# MDS multi-twisted Reed-Solomon codes with small dimensional hull 

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#### Abstract

In this paper, we find a necessary and sufficient condition for multi-twisted Reed-Solomon codes to be MDS. In particular, we introduce a new class of MDS double-twisted Reed-Solomon codes $\mathcal{C}_{\alpha, t, \boldsymbol{h}, \boldsymbol{\eta}}$ with twists $\boldsymbol{t}=(1,2)$ and hooks $\boldsymbol{h}=(0,1)$ over the finite field $\mathbb{F}_{q}$, providing a non-trivial example over $\mathbb{F}_{16}$ and enumeration over the finite fields of size up to 17 . Moreover, we obtain necessary conditions for the existence of multi-twisted Reed-Solomon codes with small dimensional hull. Consequently, we derive conditions for the existence of MDS multi-twisted Reed-Solomon codes with small dimensional hull.


Keywords: Reed-Solomon codes, MDS codes, LCD codes, one-dimensional hull, twisted Reed-Solomon codes
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## 1 Introduction

A linear $[n, k, d]$ code is said to be maximum distance separable (MDS) code if its parameters achieve the Singleton bound [18, i.e. $d=n-k+1$. Reed-Solomon (RS) and generalized Reed-Solomon (GRS) codes [20] are famous examples of MDS codes. For other families of MDS codes, we refer to [18, 21] and [22]. In [3], Beelen at el. introduced MDS single-twisted RS codes as a generalization of RS codes. The idea was to choose special type polynomials of degree $k$ with twist $t=1$ and hook $h=0, k-1$ such that those polynomials could have at most $k-1$ roots among the evaluation points. Further, in [1], Beelen at al. extended the notion of single-twisted RS codes to multi-twisted RS codes by adding extra monomials (or twists) in the polynomials. In recent years, applications of multi-twisted RS codes in code-based cryptography are studied by many authors (see: [1], 11]). Hence, it is significant to explore algebraic criteria of multi-twisted RS codes to be MDS.

We know that dual $\mathcal{C}^{\perp}$ of a linear code $\mathcal{C}$ is also a linear code over the same finite field. The hull of $\mathcal{C}$ is denoted by $\operatorname{Hull}(\mathcal{C})$ and defined as $\operatorname{Hull}(\mathcal{C})=\mathcal{C} \cap \mathcal{C}^{\perp}$. It is clear that $\operatorname{Hull}(\mathcal{C})$ is also a linear code. There are many applications of the hull, e.g. determining the complexity of algorithms for checking equivalence of linear codes, computing the automorphism group of a linear code. It is also useful in side-channel attacks and fault injection attacks (see: [5, 12, 13, 17, 23, 24, 25]). In [19], Massey introduced linear codes with complementary dual (LCD), i.e. the codes having zero-dimensional hull. For other constructions of LCD codes we refer to [29, 2, 6]. Moreover, Liu and Liu studied MDS LCD single-twisted RS codes and LCD multi-twisted RS codes (under certain assumptions) in [15]. Linear codes with one-dimensional hull were studied in [14]. In [28], Wu studied single-twisted RS codes with one-dimensional hull.

Motivated by their work, in this paper, we study the multi-twisted RS codes with general twists and hooks having zero and one-dimensional hull. First, we give necessary and sufficient condition for multi-twisted RS code to be MDS. Then we introduce new $k$-dimensional MDS double-twisted $\operatorname{RS}$ codes $\mathcal{C}_{\boldsymbol{\alpha}, \boldsymbol{t}, \boldsymbol{h}, \boldsymbol{\eta}}$ by choosing suitable $\boldsymbol{\eta}$ in twisted polynomials of degree $k+1$ with twists $\boldsymbol{t}=(1,2)$ and hooks $\boldsymbol{h}=(0,1)$ such that those polynomials could have at most $k-1$ roots among the evaluation points $E_{\boldsymbol{\alpha}}$. Moreover, we obtain necessary conditions for the existence of multi-twisted Reed-Solomon codes with small dimensional hull. Then as a consequence, we derive conditions for the existence of MDS multi-twisted RS codes with small dimensional hull.

This paper is organized as follows: in Section 2, we recall some basic terminologies and results which are needed for this paper. In Section 3, we describe a necessary and sufficient condition for a multi-twisted RS code to be MDS. In Section 4 we explicitly describe the algebraic properties of MDS double-twisted RS codes

[^0]over the finite fields $\mathbb{F}_{q}$, providing non-trivial examples over $\mathbb{F}_{16}$ and enumeration over the finite fields of order up to 17. In Section 5, we study multi-twisted RS codes with small dimensional hull and provide examples. Also, we obtain necessary and sufficient conditions for the existence of MDS multi-twisted RS codes with small dimensional hull. Finally, in Section 6, we conclude with some future directions and compare these results with the existing results.

## 2 Preliminaries

In this section, we recall some fundamental concepts from literature which are needed for this paper.
Definition 2.1. [18] Let $\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n} \in \mathbb{F}_{q}$ be distinct elements and $0 \leq k<n$. Then $R S$ code is defined as

$$
\begin{equation*}
\mathcal{C}_{n, k}^{R S}=\left\{\left(f\left(\alpha_{1}\right), f\left(\alpha_{2}\right), \ldots, f\left(\alpha_{n}\right)\right)\right\} \tag{2.1}
\end{equation*}
$$

where $f(x) \in \mathbb{F}_{q}[x]$ with $\operatorname{deg}(f(x))<k$.
Here $\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}$ are called evaluation points for the code. Clearly, $\mathcal{C}_{n, k}^{R S}$ is an $\mathbb{F}_{q}$-linear subspace of $\mathbb{F}_{q}^{n}$ having dimension $k$, i.e. $\mathcal{C}_{n, k}^{R S}$ is a linear code. Since the polynomial $f(x)$ has at most $k-1$ roots in the field, RS codes achieve the equality in the Singleton bound.

In [3], Beelen et al. introduced the notion of twisted polynomials and using such polynomials they introduced single-twisted RS code. Let $0 \leq h<k \leq q-1$ and $\eta$ be a non-zero element of a finite field $\mathbb{F}_{q}$. Then the set of $(k, t, h, \eta)$-twisted polynomials over $\mathbb{F}_{q}$, is defined as

$$
\mathcal{V}_{k, t, h, \eta}:=\left\{f=\sum_{i=0}^{k-1} a_{i} x^{i}+\eta a_{h} x^{k-1+t}: a_{i} \in \mathbb{F}_{q} \text { for each } i\right\}
$$

where $k, t, h \in \mathbb{N}$.
Definition 2.2. [3] Let $\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n} \in \mathbb{F}_{q}$ be distinct elements and $\boldsymbol{\alpha}=\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}\right)$. Let $k<n$ and $0<t \leq n-k$. Then the single-twisted $R S$ code is defined as

$$
\begin{equation*}
\mathcal{C}_{k}(\boldsymbol{\alpha}, t, h, \eta)=\left\{\left(f\left(\alpha_{1}\right), f\left(\alpha_{2}\right), \ldots, f\left(\alpha_{n}\right)\right)\right\} \tag{2.2}
\end{equation*}
$$

