Preferred Z-Complementary Pairs and Their Application in Doppler Resilient Waveform Design

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Abstract Z-complementary pairs (ZCPs) are well-known, but few work has dedicated to their aperiodic cross-correlation. One objective of this paper is to propose a novel class of sequence pairs, called "preferred Z-complementary pairs (PZCPs)", where each sequence pair has Z-complementary property, and the aperiodic cross-correlation between the two sequences in each pair are zeros within a certain region. Some constructions of PZCPs from Golay complementary pairs (GCPs) are presented. Another objective of this paper is to apply PZCPs to design Doppler resilient waveforms combined with equal sums of powers (ESPs). Simulation results show that the proposed waveform has good Doppler tolerance.

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1 Introduction

In 1950's, when Golay studied multislit spectrometry, he first introduced the concept of Golay complementary pairs (GCPs). Subsequently, binary GCPs are introduced in detail in [1]. Due to its good aperiodic auto-correlation,

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researchers have done a lot of work in studying the properties and systematic constructions of GCPs [2–5]. Especially in [4], Davis and Jedwab proposed a direct construction of GCPs with length 2^m based on generalized Boolean functions (GBFs). This is a pioneering work. Later, their construction was extended to GCPs over QAM constellations by using weighted sum of several QPSK GCPs [6]. However, the length of GCPs is very limited. For example, binary GCPs are available only for lengths of form $2^{\alpha}10^{\beta}26^{\gamma}$ (where α, β and γ are non-negative integers) [5].

Z-complementary pair (ZCP) is a natural extension of GCPs [7]. It requires the sums of aperiodic auto-correlation functions to be zero for each time shifts within a certain region around the in-phase position, called the zero correlation zone (ZCZ). A lot of systematic constructions of ZCPs can be found in [8–18]. Another extension of GCPs is the complementary sequence set (CSS), which contains multiple sequences and the sum of their aperiodic auto-correlation functions is a Dirac delta function [19]. For more information CSS, please refer to the survey papers [20, 21]. ZCPs and CSSs have a wider lengths as compared to GCPs, and they have all been widely used in communications and radar.

For a long time, research related to complementary pairs has studied the complementarity of their aperiodic auto-correlation, and few people have studied the correlation of a single complementary sequence. Until 2013, Gong *et al.* considered the periodic auto-correlation of a single Golay complementary sequence, and gave a systematic construction such that each of sequences has a zero auto correlation zone [22]. In 2021, Hu *et al.* extended this result to odd periodic auto-correlation [23]. This property plays a very important role in the synchronization and channel estimation.

It should be noted that each GCP, ZCP or CSS, is defined by their aperiodic auto-correlation sums only. Within each GCP, ZCP or CSS, it is assumed that separate non-interfering channels are used for the transmission of the constituent sequences. In complementary sequence, concerning the consideration of cross-correlation, there are very few at present, only mentioned in the literature [24, 25]. Liu *et al.* [24] considered that the aperiodic cross-correlation sums of ZCPs $(\mathbf{a}, \mathbf{b})$ has a ZCZ at the end, i.e., $R_{\mathbf{a},\mathbf{b}}(\tau) + R_{\mathbf{b},\mathbf{a}}(\tau) = 0$ for $N - Z + 1 \leq \tau \leq N - 1$. Wang *et al.* [25] directly considered the aperiodic cross-correlation of $(\mathbf{a}, \mathbf{b})$ at the front end, i.e., $|R_{\mathbf{a},\mathbf{b}}(\tau)| < \epsilon$ for $0 \leq \tau \leq Z-1$, where ϵ is a very small number. However, due to the direct consideration of cross-correlation, it is very difficult to construct such sequence pairs via mathematical tools when ϵ is very small. Therefore, they used the majorization minimization (MM) algorithm to search for sequence pairs with such properties in [25]. So far, no one considered the aperiodic cross-correlation sums of ZCPs at the front end. In this paper, we will propose the concept of PZCPs, which will be introduced later.

GCPs have been widely used in various fields, and radar waveform design is one of the important application scenarios. In the non-coherent radar pulse compression system without considering Doppler, GCPs can achieve very low range sidelobe [26]. However, once Doppler is considered, GCPs will also have high range sidelobes. In order to solve this problem, Pezeshki *et al.* combined GCPs and Prouhet-Thue-Morse (PTM) sequence to design a Doppler resilient waveform and combined with the Alamouti matrix, the designed waveform is also applicable to the fully polarimetric radar systems [27]. In 2014, Tang *et al.* extended this idea to the MIMO radar system based on complete complementary codes (CCC) and generalized PTM sequences [28]. The use of PTM sequence requires that the number of transmitted pulses must be the power of two [27]. To solve this problem, in [29, 30], they designed Doppler tolerant waveforms based on equal sums of power (ESP) sequence and GCPs (or ZCPs), the number of transmitted pulses on each antenna is small and may not be limited to power of two. Note that these two papers did not consider the cross ambiguity function in multi antenna transmission, in this paper we will propose a new concept of sequences, give some systematic constructions, and use them to design doppler resilient waveform combined with ESP sequences. Simulation results show that the proposed waveform is feasible.

The rest of the paper is organised as follows. In Section 2, along with the preliminaries the definition of the PZCP is proposed. In Section 3, some constructions of PZCPs are proposed, and the PZCP is used in Doppler resilient waveform design in Section 4. Finally, we give some concluding remarks in Section 5.

2 Preliminaries

Throughout this paper, x^* denotes the conjugate of complex number x. $\xi_q = e^{\sqrt{-1}\frac{2\pi}{q}}$ denotes the q-th root of unity. A sequence $\mathbf{a} = (a_0, a_1, \cdots, a_{L-1})$ is called a q-ary sequence if $a_i \in \{\xi_q^k : k = 0, 1, \cdots q-1\}$ for all i. $\otimes$ denotes the Kronecker product of the sequences. $\overleftarrow{\mathbf{a}}$ denotes the conjugate reverse of sequence $\mathbf{a}$. $\mathbf{a}|\mathbf{b}$ denotes the horizontal concatenation of sequences $\mathbf{a}$ and $\mathbf{b}$.

Definition 1 *Let $\mathbf{a} = (a_0, a_1, \cdots, a_{L-1})$ and $\mathbf{b} = (b_0, b_1, \cdots, b_{L-1})$ be two q-ary sequences of equal length L. Their aperiodic cross-correlation function (ACCF) is defined as follows*

$$R_{\mathbf{a},\mathbf{b}}(\tau) = \begin{cases} \sum\limits_{i=0}^{L-1-\tau} a_i b_{i+\tau}^*, & 0 \le \tau < L, \\ \sum\limits_{i=0}^{L-1+\tau} a_{i-\tau} b_i^*, & -L < \tau < 0, \\ 0, & \textit{otherwise}, \end{cases} \tag{1}$$

and $R_{\mathbf{a}}(\tau) = R_{\mathbf{a},\mathbf{a}}(\tau)$ is the called aperiodic auto-correlation function (AACF).

