Algebraic Manipulation Detection Codes via Highly Nonlinear Functions

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Abstract

In this paper, we study the relationship between algebraic manipulation detection (AMD) codes and highly nonlinear functions. As applications, on one hand, a generic construction for systematic AMD codes is introduced based on highly nonlinear functions. Systematic AMD codes with new parameters can be generated from known highly nonlinear functions. Especially, several infinite classes of optimal systematic AMD codes, some with asymptotically optimal tag size, can be constructed. On the other hand, systematic AMD codes are used to construct highly nonlinear functions. The known construction by Cramer *et al.* [10] for systematic AMD codes turns out to be based on a special kind of functions with high nonlinearity.

Keywords: Algebraic manipulation detection code, highly nonlinear function, nonlinearityp, partial nonlinearity.

I. Introduction

Algebraic manipulation detection (AMD) codes were first introduced by Cramer *et al.* [10] to convert linear secret sharing schemes into robust secret sharing schemes and build nearly optimal robust fuzzy extractors. An AMD code can be viewed as an authentication code without a secret key. Generally speaking, an AMD code consists of a probabilistic encoding map E and a determined decoding function Dec, where E encodes a plaintext s into a ciphertext $g \in_R G_s$ such that any tampering will be detected, except with a small constant error probability. For AMD codes, we consider the attack model such that an adversary may manipulate the valid message g = E(s) by adding some offset Δ of his choice. The attack model is divided into two sub-models by distinguishing two different settings: the adversary has full knowledge of the source (the strong attack model) and the adversary has no knowledge about the source (the weak attack model). The main objective for an AMD code is, for a given tag size, to minimize the success probability of an adversary such that $g + \Delta \in G_{s'}$, i.e., $Dec(g') = s' \neq Dec(g) = s$. Here the tag size denotes the difference between the length of the plaintext and its corresponding ciphertext.

Upon to now, there are mainly two kinds of known constructions for AMD codes. One is via algebraic methods. In [10], Cramer *et al.* proposed a construction of AMD codes with nearly optimal tag size based on polynomial evaluations. In [21], Reed-Muller codes were included to construct AMD codes. Later in [12], linear codes such as BCH codes were included to generate AMD codes with small tag size. The other one is via combinatorial method, which generates AMD codes by carefully designing the combinatorial structures behind, i.e., the structure of image sets of the probabilistic encoding map *E*. In [11], Cramer *et al.* first introduced a kind of differential structures to construct AMD codes. In [30], Paterson and Stinson characterized optimal AMD codes with various types of generalized external difference families (say, strong external difference families) for different merits of optimality, respectively. In [18] and [32], for the weak attack model, combinatorial characterizations were given for AMD codes via weighted external difference families. In the past few years, many efforts have been devoted to construct AMD codes and their corresponding generalized external difference families (see [19], [23], [24], [30], [36], [37], and the references therein).

In cryptography, another interesting topic closely related to authentication codes are functions with high nonlinearity [6]. Highly nonlinear functions such as bent functions received much attention not only for their applications in cryptography (for instances, combining keystream generators for stream ciphers [13], S-boxes for block ciphers [8], [28], functions with optimal

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algebraic immunity [33], and authentication codes [7], [16]), sequences design [15], [29] and coding theory [5], [38], but also for their close relationship with combinatorial designs [14], [31].

In this paper, we study the relationship between AMD codes and highly nonlinear functions. On one hand, we propose a generic construction of AMD codes via functions with high nonlinearity. By choosing special highly nonlinear functions such as perfect nonlinear functions, a few infinite classes of AMD codes with new parameters can be generated for both weak and strong attack models. For the weak attack model, *R*-optimal AMD codes have asymptotically optimal tag size can be constructed. For the strong attack model, some AMD codes generated by our construction are proved to have the minimum possible probability of successful tampering. On the other hand, we try to construct highly nonlinear functions from known AMD codes. Based on a subclass of AMD codes with more strict assumptions, highly nonlinear functions can be generated, where their nonlinearities are determined by the parameters of the corresponding AMD codes. Especially, we prove that the known construction in [10, Theorem 2] can also be explained by highly nonlinear functions.

The remainder of this paper is organized as follows. In Section II, we introduce some preliminaries about AMD codes. In Section III, we construct systematic AMD codes via highly nonlinear functions for the weak attack model, whereas Section IV is devoted to construct systematic AMD codes under the strong attack model. In Section V, highly nonlinear functions are constructed based on known systematic AMD codes. Conclusion is drawn in Section VI.

II. PRELIMINARIES

In this section, we recap some notation, definitions and results about AMD codes.

Definition 1: Let S be a set of plaintext messages with size m termed the *source space*, and G be the *encoded message space*, which is usually an Abelian group of order n. Consider a pair of a probabilistic encoding map $E: S \to G$ and a deterministic decoding function $\mathrm{Dec}: G \to S \cup \{\bot\}$ such that $\mathrm{Dec}(E(s)) = s$ with probability 1 for any $s \in S$. Let G_s be the set of *valid* encodings of $s \in S$, i.e., $G_s \triangleq \{g \in G : \mathrm{Dec}(g) = s\}$.

- (1) The pair (E, Dec) is called a $strong\ (m, n, \rho)$ algebraic manipulation detection (AMD) code if for any $s \in S, \Delta \in G \setminus \{0\}$, the probability of $\mathrm{Dec}(E(s) + \Delta) \not\in \{s, \bot\}$ is at most ρ , i.e., $\mathrm{Pr}(\mathrm{Dec}(E(s) + \Delta) \not\in \{s, \bot\}) \leq \rho \leq 1$.
- (2) The pair (E, Dec) is called a weak (m, n, ρ) -AMD code if for any $\Delta \in G \setminus \{0\}$ and any random $s \in_R S$ rather than an arbitrary one, the probability $\sum_{s \in S} \Pr(s) \sum_{g \in G_s} \Pr(E(s) = g) \Pr(\mathrm{Dec}(g + \Delta)) \notin \{s, \bot\}) \leq \rho$.
- (3) An AMD code (E, Dec), whether strong or weak, is called *systematic* if $S = A_1$ is an Abelian group, G is an Abelian group $A_1 \times A_2 \times B$, and the encoding has the form

$$E: A_1 \to A_1 \times A_2 \times B \quad \text{with } E(s) = (s, x, f(s, x)) \tag{1}$$

for some function $f: A_1 \times A_2 \to B$ and $x \in_R A_2$. For a systematic AMD code, the decoding function is naturally given by

$$Dec(s, x, t) = \begin{cases} s, & \text{if } t = f(s, x), \\ \bot, & \text{otherwise.} \end{cases}$$
 (2)

Note that by randomly adding $(\delta_1, \delta_2, \delta_3) \in A_1 \times A_2 \times A_3$ to (s, x, f(s, x)), we can make (s, x, f(s, x)) unreadable to the adversary.