where $f(x) \in \mathcal{V}_{k, t, h, \eta}$ with $\operatorname{deg}(f(x)) \leq k-1+t<n$.
Note that, when $\eta=0$, then single-twisted RS codes become RS codes. Since $\operatorname{deg}(f(x)) \nless k$, single-twisted RS codes are not MDS in general. However, in [3], Beelen et al. obtained a necessary and sufficient condition for single-twisted RS codes to be MDS. Using SageMath implementations, they concluded that many long MDS single-twisted RS codes can be obtained using twisted polynomials for particular values of $q, n, k, h$ and $t$; but not for general values. In addition, they claimed that in order to find explicit constructions of such long MDS single-twisted RS codes is difficult; however, they gave following result in favour to the existence.
Lemma 2.3. ([3], Theorem 17) Let $\mathbb{F}_{s} \subset \mathbb{F}_{q}$ be a proper subfield and $\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n} \in \mathbb{F}_{s}$. If $\eta \in \mathbb{F}_{q} \backslash \mathbb{F}_{s}$, then the single-twisted code $\mathcal{C}_{k}(\boldsymbol{\alpha}, t, h, \eta)$ is MDS.

Further, in [1], Beelen at el. introduced multi-twisted RS codes as a generalization of single-twisted RS codes. They extended the twists and hooks: let $0 \leq h_{i}<h_{i+1}<k \leq q$ and $1 \leq t_{i}<t_{i+1} \leq n-k$ for each $i$. Then, for $\boldsymbol{t}=\left(t_{1}, t_{2}, \ldots, t_{\ell}\right), \boldsymbol{h}=\left(h_{1}, h_{2}, \ldots, h_{\ell}\right)$ and $\boldsymbol{\eta}=\left(\eta_{1}, \eta_{2}, \ldots, \eta_{\ell}\right) \in\left(\mathbb{F}_{q}^{*}\right)^{\ell}$, set of $(k, \boldsymbol{t}, \boldsymbol{h}, \boldsymbol{\eta})$-twisted polynomials over $\mathbb{F}_{q}$, is defined as

$$
\mathcal{V}_{k, \boldsymbol{t}, \boldsymbol{h}, \boldsymbol{\eta}}:=\left\{f=\sum_{i=0}^{k-1} a_{i} x^{i}+\sum_{j=1}^{\ell} \eta_{j} a_{h_{j}} x^{k-1+t_{j}}: a_{i} \in \mathbb{F}_{q} \text { for each } i\right\} .
$$

Definition 2.4. [1] Let $\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}$ be distinct elements of a finite field $\mathbb{F}_{q}$ and $k<n$. Then multi-twisted $R S$ code is defined as

$$
\begin{equation*}
\mathcal{C}_{k}(\boldsymbol{\alpha}, \boldsymbol{t}, \boldsymbol{h}, \boldsymbol{\eta}):=\left\{\left(f\left(\alpha_{1}\right), f\left(\alpha_{2}\right), \ldots, f\left(\alpha_{n}\right)\right)\right\} \tag{2.3}
\end{equation*}
$$

where $f(x) \in \mathcal{V}_{k, \boldsymbol{t}, \boldsymbol{h}, \boldsymbol{\eta}}$ with $\operatorname{deg}(f(x)) \leq k-1+t_{\ell}<n$.

As per the Definition 2.4 the generator matrix of multi-twisted RS code $\mathcal{C}_{k}(\boldsymbol{\alpha}, \boldsymbol{t}, \boldsymbol{h}, \boldsymbol{\eta})$ is

$$
\left(\begin{array}{cccc}
1 & 1 & \cdots & 1  \tag{2.4}\\
\alpha_{1} & \alpha_{2} & \cdots & \alpha_{n} \\
\vdots & \vdots & \cdots & \vdots \\
\alpha_{1}^{h_{1}}+\eta_{1}\left(\alpha_{1}\right)^{k-1+t_{1}} & \alpha_{2}^{h_{1}}+\eta_{1}\left(\alpha_{2}\right)^{k-1+t_{1}} & \cdots & \alpha_{n}^{h_{1}}+\eta_{1}\left(\alpha_{n}\right)^{k-1+t_{1}} \\
\left(\alpha_{1}\right)^{h_{1}+1} & \left(\alpha_{2}\right)^{h_{1}+1} & \cdots & \left(\alpha_{n}\right)^{h_{1}+1} \\
\vdots & \vdots & \vdots & \vdots \\
\alpha_{1}^{h_{2}}+\eta_{2}\left(\alpha_{1}\right)^{k-1+t_{2}} & \alpha_{2}^{h_{2}}+\eta_{2}\left(\alpha_{2}\right)^{k-1+t_{2}} & \cdots & \alpha_{n}^{h_{2}}+\eta_{2}\left(\alpha_{n}\right)^{k-1+t_{2}} \\
\left(\alpha_{1}\right)^{h_{2}+1} & \left(\alpha_{2}\right)^{h_{2}+1} & \cdots & \left(\alpha_{n}\right)^{h_{2}+1} \\
\vdots & \vdots & \vdots & \vdots \\
\alpha_{1}^{h_{\ell}}+\eta_{\ell}\left(\alpha_{1}\right)^{k-1+t_{\ell}} & \alpha_{2}^{h_{\ell}}+\eta_{\ell}\left(\alpha_{2}\right)^{k-1+t_{\ell}} & \cdots & \alpha_{n}^{h_{\ell}}+\eta_{\ell}\left(\alpha_{n}\right)^{k-1+t_{\ell}} \\
\left(\alpha_{1}\right)^{h_{\ell}+1} & \left(\alpha_{2}\right)_{\ell}+1 & \cdots & \left(\alpha_{n}\right)_{\ell}^{h_{\ell}+1} \\
\vdots & \vdots & \vdots & \vdots \\
\left(\alpha_{1}\right)^{k-1} & \left(\alpha_{2}\right)^{k-1} & \cdots & \left(\alpha_{n}\right)^{k-1}
\end{array}\right)_{k \times\left(h_{1}+1\right)^{t h} \text { row }} \quad \leftarrow\left(h_{2}+1\right)^{t h} \text { row }
$$

## 3 MDS multi-twisted RS codes

In this section, we obtain a necessary and sufficient condition for a multi-twisted RS code to be MDS. In 3], authors gave the criteria for a single-twisted $(\ell=1) \mathrm{RS}$ code $\mathcal{C}_{k}(\boldsymbol{\alpha}, t, h, \eta)$ to be MDS. We have extended their study for the multi-twisted RS code $(\ell>1)$ and give a more generalized structure as follows:

