A lower bound on the 2-adic complexity of modified Jacobi sequence

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Abstract

Let p,q be distinct primes satisfying $\gcd(p-1,q-1)=d$ and let $D_i, i=0,1,\cdots,d-1$, be Whiteman's generalized cyclotomic classes with $Z_{pq}^*=\cup_{i=0}^{d-1}D_i$. In this paper, we give the values of Gauss periods based on the generalized cyclotomic sets $D_0^*=\sum_{i=0}^{\frac{d}{2}-1}D_{2i}$ and $D_1^*=\sum_{i=0}^{\frac{d}{2}-1}D_{2i+1}$. As an application, we determine a lower bound on the 2-adic complexity of modified Jacobi sequence. Our result shows that the 2-adic complexity of modified Jacobi sequence is at least pq-p-q-1 with period N=pq. This indicates that the 2-adic complexity of modified Jacobi sequence is large enough to resist the attack of the rational approximation algorithm (RAA) for feedback with carry shift registers (FCSRs).

Key words: Gaussian period, generalized cyclotomic class, modified Jacobi sequence, 2-adic complexity.

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1 Introduction

Pseudo-random sequences with good statistical property are widely used as basic blocks for constructing stream ciphers. Any key stream generators could be implemented by both linear feedback shift registers (LFSRs) and feedback with carry shift registers (FCSRs). However, after the Berlekamp-Massey algorithm (BMA) for LFSRs [12] and the rational approximation algorithm for FCSRs [13] were presented, linear complexity and 2-adic complexity of the key stream sequence have been regarded as critical security criteria and both are required to be no less than one half of the period.

Sequences from cyclotomic and generalized cyclotomic classes are large and important sequence families for constructing codebooks [10], [9], Frequency Hopping Sequences [22], Optical Orthogonal Codes [4], [5], [2] and cyclic codes [6]-[7] and most of cyclotomic sequences and generalized cyclotomic sequences have been proved to be with large linear complexity [8], [1], [21], [11]. However, there are a handful research papers that focus on 2-adic complexities of these sequences. In fact, although the concept of 2-adic has been presented for more than two decades, there are only a few kinds of sequences whose 2-adic complexities have been completely determined. For example, in 1997, Klapper has pointed out that an m-sequence with prime period has maximal 2-adic complexity [13]. In 2010, Tian and Qi showed that the 2-adic complexity of all the binary m-sequences is maximal [16]. Afterwards, Xiong et al. [18] presented a new method using circulant matrices to compute the 2-adic complexities of binary sequences. They showed that all the known sequences with ideal 2-level autocorrelation have maximum 2-adic complexity. Moreover, they also proved that the 2-adic complexities of Legendre sequences and Ding-Helleseth-Lam sequences with optimal autocorrelation are also maximal. Then, using the same method as that in [18], Xiong et al. [19] pointed out that two other classes of sequences based on interleaved structure have also maximal 2-adic complexity. One of these two classes of sequences was constructed by Tang and Ding [14], which has optimal autocorrelation, the other was constructed by Zhou et al [24], which is optimal with respect to the Tang-Fan-Matsufuji bound [15].

Modified Jacobi sequence is one of sequence families constructed by Whiteman generalized cyclotomic classes. Green and Choi have proved that these sequences have large linear complexity and low autocorrelation in many cases. But, as far as the authors known, among Jacobi sequence family, there is no other result about the 2-adic complexities of these sequences other than twin-prime sequences, which has ideal autocorrelation and has been proved to be with maximal 2-adic complexity by Xiong et al [18], and another class of sequences constructed from Whiteman generalized cyclotomic classes of order 2 has been proved to be also with maximal 2-adic complexity by Zeng et al.

[22].

In this paper, we study the 2-adic complexity of modified Jacobi sequences. And we give a general lower bound on the 2-adic complexity, i.e., we will prove that the 2-adic complexity of all these sequences is lower bounded by pq - p - q - 1 with period N = pq. As a special case, we can also confirm that the twin-prime sequence has maximal 2-adic complexity.

The rest of this paper is organized as follows. Some necessary definitions, notations, and previous results are introduced in Section 2. Gauss periods based on two Whiteman generalized cyclic sets are given in section 3. And the lower bound on the 2-adic of modified Jacobi sequence is given in Section 4. Finally we summarize our results and give some remarks in Section 5.

2 Preliminaries

Let N be a positive integer, $\{s_i\}_{i=0}^{N-1}$ a binary sequence of period N, and $S(x) = \sum_{i=0}^{N-1} s_i x^i \in \mathbb{Z}[x]$. If we write

$$\frac{S(2)}{2^N - 1} = \frac{\sum_{i=0}^{N-1} s_i 2^i}{2^N - 1} = \frac{m}{n}, \ 0 \le m \le n, \ \gcd(m, n) = 1,\tag{1}$$

then the 2-adic complexity $\Phi_2(s)$ of the sequence $\{s_i\}_{i=0}^{N-1}$ is defined as the integer $\lceil \log_2 n \rceil$, i.e.,

$$\Phi_2(s) = \left[\log_2 \frac{2^N - 1}{\gcd(2^N - 1, S(2))} \right],\tag{2}$$

where $\lfloor x \rfloor$ is the greatest integer that is less than or equal to x.

