}, X^{5} Y, X^{6}, X^{6} Y, X^{7}\right\} \subset \square_{\prec}(F) .
$$

Hence, assume $a_{4}=0$ Then if $a_{12} \neq 0$ we have

$$
\left\{X^{3} Y, X^{4} Y, X^{5} Y, X^{6} Y\right\} \subset \square_{\prec w}(F) .
$$

Hence, we assume $a_{12}=0$. We again multiply by $X$ and continue the reduction

$$
X\left(a_{7} X Y^{2}+X Y\right) \xrightarrow{F(X, Y)} a_{7} X^{5}+a_{7}^{2} Y^{2}+a_{7}^{2} X^{3}+a_{7} Y+X^{2} Y .
$$

If $a_{7} \neq 0$ then

$$
\left\{X^{5}, X^{5} Y, X^{6}, X^{6} Y, X^{7}\right\} \subset \square_{\prec_{w}}(F)
$$

Finally, assume $a_{7}=0$. But then we are left with $X^{2} Y$ and therefore

$$
\left\{X^{2} Y, X^{3} Y, X^{4} Y, X^{5} Y, X^{6} Y\right\} \subset \square_{\prec w}(F)
$$

In conclusion we can always establish at least $5+\min \{2,3,3,5,4,5,4,5,4,5,4\}=$ 7 monomials in $\square_{\prec_{w}}(F)$, and we conclude that $w_{H}(\boldsymbol{c}) \geq 7$. Moreover, our analysis reveals that if $a_{1}=1$ then $w_{H}(\boldsymbol{c}) \geq 5+3=8$.

### 3.8 Leading monomial equal to $X^{3} Y^{2}$

Consider $\boldsymbol{c}=\operatorname{ev}\left(F+I_{8}\right)$ where

$$
\begin{aligned}
F(X, Y)= & X^{3} Y^{2}+a_{1} X^{6}+a_{2} X^{4} Y+a_{3} X^{2} Y^{2}+a_{4} X^{5} \\
& +a_{5} X^{3} Y+a_{6} X Y^{2}+a_{7} X^{4}+a_{8} X^{2} Y+a_{9} Y^{2} \\
& +a_{10} X^{3}+a_{11} X Y+a_{12} X^{2}+a_{13} Y+a_{14} X+a_{15} .
\end{aligned}
$$

For sure

$$
\left\{X^{3} Y^{2}, X^{4} Y^{2}, X^{5} Y^{2}, X^{6} Y^{2}\right\} \subset \square_{\prec w}(F)
$$

We next consider an exhaustive series of conditions under which we establish more monomials in $\square_{\prec_{w}}(F)$. We have

$$
\begin{aligned}
& Y F(X, Y) \\
Y^{3}+X^{3} Y+X & \left(1+a_{1}\right) X^{6} Y+a_{2} X^{4} Y^{2}+a_{3} X^{2} Y^{3}+a_{4} X^{5} Y+a_{5} X^{3} Y^{2} \\
& +a_{6} X Y^{3}+a_{7} X^{4} Y+a_{8} X^{2} Y^{2}+a_{9} Y^{3}+a_{10} X^{3} Y+a_{11} X Y^{2} \\
& +X^{4}+a_{12} X^{2} Y+a_{13} Y^{2}+a_{14} X Y+a_{15} Y .
\end{aligned}
$$

If $a_{1} \neq 1$ then

$$
\left\{X^{6} Y\right\} \subset \square_{\prec w}(F) .
$$

Hence, assume $a_{1}=1$ and continue the reduction.