Theorem 3.1. Let $\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n} \in \mathbb{F}_{q}$ be distinct and for $k<n<q$ and $\boldsymbol{t}=\left(t_{1}, t_{2}, \ldots, t_{\ell}\right)$, $\boldsymbol{h}=\left(h_{1}, h_{2}, \ldots, h_{\ell}\right)$ and $\boldsymbol{\eta}=\left(\eta_{1}, \eta_{2}, \ldots, \eta_{\ell}\right)$ be as defined in previous section. Then the multi-twisted $R S$ code $\mathcal{C}_{k}(\boldsymbol{\alpha}, \boldsymbol{t}, \boldsymbol{h}, \boldsymbol{\eta})$ is MDS if and only if for each $\mathcal{I} \subset\{1,2, \ldots, n\}$ with cardinality $k$ correspondingly the polynomial $\prod_{i \in \mathcal{I}}\left(x-\alpha_{i}\right)=\sum_{i} \sigma_{i} x^{i}$ with $\sigma_{i}=0$ for $i<0$, the matrix

$$
\left.\underset{\substack{\uparrow \\ t_{\ell}^{t h}}}{\left(\eta_{\ell}^{-1}, 1, \ldots, 1, \eta_{\ell-1}^{\uparrow},\right.}, \underset{\substack{\uparrow \\ t_{\ell-1}^{t h}}}{\eta_{1}^{-1}}, 1, \ldots, \underset{\substack{t_{1}^{t h}}}{\eta_{1}^{-1}}, 1, \ldots, \underset{\substack{\text { st }}}{1}\right) \cdot A_{\mathcal{I}}+B_{\mathcal{I}}
$$

is non-singular. Here, $A_{\mathcal{I}}$ and $B_{\mathcal{I}}$ are lower and upper-triangular $t_{\ell} \times t_{\ell}$ matrices, respectively, given by:

$$
A_{\mathcal{I}}=\left(\begin{array}{ccccc}
1 & 0 & 0 & \cdots & 0 \\
\sigma_{k-1} & 1 & 0 & \cdots & 0 \\
\sigma_{k-2} & \sigma_{k-1} & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\sigma_{k-t_{\ell}+1} & \cdots & \cdots & \sigma_{k-1} & 1
\end{array}\right) \text { and }
$$

Proof. It suffices to show that the codeword with at least $k$ zero positions is precisely the zero codeword. Let $f \in \mathcal{V}_{k, \boldsymbol{t}, \boldsymbol{h}, \boldsymbol{\eta}}$ corresponds to the codeword with at least $k$ zero positions, i.e. let $f$ have $k$ roots among the evaluation points. Then we can write

$$
\begin{equation*}
f(x)=\sigma(x) g(x) \tag{3.1}
\end{equation*}
$$

where, $\sigma(x)=\prod_{i \in \mathcal{I}}\left(x-\alpha_{i}\right)=\sum_{i=0}^{k} \sigma_{i} x^{i}$ for some $\mathcal{I} \subset\{1,2, \ldots, n\}$ with cardinality $k$ and $g(x)=\sum_{i=0}^{t_{e}-1} g_{i} x^{i} \in \mathbb{F}_{q}[x]$. Note that $\sigma_{k}=1$ and all the coefficients of $x^{k}$ to $x^{k+t_{\ell}-1}$ are zero except the coefficients of $x^{k+t_{1}-1}, x^{k+t_{2}-1}, \ldots$, $x^{k+t_{\ell-1}-1}, x^{k+t_{\ell}-1}$ in $f$. Consequently, we obtain the following system of $\left(t_{\ell}-\ell\right)$ equations in $g_{j} \mathrm{~s}$ :

$$
\begin{equation*}
\sum_{j=0}^{i} \sigma_{i-j} g_{j}=0 \tag{3.2}
\end{equation*}
$$

where $i \in\left\{k, k+1, \ldots, k+t_{\ell}-2\right\} \backslash\left\{k+t_{1}-1, k+t_{2}-1, \ldots, k+t_{\ell-1}-1\right\}$ and $g_{j}=0$ when $j \notin\left\{0,1, \ldots, t_{\ell}-1\right\}$ and $\sigma_{j}=0$ when $j \notin\{0,1, \ldots, k\}$. Comparing the coefficients of $x^{k+t_{s}-1}$ in both sides of 3.1 and substituting the value of $a_{h_{s}}=\sum_{j=0}^{h_{s}} \sigma_{h_{s}-j} g_{j}$, we obtain the following system of $\ell$ equations in $g_{j} \mathrm{~s}$ :

$$
\begin{equation*}
\eta_{s}^{-1}\left(\sum_{i=0}^{k} \sigma_{k-i} g_{t_{s}-1+i}\right)-\sum_{j=0}^{h_{s}} \sigma_{h_{s}-j} g_{j}=0 \tag{3.3}
\end{equation*}
$$

for $s=1,2, \ldots, \ell$. From (3.2) and (3.3), we have a homogeneous system of $t_{\ell}$ equations in $t_{\ell}$ variables. The code $\mathcal{C}_{k}(\boldsymbol{\alpha}, \boldsymbol{t}, \boldsymbol{h}, \boldsymbol{\eta})$ is MDS if and only if $f$ is zero polynomial, i.e. the above homogeneous system has only the zero vector as solution for all choices of $\mathcal{I}$. This completes the proof.

Example 3.2. Consider the finite field $\mathbb{F}_{2^{4}}=\mathbb{F}_{2}(\alpha)$ with $\alpha^{4}+\alpha+1=0$. Let $n=5, k=3, \boldsymbol{\alpha}=\left(0, \alpha^{2}, \alpha+\right.$ $\left.1, \alpha^{2}+\alpha, \alpha^{3}+\alpha+1\right), \boldsymbol{t}=(1,2)$ and $\boldsymbol{h}=(0,1)$. Let $\mathcal{C}_{k}(\boldsymbol{\alpha}, \boldsymbol{t}, \boldsymbol{h}, \boldsymbol{\eta})$ be a multi-twisted $R S$ code having $\ell=2$ twists with $\boldsymbol{\eta}=\left(\eta_{1}, \eta_{2}\right)$, where $\eta_{1}=\alpha^{3}+\alpha^{2}$ and $\eta_{2} \in\left\{1, \alpha^{2}+1, \alpha^{2}+\alpha+1, \alpha^{3}, \alpha^{3}+\alpha^{2}, \alpha^{3}+\alpha^{2}+\alpha\right\}$. Now for each $\mathcal{I} \subset\{1,2,3,4,5\}$ with cardinality 3 correspondingly polynomial $\prod_{i \in \mathcal{I}}\left(x-\alpha_{i}\right)=\sum_{i} \sigma_{i} x^{i}$, the matrix

$$
\left[\begin{array}{cc}
\eta_{2}^{-1} & 0 \\
0 & \eta_{1}^{-1}
\end{array}\right] \cdot\left[\begin{array}{cc}
1 & 0 \\
\sigma_{2} & 1
\end{array}\right]+\left[\begin{array}{cc}
-\sigma_{0} & -\sigma_{1} \\
0 & -\sigma_{0}
\end{array}\right]
$$

is non-singular. Then by Theorem 3.1 the $\operatorname{code} \mathcal{C}_{k}(\boldsymbol{\alpha}, \boldsymbol{t}, \boldsymbol{h}, \boldsymbol{\eta})$ is MDS. For detailed implementation we refer to Appendix A.