Let p = df + 1 and q = df' + 1 (p < q) be two odd primes with $\gcd(p-1, q-1) = d$. Define N = pq, $\mathcal{L} = (p-1)(q-1)/d$. The Chinese Remainder Theorem guarantees that there exists a common primitive root g of both p and q. Then the order of g modulo N is \mathcal{L} . Let x be an integer satisfying $x \equiv g \pmod{p}$, $x \equiv 1 \pmod{q}$. The existence and uniqueness of $x \pmod{N}$ are also guaranteed by the Chinese Remainder Theorem. Whiteman [17] presented the definition of the following generalized cyclotomic classes

$$D_i = \{g^t x^i : t = 0, 1, \dots, \mathcal{L} - 1\}, \ i = 0, 1, \dots, d - 1$$
(3)

of order d. And he also has proved

$$Z_N^* = \bigcup_{i=0}^{d-1} D_i, \ D_i \cap D_j = \emptyset \text{ for } i \neq j,$$

where \emptyset denotes the empty set. The corresponding generalized cyclotomic numbers of order d are defined by

$$(i,j) = |(D_i + 1) \cap D_j|$$
, for all $i, j = 0, 1, \dots, d - 1$.

And the following properties of the generalized cyclotomic numbers have also been given by Whiteman:

$$(i,j) = (d-i,j-i),$$
 (4)

$$(i,j) = \begin{cases} (j + \frac{d}{2}, i + \frac{d}{2}), & \text{if } ff' \text{ is even,} \\ (j,i), & \text{if } ff' \text{ is odd,} \end{cases}$$
 (5)

$$\sum_{i=0}^{d-1} (i,j) = \frac{(p-2)(q-2)-1}{d} + \delta_i, \tag{6}$$

where

$$\delta_i = \begin{cases} 1, & \text{if } ff' \text{ is even and } i = \frac{d}{2}, \text{ or if } ff' \text{ is odd and } i = 0, \\ 0, & \text{otherwise.} \end{cases}$$

If we denote

$$P = \{p, 2p, \dots, (q-1)p\}, \ Q = \{q, 2q, \dots, (p-1)q\}, \ R = \{0\},\$$

$$C_0 = \bigcup_{i=0}^{\frac{d}{2}-1} D_{2i} \cup Q \cup R, \ C_1 = \bigcup_{i=0}^{\frac{d}{2}-1} D_{2i+1} \cup P,$$

$$D_0^* = \bigcup_{i=0}^{\frac{d}{2}-1} D_{2i}, \ D_1^* = \bigcup_{i=0}^{\frac{d}{2}-1} D_{2i+1},$$

then we have $\mathbb{Z}_N = C_0 \cup C_1$, $\mathbb{Z}_N^* = D_0^* \cup D_1^*$ and it is easy to see that D_0^* is a subgroup of \mathbb{Z}_N^* . The modified Jacobi sequence $\{s_i\}_{i=0}^{N-1}$ is defined by

$$s_i = \begin{cases} 0, & \text{if } i \pmod{N} \in C_0, \\ 1, & \text{if } i \pmod{N} \in C_1. \end{cases}$$
 (7)

Moreover, it is not difficult to verify that modified Jacobi sequence is also equivalent to the following definition

$$s_i = \begin{cases} 1, & \text{if } i \pmod{N} \in P, \\ \frac{1 - \left(\frac{i}{p}\right)\left(\frac{i}{q}\right)}{2}, & \text{if } i \in Z_N^*, \\ 0, & \text{otherwise,} \end{cases}$$

where (\cdot) is the Legendre symbol. But, in this paper, we will discuss its 2-adic complexity using properties of generalized cyclotomic classes.

From the above argument, we have known that D_0^* is a subgroup of \mathbb{Z}_N^* , then D_0^* and D_1^* are also generalized cyclotomic classes. In this paper, we will determine the Gauss periods based on D_0^* and D_1^* . Then, using these gauss periods, a lower bound on the 2-adic of modified Jacobi sequence will be given. To this end, we will first list the following properties of the above sets (The proofs of these properties can also be found in many literatures, for example, see [20]).

Lemma 1 For any $a \in \mathbb{Z}_N$ and $B \subseteq \mathbb{Z}_N$, we denote $aB = \{ab | b \in B\}$. Then we have the following properties.