Throughout this paper, we fix the following notation for AMD codes.

- An (m, n, ρ) -AMD code is said to have equiprobable sources if $\Pr(s) = \frac{1}{m}$ for any $s \in S$.
- An (m, n, ρ) -AMD code is said to be equiprobable encoding if $\Pr(E(s) = g) = \frac{1}{|G_s|}$ for any $s \in S$ and $g \in G_s$.
- An (m, n, ρ) -AMD code is said to be *uniform* if $|G_s|$ is constant for any $s \in S$.
- A uniform (m, n, ρ) -AMD code with $|G_s| = t$ for $s \in S$ is said to be *t-regular* if it has equiprobable sources and equiprobable encoding.

For a systematic AMD code, if $\Pr(E(s) = (s, x = x_0, f(s, x_0)) \neq 0$ for any $s \in A_1$ and $x \in_R A_2$ that is for any $x_0 \in A_2$, $\Pr(x = x_0) > 0$, then it is $|A_2|$ -uniform. Thus, an equiprobable encoding systematic AMD code with equiprobable sources is $|A_2|$ -regular.

Definition 2 ([10]): The tag size of an (m, n, ρ) -AMD code is $\varpi = \log |G| - \log |S| = \log n - \log m$.

For the convenience of theoretic analysis, for any $u, k \in \mathbb{N}$, define effective tag size as $\varpi^*(k, u) = \min\{\log |G|\} - u$, where the minimum is over all $(|S|, |G|, \rho)$ -AMD codes such that $|S| \ge 2^u$ and $\rho \le 2^{-k}$. In [10], Cramer et al. derived a lower bound for $\varpi^*(k, u)$ as follows.

Lemma 1 ([10]): For any $u, k \in \mathbb{N}$, the effective tag size is lower bounded by

$$\varpi^*(k, u) \ge 2k - 2^{-u+1} \ge 2k - 1$$

for strong AMD codes, and

$$\varpi^*(k,u) \ge k - 2^{-u+1} \ge k - 1$$

for weak AMD codes, respectively.

Besides the above bound for the effective tag size of an AMD code, the following theoretic bounds on the parameters are also known.

Lemma 2 ([30]): For any weak (m, n, ρ) -AMD code, we have

$$\rho \ge \frac{a(m-1)}{m(n-1)},$$

where $a = \sum_{s \in S} |G_s|$.

Especially, for t-regular weak AMD codes, the probability of successful tampering is lower bounded as follows.

Lemma 3 ([30], [32]): For any t-regular weak (m, n, ρ) -AMD codes, we have

$$\rho \ge \left\lceil \frac{t^2 m(m-1)}{n-1} \right\rceil \frac{1}{tm}.\tag{3}$$

Definition 3 ([30], [32]): A weak AMD code is R-optimal if its parameters meet the bound in Lemma 2 with equality (or Lemma 3 for the t-regular case). Here, "R" indicates that random choosing Δ is an optimal strategy for the adversary.

Lemma 4 ([30]): For any strong (m, n, ρ) -AMD code, we have

$$\rho_s \ge \frac{1}{|G_s|} \tag{4}$$

for any source $s \in S$, where ρ_s is the probability of successful tampering given the source $s \in S$ for a random chosen Δ .

Definition 4 ([30]): A strong AMD code is G-optimal if its parameters meet the bound in Lemma 4 with equality. Here, "G" indicates that guessing the most likely encoding is an optimal strategy for the adversary.

III. WEAK ALGEBRAIC MANIPULATION DETECTION CODES FROM HIGHLY NONLINEAR FUNCTIONS

In this section, we propose a construction for systematic weak AMD codes via highly nonlinear functions. Systematic weak AMD codes with asymptotically optimal effective tag size are constructed in Corollaries 1 and 2, and *R*-optimal systematic weak AMD codes are constructed in Corollaries 3 and 5.

First of all, we recall some necessary definitions about nonlinearity of functions.

Let (A, +) and (B, +) be two Abelian groups with order n and m, respectively. Let f be a function from A to B. One robust way to measure the nonlinearity of a function f from A to B is to use the derivatives $D_a(f(x)) = f(x+a) - f(x)$ for $a \in A$, which is closely related to differential cryptanalysis [4], [28].

Definition 5 ([28]): The nonlinearity P_f of a function f from A to B is defined as

$$P_f \triangleq \max_{a \in A \setminus \{0\}} \max_{b \in B} \Pr(D_a(f(x)) = b) = \max_{a \in A \setminus \{0\}} \max_{b \in B} \frac{|\{x \in A : D_a(f(x)) = b\}|}{|A|}, \tag{5}$$

where $Pr(D_a(f(x)) = b)$ denotes the probability of the occurrence of the event $D_a(f(x)) = b$.

Remark 1: The *Hamming distance* between two functions f and g from A to B is defined to be $d(f,g) = |\{x \in A : f(x) \neq g(x)\}|$. A function f is *linear* if and only if f(x+y) = f(x) + f(y) for all $x, y \in A$. A function g is affine if and

only if g = f + b, where f is linear and b is a constant. An alternative method of measuring the nonlinearity of a function $f: A \to B$ is given by the minimum Hamming distance between f and all possible affine functions from f to f [29]. This measure of nonlinearity is closely related to linear cryptanalysis [25]. For the relationship between these two definitions of nonlinearity, the reader is referred to [6], [9], for instances. In this paper, the former definition of nonlinearity is used.

It is easy to check (see, for example, [6]) that $P_f = 1$ if f is a linear function from A to B, and $P_f \ge \frac{1}{|B|}$ for any function f from A to B. The smaller the value of P_f , the higher the corresponding nonlinearity of f.

Definition 6 ([28]): A function f from A to B is said to have perfect nonlinearity if $P_f = \frac{1}{|B|}$.

Construction A: Let f be a function from $A = A_1 \times A_2$ to B and let $S = A_1$ be a subgroup of A. Define a probabilistic encoding map $E_f: A_1 \to G = A \times B = A_1 \times A_2 \times B$ as

$$E_f(S_1) = (S_1, S_2, f(S_1, S_2)), (6)$$

where $S_2 \in_R A_2$.