Definition 2 ([27]) *Let $\mathbf{P} = \{\mathbf{p}_0, \mathbf{p}_1, \cdots, \mathbf{p}_{K-1}\}$, $\mathbf{Q} = \{\mathbf{q}_0, \mathbf{q}_1, \cdots, \mathbf{q}_{K-1}\}$ be two pulse trains, where $\mathbf{p}_k, \mathbf{q}_k$ are complex-valued sequences of length L. The cross ambiguity function (CAF) is defined as follows*

$$A_{\mathbf{P},\mathbf{Q}}(\tau, \theta) = \sum_{k=0}^{K-1} R_{\mathbf{p}_k,\mathbf{q}_k}(\tau) e^{jk\theta}, \tag{2}$$

and $A_{\mathbf{P}}(\tau,\theta)=A_{\mathbf{P},\mathbf{P}}(\tau,\theta)$ *is called auto ambiguity function (AAF).*

Let $\mathbf{X}=\{\mathbf{x}_0,\mathbf{x}_1,\cdots,\mathbf{x}_{K-1}\}$ be a train of pulses, then the Taylor expansion of its AAF at $\theta=0$ is

$$A_{\mathbf{X}}(\tau,\theta)=\sum_{m=0}^{\infty}\frac{c_m(\tau)(j\theta)^m}{m!} \tag{3}$$

where the Taylor coefficients $c_m(\tau)$ is given by

$$c_m(\tau)=\sum_{k=0}^{K-1}k^m R_{\mathbf{x}_k}(\tau). \tag{4}$$

If the Taylor coefficients vanish at all nonzero delays, then a doppler resilient waveform can be realized.

Definition 3 *A pair of length-L sequences $(\mathbf{x},\mathbf{y})$ is called an (L,Z)-PZCP, if*

$$C1\text{: } R_{\mathbf{x}}(\tau)+R_{\mathbf{y}}(\tau)=0,\ \text{for any } 0<|\tau|<Z, \tag{5}$$
$$C2\text{: } R_{\mathbf{x},\mathbf{y}}(\tau)+R_{\mathbf{y},\mathbf{x}}(\tau)=0,\ \text{for any } 0\le|\tau|<Z. \tag{6}$$

Remark 1 *Compared with the definition in [7, 24], we have*

- *PZCPs in this paper are a special class of ZCPs 2007 [7], and Z is the width of the ZCZ of ZCPs. A ZCP is called a GCP if $Z=L$.*
- *The cross-correlation sums at shift τ of a PZCP in this paper equals to zero for $\tau\in[0,Z)$, while the cross-correlation sums at shift τ of a CZCP in [24] equals to zero for $\tau\in[N-Z+1,N-1]$.*

According to the definition of PZCPs, binary PZCPs have the following property.

Property 1 *Binary PZCPs of odd length does not exist.*

Proof Suppose that there is a binary PZCP $(\mathbf{a},\mathbf{b})$ with odd length L, then we have

$$\begin{aligned} R_{\mathbf{a},\mathbf{b}}(0)&=\sum_{i=0}^{L-1}a_ib_i\\ &=L-2|\{0\le i\le L-1: a_ib_i=-1\}|\\ &\equiv 1 \pmod 2\end{aligned} \tag{7}$$

which illustrates that $R_{\mathbf{a},\mathbf{b}}(0)$ is an odd integer. Furthermore, $R_{\mathbf{a},\mathbf{b}}(0)+R_{\mathbf{b},\mathbf{a}}(0)=2R_{\mathbf{a},\mathbf{b}}(0)\neq 0$, which is a contradiction with the definition of PZCP. Therefore, there is no binary PZCP of odd length.

2.1 Generalized Boolean function

A GBF $f(\mathbf{x})$ of m variables $\mathbf{x} = (x_1, x_2, \cdots, x_m)$ is defined as a mapping form $\mathbb{Z}_2^m$ to $\mathbb{Z}_q$, where $x_i \in \mathbb{Z}_2,\ \mathbb{Z}_q = \{0, 1, \cdots, q-1\}$. The $f(\mathbf{x})$ can determine a sequence $\mathbf{f} = (f_0, f_1, \cdots, f_{2^m-1})$ of length 2^m, where $f_i = f(i_1, i_2, \cdots, i_m)$ and $(i_1, i_2, \cdots, i_m)$ is the binary representation of integer $i = \sum_{j=1}^{m} i_j 2^{j-1}$. Taking $q = 2$ and $m = 3$ for example, the sequence corresponding to the generalized Boolean function x_1x_2 is $\mathbf{x}_1\mathbf{x}_2 := (0, 0, 0, 1, 0, 0, 0, 1)$.

Davis and Jedwab first gave the direct construction of GCPs based on GBFs in 1999 [4].

Lemma 1 ([4]) *Let*

$$f(\mathbf{x}) = \frac{q}{2}\sum_{k=1}^{m-1} x_{\pi(k)}x_{\pi(k+1)} + \sum_{k=1}^{m} c_k x_k + c', \tag{8}$$

$$g(\mathbf{x}) = f(\mathbf{x}) + \frac{q}{2}x_{\pi(1)}, \tag{9}$$

where q is an even integer, π is a permutation of $\{1, 2, \cdots, m\}$, $c_k,\ c' \in \mathbb{Z}_q$. Then the sequence pair $(\psi(\mathbf{f}), \psi(\mathbf{g})) = (\xi_q^{\mathbf{f}}, \xi_q^{\mathbf{g}})$ is a GCP of length 2^m, where $\xi_q^{\mathbf{f}} = (\xi_q^{f_0}, \xi_q^{f_1}, \cdots, \xi_q^{f_{2^m-1}})$.

Later, the GCP constructed by *Lemma* 1 is called the standard GCP.

3 The Constructions of PZCPs

3.1 Direct Constructions

In this subsection, we propose a direct construction of PZCPs from the standard GCPs.