$$
\begin{aligned}
Y^{3}+X^{3} Y+X & a_{2} X^{4} Y^{2}+\left(a_{3}+a_{4}\right) X^{5} Y+a_{5} X^{3} Y^{2}+\left(a_{6}+a_{7}\right) X^{4} Y \\
& +a_{8} X^{2} Y^{2}+\left(a_{9}+a_{10}\right) X^{3} Y+a_{11} X Y^{2}+X^{4}+a_{12} X^{2} Y \\
& +a_{13} Y^{2}+a_{3} X^{3}+a_{14} X Y+a_{6} X^{2}+a_{15} Y+a_{9} X \\
\xrightarrow{F(X, Y)} & a_{2} X^{7}+\left(a_{2}^{2}+a_{3}+a_{4}\right) X^{5} Y+\left(a_{2} a_{3}+a_{5}\right) X^{3} Y^{2}+a_{2} a_{4} X^{6} \\
& +\left(a_{2} a_{5}+a_{6}+a_{7}\right) X^{4} Y+\left(a_{2} a_{6}+a_{8}\right) X^{2} Y^{2}+a_{2} a_{7} X^{5} \\
& +\left(a_{2} a_{8}+a_{9}+a_{10}\right) X^{3} Y+\left(a_{2} a_{9}+a_{11}\right) X Y^{2}+\left(a_{2} a_{10}+1\right) X^{4} \\
& +\left(a_{2} a_{11}+a_{12}\right) X^{2} Y+a_{13} Y^{2}+\left(a_{2} a_{12}+a_{3}\right) X^{3}+\left(a_{2} a_{13}+a_{14}\right) X Y \\
& +\left(a_{2} a_{14}+a_{6}\right) X^{2}+a_{15} Y+\left(a_{2} a_{15}+a_{9}\right) X .
\end{aligned}
$$

If $a_{2} \neq 0$ then

$$
\left\{X^{7}\right\} \subset \square_{\prec_{w}}(F) .
$$

Hence, assume $a_{2}=0$. If $a_{3} \neq a_{4}$ then

$$
\left\{X^{5} Y, X^{6} Y\right\} \subset \square_{\prec w}(F) .
$$

Hence, assume $a_{3}=a_{4}$ and continue the reduction.

$$
\begin{aligned}
\stackrel{F(X, Y)}{\longrightarrow} & a_{5} X^{6}+\left(a_{6}+a_{7}\right) X^{4} Y+\left(a_{4} a_{5}+a_{8}\right) X^{2} Y^{2}+a_{4} a_{5} X^{5} \\
& +\left(a_{5}^{2}+a_{9}+a_{10}\right) X^{3} Y+\left(a_{5} a_{6}+a_{11}\right) X Y^{2}+\left(a_{5} a_{7}+1\right) X^{4} \\
& +\left(a_{5} a_{8}+a_{12}\right) X^{2} Y+\left(a_{5} a_{9}+a_{4}\right) Y^{2}+\left(a_{5} a_{10}+a_{4}\right) X^{3} \\
& +\left(a_{5} a_{11}+a_{14}\right) X Y+\left(a_{5} a_{12}+a_{6}\right) X^{2}+\left(a_{5} a_{13}+a_{15}\right) Y \\
& +\left(a_{5} a_{14}+a_{9}\right) X+a_{5} a_{15} .
\end{aligned}
$$

If $a_{5} \neq 0$ then

$$
\left\{X^{6}, X^{6} Y, X^{7}\right\} \subset \square_{\prec w}(F)
$$

Hence, assume $a_{5}=0$. But then if $a_{6} \neq a_{7}$

$$
\left\{X^{4} Y, X^{5} Y, X^{6} Y\right\} \subset \square_{\prec w}(F) .
$$

Hence, assume $a_{6}=a_{7}$. But then if $a_{8} \neq 0$ we obtain

$$
\left\{X^{2} Y^{2}\right\} \subset \square_{\prec_{w}}(F) .
$$

Actually, this result could be improved to

$$
\left\{X^{2} Y^{2}, X^{6}, X^{6} Y, X^{7}\right\} \subset \square_{\prec_{w}}(F)
$$

if we multiply the above polynomial by $X$ and reduce it modulo $F(X, Y)$. The details are left for the reader. Next assume $a_{8}=0$. But then if $a_{9} \neq a_{10}$ we get

$$
\left\{X^{3} Y, X^{4} Y, X^{5} Y, X^{6} Y\right\} \subset \square_{\prec w}(F) .
$$