By the probabilistic encoding map E_f and the corresponding deterministic decoding function given by (2), we can define a systematic AMD code (E_f, Dec) from A_1 to $G = A \times B = A_1 \times A_2 \times B$. Note that a possible successful tampering should satisfy $\Delta \in (A_1 \setminus \{0\}) \times A_2 \times B$. However, for the nonlinearity, we should consider all possible $\Delta \in A_1 \times A_2 \times B \setminus \{(0,0,0)\}$. Thus, to the convenience of analysis, we introduce the partial nonlinearity of a function, which only considers the case $\Delta \in (A_1 \setminus \{0\}) \times A_2 \times B$.

Definition 7: The partial nonlinearity $\Psi_f(A_1)$ of a function f from $A = A_1 \times A_2$ to B is defined as

$$\Psi_f(A_1) \triangleq \max_{a_1 \in A_1 \setminus \{0\}} \max_{a_2 \in A_2} \max_{b \in B} \Pr\left(D_{(a_1, a_2)}(f(x)) = b\right) = \max_{a_1 \in A_1 \setminus \{0\}} \max_{a_2 \in A_2} \max_{b \in B} \frac{|\{x \in A : D_{(a_1, a_2)}(f(x)) = b\}|}{|A|}, \quad (7)$$

where A_1 is a subgroup of A and $\Pr\left(D_{(a_1,a_2)}(f(x))=b\right)$ denotes the probability of the occurrence of the event $D_{(a_1,a_2)}(f(x))=b$.

Remark 2: By Definitions 5 and 7, we have $P_f \ge \Psi_f(A_1)$ for any subgroup A_1 of A.

The parameters of the constructed AMD code have the following relationship with the nonlinearity of f.

Theorem 1: If the function f from $A = A_1 \times A_2$ to B with partial nonlinearity $\Psi_f(A_1)$ has equiprobable sources and E_f is equiprobable encoding, then the systematic weak AMD code (E_f, Dec) generated by Construction A has parameters $(n_1, n_1 n_2 m, \Psi_f(A_1) \leq P_f)$, where $|A_1| = n_1$, $|A_2| = n_2$, |B| = m and P_f denotes the nonlinearity of f.

Proof. By Construction A, we only need to prove that the probability of successful tampering is upper bounded by $\Psi_f(A_1)$. For any $\Delta = (a_1, a_2, b) \in G = A_1 \times A_2 \times B$ with $a_1 \in A_1 \setminus \{0\}$, $a_2 \in A_2$ and $b \in B$,

$$\sum_{S_{1} \in A_{1}} \Pr(S' = S_{1}) \sum_{S_{2} \in A_{2}} \Pr(E_{f}(S_{1}) = (S_{1}, S_{2}, f(S_{1}, S_{2})) \Pr(\operatorname{Dec}((S_{1}, S_{2}, f(S_{1}, S_{2})) + \Delta) \notin \{S_{1}, \bot\})$$

$$= \sum_{S_{1} \in A_{1}} \Pr(S' = S_{1}) \sum_{S_{2} \in A_{2}} \Pr(S^{*} = S_{2}) \Pr(\operatorname{Dec}((S_{1}, S_{2}, f(S_{1}, S_{2})) + \Delta) \notin \{S_{1}, \bot\})$$

$$= \sum_{S_{1} \in A_{1}} \frac{1}{|A_{1}|} \sum_{S_{2} \in A_{2}} \frac{1}{|A_{2}|} \Pr(f(S_{1} + a_{1}, S_{2} + a_{2}) = f(S_{1}, S_{2}) + b)$$

$$= \frac{1}{|A_{1}||A_{2}|} \sum_{S_{1} \in A_{1}} \sum_{S_{2} \in A_{2}} \Pr(f(S_{1} + a_{1}, S_{2} + a_{2}) = f(S_{1}, S_{2}) + b)$$

$$= \frac{1}{|A_{1}||A_{2}|} |\{(S', S^{*}) \in A : f(S' + a_{1}, S^{*} + a_{2}) - f(S', S^{*}) = b\}|$$

$$\leq \Psi_{f}(A_{1}) \leq P_{f}.$$
(8)

According to Definitions 1 and 7, we know that the systematic weak AMD code (E_f, Dec) generated by Construction A has parameters $(n_1, n_1 n_2 m, \Psi_f(A_1) \leq P_f)$, which completes the proof.

In what follows, we list some well-known highly nonlinear functions and their corresponding systematic AMD codes as applications of Construction A.

A. Linear functions

One simple but useful way to obtain functions with high nonlinearity is to use linear functions from $(\mathbb{F}_{q^r},+)$ to $(\mathbb{F}_q,+)$ as functions from $(\mathbb{F}_{q^r}^*,\times)\cong(\mathbb{Z}_{q^r-1},+)$ to $(\mathbb{F}_q,+)$.

Lemma 5 ([6]): Any nonzero linear function L from $(\mathbb{F}_{q^r},+)$ to $(\mathbb{F}_q,+)$ is a function from $(\mathbb{F}_{q^r}^*,\times)$ to $(\mathbb{F}_q,+)$ with nonlinearity $P_f=\frac{1}{q}+\frac{1}{q(q^r-1)}$.

Applying the highly nonlinear functions in Lemma 5, the following corollary follows directly from Construction A and Theorem 1.

Corollary 1: Let $A = (\mathbb{Z}_{q^r-1}, +)$ and $B = (\mathbb{F}_q, +)$. Further let $A_1 = (\mathbb{Z}_{m_1}, +)$ and $A_2 = (\mathbb{Z}_{m_2}, +)$ be two subgroups of A with order m_1 and $m_2 = \frac{q^r-1}{m_1}$, respectively. If $\gcd(m_1, m_2) = 1$, then $A \cong A_1 \times A_2$. Define the probabilistic encoding map E_L from A_1 to $G = A_1 \times A_2 \times B$ as

$$E_L(s_1) = (s_1, s_2, L(\Phi(s_1, s_2))), \tag{9}$$

where $s_2 \in_R A_2$, Φ is an isomorphism from \mathbb{Z}_{q^r-1} to $(\mathbb{F}_{q^r}^*, \times)$, and L(x) is a nonzero linear function from $(\mathbb{F}_{q^r}^*, \times)$ to $(\mathbb{F}_q, +)$. If the systematic weak AMD code (E_L, Dec) given by (9) and (2) has equiprobable sources and E_L is equiprobable encoding, then it is an m_2 -regular $(m_1, (q^r - 1)q, \frac{1}{q} + \frac{1}{q(q^r - 1)})$ -AMD code.