- (1) For each fixed $a \in D_i$, we have $aD_j = D_{(i+j) \pmod{d}}$, aP = P and aQ = Q, where $i, j = 0, 1, \dots, d-1$.
- (2) For each fixed $a \in P$, if b runs through each element of D_i , $i = 0, 1, \dots, d-1$, then ab runs exactly each element of P $\frac{p-1}{2}$ times. Symmetrically, for each fixed $a \in Q$, if b runs through each element of D_i , $i = 0, 1, \dots, d-1$, then ab runs exactly each element of Q $\frac{q-1}{2}$ times.
- (3) For each fixed $a \in P$, we have aP = P, aQ = R. Symmetrically, for each fixed $a \in Q$, we have aQ = Q, aP = R.
- (4) For each fixed $a \in D_i^*$, we have $aD_j^* = D_{(i+j) \pmod{2}}^*$, aP = P and aQ = Q, where i, j = 0, 1.

Let $\omega_N = e^{2\pi\sqrt{-1}/N}$ be a Nth complex primitive root of unity. Then the additive character χ of \mathbb{Z}_N is given by

$$\chi(x) = \omega_N^x, \ x \in \mathbb{Z}_N \tag{8}$$

and Gaussian periods of order d are defined by

$$\eta_i = \sum_{x \in D_i} \chi(x), \ i = 0, 1, \dots, d - 1.$$

It is well-known that

$$\sum_{x \in P} \chi(x) = -1,\tag{9}$$

$$\sum_{x \in Q} \chi(x) = -1,\tag{10}$$

and

$$\sum_{i=0}^{d-1} \eta_i = 1. (11)$$

Moreover, the following results, which have also been proved by Whiteman [17], will be useful.

Lemma 2 The element $-1 \in \mathbb{Z}_N^*$ satisfies

$$-1 \equiv \begin{cases} g^{\delta} x^{\frac{d}{2}} \pmod{N}, & \text{if } ff' \text{ is even,} \\ g^{\frac{\mathcal{L}}{2}} \pmod{N}, & \text{if } ff' \text{ is odd,} \end{cases}$$

where δ is some fixed integer such that $0 \le \delta \le \mathcal{L} - 1$

Lemma 3 For each $u \in P \cup Q$,

$$|D_i \cap (D_j + u)| = \begin{cases} \frac{(p-1)(q-1)}{d^2}, & \text{if } i \neq j, \\ \frac{(p-1)(q-1-d)}{d^2}, & \text{if } i = j \text{ and } u \in P, \\ \frac{(p-1-d)(q-1)}{d^2}, & \text{if } i = j \text{ and } u \in Q. \end{cases}$$

3 Gaussian periods of Whiteman generalized cyclotomic classes

Let $\Omega_0 = \sum_{x \in D_0^*} \chi(x) = \sum_{i=0}^{\frac{d}{2}-1} \eta_{2i}$ and $\Omega_1 = \sum_{x \in D_1^*} \chi(x) = \sum_{i=0}^{\frac{d}{2}-1} \eta_{2i+1}$. In this section, we will determine the values of Ω_0 and Ω_1 .

Theorem 1 Let p = df + 1, q = df' be distinct primes satisfying gcd(p-1, q-1) = d and let D_i^* be the generalized cyclotomic set be defined in section 2 and Ω_0 the gauss period based on D_i^* , where i = 0, 1. Then the Gauss periods are given by

$$\Omega_0 = \sum_{x \in D_0^*} \chi(x) = \begin{cases}
\frac{1 \pm \sqrt{pq}}{2}, & \text{if } ff' \text{ is odd, or if} \\
ff' \text{ is even and } d \equiv 0 \pmod{4}, \\
\frac{1 \pm \sqrt{-pq}}{2}, & \text{if } ff' \text{ is even and } d \equiv 2 \pmod{4},
\end{cases}$$

$$\Omega_1 = 1 - \Omega_0. \tag{13}$$

Proof. Above all, from the definition of generalized cyclotomic class, for any $\tau \in D_k$, it can be easy verify that $\tau^{-1} \in D_{(d-k) \pmod{d}}$. Then, by Lemma 1, for any $0 \le i, j, k \le d-1$, we have

$$|(D_i + \tau) \cap D_j| = |(\tau^{-1}D_i + 1) \cap \tau^{-1}D_j|$$

= $((i + d - k) \pmod{d}, (j + d - k) \pmod{d}).$

If ff' is odd, then, by Lemma 2, we have $-1 \in D_0$. Therefore, using Lemma 3, we get

$$(\Omega_{0})^{2} = \left(\sum_{i=0}^{\frac{d}{2}-1} \eta_{2i}\right)^{2} = \sum_{i=0}^{\frac{d}{2}-1} \eta_{2i}^{2} + \sum_{i=0}^{\frac{d}{2}-1} \sum_{j\neq i,j=0}^{\frac{d}{2}-1} \eta_{2i} \eta_{2j}$$

$$= \sum_{i=0}^{\frac{d}{2}-1} \left(\sum_{x\in D_{2i}} \sum_{y\in D_{2i}} \omega_{N}^{y-x}\right) + \sum_{i=0}^{\frac{d}{2}-1} \sum_{j\neq i,j=0}^{\frac{d}{2}-1} \left(\sum_{x\in D_{2i}} \sum_{y\in D_{2j}} \omega_{N}^{y-x}\right)$$