Theorem 1 *In the context of* Lemma *1, let $\pi(m) = m$, then $(\psi(\mathbf{f}), \psi(\mathbf{g}))$ forms a $(2^m, 2^{\pi(1)-1}+1)$-PZCP.*

Proof Since $(\psi(\mathbf{f}), \psi(\mathbf{g}))$ is a GCP, it is obvious that the sequence pair satisfies the condition C1. For $0 \le \tau \le 2^{\pi(1)-1}$, we have to show that

$$\begin{aligned} s(\tau) =& R_{\psi(\mathbf{f}),\psi(\mathbf{g})}(\tau) + R_{\psi(\mathbf{g}),\psi(\mathbf{f})}(\tau) \\ =& \sum_{i=0}^{2^m-1-\tau} \left(\xi_q^{f_i - g_{i+\tau}} + \xi_q^{g_i - f_{i+\tau}}\right) \\ =& 0. \end{aligned} \tag{10}$$

Let $j = i + \tau$, also let $(i_1, i_2, \cdots, i_m)$ and $(j_1, j_2, \cdots, j_m)$ be the binary representations of i and j, respectively. For easy to describe, we divide the set $J = \{i : 0 \le i \le 2^m - 1 - \tau\}$ into four subsets: $J_1(\tau) = \{i \in J : i_{\pi(1)} \neq j_{\pi(1)}\}$,

$J_2(\tau) = \{i \in J : i_{\pi(1)} = j_{\pi(1)}, i_m = j_m = 0\}$, $J_3(\tau) = \{i \in J : i_{\pi(1)} = j_{\pi(1)}, i_m = j_m = 1\}$, and $J_4(\tau) = \{i \in J : i_{\pi(1)} = j_{\pi(1)}, i_m \neq j_m\}$. Then

$$\begin{aligned} s(\tau) = &\sum_{i\in J_1(\tau)} \left(\xi_q^{f_i-g_j} + \xi_q^{g_i-f_j}\right) + \sum_{i\in J_2(\tau)} \left(\xi_q^{f_i-g_j} + \xi_q^{g_i-f_j}\right) \\ &+ \sum_{i\in J_3(\tau)} \left(\xi_q^{f_i-g_j} + \xi_q^{g_i-f_j}\right) + \sum_{i\in J_4(\tau)} \left(\xi_q^{f_i-g_j} + \xi_q^{g_i-f_j}\right). \end{aligned} \tag{11}$$

For the first term of $s(\tau)$, we have

$$\begin{aligned} \sum_{i\in J_1(\tau)} \left(\xi_q^{f_i-g_j} + \xi_q^{g_i-f_j}\right) &= \sum_{i\in J_1(\tau)} \left(\xi_q^{(f_i-g_j)-(g_i-f_j)} + 1\right) \xi_q^{g_i-f_j} \\ &= \sum_{i\in J_1(\tau)} \left((-1)^{i_{\pi(1)}+j_{\pi(1)}} + 1\right) \xi_q^{g_i-f_j} \\ &= 0. \end{aligned} \tag{12}$$

For the second term of $s(\tau)$, since $\pi(m) = m$, (f_i, g_i) can be expressed as

$$f_i = \sum_{k=1}^{m-2} i_{\pi(k)} i_{\pi(k+1)} + \sum_{k=1}^{m-1} c_k i_k + c', \tag{13}$$

$$g_i = f_i + \frac{q}{2} i_{\pi(1)}. \tag{14}$$

Let t be the smallest integer such that $i_{\pi(t)} \neq j_{\pi(t)}$. Let i' and j' be integers that differ from i and j in only one position $\pi(t-1)$, i.e., $i'_{\pi(t-1)} = 1 - i_{\pi(t-1)}$ and $j'_{\pi(t-1)} = 1 - j_{\pi(t-1)}$, respectively, such that $j' = i' + \tau$. Since $i_m = j_m = 0$, one has $0 \leq i', j' \leq 2^{m-1}$, then $i' \in J_2(\tau)$. As (i, i') and (j, j') differ in only one position $\pi(t-1)$, we have

$$\begin{aligned} &(f_i - g_j) - (f_{i'} - g_{j'}) \\ =\ & q\big(i_{\pi(t-2)} i_{\pi(t-1)} + i_{\pi(t-1)} i_{\pi(t)}\big) \\ & - \frac{q}{2} i_{\pi(t-2)} - \frac{q}{2} i_{\pi(t)} - c_{\pi(t-1)} + 2c_{\pi(t-1)} i_{\pi(t-1)} \\ & - q\big(j_{\pi(t-2)} j_{\pi(t-1)} + j_{\pi(t-1)} j_{\pi(t)}\big) \\ & + \frac{q}{2} j_{\pi(t-2)} + \frac{q}{2} j_{\pi(t)} + c_{\pi(t-1)} - 2c_{\pi(t-1)} j_{\pi(t-1)} \\ \equiv\ & \frac{q}{2} \pmod q, \end{aligned} \tag{15}$$

which implies $\xi_q^{f_i-g_j} + \xi_q^{f_{i'}-g_{j'}} = 0$, i.e., $\sum_{i\in J_2(\tau)} \xi_q^{f_i-g_j} = 0$. Similarly, it can be shown that $\sum_{i\in J_2(\tau)} \xi_q^{g_i-f_j} = 0$.

For the third term of $s(\tau)$, since $i_m = j_m = 1$, we have $2^{m-1} \leq i, j \leq 2^m - 1$. Let t be the smallest integer such that $i_{\pi(t)} \neq j_{\pi(t)}$, then we have $t \leq m-1$. Let i' and j' be integers that differ from i and j in only one position $\pi(t-1)$, i.e., $i'_{\pi(t-1)} = 1 - i_{\pi(t-1)}$ and $j'_{\pi(t-1)} = 1 - j_{\pi(t-1)}$, respectively, such that $j' = i' + \tau$,

$2^{m-1} \le i', j' \le 2^m - 1$, and $i' \in J_3(\tau)$. Similar to the proof of the second term, there exist i', j' such that $\xi_q^{f_i - g_j} + \xi_q^{f_{i'} - g_{j'}} = 0$, i.e., $\sum_{i \in J_3(\tau)} \xi_q^{f_i - g_j} = 0$. Similarly, it can be shown that $\sum_{i \in J_3(\tau)} \xi_q^{g_i - f_j} = 0$.