Corollary 2: Let $r \in \mathbb{N}$ and m_2 be a factor of $q^r - 1$. Further let $m_1 = \frac{q^r - 1}{m_2}$, $u = \lfloor \log m_1 \rfloor$, and $k = \lfloor \log \frac{q^r - 1}{q^{r-1}} \rfloor$. If $\gcd(m_1, m_2) = 1$, then the effective tag size $\varpi^*(k, u)$ for weak AMD codes satisfies

$$k-1 \le \varpi^*(k,u) < k+1 + \log \frac{m_2 q^r}{q^r - 1}.$$

The systematic weak AMD code in Corollary 1 has an asymptotically optimal effective tag size with respect to the bound in Lemma 1, i.e., $\lim_{k\to\infty} \frac{\log |G| - \log |A_1|}{k-1} = 1$.

Proof: By Corollary 1, there exists a systematic weak AMD code with $|A_1| = \frac{q^r - 1}{m_2} \ge 2^u$, $\rho = \frac{q^{r-1}}{q^r - 1} \le 2^{-k}$, and the tag size

$$\varpi = \log |G| - \log |A_1| = \log(m_2 q) = \log m_2 + \log q.$$

Note that

$$\varpi - k = \log m_2 + \log q - \left| \log \frac{q^r - 1}{q^{r-1}} \right| < 1 + \log m_2 + \log \frac{q^r}{q^r - 1}.$$

The first conclusion then follows from the fact that for any $k, u \in \mathbb{N}$, we have $k-1 \le \varpi^*(k, u) \le \varpi$. The second conclusion can be derived by the fact that

$$\lim_{k \to \infty} \frac{\log |G| - \log |A_1|}{k - 1} = \lim_{q \to \infty} \frac{1 + \log m_2 + \log \frac{q^r}{q^r - 1} + k}{k - 1} = 1$$

by noting that $m_2 \in \mathbb{N}$ is a constant.

B. Maiorana-McFarland's class of functions

Let $r \in \mathbb{N}$ and q be a prime power. Define a function $f: (\mathbb{F}_q^{2r}, +) \to (\mathbb{F}_q, +)$ as

$$f(x_1, x_2, \dots, x_{2r}) = \sum_{1 \le i \le r} x_i x_{i+r}.$$
 (10)

Lemma 6 ([26]): The function $f(x_1, x_2, \ldots, x_{2r})$ defined by (10) has perfect nonlinearity $P_f = \frac{1}{q}$.

Corollary 3: Let $A_1 = (\mathbb{F}_{q^{2r-1}}, +)$ and $A_2 = B = (\mathbb{F}_q, +)$, where we regard an element of $\mathbb{F}_{q^{2r-1}}$ as a vector in \mathbb{F}_q^{2r-1} . Define the probabilistic encoding map E_f from A_1 to $G = A_1 \times A_2 \times B$ as

$$E_f(S_1) = (S_1, s_2, f(S_1, s_2)) = \left(x_1, x_2, \dots, x_{2r-1}, s_2, x_r s_2 + \sum_{1 \le i \le r-1} x_i x_{i+r}\right),\tag{11}$$

where $S_1 = (x_1, x_2, \dots, x_{2r-1}) \in A_1$, $s_2 \in_R A_2$, and f is defined by (10). If the systematic weak AMD code (E_f, Dec) given by (11) and (2) has equiprobable sources and E_f is equiprobable encoding, then it is an R-optimal q-regular $(q^{2r-1}, q^{2r+1}, \frac{1}{q})$ -AMD code with respect to the bound in Lemma 3.

Proof. The statement that the constructed AMD code (E_f, Dec) has parameters $(q^{2r-1}, q^{2r+1}, \frac{1}{q})$ directly follows from Theorem 1, Lemma 6, and the fact that it is q-regular. According to Lemma 3, we should have

$$\rho \ge \left\lceil \frac{q^2 q^{2r-1} (q^{2r-1} - 1)}{q^{2r+1} - 1} \right\rceil \frac{1}{q^{2r}} = \frac{1}{q},$$

which means that the constructed AMD code is R-optimal.

Corollary 4: For any $k, u \in \mathbb{N}$, the effective tag size $\varpi^*(k, u)$ for weak AMD codes is bounded as follows:

$$k - 1 \le \varpi^*(k, u) \le 2k.$$

Proof: For any given k and u, choose $q=2^k$ and r to be the smallest positive integer such that $u \leq k(2r-1)$. According to Corollary 3, there exists a systematic weak AMD code with $|A_1| = q^{2r-1} \geq 2^u$, $\rho = \frac{1}{q} \leq 2^{-k}$, and the tag size $\varpi = \log |G| - \log |A_1| = 2 \log q = 2k$. Then the claim follows from the fact that $k-1 \leq \varpi^*(k,u) \leq \varpi$.

C. Dillon's class of functions

In this subsection, we recall the well-known Dillon's class of functions with perfect nonlinearity. A function $g:A\to B$ is balanced if the size of $g^{-1}(b)$ is the same for every $b\in B$, which is |A|/|B|. It is known (see, for example, [6]) that g has perfect nonlinearity if and only if for every $a\in A\setminus\{0\}$, the derivative $D_a(g(x))$ is balanced, and this is possible only when |B| divides |A|.

Lemma 7 ([14]): For any $r \in \mathbb{N}$, let \mathbb{F}_q^r be identified with the finite field \mathbb{F}_{q^r} and let g be any balanced function from \mathbb{F}_{q^r} to \mathbb{F}_q . Then the function $f: (\mathbb{F}_{q^{2r}}, +) \to (\mathbb{F}_q, +)$ defined by

$$f(x,y) = g(xy^{q^r-2}), \quad x, y \in \mathbb{F}_{q^r}$$

has perfect nonlinearity $P_f = \frac{1}{a}$.