$$= \sum_{i=0}^{\frac{d}{2}-1} \left(\sum_{k=0}^{d-1} (2i-k,2i-k)\eta_{k} + \frac{(p-1)(q-1)}{d} - \frac{(p-1)(q-1-d)}{d^{2}}\right)$$

$$- \frac{(p-1-d)(q-1)}{d^{2}}\right)$$

$$+ \sum_{i=0}^{\frac{d}{2}-1} \sum_{j\neq i,j=0}^{\frac{d}{2}-1} \left(\sum_{k=0}^{d-1} (2i-k,2j-k)\eta_{k} - \frac{(p-1)(q-1)}{d^{2}} - \frac{(p-1)(q-1)}{d^{2}}\right)$$

$$= \sum_{k=0}^{d-1} \left(\sum_{i=0}^{\frac{d}{2}-1} \sum_{j=0}^{\frac{d}{2}-1} (2i-k,2j-k)\right) \eta_{k} + \frac{p-1}{2} + \frac{q-1}{2}$$

$$= \sum_{k=0}^{d-1} \left(\sum_{i=0}^{\frac{d}{2}-1} \sum_{j=0}^{\frac{d}{2}-1} (d-(2i-k),2(j-i))\right) \eta_{k} + \frac{p-1}{2} + \frac{q-1}{2}$$

$$= \sum_{k=0}^{d-1} \left(\sum_{j'=0}^{\frac{d}{2}-1} \sum_{j=0}^{\frac{d}{2}-1} (d-(2(j-j')-k),2j')\right) \eta_{k} + \frac{p-1}{2} + \frac{q-1}{2},$$

$$(14)$$

where Eq. (14) comes from Eq.(4). Similarly, we can get

$$(\Omega_1)^2 = \left(\sum_{i=0}^{\frac{d}{2}-1} \eta_{2i+1}\right)^2 = \sum_{i=0}^{\frac{d}{2}-1} \eta_{2i+1}^2 + \sum_{i=0}^{\frac{d}{2}-1} \sum_{j\neq i,j=0}^{\frac{d}{2}-1} \eta_{2i+1} \eta_{2j+1}$$
$$= \sum_{k=0}^{d-1} \left(\sum_{j'=0}^{\frac{d}{2}-1} \sum_{j=0}^{\frac{d}{2}-1} \left(d - \left(2(j-j') + 1 - k\right), 2j'\right)\right) \eta_k + \frac{p-1}{2} + \frac{q-1}{2}.$$

Then,

$$(\Omega_0)^2 + (\Omega_1)^2 = \sum_{k=0}^{d-1} \left(\sum_{j'=0}^{\frac{d}{2}-1} \left(\sum_{j=0}^{\frac{d}{2}-1} \left((d - (2(j-j') - k), 2j') + (d - (2(j-j') + 1 - k), 2j')) \right) \eta_k \right)$$

$$+ (p-1) + (q-1)$$

$$= \sum_{k=0}^{d-1} \left(\sum_{j'=0}^{\frac{d}{2}-1} \left(\sum_{j=0}^{d-1} (j, 2j') \right) \right) \eta_k + (p-1) + (q-1)$$

$$= \sum_{j'=0}^{\frac{d}{2}-1} \left(\sum_{j=0}^{d-1} (j, 2j') \right) + (p-1) + (q-1)$$

$$= \sum_{j=0}^{d-1} (j, 0) + \sum_{j'=1}^{\frac{d}{2}-1} \left(\sum_{j=0}^{d-1} (j, 2j') \right) + (p-1) + (q-1)$$

$$= \sum_{j=0}^{d-1} (0, j) + \sum_{j'=1}^{\frac{d}{2}-1} \left(\sum_{j=0}^{d-1} (2j', j) \right) + (p-1) + (q-1)$$

$$= 1 + \frac{d}{2} \times \frac{(p-2)(q-2) - 1}{d} + (p-1) + (q-1)$$

$$= 1 + \frac{pq-1}{2},$$

$$(15)$$

Where Eq. (15) is by Eq. (11), Eq. (16) is by Eq. (5), and Eq. (17) hold because of Eq. (6). Moreover, from Eq.(11), we know that $\Omega_0 + \Omega_1 = 1$ and

$$(\Omega_0)^2 + (\Omega_1)^2 = (\Omega_0 + \Omega_1)^2 - 2\Omega_0\Omega_1$$

= 1 - 2\Omega_0\Omega_1

Then we obtain $\Omega_0\Omega_1 = \frac{1-pq}{4}$, which implies $\Omega_0 = \frac{1\pm\sqrt{pq}}{2}$ and $\Omega_1 = \frac{1\mp\sqrt{pq}}{2}$.