Remark 3.3. By SageMath implementations, similar to the existence of MDS single-twisted $R S$ codes one can see that many long MDS multi-twisted $R S$ codes can be obtained using twisted polynomials for particular values
of $q, n, k, \boldsymbol{h}$ and $\boldsymbol{t}$; but not for general values. Also, it seems hard to find explicit constructions of such long MDS multi-twisted RS codes. However, the existence of MDS multi-twisted RS code given by Beelen et al. [1], uses the fact that $\mathcal{C}_{k}(\boldsymbol{\alpha}, \boldsymbol{t}, \boldsymbol{h}, \boldsymbol{\eta})$ is MDS if and only if every $k$ columns of generator matrix (2.4) are linearly independent over $\mathbb{F}_{q}$. For this, the authors assumed some constraints on the evaluation vector $\boldsymbol{\alpha}$ and the vector $\boldsymbol{\eta}$. Now, we give an alternate proof of the same result as a consequence of the Theorem 3.1.

Theorem 3.4. [1] Let $\mathbb{F}_{q_{0}} \mp \mathbb{F}_{q_{1}} q \mathbb{F}_{q_{2}} \mp \cdots q \mathbb{F}_{q_{\ell}}=\mathbb{F}_{q}$ be a proper chain of subfields and $\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n} \in \mathbb{F}_{q_{0}}$. If $\eta_{i} \in \mathbb{F}_{q_{i}} \backslash \mathbb{F}_{q_{i-1}}$, for all $1 \leq i \leq \ell$. Then the multi-twisted RS code $\mathcal{C}_{k}(\boldsymbol{\alpha}, \boldsymbol{t}, \boldsymbol{h}, \boldsymbol{\eta})$ is MDS

Proof. Let $\mathcal{I} \subset\{1,2, \ldots, n\}$ with cardinality $k$ and $W=\operatorname{diag}\left(\eta_{\ell}^{-1}, 1, \ldots, 1, \eta_{\ell-1}^{-1}, 1, \ldots, 1, \eta_{1}^{-1}, 1, \ldots, 1\right) \cdot A_{\mathcal{I}}+B_{\mathcal{I}}$ be the corresponding matrix as described in Theorem 3.1. Since $\alpha_{i} \in \mathbb{F}_{q_{0}}$ for each $i, \sigma_{j}$ also belongs to $\mathbb{F}_{q_{0}}$ for each $j$. In particular, $\sigma_{0} \in \mathbb{F}_{q_{0}}$. Further, $\eta_{j}^{-1} \in \mathbb{F}_{q_{j}} \backslash \mathbb{F}_{q_{j-1}}$ implies that $\eta_{j}^{-1}-\sigma_{j} \neq 0$. Hence, the diagonal entries in $W$ are all non-zero and on applying elementary row operations, we can convert this matrix to lower triangular matrix with diagonal elements $\eta_{\ell}^{-1}+T_{\ell}, 1, \ldots, 1, \eta_{\ell-1}^{-1}+T_{\ell-1}, 1, \ldots, \ldots, 1, \eta_{1}^{-1}+T_{1}, 1, \ldots, 1$ for some $T_{i} \in \mathbb{F}_{q_{i-1}}$. Hence, corresponding matrix is non-singular. Thus by Theorem 3.1. $\mathcal{C}_{k}(\boldsymbol{\alpha}, \boldsymbol{t}, \boldsymbol{h}, \boldsymbol{\eta})$ is MDS. This completes the proof.

## 4 MDS double-twisted RS codes

In this section, we give algebraic criteria for double-twisted RS codes having twists $\boldsymbol{t}=(1,2)$ and hooks $\boldsymbol{h}=(0,1)$ to be MDS with enumeration. For these double-twisted RS codes, we assume $\boldsymbol{\eta}=\left(\eta_{1}, \eta_{2}\right) \in\left(\mathbb{F}_{q}^{*}\right)^{2}$. Let $k<n<q$ then we have $\mathcal{V}_{k, \boldsymbol{t}, \boldsymbol{h}, \boldsymbol{\eta}}:=\left\{a_{0}+a_{1} x+\cdots+a_{k-1} x^{k-1}+\eta_{1} a_{0} x^{k}+\eta_{2} a_{1} x^{k+1}: a_{i} \in \mathbb{F}_{q}\right.$ for each $\left.i\right\}$. Let $\boldsymbol{\alpha}=\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}\right) \in \mathbb{F}_{q}^{n}$ be an evaluation vector and $E_{\boldsymbol{\alpha}}=\left\{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}\right\} \subset \mathbb{F}_{q}$ be set of evaluation points, where $\alpha_{i} \neq \alpha_{j}$ for each $i \neq j$.

Proposition 4.1. The polynomial $f(x)=a_{0}+a_{1} x+\cdots+a_{k-1} x^{k-1}+\eta_{1} a_{0} x^{k}+\eta_{2} a_{1} x^{k+1} \in \mathcal{V}_{k, \boldsymbol{t}, \boldsymbol{h}, \boldsymbol{\eta}}$ has at most $k-1$ roots among $E_{\boldsymbol{\alpha}}$ if any of the following conditions hold:
(i) $a_{0}=a_{1}=0$;
(ii) $a_{0} \neq 0, a_{1}=0$, and $\eta_{1} \neq \frac{(-1)^{k}}{\prod_{i \in \mathcal{J}_{k}} \alpha_{i}}$;
(iii) $a_{0} \neq 0, a_{1} \neq 0, \eta_{2} \neq \frac{(-1)^{k}}{\prod_{i \in \mathcal{J}_{k}} \alpha_{i}}$, and $\eta_{1} \neq \frac{\left(\sum_{i \in \mathcal{J}_{k}} \prod_{\substack{ \\j \neq \mathcal{J}_{k}}} \alpha_{j}\right)\left(\sum_{i \in \mathcal{J}_{k}} \alpha_{i}\right)+\left(\frac{(-1)^{k}}{\eta_{2}}-\prod_{i \in \mathcal{J}_{k}} \alpha_{i}\right)}{(-1)^{k}\left(\prod_{i \in \mathcal{J}_{k}} \alpha_{i}\right)\left(\frac{(-1)^{k}}{\eta_{2}}-\prod_{i \in \mathcal{J}_{k}} \alpha_{i}\right)}$;
(iv) $a_{0}=0, a_{1} \neq 0$, and $\eta_{2} \neq \frac{(-1)^{k-1}}{\left(\sum_{j \in \mathcal{J}_{k-1}} \alpha_{j}\right)\left(\prod_{j \in \mathcal{J}_{k-1}} \alpha_{j}\right)}$;
where $\mathcal{J}_{s} \subset\{1,2, \ldots, n\}$ having cardinality $s$ such that $\alpha_{j} \neq 0$ for each $j \in \mathcal{J}_{s}$.
Proof. It is immediate to observe that if $(i)$ or (ii) holds, then $f$ has at most $k-1$ roots among $E_{\boldsymbol{\alpha}}$. If (iii) holds then since $a_{0} \neq 0, f$ doesn't have 0 among its roots. Now, suppose roots of $f$ are $\alpha_{i_{1}}, \alpha_{i_{2}}, \ldots, \alpha_{i_{k}}$ and $\beta$, where $\alpha_{i_{j}} \in E_{\boldsymbol{\alpha}}$ for $1 \leq j \leq k$. Then $f(x)=\eta_{2} a_{1}(x-\beta) \prod_{j=1}^{k}\left(x-\alpha_{i_{j}}\right)$, where $\eta_{2} \neq 0$. Now the product of roots of $f(x)$ is

$$
\begin{equation*}
\beta \prod_{j=1}^{k} \alpha_{i_{j}}=\frac{(-1)^{k+1} a_{0}}{\eta_{2} a_{1}} \tag{4.1}
\end{equation*}
$$

and, the sum of product of $k$-roots of $f$, i.e.