Corollary 5: Let $A_1 = (\mathbb{F}_{q^{2r-1}}, +)$ and $A_2 = B = (\mathbb{F}_q, +)$, where we regard an element of $\mathbb{F}_{q^{2r-1}}$ as a vector in \mathbb{F}_q^{2r-1} . Let $g: \mathbb{F}_{q^r} \to \mathbb{F}_q$ be a balanced function. Define the probabilistic encoding map E_f from A_1 to $G = A_1 \times A_2 \times B$ as

$$E_f(S_1) = (S_1, y_r, f(X, Y)) = \left(x_1, x_2, \dots, x_r, y_1, y_2, \dots, y_{r-1}, y_r, g(XY^{q^r-2})\right),$$
(12)

where $S_1=(x_1,x_2,\ldots,x_r,y_1,y_2,\ldots,y_{r-1})\in A_1,\ y_r\in_R A_2,\ X=(x_1,x_2,\ldots,x_r),$ and $Y=(y_1,y_2,\ldots,y_r).$ If the systematic weak AMD code (E_f,Dec) given by (12) and (2) has equiprobable sources and E_f is equiprobable encoding, then it is an R-optimal q-regular $(q^{2r-1},q^{2r+1},\frac{1}{q})$ -AMD code with respect to the bound in Lemma 3.

The proof of Corollary 5 is similar to that of Corollary 3 so we omit it here. Note that although the parameters of the systematic weak AMD codes constructed in Corollaries 3 and 5 are the same, their probabilistic encoding maps are different.

IV. STRONG ALGEBRAIC MANIPULATION DETECTION CODES FROM HIGHLY NONLINEAR FUNCTIONS

In this section, we consider the systematic strong AMD codes generated by Construction A via highly nonlinear functions. We first analyze the relationship between the nonlinearity of the function f and the probability of successful tampering in Theorem 2. By choosing some special functions, we construct systematic strong AMD codes as examples in Corollaries 6 and 7.

By the probabilistic encoding map E_f given by (6) and the corresponding decoding function given by (2), we can define a systematic AMD code (E_f , Dec) from A_1 to $G = A \times B = A_1 \times A_2 \times B$. In what follows, we first analyze the relationship between parameters of the strong AMD code generated by Construction A and the nonlinearity of f.

Theorem 2: Let f be a function from $A = A_1 \times A_2$ to B with nonlinearity P_f and partial nonlinearity $\Psi_f(A_1)$, where $|A_1| = n_1, |A_2| = n_2$, and |B| = m. For the equiprobable encoding case, the systematic strong AMD code (E_f, Dec) generated by Construction A has parameters $(n_1, n_1 n_2 m, \rho)$ if and only if for any given $S_1 \in A_1$,

$$\frac{|\{S^* \in A_2 : f(S_1 + a_1, S^* + a_2) = f(S_1, S^*) + b\}|}{|A_2|} \le \rho \tag{13}$$

holds for any $\Delta = (a_1, a_2, b) \in (A_1 \setminus \{0\}) \times A_2 \times B$. The parameter ρ satisfies that $\rho \geq \Psi_f(A_1) \geq \frac{1}{|B|}$. Furthermore, if f is a perfect nonlinear function, then we have $\rho \geq P_f$.

Proof. By Construction A and Definition 1, the AMD code (E_f, Dec) generated by Construction A has parameters $(n_1, n_1 n_2 m, \rho)$ if and only if for any $S_1 \in A_1$,

$$\Pr(\operatorname{Dec}(E_f(S_1) + \Delta) \notin \{S_1, \bot\}) \le \rho \tag{14}$$

holds for any $\Delta = (a_1, a_2, b) \in G$ with $a \in A_1 \setminus \{0\}, a_2 \in A_2$, and $b \in B$. But

$$\Pr(\operatorname{Dec}(E_f(S_1) + \Delta) \notin \{S_1, \bot\}) = \sum_{S^* \in A_2} \Pr(S_2 = S^*) \Pr(f(S_1 + a_1, S^* + a_2) = f(S_1, S^*) + b)$$

$$= \sum_{S^* \in A_2} \frac{1}{|A_2|} \Pr(f(S_1 + a_1, S^* + a_2) = f(S_1, S^*) + b)$$

$$= \frac{|\{S^* \in A_2 : f(S_1 + a_1, S^* + a_2) = f(S_1, S^*) + b\}|}{|A_2|},$$

where the second equality follows from the fact that E_f is equiprobable encoding. Therefore, (14) is equivalent with (13).

Now we prove $\rho \geq \Psi_f(A_1)$. Clearly, there exists a fixed $\Delta = (a_1, a_2, b) \in (A_1 \setminus \{0\}) \times A_2 \times B$ such that $\Psi_f(A_1) = \Pr(D_{(a_1, a_2)}(f(S_1, S_2)) = b)$. Then

$$\begin{array}{lcl} \Psi_f(A_1) & = & \Pr(D_{(a_1,a_2)}(f(S_1,S_2)=b) \\ & = & \frac{\sum\limits_{S_1\in A_1}|\{S^*\in A_2:\ f(S_1+a_1,S^*+a_2)=f(S_1,S^*)+b\}|}{|A_1||A_2|} \\ & \leq & \frac{\sum\limits_{S_1\in A_1}\rho}{|A_1|} \\ & = & \rho. \end{array}$$

For any $a_1 \in A_1 \setminus \{0\}$ and $a_2 \in A_2$,

$$\max_{b \in B} \frac{|\{(S', S^*) \in A_1 \times A_2 : D_{(a_1, a_2)}(f(S', S^*)) = b\}|}{|A_1||A_2|}$$

$$\geq \frac{\sum_{b \in B} \frac{|\{(S', S^*) \in A_1 \times A_2 : D_{(a_1, a_2)}(f(S', S^*)) = b\}|}{|A_1||A_2|}}{|B|}$$

$$= \frac{1}{|B|}$$

implies that $\Psi_f(A_1) \geq \frac{1}{|B|}$.

At last, if f is a perfect nonlinear function, then we have $\frac{1}{|B|} = P_f \ge \Psi_f(A_1) \ge \frac{1}{|B|}$ according to Remark 2. Thus, we have $\rho \ge \Psi_f(A_1) = P_f = \frac{1}{|B|}$.

In what follows, we list a few systematic strong AMD codes with $\rho = \frac{1}{|B|}$. Especially, we include the classes of Maiorana-McFarland functions and Dillon functions to construct such AMD codes.

Based on Theorem 2 and Lemma 4, we have the following corollary. Herein we highlight that this explicit construction was first introduced in [21]. We recall it as an application of our generic construction and the prove is only for completeness.