Suppose that ff' is even then, by Lemma 2, we have $-1 \in D_{\frac{d}{2}}$. Let $d \equiv 2 \pmod{4}$, i.e., $\frac{d}{2}$ is odd. Then, for any $0 \le i, j \le \frac{d}{2} - 1$ with $i \ne j$, we know that $2i + \frac{d}{2} \not\equiv 2j \pmod{d}$ and $2i + 1 + \frac{d}{2} \not\equiv 2j + 1 \pmod{d}$. Therefore,

$$(\Omega_0)^2 = \sum_{i=0}^{\frac{d}{2}-1} \sum_{j=0}^{\frac{d}{2}-1} \left(\sum_{x \in D_{2i+\frac{d}{2}}} \sum_{y \in D_{2j}} \omega_N^{y-x} \right)$$

$$= \sum_{i=0}^{\frac{d}{2}-1} \sum_{j=0}^{\frac{d}{2}-1} \left(\sum_{k=0}^{d-1} (2i + \frac{d}{2} - k, 2j - k) \eta_k - 2 \times \frac{(p-1)(q-1)}{d^2} \right)$$

$$= \sum_{k=0}^{d-1} \left(\sum_{i=0}^{\frac{d}{2}-1} \sum_{j=0}^{\frac{d}{2}-1} (\frac{d}{2} - (2i - k), 2(j-i) + \frac{d}{2}) \right) \eta_k - \frac{(p-1)(q-1)}{2} (18)$$

$$= \sum_{k=0}^{d-1} \left(\sum_{i=0}^{\frac{d}{2}-1} \sum_{j=0}^{\frac{d}{2}-1} (2(j-i), k-2i) \right) \eta_k - \frac{(p-1)(q-1)}{2}$$

$$(19)$$

$$= \sum_{k=0}^{d-1} \left(\sum_{j'=0}^{\frac{d}{2}-1} \sum_{j=0}^{\frac{d}{2}-1} (2j', k - 2(j-j')) \right) \eta_k - \frac{(p-1)(q-1)}{2},$$

where Eq. (18) comes again from Eq. (4) and Eq. (19) comes from Eq. (5). Similarly, we can get

$$(\Omega_1)^2 = \sum_{k=0}^{d-1} \left(\sum_{j'=0}^{\frac{d}{2}-1} \sum_{j=0}^{\frac{d}{2}-1} (2j', k - 2(j-j') - 1) \right) \eta_k - \frac{(p-1)(q-1)}{2}.$$

Then we get

$$(\Omega_{0})^{2} + (\Omega_{1})^{2} = \sum_{k=0}^{d-1} \left(\sum_{j'=0}^{\frac{d}{2}-1} \left(\sum_{j=0}^{\frac{d}{2}-1} \left((2j', k - 2(j - j')) + (2j', k - 2(j - j') - 1) \right) \right) \right) \eta_{k}$$

$$- (p-1)(q-1)$$

$$= \sum_{k=0}^{d-1} \left(\sum_{j'=0}^{\frac{d}{2}-1} \left(\sum_{j=0}^{d-1} (2j', j) \right) \right) \eta_{k} - (p-1)(q-1)$$

$$= \sum_{j'=0}^{\frac{d}{2}-1} \left(\sum_{j=0}^{d-1} (2j', j) \right) - (p-1)(q-1)$$

$$= \frac{d}{2} \times \frac{(p-2)(q-2) - 1}{d} - (p-1)(q-1)$$

$$= 1 - \frac{pq+1}{2},$$
(21)

where Eq. (20) is by Eq. (11) and Eq. (21) is by Eq. (6). Similar argument to the above, we can get $\Omega_0\Omega_1=\frac{1+pq}{4}$, which implies $\Omega_0=\frac{1\pm\sqrt{-pq}}{2}$ and $\Omega_1=\frac{1\mp\sqrt{-pq}}{2}$. For the case of even ff' and $d\equiv 0\pmod{4}$, the result can be similarly obtained.

4 A lower bound on the 2-adic complexity of Jacobi sequence

Let $\{s_i\}_{i=0}^{N-1}$ be a binary sequence with period N and let $A = (a_{i,j})_{N \times N}$ be the matrix defined by $a_{i,j} = s_{i-j \pmod{N}}$. In this section, using the periods which have been determined in Section 3, we will give a lower bound on the 2-adic complexity of modified Jacobi sequence. In order to derive the lower bound, the following two results will be useful, which can be found in [18] and [3] respectively.

Lemma 4 [18] Viewing A as a matrix over the rational fields \mathbb{Q} , if $det(A) \neq 0$, then

$$\gcd\left(S(2), 2^N - 1\right) |\gcd\left(\det(A), 2^N - 1\right). \tag{22}$$

Lemma 5 [3] $\det(A) = \prod_{a=0}^{N-1} S(\omega_N^a)$, where ω_N is defined as in Eq. (8).