For the fourth term of $s(\tau)$, since $i_m \ne j_m$ and $i \le j$, we have $i_m = 0$ and $j_m = 1$. Let t be the smallest integer such that $i_{\pi(t)} \ne j_{\pi(t)}$, and let i' and j' be integers that differ from i and j in only one position $\pi(t-1)$, i.e., $i'_{\pi(t-1)} = 1 - i_{\pi(t-1)}$ and $j'_{\pi(t-1)} = 1 - j_{\pi(t-1)}$, respectively, such that $j' = i' + \tau$. We can obtain $\pi(t) < \pi(1)$. Suppose not, we assume $\pi(t) \ge \pi(1)$, then

$$\begin{aligned} \tau &= j - i \\ &= 2^{m-1} + \sum_{k=1, k \ne \pi(t)}^{m-1} (j_k - i_k) 2^{k-1} \\ &\ge 2^{m-1} - \sum_{k=1}^{m-1} 2^{k-1} + 2^{\pi(t)-1} \\ &\ge 2^{m-1} - (2^{m-1} - 1) + 2^{\pi(1)-1} \\ &= 2^{\pi(1)-1} + 1 \end{aligned} \tag{16}$$

which contradicts the assumption that $0 \le \tau \le 2^{\pi(1)-1}$. Thus we have $\pi(t) < \pi(1)$. Further, we know that $i'_m = 0$ and $j'_m = 1$, so $i' \in J_4(\tau)$. Similar to the proof of the second term, there exist two integers i', j' such that $\xi_q^{f_i - g_j} + \xi_q^{f_{i'} - g_{j'}} = 0$, i.e., $\sum_{i \in J_4(\tau)} \xi_q^{f_i - g_j} = 0$. Similarly, it can be shown that $\sum_{i \in J_4(\tau)} \xi_q^{g_i - f_j} = 0$.

Based on the above discussion, we can say that $(\psi(\mathbf{f}), \psi(\mathbf{g}))$ is a $(2^m, 2^{\pi(1)-1}+1)$-PZCP. Now we finish the proof.

In the following, we give an example to illustrate the result of Theorem 1.

Example 1 *Let $m = 4$, $q = 4$, $\pi(1,2,3,4) = (3,1,2,4)$, $(c_1, c_2, c_3, c_4) = (1,0,2,3)$, $c' = 0$. Then we get a $(16,5)$-PZCP $(\psi(\mathbf{f}), \psi(\mathbf{g}))$ from* Theorem *1, i.e.,*

$$\begin{aligned} \mathbf{f} &= (0,0,2,0,1,3,3,3,3,3,3,1,0,2,0,0), \\ \mathbf{g} &= (0,0,2,0,3,1,1,1,3,3,3,1,2,0,2,2). \end{aligned} \tag{17}$$

Then

$$\left(R_{\psi(\mathbf{f})}(\tau) + R_{\psi(\mathbf{g})}(\tau)\right)_{\tau=0}^{15} = (32, \mathbf{0}_{15}), \tag{18}$$

and

$$\left(R_{\psi(\mathbf{f}),\psi(\mathbf{g})}(\tau) + R_{\psi(\mathbf{g}),\psi(\mathbf{f})}(\tau)\right)_{\tau=0}^{15} = (\mathbf{0}_5, 4, 0, 4, 0, 12, 0, 4, \mathbf{0}_4). \tag{19}$$

Corollary 1 *Let $f(\mathbf{x})$ be as shown in* Lemma *1, $g(\mathbf{x}) = f(\mathbf{x}) + \frac{q}{2} x_{\pi(m)}$. If $\pi(m) = 1$, then $(\psi(\mathbf{f}), \psi(\mathbf{g}))$ forms a $(2^m, 2^{\pi(1)-1}+1)$-PZCP.*

Proof The proof is identical to Theorem 1, and is omitted here.

In [14], Adhikary *et. al.* proposed a generic construction of ZCPs of length $2^{m-1}+2$ based on generalized Boolean functions.

Lemma 2 *Let $m \geq 3$ be an integer, q be an even number, π be a permutation of $\{1, 2, \cdots, m-2\}$. Let*

$$\begin{aligned} f(\mathbf{x}) = \frac{q}{2}\Big[&x_{m-1}x_m + x_{m-1}(1-x_m)\sum_{k=1}^{m-3} x_{\pi(k)}x_{\pi(k+1)} \\ &+ (1-x_{m-1})x_m \sum_{k=1}^{m-3}(1-x_{\pi(k)})(1-x_{\pi(k+1)})\Big] + \sum_{k=1}^{m-2} e_k x_k, \end{aligned} \quad (20)$$

$$g(\mathbf{x}) = f(\mathbf{x}) + (1-x_{m-1})x_m x_{\pi(m-3)}, \quad (21)$$

$$h(\mathbf{x}) = f(\mathbf{x}) + x_{m-1}(1-x_m)x_{\pi(m-3)} + (1-x_{m-1})(1-x_m). \quad (22)$$

Let $(\mathbf{a}, \mathbf{b})$ be a sequence pair of length $2^{m-1}+2$ by eliminating the first and last $2^{m-2}-1$ elements of sequence pair $(\mathbf{g}, \mathbf{h})$. Then the sequence pair $(\psi(\mathbf{a}), \psi(\mathbf{b})) = (\xi_q^{\mathbf{a}}, \xi_q^{\mathbf{b}})$ is a ZCP of length $2^{m-1}+2$ with ZCZ width $Z = 2^{\pi(m-2)}+1$.

Theorem 2 *In fact, the sequence pair $(\mathbf{a}, \mathbf{b})$ of* Lemma *2 is also a PZCP, and its parameters are as follows*

$$\begin{cases} (2^{m-1}+2, 2^{\pi(m-2)-1}+1), & \text{if } m \geq 4; \\ (6, 4), & \text{if } m = 3. \end{cases} \quad (23)$$

The proof of Theorem 2 is similar to Theorem 1, and we omit it here.

3.2 Indirect Constructions

In this subsection, we propose some indirect constructions of PZCPs by concatenation.

Theorem 3 *Let $(\mathbf{a}, \mathbf{b})$ and $(\mathbf{c}, \mathbf{d})$ be two GCPs of equal length L. Define*

$$\begin{aligned} \mathbf{e} &= (\mathbf{a}|\mathbf{c}|\mathbf{b}|\mathbf{d}), \\ \mathbf{f} &= (\mathbf{a}|-\mathbf{c}|\mathbf{b}|-\mathbf{d}). \end{aligned} \quad (24)$$

Then $(\mathbf{e}, \mathbf{f})$ forms a $(4L, L+1)$-PZCP.