$$
\prod_{j=1}^{k} \alpha_{i_{j}}+\beta\left(\alpha_{i_{1}} \alpha_{i_{2}} \cdots \alpha_{i_{k-1}}+\alpha_{i_{1}} \cdots \alpha_{i_{k-2}} \alpha_{i_{k}}+\cdots+\alpha_{i_{2}} \alpha_{i_{3}} \cdots \alpha_{i_{k}}\right)=\frac{(-1)^{k}}{\eta_{2}}
$$

Now if $\alpha_{i_{1}} \alpha_{i_{2}} \cdots \alpha_{i_{k-1}}+\alpha_{i_{1}} \cdots \alpha_{i_{k-2}} \alpha_{i_{k}}+\cdots+\alpha_{i_{2}} \alpha_{i_{3}} \cdots \alpha_{i_{k}}=0$, then $\eta_{2}=\frac{(-1)^{k}}{\prod_{j=1}^{k} \alpha_{i_{j}}}$, which contradicts the assumption (iii). Therefore

$$
\begin{equation*}
\beta=\frac{\left(\frac{(-1)^{k}}{\eta_{2}}-\prod_{j=1}^{k} \alpha_{i_{j}}\right)}{\left(\alpha_{i_{1}} \alpha_{i_{2}} \cdots \alpha_{i_{k-1}}+\alpha_{i_{1}} \cdots \alpha_{i_{k-2}} \alpha_{i_{k}}+\cdots+\alpha_{i_{2}} \alpha_{i_{3}} \cdots \alpha_{i_{k}}\right)} \tag{4.2}
\end{equation*}
$$

Also the sum of roots of $f$ is

$$
\begin{equation*}
\sum_{j=1}^{k} \alpha_{i_{j}}+\beta=\frac{-\eta_{1} a_{0}}{\eta_{2} a_{1}} \tag{4.3}
\end{equation*}
$$

Using 4.1) and 4.3, we obtain

$$
\begin{equation*}
\sum_{j=1}^{k} \alpha_{i_{j}}+\beta=(-1)^{k} \eta_{1}\left(\beta \prod_{j=1}^{k} \alpha_{i_{j}}\right) \tag{4.4}
\end{equation*}
$$

Using (4.2) in 4.4, we obtain

$$
\begin{equation*}
\eta_{1}=\frac{\left(\alpha_{i_{1}} \alpha_{i_{2}} \cdots \alpha_{i_{k-1}}+\alpha_{i_{1}} \cdots \alpha_{i_{k-2}} \alpha_{i_{k}}+\cdots+\alpha_{i_{2}} \alpha_{i_{3}} \cdots \alpha_{i_{k}}\right)\left(\sum_{j=1}^{k} \alpha_{i_{j}}\right)+\left(\frac{(-1)^{k}}{\eta_{2}}-\prod_{j=1}^{k} \alpha_{i_{j}}\right)}{(-1)^{k}\left(\prod_{j=1}^{k} \alpha_{i_{j}}\right)\left(\frac{(-1)^{k}}{\eta_{2}}-\prod_{j=1}^{k} \alpha_{i_{j}}\right)} \tag{4.5}
\end{equation*}
$$

but this contradicts the assumption (iii). Note that this contradiction is free from the choices of $\beta$. Observe that in 4.5 denominator is non-zero by 4.1 and 4.2. Hence $f$ can have at most $k-1$ roots among $E_{\boldsymbol{\alpha}}$. Lastly, assume (iv) holds then since $a_{0}=0$, one of the roots of $f$ is definitely 0 . This gives a factor of $f$; denoted by $f / x$ as $a_{1}+a_{2} x+\cdots+a_{k-1} x^{k-2}+\eta_{2} a_{1} x^{k}$. Therefore, we have following two cases:

- $0 \notin E_{\boldsymbol{\alpha}}$ : Let roots of $f$ be $0, \alpha_{i_{2}}, \alpha_{i_{3}}, \ldots, \alpha_{i_{k+1}}$, then $\sum_{j=2}^{k+1} \alpha_{i_{j}}=0$ since sum of roots of $f$ is zero. That means

$$
\begin{equation*}
\alpha_{i_{2}}=-\sum_{j=3}^{k+1} \alpha_{i_{j}} . \tag{4.6}
\end{equation*}
$$

Assume $\alpha_{i_{j}} \in E_{\boldsymbol{\alpha}}$ for each $j$, then $f / x$ also has these $k$ roots in $E_{\boldsymbol{\alpha}}$. Then the product of roots of $f / x$ is

$$
\begin{equation*}
\frac{(-1)^{k}}{\eta_{2}}=\prod_{j=2}^{k+1} \alpha_{i_{j}} . \tag{4.7}
\end{equation*}
$$

Using $\sqrt{4.6}$ in 4.7 , we get $\eta_{2}=\frac{(-1)^{k-1}}{\left(\sum_{j=3}^{k+1} \alpha_{i_{j}}\right)\left(\prod_{j=3}^{k+1} \alpha_{i_{j}}\right)}$. In general we obtain $\eta_{2}=\frac{(-1)^{k-1}}{\left(\sum_{j \in \mathcal{J}_{k-1}} \alpha_{j}\right)\left(\prod_{j \in \mathcal{J}} \prod_{k-1} \alpha_{j}\right)}$ where $\mathcal{J}_{s} \subset\{1,2, \ldots, n\}$ having cardinality $s$ such that $\alpha_{j} \neq 0$ for each $j \in \mathcal{J}_{s}$. This contradicts the assumption.

- $0 \in E_{\boldsymbol{\alpha}}$ : Let roots of $f / x$ be $\alpha_{i_{1}}, \alpha_{i_{2}}, \ldots, \alpha_{i_{k-1}}, \beta$ where $\alpha_{i_{j}} \in E_{\boldsymbol{\alpha}}$. The product and sum of roots of $f / x$ are given by

$$
\beta \prod_{j=1}^{k-1} \alpha_{i_{j}}=\frac{(-1)^{k}}{\eta_{2}} \text { and } \beta+\sum_{j=1}^{k-1} \alpha_{i_{j}}=0, \text { respectively. }
$$

Now eliminating $\beta$ we obtain a contradiction to the condition in (iv). Again note that this contradiction is free from the choices of $\beta$. Hence $f / x$ can have at most $k-2$ roots among $E_{\boldsymbol{\alpha}}$.