Lemma 6 Let $\{s_i\}_{i=0}^{N-1}$ be the modified Jacobi sequence with period N = pq. Then we have

$$S(\omega_N^a) = \begin{cases} \frac{(p+1)(q-1)}{2}, & \text{if } a \in R, \\ -\Omega_0, & \text{if } a \in D_0^*, \\ -\Omega_1, & \text{if } a \in D_1^*, \\ -\frac{p+1}{2}, & \text{if } a \in P, \\ \frac{q-1}{2}, & \text{if } a \in Q. \end{cases}$$
(23)

Proof. Recall that

$$S(\omega_N^a) = \sum_{k \in C_1} (\omega_N^a)^k = \sum_{k \in D_1^*} (\omega_N^a)^k + \sum_{k \in P} (\omega_N^a)^k$$
$$= \sum_{k \in aD_1^*} \omega_N^k + \sum_{k \in aP} \omega_N^k.$$

Firstly, if a=0, then it is easy to see $S(\omega_N^a)=\frac{(p-1)(q-1)}{2}+(q-1)=\frac{(p+1)(q-1)}{2}$. Secondly, if $a\in D_0^*$, by Lemma 1 and Eq. (9), then we have

$$S(\omega_N^a) = \sum_{k \in aD_1^*} \omega_N^k + \sum_{k \in aP} \omega_N^k = \sum_{k \in D_1^*} \omega_N^k + \sum_{k \in P} \omega_N^k$$
$$= \Omega_1 - 1 = -\Omega_0 \quad \text{(for } \Omega_0 + \Omega_1 = 1 \text{ by Eq.}(11)).$$

Similarly, if $a \in D_1^*$, we have

$$S(\omega_N^a) = \sum_{k \in aD_1^*} \omega_N^k + \sum_{k \in aP} \omega_N^k = \sum_{k \in D_0^*} \omega_N^k + \sum_{k \in P} \omega_N^k$$
$$= \Omega_0 - 1 = -\Omega_1.$$

If $a \in P$, again by Lemma 1, we have

$$S(\omega_N^a) = \sum_{k \in aC_1} \omega_N^k + \sum_{k \in aP} \omega_N^k = \frac{p-1}{2} \sum_{k \in P} \omega_N^k + \sum_{k \in P} \omega_N^k$$
$$= -\frac{p-1}{2} - 1 = -\frac{p+1}{2}.$$

Similarly, if $a \in Q$ we have

$$S(\omega_N^a) = \sum_{k \in aC_1} \omega_N^k + \sum_{k \in aP} \omega_N^k = \frac{q-1}{2} \sum_{k \in Q} \omega_N^k + (q-1)$$
$$= -\frac{q-1}{2} + (q-1) = \frac{q-1}{2}.$$

The result follows.

Lemma 7 Let p and q be two distinct odd primes and N = pq. Then we have $\gcd(2^p - 1, \frac{2^N - 1}{2^p - 1}) = \gcd(2^p - 1, q)$ and $\gcd(2^q - 1, \frac{2^N - 1}{2^q - 1}) = \gcd(2^q - 1, p)$. Particularly, if q > p, we have $\gcd(2^q - 1, p) = 1$, i.e., $\gcd(2^q - 1, \frac{2^N - 1}{2^q - 1}) = 1$.

Proof. Note that

$$2^{N} - 1 = 2^{pq} - 1 = (2^{p} - 1)(2^{p(q-1)} + 2^{p(q-2)} + \dots + 2^{p} + 1)$$
$$= 2^{pq} - 1 = (2^{q} - 1)(2^{q(p-1)} + 2^{q(p-2)} + \dots + 2^{q} + 1).$$

Then we get $\frac{2^{pq}-1}{2^p-1} \equiv q \pmod{2^p-1}$ and $\frac{2^{pq}-1}{2^q-1} \equiv p \pmod{2^q-1}$, which imply $\gcd(2^p-1,\frac{2^N-1}{2^p-1}) = \gcd(2^p-1,q)$ and $\gcd(2^q-1,\frac{2^N-1}{2^q-1}) = \gcd(2^q-1,p)$. Particularly, for q>p, if $\gcd(2^q-1,p)>1$, i.e., $p|2^q-1$, then the multiplicative order of 2 modular p, denoted as $\operatorname{Ord}_p(2)$, is a divisor of q. But q is a prime, then $\operatorname{Ord}_p(2)=q$. By Fermat Theorem, we know that $p|2^{p-1}-1$. Therefore, we have $q\leq p-1$, which contradicts to the fact p< q. The desired result follows.

Theorem 2 Let p = df + 1 and q = df' + 1 be two odd primes satisfying gcd(p-1,q-1) = d and p < q. Suppose $\{s_i\}_{i=0}^{N-1}$ is the modified Jacobi sequence with period N = pq. Then the 2-adic complexity $\phi_2(s)$ of $\{s_i\}_{i=0}^{N-1}$ is bounded by

$$\phi_2(s) \ge pq - p - q - 1. \tag{24}$$

Specially, if q = p + 2, then the 2-adic complexity of $\{s_i\}_{i=0}^{N-1}$ is maximal.