Hence, assume $a_{9}=a_{10}$. If $a_{11} \neq 0$ then

$$
\left\{X Y^{2}, X^{2} Y^{2}\right\} \subset \square_{\prec_{w}}(F)
$$

Finally, assume $a_{11}=0$. But then $X^{4}$ is the leading monomial and we obtain

$$
\left\{X^{4}, X^{4} Y, X^{5}, X^{5} Y, X^{6}, X^{6} Y, X^{7}\right\} \subset \square_{\prec w}(F) .
$$

In conclusion we can always establish at least $4+\min \{1,1,2,3,3,4,4,2,7\}=$ 5 monomials in $\square_{\prec_{w}}(F)$, and we conclude that $w_{H}(\boldsymbol{c}) \geq 5$. Moreover, if $a_{1}=1$ and $a_{2}=0$ then $w_{H}(\boldsymbol{c}) \geq 4+2=6$.

### 3.9 Leading monomial equal to $X^{7}$

Consider $\boldsymbol{c}=\operatorname{ev}\left(F+I_{8}\right)$ where

$$
\begin{aligned}
F(X, Y)= & X^{7}+a_{1} X^{5} Y+a_{2} X^{3} Y^{2}+a_{3} X^{6}+a_{4} X^{4} Y+a_{5} X^{2} Y^{2} \\
& +a_{6} X^{5}+a_{7} X^{3} Y+a_{8} X Y^{2}+a_{9} X^{4}+a_{10} X^{2} Y+a_{11} Y^{2} \\
& +a_{12} X^{3}+a_{13} X Y+a_{14} X^{2}+a_{15} Y+a_{16} X+a_{17} .
\end{aligned}
$$

Observe that among the 22 affine roots over $\mathbb{F}_{8}$ of $Y^{3}+X^{3} Y+X$ the only point having the first coordinate equal to 0 is $(0,0)$. Hence, ev $\left(X^{7}+1\right)$ is of Hamming weight 1 meaning that $w_{H}(\boldsymbol{c})=1$ when $a_{1}=\cdots=a_{16}=0$ and $a_{17}=1$. In the following we show that for all other choices of $a_{i}$ the Hamming weight becomes at least 3 . We first observe, that

$$
\left\{X^{7}\right\} \subset \square_{\prec_{w}}(F) .
$$

Now consider

$$
\begin{aligned}
& Y F(X, Y) \\
\xrightarrow{X^{7} Y+Y} & a_{1} X^{5} Y^{2}+a_{2} X^{3} Y^{3}+a_{3} X^{6} Y+a_{4} X^{4} Y^{2}+a_{5} X^{2} Y^{3} \\
& +a_{6} X^{5} Y+a_{7} X^{3} Y^{2}+a_{8} X Y^{3}+a_{9} X^{4} Y+a_{10} X^{2} Y^{2} \\
& +a_{11} Y^{3}+a_{12} X^{3} Y+a_{13} X Y^{2}+a_{14} X^{2} Y \\
& +a_{15} Y^{2}+a_{16} X Y+\left(a_{17}+1\right) Y \\
\xrightarrow{Y^{3}+X^{3} Y+X} & a_{1} X^{5} Y^{2}+\left(a_{2}+a_{3}\right) X^{6} Y+a_{4} X^{4} Y^{2}+\left(a_{5}+a_{6}\right) X^{5} Y \\
& +a_{7} X^{3} Y^{2}+\left(a_{8}+a_{9}\right) X^{4} Y+a_{10} X^{2} Y^{2}+\left(a_{11}+a_{12}\right) X^{3} Y \\
& +a_{13} X Y^{2}+a_{2} X^{4}+a_{14} X^{2} Y+a_{15} Y^{2}+a_{5} X^{3}+a_{16} X Y \\
& +a_{8} X^{2}+\left(a_{17}+1\right) Y+a_{11} X .
\end{aligned}
$$

If the above polynomial is non-zero then going through all possible leading monomials we see that we can always establish at least two more monomials in $\square_{\prec_{w}}(F)$ in addition to $X^{7}$. For instance if $a_{1} \neq 0$ then we can add $\left\{X^{5} Y^{2}, X^{6} Y^{2}\right\}$. If $a_{1}=0$ and $a_{2} \neq a_{3}$ then we can add $\left\{X^{6} Y, X^{6} Y^{2}\right\}$ and so on. By inspection the above polynomial equals the zero polynomial if and only if $F(X, Y)=X^{7}+1$ and we are through.