This completes the proof.
Theorem 4.2. Let $k<n$ and $\boldsymbol{\alpha}=\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}\right) \in \mathbb{F}_{q}^{n}$ where $\alpha_{i} \neq \alpha_{j}$ for each $i \neq j, \boldsymbol{t}=(1,2), \boldsymbol{h}=(0,1)$ and $\boldsymbol{\eta}=\left(\eta_{1}, \eta_{2}\right) \in\left(\mathbb{F}_{q}^{*}\right)^{2}$. Then the double-twisted RS code

$$
\mathcal{C}_{k}(\boldsymbol{\alpha}, \boldsymbol{t}, \boldsymbol{h}, \boldsymbol{\eta})=\left\{\left(f\left(\alpha_{1}\right), f\left(\alpha_{2}\right), \ldots, f\left(\alpha_{n}\right)\right) \in \mathbb{F}_{q}^{n}: f \in \mathcal{V}_{k, \boldsymbol{t}, \boldsymbol{h}, \boldsymbol{\eta}}\right\}
$$

is MDS if and only if the following conditions hold:
(i) $\eta_{1} \neq \frac{(-1)^{k}}{\prod_{i \in \mathcal{J}_{k}} \alpha_{i}}$ whenever $\sum_{i \in \mathcal{J}_{k}} \prod_{\substack{j \in \mathcal{J}_{k} \\ j \neq i}} \alpha_{j}=0$;
(ii) $\eta_{1} \neq \frac{\left(\sum_{i \in \mathcal{J}_{j}} \prod_{\substack{\in \in \mathcal{J}_{k} \\ j \neq i}} \alpha_{j}\right)\left(\sum_{i \in \mathcal{J}_{k}} \alpha_{i}\right)+\left(\frac{(-1)^{k}}{\eta_{2}}-\prod_{i \in \mathcal{J}_{k}} \alpha_{i}\right)}{(-1)^{k}\left(\prod_{i \in \mathcal{J}_{k}} \alpha_{i}\right)\left(\frac{(-1)^{k}}{\eta_{2}}-\prod_{i \in \mathcal{J}_{k}} \alpha_{i}\right)}$ whenever $\eta_{2} \neq \frac{(-1)^{k}}{\prod_{j \in \mathcal{J}_{k}} \alpha_{j}}$;
(iii) $\eta_{2} \neq \frac{(-1)^{k-1}}{\left(\sum_{j \in \mathcal{J}_{k-1}} \alpha_{j}\right)\left(\prod_{j \in \mathcal{J}_{k-1}} \alpha_{j}\right)}$ whenever $\prod_{i \in \mathcal{I}_{k}} \alpha_{i}=0$
where $\mathcal{I}_{s}$ and $\mathcal{J}_{s} \subset\{1,2, \ldots, n\}$ having cardinality s such that $\alpha_{j} \neq 0$ for each $j \in \mathcal{J}_{s}$.
Proof. The code $\mathcal{C}_{k}(\boldsymbol{\alpha}, \boldsymbol{t}, \boldsymbol{h}, \boldsymbol{\eta})$ is MDS if and only if every polynomial $f \in \mathcal{V}_{k, \boldsymbol{t}, \boldsymbol{h}, \boldsymbol{\eta}}$ has at most $k-1$ roots among the evaluation points $E_{\alpha}$. The proof of if part follows by Proposition 4.1. On the other hand, if any of the conditions (i)-(iii) do not hold then by similar approach to [3] one can find a polynomial $f \in \mathcal{V}_{k, t, \boldsymbol{h}, \boldsymbol{\eta}}$ having at least $k$ roots among the evaluation points $E_{\boldsymbol{\alpha}}$. Thus we get required proof.

Example 4.3. Consider a finite field $\mathbb{F}_{2^{4}}=\frac{\mathbb{F}_{2}[x]}{\left\langle x^{4}+x+1\right\rangle}=\mathbb{F}_{2}(\alpha)$. Let $n=5, k=3, \boldsymbol{t}=(1,2), \boldsymbol{h}=(0,1)$, and $\boldsymbol{\eta}=$ $\left(\eta_{1}, \eta_{2}\right) \in \mathbb{F}_{2}(\alpha)^{2}$, then we have the set of double-twisted polynomials $\mathcal{V}_{k, \boldsymbol{t}, \boldsymbol{h}, \boldsymbol{\eta}}=\left\{a_{0}+a_{1} x+a_{2} x^{2}+\eta_{1} a_{0} x^{3}+\eta_{2} a_{1} x^{4}\right.$ : $a_{i} \in \mathbb{F}_{2}(\alpha)$ for each $\left.i\right\}$. Then every $f \in \mathcal{V}_{k, \boldsymbol{t}, \boldsymbol{h}, \boldsymbol{\eta}}$ has at most 2 roots among $\left\{0, \alpha^{3}+\alpha^{2}, \alpha^{3}+\alpha^{2}+\alpha+1, \alpha^{3}+1,1\right\} \subset$ $\mathbb{F}_{2}(\alpha)$ for $\eta_{1}=\alpha^{2}+\alpha$ and $\eta_{2} \in\left\{1, \alpha, \alpha^{2}+\alpha, \alpha^{3}, \alpha^{3}+\alpha, \alpha^{3}+\alpha^{2}\right\}$. Thus the double-twisted RS code $\mathcal{C}_{k}(\boldsymbol{\alpha}, \boldsymbol{t}, \boldsymbol{h}, \boldsymbol{\eta})$ is MDS.

Remark 4.4. If we apply the similar approach as Theorem 3.3 of [26, 27] and Theorem 3.2 of [9], we obtain that the double-twisted $R S$ code $\mathcal{C}_{k}(\boldsymbol{\alpha}, \boldsymbol{t}, \boldsymbol{h}, \boldsymbol{\eta})$ in Theorem 4.2 is MDS if and only if $\boldsymbol{\eta}=\left(\eta_{1}, \eta_{2}\right) \in\left(\mathbb{F}_{q}^{*}\right)^{2}$ satisfies

$$
1-\eta_{1}(-1)^{k}\left(\prod_{i \in \mathcal{I}_{k}} \alpha_{i}\right)+\eta_{2}(-1)^{k}\left(\left(\sum_{j \in \mathcal{I}_{k}} \prod_{\substack{i \neq j \\ i \in \mathcal{I}_{k}}} \alpha_{i}\right)\left(\sum_{i \in \mathcal{I}_{k}} \alpha_{i}\right)-\left(\prod_{j \in \mathcal{I}_{k}} \alpha_{j}\right)\right)+\eta_{1} \eta_{2}\left(\prod_{i \in \mathcal{I}_{k}} \alpha_{i}^{2}\right) \neq 0
$$

for every $\mathcal{I}_{k} \subseteq\{1,2, \ldots, n\}$ of cardinality $k$. One can easily verify that $\eta_{1}$ and $\eta_{2}$ obtained by Theorem 4.2 satisfy the above condition and vice-versa. However, Theorem 4.2 gives an approach to discard some choices for $\eta_{1}$ and $\eta_{2}$, which is better than exhaustive search.