Proof. By Lemmas 5 and 6, we can get

$$\det(A) = \prod_{a=0}^{N-1} S(\omega_N^a)$$

$$= \prod_{a \in R} S(\omega_N^a) \prod_{a \in D_0^*} S(\omega_N^a) \prod_{a \in D_1^*} S(\omega_N^a) \prod_{a \in P} S(\omega_N^a) \prod_{a \in Q} S(\omega_N^a)$$

$$= 2\left(\frac{p+1}{2}\right)^q \left(\frac{q-1}{2}\right)^p (\Omega_0 \Omega_1)^{\frac{(p-1)(q-1)}{2}}.$$

From Theorem 1, we know that $\Omega_0\Omega_1 = \frac{1-pq}{4}$ if ff' is odd or if ff' is even and $d \equiv 0 \pmod{4}$, and $\Omega_0\Omega_1 = \frac{1+pq}{4}$ if ff' is even and $d \equiv 2 \pmod{4}$. Then

we have

$$\det(A) = \begin{cases} 2\left(\frac{p+1}{2}\right)^q \left(\frac{q-1}{2}\right)^p \left(\frac{pq-1}{4}\right)^{\frac{(p-1)(q-1)}{2}}, & \text{if } ff' \text{ is odd,} \\ & \text{or if } ff' \text{ is even and } d \equiv 0 \pmod{4} \\ 2\left(\frac{p+1}{2}\right)^q \left(\frac{q-1}{2}\right)^p \left(\frac{pq+1}{4}\right)^{\frac{(p-1)(q-1)}{2}}, & \text{if } ff' \text{ is} \\ & \text{even and } d \equiv 2 \pmod{4}. \end{cases}$$
 (25)

Now, let r be an any prime factor of 2^N-1 and $\operatorname{Ord}_r(2)$ the multiplicative order of 2 modular r. Then we get $\operatorname{Ord}_r(2)|N$. Note that N=pq. Therefore, $\operatorname{Ord}_r(2)=pq$, p or q. Next, we will prove $\gcd(\det(A),2^N-1)\leq (2^p-1)(2^q-1)$. To this end, we need discuss the following three cases.

- Case 1. $\operatorname{Ord}_r(2) = pq$. By Fermat Theorem, we know that $r|2^{r-1} 1$, which implies $\operatorname{Ord}_r(2) \leq r 1$. Since $\operatorname{Ord}_r(2) = pq$, then we have $pq \leq r 1$, i.e., $r \geq pq + 1$. But we know that $\frac{p+1}{2} < pq + 1$, $\frac{q-1}{2} < pq + 1$, and $\frac{pq\pm 1}{4} < pq + 1$, then $\gcd(\det(A), 2^N 1) = 1 < (2^p 1)(2^q 1)$.
- Case 2. $\operatorname{Ord}_r(2) = p$. Similar argument to that in Case 1, we can get $r \geq p+1$, which implies $\gcd(\frac{p+1}{2}, 2^N 1) = 1$. It is obvious that $r|2^p 1$. If $r|\frac{q-1}{2}$ or $r|\frac{pq\pm 1}{4}$ (Here $\frac{pq\pm 1}{4}$ corresponds to the cases of odd ff' and even ff'), then we have $\gcd(r,q) = 1$. Furthermore, by Lemma 7, we know $\gcd(2^p 1, \frac{2^N 1}{2^p 1}) = \gcd(2^p 1, q)$. Thus we have $\gcd(r, \frac{2^N 1}{2^p 1}) = 1$.
- $\gcd(2^p-1,q)$. Thus we have $\gcd(r,\frac{2^N-1}{2^p-1})=1$. Case 3. $\operatorname{Ord}_r(2)=q$. It is obvious that $r|2^q-1$. From Lemma 7, we know that $\gcd(2^q-1,\frac{2^N-1}{2^q-1})=1$, which implies that $\gcd(r,\frac{2^N-1}{2^q-1})=1$.

Combining Case 2 and Case 3, no matter $\operatorname{Ord}_r(2) = p$ or $\operatorname{Ord}_r(2) = q$, we always have $\gcd(\det(A), 2^N - 1) | (2^p - 1)(2^q - 1)$ and $\gcd(\det(A), \frac{2^N - 1}{(2^p - 1)(2^q - 1)}) = 1$, which implies that $\gcd(\det(A), 2^N - 1) \leq (2^p - 1)(2^q - 1)$. From Lemma 4, we have

$$\gcd(S(2), 2^N - 1) \le \gcd(\det(A), 2^N - 1) \le (2^p - 1)(2^q - 1).$$

Then, by Eq. (2), we get

$$\phi_2(s) = \left\lfloor \log_2 \frac{2^N - 1}{\gcd(2^N - 1, S(2))} \right\rfloor \ge \left\lfloor \log_2 \frac{2^{pq} - 1}{(2^p - 1)(2^q - 1)} \right\rfloor \ge pq - p - q - 1.$$

Specially, if q=p+2, then we know that ff' is even, which implies that $\det(A)=2\left(\frac{p+1}{2}\right)^{p(p+2)+1}$. From the above argument, we know that

$$\gcd\left(\frac{p+1}{2}, 2^{p(p+2)} - 1\right) = 1.$$

The desire result follows.