Proof For $0 < \tau \leq L$, the AACFs of $\mathbf{e}$ and $\mathbf{f}$ are as follows:

$$\begin{aligned} R_{\mathbf{e}}(\tau) = &R_{\mathbf{a}}(\tau) + R_{\mathbf{c}}(\tau) + R_{\mathbf{b}}(\tau) + R_{\mathbf{d}}(\tau) \\ &+ R^*_{\mathbf{c},\mathbf{a}}(L-\tau) + R^*_{\mathbf{b},\mathbf{c}}(L-\tau) + R^*_{\mathbf{d},\mathbf{b}}(L-\tau), \end{aligned} \quad (25)$$

$$\begin{aligned} R_{\mathbf{f}}(\tau) = &R_{\mathbf{a}}(\tau) + R_{\mathbf{c}}(\tau) + R_{\mathbf{b}}(\tau) + R_{\mathbf{d}}(\tau) \\ &- R^*_{\mathbf{c},\mathbf{a}}(L-\tau) - R^*_{\mathbf{b},\mathbf{c}}(L-\tau) - R^*_{\mathbf{d},\mathbf{b}}(L-\tau). \end{aligned} \quad (26)$$

Thus, $R_{\mathbf{e}}(\tau)+R_{\mathbf{f}}(\tau)=0$ for $1\le\tau\le L$. On the other hand, their ACCFs are as follows:

$$\begin{aligned} R_{\mathbf{e},\mathbf{f}}(\tau) &= R_{\mathbf{a}}(\tau)-R_{\mathbf{c}}(\tau)+R_{\mathbf{b}}(\tau)-R_{\mathbf{d}}(\tau)\\ &\quad - R^*_{\mathbf{c},\mathbf{a}}(L-\tau)+R^*_{\mathbf{b},\mathbf{c}}(L-\tau)-R^*_{\mathbf{d},\mathbf{b}}(L-\tau), \qquad (27)\\ R_{\mathbf{f},\mathbf{e}}(\tau) &= R_{\mathbf{a}}(\tau)-R_{\mathbf{c}}(\tau)+R_{\mathbf{b}}(\tau)-R_{\mathbf{d}}(\tau)\\ &\quad + R^*_{\mathbf{c},\mathbf{a}}(L-\tau)-R^*_{\mathbf{b},\mathbf{c}}(L-\tau)+R^*_{\mathbf{d},\mathbf{b}}(L-\tau). \qquad (28)\end{aligned}$$

Thus, $R_{\mathbf{e},\mathbf{f}}(\tau)+R_{\mathbf{f},\mathbf{e}}(\tau)=0$ for $0\le\tau\le L$. This completes the proof.

Remark 2 *When $\pi(m-2)=m-2$, $L=2^m$, the PZCP generated by Theorem 1 and Theorem 3 have the same parameters.*

Theorem 4 *Let $(\mathbf{a},\mathbf{b})$ be a GCP of length L, $\mathbf{e}=(\mathbf{a}|\mathbf{b})$, $\mathbf{f}=(\mathbf{a}|-\mathbf{b})$, $\mathbf{g}=\overleftarrow{\mathbf{f}}$ and $\mathbf{h}=-\overleftarrow{\mathbf{e}}$. Let*

$$\begin{aligned}\mathbf{p}&=(x_1|\mathbf{e}|\mathbf{g}|y_1),\\ \mathbf{q}&=(x_2|\mathbf{f}|\mathbf{h}|y_2).\end{aligned} \qquad (29)$$

Then $(\mathbf{p},\mathbf{q})$ is a $(4L+2,L+1)$-PZCP if $x_1=-x_2$, $y_1=y_2$, where $x_1,x_2,y_1,y_2\in\{t\in\mathbb{C}:|t|=1\}$.

Before proving Theorem 4, we give the following lemma.

Lemma 3 *Let $(\mathbf{a},\mathbf{b})$ be a GCP, then $(\mathbf{a},\overleftarrow{\mathbf{b}})$, $(\overleftarrow{\mathbf{a}},\mathbf{b})$ and $(\overleftarrow{\mathbf{a}},\overleftarrow{\mathbf{b}})$ are GCPs.*

Proof When $0<\tau\le L$, our calculations yield the following result

$$\begin{aligned} R_{\mathbf{p}}(\tau) &= x_1a^*_{\tau-1}+a^*_{\tau-1}y_1^*+R_{\mathbf{a}}(\tau)+R_{\mathbf{b}}(\tau)+R_{\overleftarrow{\mathbf{b}}}(\tau)+R_{\overleftarrow{\mathbf{a}}}(\tau)\\ &\quad + R^*_{\mathbf{b},\mathbf{a}}(L-\tau)-R^*_{\overleftarrow{\mathbf{b}},\mathbf{b}}(L-\tau)-R^*_{\overleftarrow{\mathbf{a}},\mathbf{b}}(L-\tau), \qquad (30)\\ R_{\mathbf{q}}(\tau) &= x_2a^*_{\tau-1}-a^*_{\tau-1}y_2^*+R_{\mathbf{a}}(\tau)+R_{\mathbf{b}}(\tau)+R_{\overleftarrow{\mathbf{b}}}(\tau)+R_{\overleftarrow{\mathbf{a}}}(\tau)\\ &\quad - R^*_{\mathbf{b},\mathbf{a}}(L-\tau)+R^*_{\overleftarrow{\mathbf{b}},\mathbf{b}}(L-\tau)+R^*_{\overleftarrow{\mathbf{a}},\mathbf{b}}(L-\tau). \qquad (31)\end{aligned}$$

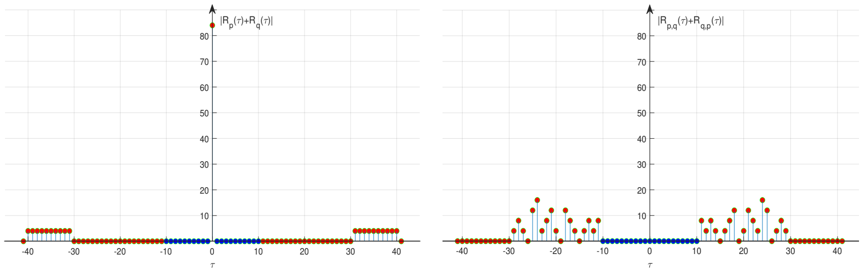
Further, $R_{\mathbf{p}}(\tau)+R_{\mathbf{q}}(\tau)=0$ for $1\le\tau\le L$. On the other hand, when $\tau=0$, we have

$$\begin{aligned} R_{\mathbf{p},\mathbf{q}}(\tau)+R_{\mathbf{q},\mathbf{p}}(\tau) &= 2(R_{\mathbf{a}}(0)-R_{\mathbf{b}}(0)+R_{\overleftarrow{\mathbf{b}}}(0)-R_{\overleftarrow{\mathbf{a}}}(0))\\ &\quad + x_1x_2^*+y_1y_2^*+x_2x_1^*+y_2y_1^*\\ &= -|x_1|^2+|y_1|^2-|x_1|^2+|y_1|^2\\ &= 0.\end{aligned} \qquad (32)$$