Note that the MDS codes described in Theorem 4.2 are double-twisted RS codes with twists $\boldsymbol{t}=(1,2)$ and hooks $\boldsymbol{h}=(0,1)$, which are particular multi-twisted RS codes with $\ell=2$, and different from the MDS doubletwisted RS codes obtained in [27] and [9]. Now we give the enumeration of such classes of MDS double-twisted RS codes over finite fields of size up to 17 in Table 1. By SageMath implementations, one can see that many long MDS double-twisted RS codes can be obtained using twisted polynomials for particular values of $q, n, k, \boldsymbol{h}$ and $\boldsymbol{t}$; but not for general values. Also, it seems hard to find explicit constructions of such long MDS double-twisted RS codes. However, if we include additional conditions defined in Theorem 3.4 then MDS double-twisted RS codes exist always.

| $\boldsymbol{q}$ | $n$ | $k=1$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 3 | 2 | 8 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| 4 | $\begin{aligned} & 2 \\ & 3 \end{aligned}$ | $\begin{aligned} & 39 \\ & 24 \end{aligned}$ | 15 |  |  |  |  |  |  |  |  |  |  |  |  |  |
| 5 | $\begin{aligned} & 2 \\ & 3 \\ & 4 \end{aligned}$ | $\begin{aligned} & 116 \\ & 100 \\ & 44 \\ & \hline \end{aligned}$ | $\begin{aligned} & 82 \\ & 28 \\ & \hline \end{aligned}$ | 36 |  |  |  |  |  |  |  |  |  |  |  |  |
| 7 | $\begin{aligned} & 2 \\ & 3 \\ & 4 \\ & 5 \\ & 6 \end{aligned}$ | $\begin{aligned} & 588 \\ & 870 \\ & 780 \\ & 426 \\ & 132 \end{aligned}$ | $\begin{gathered} 777 \\ 513 \\ 213 \\ 54 \\ \hline \end{gathered}$ | $\begin{gathered} 704 \\ 186 \\ 32 \end{gathered}$ | $\begin{gathered} 339 \\ 6 \end{gathered}$ | 84 |  |  |  |  |  |  |  |  |  |  |
| 8 | $\begin{aligned} & 2 \\ & 3 \\ & 4 \\ & 5 \\ & 6 \\ & 6 \\ & \hline \end{aligned}$ | 1099 1988 2275 1694 805 224 | $\begin{gathered} 2009 \\ 1883 \\ 1029 \\ 322 \\ 49 \end{gathered}$ | $\begin{gathered} 1764 \\ 420 \\ 0 \\ 0 \\ \hline \end{gathered}$ | $\begin{gathered} 1281 \\ 70 \\ 0 \end{gathered}$ | $\begin{gathered} 630 \\ 63 \end{gathered}$ | 77 |  |  |  |  |  |  |  |  |  |
| 9 | $\begin{aligned} & \hline 2 \\ & 3 \\ & 4 \\ & 5 \\ & 5 \\ & 6 \\ & 7 \\ & 8 \\ & \hline \end{aligned}$ | $\begin{aligned} & \hline 1880 \\ & 3976 \\ & 5432 \\ & 4984 \\ & 3080 \\ & 1240 \\ & 296 \\ & \hline \end{aligned}$ | $\begin{gathered} 3812 \\ 4072 \\ 2592 \\ 1044 \\ 292 \\ 52 \end{gathered}$ | $\begin{gathered} 4712 \\ 1840 \\ 136 \\ 0 \\ 0 \\ \hline \end{gathered}$ | $\begin{gathered} 4574 \\ 716 \\ 40 \\ 4 \end{gathered}$ | $\begin{gathered} 2312 \\ 8 \\ 0 \end{gathered}$ | $\begin{gathered} 948 \\ 4 \end{gathered}$ | 152 |  |  |  |  |  |  |  |  |
| 11 | $\begin{gathered} \hline 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 9 \\ 10 \\ \hline \end{gathered}$ | 4640 12810 23640 30660 28560 19140 9060 2890 560 | $\begin{aligned} & 12525 \\ & 19260 \\ & 19420 \\ & 13520 \\ & 6670 \\ & 2435 \\ & 665 \\ & 120 \\ & \hline \end{aligned}$ | $\begin{gathered} 22180 \\ 16450 \\ 5190 \\ 610 \\ 20 \\ 0 \\ 0 \\ \hline \end{gathered}$ | $\begin{gathered} 28360 \\ 9890 \\ 760 \\ 10 \\ 0 \\ 0 \end{gathered}$ | $\begin{gathered} 24892 \\ 3260 \\ 60 \\ 0 \\ 0 \end{gathered}$ | $\begin{gathered} 15870 \\ 565 \\ 0 \\ 0 \end{gathered}$ | $\begin{gathered} 6630 \\ 40 \\ 0 \end{gathered}$ | $\begin{gathered} 1915 \\ 10 \end{gathered}$ | 240 |  |  |  |  |  |  |
| 13 | $\begin{gathered} \hline 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 7 \\ 9 \\ 9 \\ 10 \\ 11 \\ 12 \\ \hline \end{gathered}$ | 9708 33132 77220 129888 162360 152856 108504 57420 22044 5820 948 | $\begin{aligned} & 32538 \\ & 64626 \\ & 86082 \\ & 80376 \\ & 54708 \\ & 28206 \\ & 11442 \\ & 3738 \\ & 942 \\ & 156 \\ & \hline \end{aligned}$ | $\begin{gathered} 74368 \\ 80676 \\ 44200 \\ 10616 \\ 1092 \\ 52 \\ 0 \\ 0 \\ 0 \\ \hline \end{gathered}$ | $\begin{gathered} 122991 \\ 68004 \\ 10776 \\ 426 \\ 18 \\ 0 \\ 0 \\ 0 \\ \hline \end{gathered}$ | $\begin{gathered} 149304 \\ 34368 \\ 312 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hline \end{gathered}$ | $\begin{gathered} 135396 \\ 13086 \\ 40 \\ 0 \\ 0 \\ 0 \\ \hline \end{gathered}$ | $\begin{gathered} 90552 \\ 1860 \\ 0 \\ 0 \\ 0 \\ \hline \end{gathered}$ | $\begin{gathered} 44247 \\ 462 \\ 0 \\ 0 \\ \hline \end{gathered}$ | $\begin{gathered} 14968 \\ 48 \\ 0 \\ \hline \end{gathered}$ | $\begin{gathered} 3450 \\ 12 \\ \hline \end{gathered}$ | 348 |  |  |  |  |
| 16 | 12 2 3 4 5 6 7 8 9 10 11 12 13 14 15 | 23955 105420 323505 734370 1276275 1733160 1859715 1583010 1066065 562380 27955 68670 14505 1920 | $\begin{aligned} & 105885 \\ & 290295 \\ & 554265 \\ & 762255 \\ & 771105 \\ & 581595 \\ & 329895 \\ & 141405 \\ & 45795 \\ & 11085 \\ & 1935 \\ & 225 \\ & 15 \end{aligned}$ | 312560 494760 443560 212320 51900 5700 100 0 0 0 0 0 | $\begin{gathered} 698790 \\ 613560 \\ 176625 \\ 10530 \\ 120 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hline \end{gathered}$ | $\begin{gathered} 1210998 \\ 570675 \\ 34305 \\ 510 \\ 39 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hline \end{gathered}$ | $\begin{gathered} 1606060 \\ 365085 \\ 2220 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{gathered}$ | $\begin{gathered} 1652925 \\ 145755 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{gathered}$ | $\begin{gathered} 1349640 \\ 40695 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hline \end{gathered}$ | $\begin{gathered} 862810 \\ 14085 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hline \end{gathered}$ | $\begin{gathered} 415848 \\ 4185 \\ 0 \\ 0 \\ 0 \\ \hline \end{gathered}$ | $\begin{gathered} 150300 \\ 165 \\ 0 \\ 0 \end{gathered}$ | $\begin{gathered} 40760 \\ 0 \\ 0 \end{gathered}$ | $\begin{gathered} 6495 \\ 210 \end{gathered}$ | 345 |  |
| 17 | $\begin{gathered} \hline 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \\ 15 \\ 16 \end{gathered}$ | $\begin{gathered} \hline 31088 \\ 146960 \\ 486640 \\ 1198288 \\ 2272816 \\ 3393104 \\ 4038320 \\ 3855280 \\ 2953808 \\ 1806986 \\ 872144 \\ 325360 \\ 90640 \\ 17776 \\ 2192 \end{gathered}$ | $\begin{aligned} & 145928 \\ & 429648 \\ & 890464 \\ & 1358488 \\ & 1581464 \\ & 1462008 \\ & 1062512 \\ & 638104 \\ & 315704 \\ & 128704 \\ & 42832 \\ & 11272 \\ & 2184 \\ & 272 \end{aligned}$ | $\begin{gathered} 475184 \\ 839968 \\ 852864 \\ 460080 \\ 121280 \\ 13632 \\ 544 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{gathered}$ | 1164604 1214568 473896 46008 932 0 0 0 0 0 0 0 | $\begin{gathered} 2174976 \\ 1216256 \\ 106128 \\ 512 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{gathered}$ | 3197848 895712 10680 0 0 0 0 0 0 0 0 | $\begin{gathered} 3710016 \\ 451648 \\ 480 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{gathered}$ | $\begin{gathered} 3431638 \\ 174648 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{gathered}$ | $\begin{gathered} 2511296 \\ 49744 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{gathered}$ | $\begin{gathered} 1449416 \\ 13488 \\ 0 \\ 0 \\ 0 \\ 0 \end{gathered}$ | $\begin{gathered} 644224 \\ 2864 \\ 0 \\ 0 \\ 0 \end{gathered}$ | $\begin{gathered} 217084 \\ 600 \\ 0 \\ 0 \end{gathered}$ | $\begin{gathered} 51840 \\ 80 \\ 0 \end{gathered}$ | $\begin{gathered} 8680 \\ 16 \end{gathered}$ | 624 |