Corollary 6 ([21]): Let $A_1 = A_2 = \mathbb{F}_{q^r}$ and $B = \mathbb{F}_q$, where we regard an element of \mathbb{F}_{q^r} as a vector in \mathbb{F}_q^r . Define the probabilistic encoding map E_f from A_1 to $G = A_1 \times A_2 \times B$ as

$$E_f(S_1) = (S_1, S_2, f(S_1, S_2)) = \left(x_1, x_2, \dots, x_r, x_{r+1}, \dots, x_{2r}, \sum_{1 \le i \le r} x_i x_{i+r}\right),$$

where $S_1=(x_1,x_2,\ldots,x_r)\in A_1$, $S_2=(x_{r+1},x_{r+2},\ldots,x_{2r})\in R$ A_2 , and f is defined by (10). Then, for the equiprobable encoding case, the systematic strong AMD code given by E_f has parameters $(q^r,q^{2r+1},\frac{1}{q})$, where $\rho=\frac{1}{q}$ is minimum with respect to Theorem 2. Especially, when r=1, the q-regular AMD code is G-optimal with respect to the bound in Lemma 4.

Proof: We first prove $\rho = P_f = \frac{1}{q}$. By Theorem 2, we only need to prove that for any $S_1 \in \mathbb{F}_{q^r}$, $a_1 \in \mathbb{F}_{q^r} \setminus \{0\}$, $a_2 \in \mathbb{F}_{q^r}$, and $b \in \mathbb{F}_q$,

$$\frac{|\{S_2 \in \mathbb{F}_{q^r} : f(S_1 + a_1, S_2 + a_2) = f(S_1, S_2) + b\}|}{q^r} \le \frac{1}{q},\tag{15}$$

i.e., $f(S_1 + a_1, S_2 + a_2) = f(S_1, S_2) + b$ has at most q^{r-1} solutions for $S_2 \in \mathbb{F}_{q^r}$. Since $a_1 \neq 0$, without loss of generality, we may assume $a_1 = (a_{11}, a_{12}, \dots, a_{1r})$ with $a_{1r} \neq 0$. Note that

$$f(S_1 + a_1, S_2 + a_2) - f(S_1, S_2) - b = h(S + \Delta) - h(S) + (x_r + a_{1r})(x_{2r} + a_{2r}) - x_r x_{2r} - b$$

$$= h(S + \Delta) - h(S) + a_{1r} x_{2r} + a_{2r} x_r + a_{1r} a_{2r} - b,$$
(16)

where

$$h(S) = h(S_1, S_2) = h(x_1, x_2, \dots, x_{2r}) \triangleq \begin{cases} \sum_{1 \le i \le r-1} x_i x_{i+r}, & r \ge 2, \\ 0, & r = 1, \end{cases}$$

 $\Delta = (a_1, a_2) = (a_{11}, a_{12}, \dots, a_{1r}, a_{21}, a_{22}, \dots a_{2r}), \text{ and } a_2 = (a_{21}, a_{22}, \dots, a_{2r}). \text{ For any given } (x_1, x_2, \dots, x_{2r-1}) \in \mathbb{F}_q^{2r-1}, \\ a_1 \in \mathbb{F}_{q^r} \setminus \{0\}, \text{ and } a_2 \in \mathbb{F}_{q^r}, \text{ the fact } h(S + \Delta) - h(S) + a_{1r}x_{2r} + a_{2r}x_r + a_{1r}a_{2r} - b = 0 \text{ has at most one solution } x_{2r} \in \mathbb{F}_q \text{ implies that (16) has at most } q^{r-1} \text{ solutions for all possible } S_2 \in \mathbb{F}_{q^r}, \text{ i.e., (15) holds. Then } \rho = P_f = \frac{1}{q} \text{ by Theorem 2.}$

The second assertion is obvious from the definitions.

Remark 3: (1) For more general form of functions with perfect nonlinearity, similar to f in (10), the interested reader is referred to [9], [20], [22], [26].

Recalling the well-known Dillon's class of functions with perfect nonlinearity in Lemma 7, we have the following corollary. Note that the trace function $\operatorname{Tr}_q^{q^r}: \mathbb{F}_{q^r} \to \mathbb{F}_q$ defined by $\operatorname{Tr}_q^{q^r}(x) = \sum_{0 \le i \le r-1} x^{q^i}$ is a balanced function.

Corollary 7: Let $A_1 = A_2 = \mathbb{F}_{q^r}$ and $B = \mathbb{F}_q$, where we regard an element of \mathbb{F}_{q^r} as a vector in \mathbb{F}_q^r . Let $\{\alpha_1, \alpha_2, \dots, \alpha_r\}$ and $\{\beta_1, \beta_2, \dots, \beta_r\}$ be a pair of dual bases of \mathbb{F}_{q^r} over \mathbb{F}_q , that is,

$$\operatorname{Tr}_{q}^{q^{r}}(\alpha_{i}\beta_{j}) = \begin{cases} 1, & i = j, \\ 0, & \text{otherwise.} \end{cases}$$
 (17)

Define $f: (\mathbb{F}_{q^{2r}}, +) \to (\mathbb{F}_q, +)$ as $f(x, y) = \operatorname{Tr}_q^{q^r}(\hat{x}^{q^r-2}\hat{y})$, where $x = (x_1, x_2, \dots, x_r) \in \mathbb{F}_{q^r}, \ y = (y_1, y_2, \dots, y_r) \in \mathbb{F}_{q^r}$, $\hat{x} = \sum_{1 \le i \le r} x_i \alpha_i$, and $\hat{y} = \sum_{1 \le i \le r} y_i \beta_i$. Define the probabilisitic encoding map E_f from A_1 to $G = A_1 \times A_2 \times B$ as

$$E_f(S_1) = (S_1, S_2, f(S_1, S_2)),$$

where $S_1=(x_1,x_2,\ldots,x_r)\in A_1,\ S_2=(y_1,y_2,\ldots,y_r)\in_R A_2$. Then, for the equiprobable encoding case, the systematic strong AMD code given by E_f has parameters $(q^r,q^{2r+1},\frac{1}{q})$, where $\rho=\frac{1}{q}$ is minimum with respect to Theorem 2. Especially, when r=1, the q-regular AMD code is G-optimal with respect to the bound in Lemma 4.