When $1\le\tau\le L$, their ACCFs are as follows:

$$\begin{aligned} R_{\mathbf{p},\mathbf{q}}(\tau) &= x_1a^*_{\tau-1}+a^*_{\tau-1}y_2^*+R_{\mathbf{a}}(\tau)-R_{\mathbf{b}}(\tau)+R_{\overleftarrow{\mathbf{b}}}(\tau)-R_{\overleftarrow{\mathbf{a}}}(\tau)\\ &\quad - R^*_{\mathbf{b},\mathbf{a}}(L-\tau)-R_{\overleftarrow{\mathbf{b}},\mathbf{b}}(L-\tau)+R_{\overleftarrow{\mathbf{a}},\overleftarrow{\mathbf{b}}}(L-\tau), \qquad (33)\\ R_{\mathbf{q},\mathbf{p}}(\tau) &= x_2a^*_{\tau-1}-a^*_{\tau-1}y_1^*+R_{\mathbf{a}}(\tau)-R_{\mathbf{b}}(\tau)+R_{\overleftarrow{\mathbf{b}}}(\tau)-R_{\overleftarrow{\mathbf{a}}}(\tau)\\ &\quad + R^*_{\mathbf{b},\mathbf{a}}(L-\tau)+R_{\overleftarrow{\mathbf{b}},\mathbf{b}}(L-\tau)-R_{\overleftarrow{\mathbf{a}},\overleftarrow{\mathbf{b}}}(L-\tau). \qquad (34)\end{aligned}$$

After simple calculation, we have $R_{\mathbf{p},\mathbf{q}}(\tau)+R_{\mathbf{q},\mathbf{p}}(\tau)=0$ for $1\le\tau\le L$. To sum up, $R_{\mathbf{p},\mathbf{q}}(\tau)+R_{\mathbf{q},\mathbf{p}}(\tau)=0$ holds for all $0\le\tau\le L$.



(a) Magnitudes of AACF sums (b) Magnitudes of ACCF sums

Fig. 1 The magnitudes of AACF sums and ACCF sums of PZCP in *Example* 2.

Next, we will use an example to illustrate Theorem 4.

Example 2 *Step 1: Let* $(\mathbf{a}, \mathbf{b})$ *be a binary GCP of length* 10*, i.e.,*

$$\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} = \begin{pmatrix} + - - + - + + + + + \\ - + - + + + - - + + \end{pmatrix}, \tag{35}$$

where $+, -$ *denotes* $1, -1$ *respectively.*

Step 2: Let $\mathbf{e} = (\mathbf{a}|\mathbf{b})$ *and* $\mathbf{f} = (\mathbf{a}| - \mathbf{b})$*. Therefore,*

$$\begin{pmatrix} \mathbf{e} \\ \mathbf{f} \end{pmatrix} = \begin{pmatrix} + - - + - + + + + + - + - + + + - - + + \\ + - - + - + + + + + + - + - - - + + - - \end{pmatrix}. \tag{36}$$

Step 3: Let $\mathbf{g} = \overleftarrow{\mathbf{f}}$ *and* $\mathbf{h} = -\overleftarrow{\mathbf{e}}$*. Therefore,*

$$\begin{pmatrix} \mathbf{g} \\ \mathbf{h} \end{pmatrix} = \begin{pmatrix} - - + + - - - + - + + + + + + - + - - + \\ - - + + - - - + - + - - - - - + - + + - \end{pmatrix}. \tag{37}$$

Step 4: Let $x_1 = -1$*,* $y_1 = -1$*,* $x_2 = 1$*,* $y_2 = -1$*, and* $\mathbf{p} = (x_1|\mathbf{e}|\mathbf{g}|y_1)$*,* $\mathbf{q} = (x_2|\mathbf{f}|\mathbf{h}|y_2)$*.*

Then, the magnitudes of $R_{\mathbf{p}}(\tau) + R_{\mathbf{q}}(\tau)$ *and* $R_{\mathbf{p},\mathbf{q}}(\tau) + R_{\mathbf{p},\mathbf{q}}(\tau)$ *are shown in Fig. 1. We can see that* $(\mathbf{p}, \mathbf{q})$ *is a* $(42, 11)$*-PZCP, which is consistent with the result of Theorem 4.*

3.3 Comparison

The parameters of PZCPs proposed in this paper are listed in Table 1. In addition, we show whether known ZCP is a PZCP or not. If it is a PZCP, we will give its parameters. At present, most of the ZCPs with odd lengths are binary, such as [8, 15], we do not consider them, because they naturally do not satisfy the condition C2 in the definition of PZCP. In Table 2, we listed specific results.

Table 1 Parameters of the constructed PZCPs in this paper

Source	PZCP Length L	ZCZ Width Z	Z/L
Theorem 1	2^m	$2^{\pi(1)-1}+1$	$\leq \frac{1}{4}$
Theorem 3	$4L$	$L+1$	$\approx \frac{1}{4}$
Theorem 4	$4L+2$	$L+1$	$\approx \frac{1}{4}$

Where L is the length of GCPs.

Table 2 Known ZCPs to PZCPs conversion

Source	Parameters	Does PZCP Exist?	Maximum ZCZ of PZCP	Z_{max}/L
[9]	$(2^{m-1}+2^{m-2}, 2^{m-1})$	Yes	$Z_{max}=2^{m-4}+1$ if $m \geq 4$; $Z_{max}=4$ if $m=3$	$\begin{cases} 1/12, & \text{if } m>3 \\ 2/3, & \text{if } m=3. \end{cases}$
[10]	$(2^{m-1}+2^v, 2^{\pi(v+1)-1}+2^v)$	Yes	$Z_{max}=4$ if $v=1$; $Z_{max}=2$ if $v \geq 2$	$0,\ m \to \infty$
[11]	$(2^{m+1}+2^{m+2}+2^{m+3}, 2^{m+3})$	Yes	$Z_{max}=2^{m-1}+1$	$1/28$
[12]	$(2^{a+2}10^b26^c+2, 3\times 2^a10^b26^c+1)$	Yes	$Z_{max}=2^a10^b26^c+1$	$1/4$
[14]	$(2^{m-1}+2, 2^{\pi(m-2)}+1)$	Yes	$Z_{max}=2^{m-3}-1$ if $m \geq 4$; $Z_{max}=4$ if $m=3$	$\begin{cases} 1/4, & \text{if } m>3 \\ 2/3, & \text{if } m=3. \end{cases}$
[16]	$(2^{m-1}+\sum_{\alpha=k+1}^{m-1} a_\alpha 2^{\alpha-1}+2^v, 2^{k-1}+2^v)$	Yes	$Z_{max}=4$	$0,\ m \to \infty$
[17]	$(8\times 2^a10^b26^c+4, 5\times 2^a10^b26^c+2)$	No	–	–
[18]	$(3N,2N)$,$(5N,3N)$,$(7N,4N)$,$(9N,5N)$, $(11N,6N)$,$(12N,10N)$,$(13N,7N)$,$(14N,12N)$	Yes	Uncertain[1]	–

[1] We use many GCPs to try the constructions in [18], but the results show that although PZCP exists, the width of its ZCZ is very small, and the ratio of Z to L is much less than 1/4.