5 Summary

In this paper, we derive gauss periods of a class of generalized cyclotomic sets from Whiteman generalized cyclotomic classes. As an application, a lower bound on the 2-adic complexity of modified Jacobi sequences is determined. Our result shows that the 2-adic complexity is at least pq - p - q - 1 with period N = pq, which is obviously large enough to resist against RAA for FCSR.

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References

- [1] Bai, E., Liu, X., Xiao, G.: Linear complexity of new generalized cyclotomic sequences of order two of length pq. IEEE Trans. Inform. Theory 51, 1849-1853 (2005).
- [2] Cai, H., Liang, H., Tang, X.: Constructions of optimal 2-D optical orthogonal codes via generalized cyclotomic classes. IEEE Trans. Inform. Theory 61, 688-695 (2015).
- [3] Davis, P J.: Circulant Matrices. New York, NY, USA: Chelsea, 1994.
- [4] Ding, C., Xing, C.: Several classes of $(2^m 1, w, 2)$ optical orthogonal codes. Discrete Applied Mathematics 128, 103-120 (2003).
- [5] Ding, C., Xing, C.: Cyclotomic optical orthogonal codes of composite lengths. IEEE Trans. Inform. Theory 52, 263-268 (2004).
- [6] Ding, C.: Cyclotomic constructions of cyclic codes with length being the product of two primes. IEEE Trans. Inform. Theory 58, 2231-2236 (2012).
- [7] Ding, C.: Cyclic codes from the two-prime sequences. IEEE Trans. Inform. Theory 58, 3881-3891 (2012).
- [8] Ding, C., Helleseth, T.: On the linear complexity of Legendre sequences. IEEE Trans. Inform. Theory 44, 1693-1698 (1998).
- [9] Fan, C., Ge, G.: A unified approach to Whiteman's and Ding-Helleseth's generalized cyclotomy over residue classs rings. IEEE Trans. Information Theory 60, 1326-1336 (2014).

- [10] Hu, L., Yue, Q.: Gauss periods and codebooks from generalized cyclotomic sets of order four. Design, Codes and Cryptography 69, 233-246 (2013).
- [11] Li, X., Ma, W., Yan, T., Zhao, X.: Linear complexity of a new generalized cyclotomic sequence of order two of length pq. IEICE Transactions 96-A, 1001-1005 (2013).
- [12] Massey, J. L.: Shift-register synthesis and BCH decoding. IEEE Trans. Inform. Theory 15, 122-127 (1969).
- [13] Klapper, A., Goresky, M.: Feedback shift registers, 2-adic span, and combiners with memory. Journal of Cryptology 10, 111-147 (1997).
- [14] Tang, X., Ding, C.: New classes of balanced quaternary and almost balanced binary sequences with optimal autocorrelation Value. IEEE Trans. Inform. Theory 56, 6398-6405 (2010).
- [15] Tang, X., Fan, P., Matsufuji,S.: Lower bounds on the maximum correlation of sequences with low or zero correlation zone. Electron. Lett. 36, 551-552 (2000).
- [16] Tian, T., Qi, W.: 2-Adic complexity of binary m-sequences. IEEE Trans. Inform. Theory 56, 450-454 (2010).
- [17] Whiteman, A L.: A family of difference sets. Illinois Journal of Mathematics 6, 107-121 (1962).
- [18] Xiong, H., Qu, L., Li, C.: A new method to compute the 2-adic complexity of binary sequences. IEEE Trans. Inform. Theory 60, 2399-2406 (2014).
- [19] Xiong, H., Qu, L., Li, C.: 2-Adic complexity of binary sequences with interleaved structure. Finite Fields and Their Applications 33, 14-28 (2015).
- [20] Yan, T.: Study on Constructions and Properties of Pseudo-Random Sequence. Ph. D Thesis, 2007.
- [21] Yan, T., Du, X., Xiao, G., Huang, X.: Linear complexity of binary Whiteman generalized cyclotomic sequences of order 2^k . Information Sciences 179, 1019-1023 (2009).
- [22] Zeng, X., Cai, H., Tang, X., Yang, Y.: Optimal frequency sequences of odd length. IEEE Trans. Inform. Theory 59, 3237-3248 (2013).
- [23] Xiao, Z.,Zeng, X.: 2-Adic complexity of two classes of generalized cyclotomic binary sequences. International Journal of Foundations of Computer Science 27, 879-893 (2016).
- [24] Zhou, Z., Tang, X., Gong, G.: A new classes of sequences with zero or low correlation zone based on interleaving technique. IEEE Trans.Inform. Theory 54, 4267-4273 (2008).