### 3.10 The remaining cases

For the remaining choices of leading monomial it seems impossible to obtain better information on $\square_{\prec_{w}}(F)$ than what is derived by noting that all monomials divisible by $\operatorname{lm}(F)$ must be a leading monomial in
$\langle F\rangle+I_{8}$. In particular when the leading monomial is $X^{i}, i=0, \ldots, 7$ the information we obtain in this way can be shown to be the true Hamming weight of existing corresponding codewords. In conclusion we established the information in Figure 2,

$$
\begin{array}{cccccccc}
13 & 10 & 7 & 5 & 3 & 2 & 1 & \\
18 & 15 & 12 & 9 & 6 & 4 & 2 & \\
22 & 19 & 16 & 13 & 10 & 7 & 4 & 1
\end{array}
$$

Fig. 2. Lower bounds on $\# \square_{\prec_{w}}(F)$ where $\operatorname{lm}(F)$ are as in Figure 1

## 4 Code parameters

As code construction we use

$$
\operatorname{Span}_{\mathbb{F}_{8}}\left\{\operatorname{ev}\left(M+I_{8}\right) \mid M \in \Delta_{\prec_{w}}\left(I_{8}\right), \delta(M) \geq s\right\},
$$

where $\delta(M)$ are the estimates of $\# \square_{\prec_{w}}(F)$ as depicted in Figure 2, In this way we obtain the best possible codes, according to our estimates. The resulting parameters are shown in Table 1. In almost all cases, given a

$$
\begin{array}{lll}
{[22,1,22]_{8}} & {[22,2,19]_{8}} & {[22,3,18]_{8}} \\
{[22,4,16]_{8}} & {[22,5,15]_{8}} & {[22,7,13]_{8}} \\
{[22,8,12]_{8}} & {[22,10,10]_{8}} & {[22,11,9]_{8}} \\
{[22,13,7]_{8}} & {[22,14,6]_{8}} & {[22,15,5]_{8}} \\
{[22,17,4]_{8}} & {[22,18,3]_{8}} & {[22,20,2]_{8}}
\end{array}
$$

Table 1. Parameters $[n, k, d]_{8}$ of codes from the Klein quartic. Here, $n$ and $k$ are sharp values, whereas $d$ represents a lower bound estimate.
dimension in the table, then the corresponding estimate on the minimum distance equals the best value known to exist according to 9. The only exceptions are the dimensions $4,14,15$ and 18 where the best minimum distances known to exist are one more than we obtain. We finally remark that if we evaluate in all polynomials except those who have $X^{6} Y^{2}$ in their support then by Subsection 3.9 we get a code of dimension 21 with exactly 7 codewords of Hamming weight 1 . Hence, this code is almost as good as the $[22,21,2]]_{8}$ code, known to exist by $[9$.

## 5 Concluding remarks

In [12, Ex. 3.2] the authors estimated the minimum distances of the duals of the codes studied in the present paper using the Feng-Rao bound for dual codes. We believe that is should be possible to improve (possibly even drastic) upon their estimates of the minimum distance in the same way as we in this paper improved upon the Feng-Rao bound for primary codes. We leave this question for future research. The method of the present paper also applies to estimate higher weights (possible relative). We leave it for future research to establish examples where this gives improved information compared to what can be derived from the Feng-Rao bound. In the light of Remark 1 and the information established in Section 3, evidently our new method sometimes significantly improves upon the previous known methods. We stress that our method is very general in that it can be applied to any primary affine variety code. In particular it works for any monomial ordering and consequently also without any of the order domain conditions (Remark (1). Finding more families of good affine variety codes using our method is subject to future work.

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[^0]:    ${ }^{3}$ As we treat codes at a theoretical level we shall not need detailed information on the variety, but we find it interesting to note that besides one point being $(0,0)$ the remaining points correspond to the Fano plane by identifying each non-zero element in $\mathbb{F}_{8}$ with a vertex. Every non-zero $a$ now defines a line consisting of all $b s$ such that $(a, b)$ is in the variety.