3.4 Enumeration

In this subsection, we enumerate the results of some short-length binary PZCPs with maximum ZCZ. The parameters obtained by enumeration are as Table 3. An interesting fact is that for the short-length binary PZCPs we searched, the ratio of Z_{max} to L did not exceed 1/2 except for the length of 6.

4 Application

4.1 Application of PZCPs in doppler resilient waveform design

In this subsection, we propose a pulse train based on PZCPs and ESPs, which provide better Doppler resilient than PTM pulse train. However, it is necessary to use multiple antennas to transmit continuous pulse trains. We begin with definitions of delayed pulse trains and ESPs.

Definition 4 *Define a delayed pulse train $T(d)=\{\mathbf{x}_0, \mathbf{x}_1, \cdots, \mathbf{x}_{K-1}\}$ as one having a delay of d pulses in the sense, which AAF as follows*

$$A_T(\tau, \theta, d) = \sum_{k=0}^{K-1} R_{\mathbf{x}_k}(\tau) e^{j(k+d)\theta} \tag{38}$$

In the following, we will introduce the definition of ESP sequences which can be found in [29].

Table 3 A list of binary (L, Z)-PZCPs (with maximum Z_{max}) of length up to 20

L	$\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix}$	Z_{max}	Z_{max}/L
4	$\begin{pmatrix} ++++ \\ -+-+ \end{pmatrix}$	2	1/2
6	$\begin{pmatrix} --++++ \\ +--+-+ \end{pmatrix}$	4	2/3
8	$\begin{pmatrix} -+-+++++ \\ +--+--++ \end{pmatrix}$	3	3/8
10	$\begin{pmatrix} ++--++++++ \\ -++--+-+-+ \end{pmatrix}$	4	2/5
12	$\begin{pmatrix} -++---++++++ \\ +-+-+--+--++ \end{pmatrix}$	6	1/2
14	$\begin{pmatrix} +++-++-+++-+-+ \\ -++++--+----++ \end{pmatrix}$	6	3/7
16	$\begin{pmatrix} ++-+--+++-++++++ \\ -+++---+-++--+-+ \end{pmatrix}$	6	3/8
18	$\begin{pmatrix} ----+-+--++-+-++++ \\ +---+--++++--+---+ \end{pmatrix}$	7	7/18
20	$\begin{pmatrix} -+++-++-+---++-+++++ \\ +++--++++-+---+-+--+ \end{pmatrix}$	8	2/5

Definition 5 *Let $P = \{P_0, P_1\}$ be a 2-bolck partition of non-negative integer set S. P has equal sums of power (ESP) of degree M if*

$$\sum_{n \in P_0} n^m = \sum_{n \in P_1} n^m \tag{39}$$

for $m = 0, 1, \cdots, M$.

Many introductions of ESP can be found in [32]. Next, we illustrate ESP with an example.

Example 3 *Let $P_0 = \{0, 4, 7, 11\}$ and $P_1 = \{1, 2, 9, 10\}$ be a 2-block partition of $S = \{0, 1, 2, 4, 7, 9, 10, 11\}$, then $P = \{P_0, P_1\}$ has ESP of degree 3, since*

$$0^m + 4^m + 7^m + 11^m = 1^m + 2^m + 9^m + 10^m$$

for $m = 0, 1, 2, 3$.

In Example 3, due to the lack of numbers $3,\ 5,\ 6,\ 8$ in S, this may cause gaps in the pulse train that are not allowed by radar [29]. Based on this fact, we can add these numbers to $P_0,\ P_1$, i.e.,

$$\begin{aligned} P_0 &= \{0, 3, 4, 5, 6, 7, 8, 11\}, \\ P_1 &= \{1, 2, 3, 5, 6, 8, 9, 10\}. \end{aligned}$$

Obviously, P still has ESP of degree 3. Let $(\mathbf{x}, \mathbf{y})$ be a sequence pair, $\{\mathbf{x}_0, \mathbf{x}_1, \cdots, \mathbf{x}_{K-1}\}$ be pulse train with

$$\begin{cases} \mathbf{x}_k = \mathbf{x}, & \text{if } k \in P_0, k \notin P_1, \\ \mathbf{x}_k = \mathbf{y}, & \text{if } k \in P_1, k \notin P_0, \\ \mathbf{x}_k = \mathbf{x} + \mathbf{y}, & \text{if } k \in P_0 \cap P_1. \end{cases} \tag{40}$$

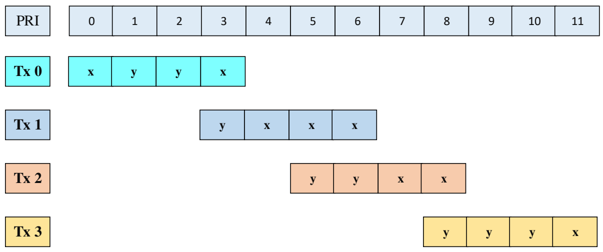


Fig. 2 Explanation of the pulse trains transmitted on the four antennas and the time of transmission

Then the pulse train is

$$\{\mathbf{x},\mathbf{y},\mathbf{y},\mathbf{x}+\mathbf{y},\mathbf{x},\mathbf{x}+\mathbf{y},\mathbf{x}+\mathbf{y},\mathbf{x},\mathbf{x}+\mathbf{y},\mathbf{y},\mathbf{y},\mathbf{x}\} \tag{41}$$

Because P_0 and P_1 are no longer two disjoint sets, this leads to the need to send both $\mathbf{x}$ and $\mathbf{y}$ at certain times, i.e., $\mathbf{x}+\mathbf{y}$. This is not allowed because the transmitted signal is not unimodular. Therefore, multi antenna transmission is used in [29], [30]. In this example, 4 pulse trains $T_0(0)$, $T_1(3)$, $T_2(5)$, $T_3(8)$ need to be transmitted[1], i.e.,

$$\begin{aligned} T_0(0) &= \{\mathbf{x},\mathbf{y},\mathbf{y},\mathbf{x}\},\\ T_1(3) &= \{\mathbf{y},\mathbf{x},\mathbf{x},\mathbf{y}\},\\ T_2(5) &= \{\mathbf{y},\mathbf{x},\mathbf{x},\mathbf{y}\},\\ T_3(8) &= \{\mathbf{x},\mathbf{y},\mathbf{y},\mathbf{x}\}. \end{aligned} \tag{42}$$