Proof: To prove $\rho = \frac{1}{q}$, according to Theorem 2, it suffices to prove that for any $S_1 \in \mathbb{F}_{q^r}$, $a_1 = (a_{11}, \dots, a_{1r}) \in \mathbb{F}_{q^r} \setminus \{(0, 0, \dots, 0)\}, a_2 = (a_{21}, \dots, a_{2r}) \in \mathbb{F}_{q^r}$, and $b \in \mathbb{F}_q$,

$$\frac{|\{S_2 \in \mathbb{F}_{q^r}: \ f(S_1 + a_1, S_2 + a_2) = f(S_1, S_2) + b\}|}{q^r} \le \frac{1}{q},$$

i.e., $f(S_1+a_1,S_2+a_2)=f(S_1,S_2)+b$ has at most q^{r-1} solutions for $S_2\in\mathbb{F}_{q^r}$. Let

$$\hat{S}_1^{q^r - 2} = \sum_{1 \le i \le r} x'_{1i} \alpha_i, \tag{18}$$

$$(\hat{S}_1 + \sum_{1 \le i \le r} a_{1i}\alpha_i)^{q^r - 2} = \sum_{1 \le i \le r} x_{1i}^*\alpha_i, \tag{19}$$

and

$$a' = (a'_{11} = x^*_{11} - x'_{11}, \dots, a'_{1r} = x^*_{1r} - x'_{1r}).$$
(20)

Since $a_1 \neq (0, 0, \dots, 0)$ and x^{q^r-2} is a non-identity permutation of \mathbb{F}_{q^r} , we have $a' \neq (0, 0, \dots, 0)$. Without loss of generality, we may assume $a'_{11} \neq 0$. By (17)-(20),

$$f(S_1 + a_1, S_2 + a_2) - f(S_1, S_2) - b$$

$$= y_1 \left(\operatorname{Tr}_q^{q^r} \left(\beta_1 \left(\hat{S}_1 + \sum_{1 \le i \le r} a_{1i} \alpha_i \right)^{q^r - 2} \right) - \operatorname{Tr}_q^{q^r} \left(\beta_1 \hat{S}_1^{q^r - 2} \right) \right) + C(S, a_1, a_2, b)$$

$$= a'_{11} y_1 + C(S, a_1, a_2, b),$$

where $S=(x_1,x_2,\ldots,x_r,y_2,\ldots,y_r)\in\mathbb{F}_{q^{2r-1}}$ and $C(S,a_1,a_2,b)$ is a constant determined by $S,\,a_1,\,a_2,$ and b. Thus, the fact $a'_{11}\neq 0$ means that $f(S_1+a_1,S_2+a_2)-f(S_1,S_2)-b=0$ has at most q^{r-1} solutions for $S_2\in\mathbb{F}_{q^r}$, which completes the proof.

V. HIGHLY NONLINEAR FUNCTIONS FROM ALGEBRAIC MANIPULATION DETECTION CODES

By Theorems 1 and 2, we can construct systematic AMD codes from known highly nonlinear functions for both weak and strong attack models. In this section, we further analyze the relationship between AMD codes and highly nonlinear functions. Specially, we try to construct highly nonlinear functions from some given systematic AMD codes. Note that a strong (m, n, ρ) -AMD code is always a weak (m, n, ρ) -AMD code. Thus, throughout this section, we only consider the functions derived from weak AMD codes.

Let A_1 , A_2 and B be Abelian groups. For a given systematic AMD code with probabilisitic encoding map $E: A_1 \to A_1 \times A_2 \times B$,

$$E(s_1) = (s_1, s_2, t_{s_1, s_2}), \quad s_1 \in A_1, \ s_2 \in A_2,$$

define a function f_E from $A_1 \times A_2$ to B as

$$f_E(s_1, s_2) = t_{s_1, s_2}. (21)$$

Theorem 3: Let $E: A_1 \to A_1 \times A_2 \times B$ be the probabilisitic encoding map of a systematic regular weak AMD code with parameters (m, n, ρ) , where $m = |A_1|$ and $n = |A_1||A_2||B|$. Then the function $f_E: A_1 \times A_2 \to B$ has nonlinearity

$$P_{f_E} \le \max\{\{\rho\} \cup \{P_{f_{E,\sigma'}}: s' \in A_1\}\},\$$

where $f_{E,s'}(x) \triangleq f_E(s',x)$ is a function from A_2 to B defined by f_E and $s' \in A_1$, and $P_{f_{E,s'}}$ denotes the nonlinearity of $f_{E,s'}$.

Proof: Let $\rho_{(\Delta_1,\Delta_2,\Delta_3)}$ denote the probability of successful tampering $(\Delta_1,\Delta_2,\Delta_3) \in (A_1 \setminus \{0\}) \times A_2 \times B$. Then

$$\begin{split} & \rho \geq \max \left\{ \rho_{(\Delta_1, \Delta_2, \Delta_3)} : \ (\Delta_1, \Delta_2, \Delta_3) \in (A_1 \setminus \{0\}) \times A_2 \times B \right\} \\ & = \max_{\Delta_1 \in A_1 \setminus \{0\}} \max_{\Delta_2 \in A_2} \max_{\Delta_3 \in B} \left\{ \sum_{s' \in A_1} \Pr(s_1 = s') \sum_{s^* \in A_2} \Pr(s_2 = s^*) \Pr(f_E(s' + \Delta_1, s^* + \Delta_2) = f_E(s', s^*) + \Delta_3) \right\} \\ & = \max_{\Delta_1 \in A_1 \setminus \{0\}} \max_{\Delta_2 \in A_2} \max_{\Delta_3 \in B} \left\{ \sum_{s' \in A_1} \frac{1}{|A_1|} \sum_{s^* \in A_2} \frac{1}{|A_2|} \Pr(f_E(s' + \Delta_1, s^* + \Delta_2) = f_E(s', s^*) + \Delta_3) \right\} \\ & = \max_{\Delta_1 \in A_1 \setminus \{0\}} \max_{\Delta_2 \in A_2} \max_{\Delta_3 \in B} \left\{ \frac{|\{(s', s^*) \in A_1 \times A_2 : \ f_E(s' + \Delta_1, s^* + \Delta_2) = f_E(s', s^*) + \Delta_3\}|}{|A_1||A_2|} \right\}. \end{split}$$

Meanwhile, for $\Delta_1 = 0$ and $\Delta_2 \in A_2 \setminus \{0\}$, we define

$$\rho_{(0,\Delta_2,\Delta_3)} \triangleq \frac{|\{(s',s^*) \in A_1 \times A_2 : f_E(s',s^* + \Delta_2) = f_E(s',s^*) + \Delta_3\}|}{|A_1||A_2|}.$$