The pulse trains transmitted and time on these four antennas are more intuitively represented in Fig. 2. Within the PRI that transmits $\mathbf{x}+\mathbf{y}$ simultaneously, two filters with filter coefficients of $\mathbf{x}$ and $\mathbf{y}$ are required. It is assumed that each antenna has the same channel response and the scattering coefficient of the target remains unchanged. If we sum the composite ambiguity functions of four pulse trains, we obtain

$$A(\tau,\theta) = B(\tau,\theta) + C(\tau,\theta) \tag{43}$$

where

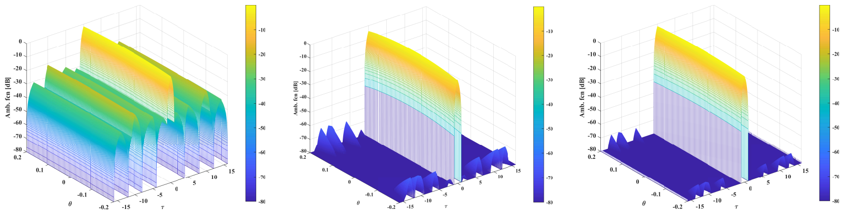
$$B(\tau,\theta) = A_{T_0}(\tau,\theta,0) + A_{T_1}(\tau,\theta,3) + A_{T_2}(\tau,\theta,5) + A_{T_3}(\tau,\theta,8), \tag{44}$$

$$C(\tau,\theta) = \sum_{k\in\{3,5,6,8\}} \left[R_{\mathbf{x},\mathbf{y}}(\tau) + R_{\mathbf{y},\mathbf{x}}(\tau)\right] e^{jk\theta}. \tag{45}$$

We compute Taylor coefficients of $B(\tau,\theta)$

$$\begin{aligned} c_m(\tau) &= \sum_{k\in P_0} k^m R_{\mathbf{x}}(\tau) + \sum_{k\in P_1} k^m R_{\mathbf{y}}(\tau)\\ &= S_m\left[R_{\mathbf{x}}(\tau) + R_{\mathbf{y}}(\tau)\right] \end{aligned} \tag{46}$$

[1] The pulse trains transmitted on each antenna are not unique, as long as the order of transmission conforms to $\{P_0, P_1\}$, it is feasible.



(a) (b) (c)

Fig. 3 Ambiguity function for different waveforms: (a) Waveform based on GCP and ESP [29]. (b) Waveform based on GCP and PTM [27]. (c) Waveform based on PZCP and ESP.

for $m = 0, 1, 2, 3$, where $S_m = \sum_{k \in P_0} k^m = \sum_{k \in P_1} k^m$.

In fact, in [29], [30], only the influence of the AAF, i.e., $B(\tau, \theta)$, is considered, and the influence of the CAF, i.e., $C(\tau, \theta)$, is not considered, which is actually not desirable. Since the receiver is equipped with two matched filters, the CAF must be considered. In this case, PZCP is obviously more practical.

4.2 Numerical experiments

In this simulation, we use a GCP (not PZCP) and a PZCP of length $L = 64$ of ZCZ width $Z = 17$ for simulation experiments, the total number of pulses is $K = 16$. In Fig. 3, we show the ambiguity function values of different schemes. Fig. 3(a) is the scheme in [29]. Because the influence of CAF is not considered, the performance becomes very poor when CAF is considered. Fig. 3(c) is a scheme based on PZCP and ESP. In the zero correlation zone, it can be seen that due to a more comprehensive consideration, the performance is better than the scheme based on GCP and PTM in the literature [27]. In Fig. 4, we compare the ambiguity function values of our proposed scheme with the scheme proposed in [27] at Doppler shift $\theta = 0.15$. Obviously, our scheme has a lower range sidelobe when the Doppler shift $\theta = 0.15$ and delay interval $[-Z + 1, Z - 1]$.

5 Conclusion

In this paper, we proposed a new class of PZCPs, which requires zero aperiodic auto-correlation sums and zero aperiodic cross-correlation sums for front-end time shifts. We also gave a construction of PZCPs from the standard GCPs and extended it to the general construction through general GCPs. In addition, combined with ESPs, PZCPs can be used for Doppler resilient waveform design. Finally, the simulation shows that the waveform based on PZCP and ESP has better performance than the waveform based on GCP and PTM.

There is still a great potential for the application of PZCPs in doppler resilient waveform design. For example, when the pulse signal power is allowed

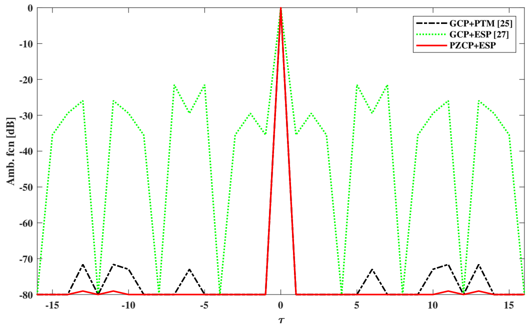


Fig. 4 Comparison of the ambiguity functions at Doppler shift $\theta = 0.15$ and delay interval $[-Z+1, Z-1]$

to be unbalanced, it is assumed that the pulse train $\mathbf{X} = \{\mathbf{x}_0, \mathbf{x}_1, \cdots, \mathbf{x}_{K-1}\}$, set $\mathbf{x}_i = \mathbf{x} + \mathbf{y}$, since $R_{\mathbf{x}+\mathbf{y}}(\tau) = R_{\mathbf{x}}(\tau) + R_{\mathbf{y}}(\tau) + R_{\mathbf{x},\mathbf{y}}(\tau) + R_{\mathbf{y},\mathbf{x}}(\tau)$, the pulse train $\mathbf{X}$ can achieve zero range sidelobes for any doppler shift θ in delay interval $[-Z+1, Z-1]$ if $(\mathbf{x}, \mathbf{y})$ is a (L, Z)-PZCP.

Declarations

- Ethical Approval and Consent to participate.
  Not applicable.
- Consent for publication.
  Yes.
- Availability of supporting data.
  Not applicable.
- Competing interests.
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- Authors' contributions.
  Bingsheng Shen proposed the main idea of paper and wrote the main manuscript text. Yang Yang polished the article and verified the computation. Pinzhi Fan and Yang Yang supplemented the details of the article and polished the article. All authors reviewed the manuscript.
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