Then

$$\max_{\Delta_2 \in A_2 \setminus \{0\}} \max_{\Delta_3 \in B} \rho_{(0,\Delta_2,\Delta_3)}$$

$$= \max_{\Delta_2 \in A_2 \setminus \{0\}} \max_{\Delta_3 \in B} \sum_{s' \in A_1} \frac{|\{(s',s^*) \in A_1 \times A_2 : f_{E,s'}(s^* + \Delta_2) = f_{E,s'}(s^*) + \Delta_3\}|}{|A_1||A_2|}$$

$$\leq \max\{P_{f_{E,s'}} : s' \in A_1\},$$

where the last inequality comes from the fact that

$$P_{f_{E,s'}} = \max_{\Delta_2 \in A_2 \setminus \{0\}} \max_{\Delta_3 \in B} \frac{|\{(s', s^*) \in A_1 \times A_2 : f_{E,s'}(s^* + \Delta_2) = f_{E,s'}(s^*) + \Delta_3\}|}{|A_2|}.$$

Therefore,

$$P_{f_E} = \max_{(\Delta_1, \Delta_2) \in A_1 \times A_2 \setminus \{(0,0)\}} \max_{\Delta_3 \in B} \frac{|\{(s', s^*) \in A_1 \times A_2 : f(s' + \Delta_1, s^* + \Delta_2) = f(s', s^*) + \Delta_3\}|}{|A_1||A_2|}$$

$$= \max\{\max\{\rho_{(\Delta_1, \Delta_2, \Delta_3)} : \Delta_1 \in A_1 \setminus \{0\}, \Delta_2 \in A_2, \Delta_3 \in B\}, \max\{\rho_{(0, \Delta_2, \Delta_3)} : \Delta_2 \in A_2 \setminus \{0\}, \Delta_3 \in B\}\}\}$$

$$\leq \max\{\{\rho\} \cup \{P_{f_{E, s'}} : s' \in A_1\}\}.$$

Generally speaking, from a systematic AMD code we can not determine the nonlinearity of the function f_E directly. It is also related with the nonlinearity of some functions with restricted input [27]. This is mainly because that in an AMD code, we do not regard the case $Dec(E(s) + \Delta) = s$ as an adversary's successful tampering, as shown in Theorem 3. However, for a stronger setting [11], [21], [34], [35] also named as *stronger AMD code*, the case $Dec(E(s) + \Delta) \neq \bot$ is regarded as an adversary's successful tampering. In this setting, we directly have the following result. The proof is similar, so we omit it here.

Theorem 4: Let $E: A_1 \to A_1 \times A_2 \times B$ be the probabilisitic encoding map of a systematic regular weak AMD code. If

$$\Pr(\operatorname{Dec}(E(s) + \Delta) \neq \bot) \leq \rho,$$

i.e., it forms a stronger AMD code, then the function $f_E:A_1\times A_2\to B$ defined as (21) has nonlinearity $P_{f_E}\le \rho$.

As an application of Theorem 3, we analyse the functions derived from the systematic q-regular strong AMD codes in [10, Theorem 2].

Corollary 8: Let q be a power of a prime p, and t > 0 be an integer such that $p \nmid (t+2)$. Let (E_h, Dec) be the known systematic q-regular strong AMD codes in [10, Theorem 2] with parameters $(q^t, q^{t+2}, \frac{t+1}{q})$, where the probabilistic encoding map $E_h : \mathbb{F}_{q^t} \to \mathbb{F}_{q^t} \times \mathbb{F}_q$ is given by

$$E_h(S = (s_1, s_2, \dots, s_t)) = (S, x, h(S, x))$$

with $x \in_R \mathbb{F}_q$ and

$$h(S,x) = x^{t+2} + \sum_{1 \le t \le t} s_i x^i.$$
 (22)

Then the function h(S,x) can be viewed as a function from $(\mathbb{F}_{q^{t+1}},+)$ to $(\mathbb{F}_q,+)$ with nonlinearity $P_h \leq \frac{t+1}{q}$, where we regard elements of $\mathbb{F}_{q^{t+1}}$ as vectors in \mathbb{F}_q^{t+1} .

Proof: According to Theorem 3, it suffices to prove that for any given $S_1 \in \mathbb{F}_{q^t}$, $P_{h_{(E,S_i)}} \leq \frac{t+1}{q}$ holds, where $h_{(E,S_1)}(x) = h(S_1,x)$ is a function from \mathbb{F}_q to \mathbb{F}_q . By (22), for any given $S_1 = (s_1,s_2,\ldots,s_t) \in \mathbb{F}_{q^t}$ and $\Delta \in \mathbb{F}_q \setminus \{0\}$, we have

$$h_{(E,S_1)}(x+\Delta) - h_{(E,S_1)}(x) = (x+\Delta)^{t+2} - x^{t+2} + \sum_{1 \le i \le t} s_i((x+\Delta)^t - x^t) = R_{(S_1,\Delta)}(x),$$

where $deg(R_{(S_1,\Delta)}(x)) = t+1$, since $p \nmid (t+2)$. Thus, for any $S_1 \in \mathbb{F}_{q^t}$,

$$P_{h_{(E,S_1)}} = \max_{\Delta \in \mathbb{F}_q \setminus \{0\}} \max_{b \in \mathbb{F}_q} \frac{|\{x \in \mathbb{F}_q : R_{(S_1,\Delta)}(x) = b\}|}{q} \le \frac{t+1}{q},$$

which completes the proof.

Remark 4: By Corollary 8, we know that the systematic AMD codes in Theorem 2 of [10] can also be explained by highly nonlinear functions.

VI. CONCLUDING REMARKS

In this paper, we investigated the relationship between systematic AMD codes and highly nonlinear functions. Highly nonlinear functions were used to construct systematic AMD codes. By carefully choosing highly nonlinear functions, optimal systematic AMD codes, some with asymptotically optimal tag size, can be generated. Highly nonlinear functions were also constructed via systematic AMD codes.

However, in general, it is still an open problem whether we can find highly nonlinear functions to generate optimal AMD codes with optimal tag size. If it is possible, then how to construct such kinds of highly nonlinear functions is another interesting topic for future research. According to Theorem 2, the probability of successful tampering for a systematic strong AMD code is lower bounded by the perfect nonlinearity of the corresponding function. For the general case, how to construct strong AMD codes via highly nonlinear functions achieving this bound is also widely open.

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