Table 1: Number of MDS double-twisted RS codes with twists $\boldsymbol{t}=(1,2)$ and hooks $\boldsymbol{h}=(0,1)$ respectively over finite fields of size $\leq 17$

## 5 Multi-twisted RS codes with small dimensional hull

In this section, we study the existence of multi-twisted RS codes with small-dimensional hull. For this, we first state some useful results based on hull.
Lemma 5.1. ([14], Proposition 1) Let $\mathcal{C}$ be an $[n, k]$ linear code over $\mathbb{F}_{q}$ with generator matrix $G$. Then code $\mathcal{C}$ has a one-dimensional hull if and only if the rank of matrix $G \cdot G^{T}$ is $k-1$. Alternately, the code $\mathcal{C}$ has a one-dimensional hull if and only if the rank of matrix $H \cdot H^{T}$ is $n-k-1$.
Proposition 5.2. 14] Let $\mathcal{C}$ be an $[n, k]$ linear code over $\mathbb{F}_{q}$ with generator matrix $G$. Then code $\mathcal{C}$ has a $\mathfrak{L}$-dimensional hull if and only if the rank of matrix $G \cdot G^{T}$ is $k-\mathfrak{L}$, where $0 \leq \mathfrak{L} \leq \min \{k, n-k\}$.

In the remaining part of this section, we fix the following notations:
$\gamma \quad$ : a primitive element of $\mathbb{F}_{q}$
$k \quad$ : a positive integer such that $k \mid(q-1)$
$\alpha_{i}:=\gamma^{\frac{q-1}{k} i}$ for $1 \leq i \leq k$
It is clear that $\gamma^{\frac{q-1}{k}}$ generates a subgroup of $\mathbb{F}_{q}^{*}$ of order $k$. Also, by [28], for any integer $m$,

$$
\theta^{m}=\alpha_{1}^{m}+\alpha_{2}^{m}+\cdots+\alpha_{k}^{m}=\left\{\begin{array}{cc}
k & \text { if } m \equiv 0 \bmod k  \tag{5.1}\\
0 & \text { otherwise }
\end{array}\right.
$$

Next, we give proof for the existence of multi-twisted RS codes as defined in Definition 2.4, with small dimensional hull in two scenarios when $q$ is even and odd.

Theorem 5.3. Let $q$ be a power of 2 , and $k>1$ be an integer such that $k \mid(q-1)$. Then there exists $a[2 k, k]$ multi-twisted $R S$ code $\mathcal{C}_{k}(\boldsymbol{\alpha}, \boldsymbol{t}, \boldsymbol{h}, \boldsymbol{\eta})$ over $\mathbb{F}_{q}$ with $\mathfrak{L}$-dimensional hull $(\mathfrak{L}>0)$ for $\boldsymbol{\alpha}=\left(\alpha_{1}, \ldots, \alpha_{k}, \gamma \alpha_{1}, \ldots, \gamma \alpha_{k}\right)$, $h_{1}>0$ and $t_{1}>1$.

Proof. Let, for $\beta \in \mathbb{F}_{q}^{*}, A_{\beta}$ be a $k \times k$ matrix over $\mathbb{F}_{q}$ given by

$$
A_{\beta}=\left(\begin{array}{cccc}
1 & 1 & \cdots & 1 \\
\beta \alpha_{1} & \beta \alpha_{2} & \cdots & \beta \alpha_{k} \\
\left(\beta \alpha_{1}\right)^{2} & \left(\beta \alpha_{2}\right)^{2} & \cdots & \left(\beta \alpha_{k}\right)^{2} \\
\vdots & \vdots & \vdots & \vdots \\
\left(\beta \alpha_{1}\right)^{k-1} & \left(\beta \alpha_{2}\right)^{k-1} & \cdots & \left(\beta \alpha_{k}\right)^{k-1}
\end{array}\right) \text {, }
$$

then, by using (5.1), we have

$$
A_{\beta} A_{\beta}^{T}=\left(\begin{array}{ccccc}
k & 0 & \cdots & 0 & 0  \tag{5.2}\\
0 & \cdots & \cdots & 0 & k \beta^{k} \\
0 & \cdots & \cdots & k \beta^{k} & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
0 & k \beta^{k} & 0 & \cdots & 0
\end{array}\right)
$$

Considering $B_{\beta}$ a $k \times k$ matrix over $\mathbb{F}_{q}$ as

$$
B_{\beta}=\left(\begin{array}{cccc}
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots \\
\eta_{1}\left(\beta \alpha_{1}\right)^{k-1+t_{1}} & \eta_{1}\left(\beta \alpha_{2}\right)^{k-1+t_{1}} & \cdots & \eta_{1}\left(\beta \alpha_{k}\right)^{k-1+t_{1}} \\
\vdots & \vdots & \vdots & \vdots \\
\eta_{2}\left(\beta \alpha_{1}\right)^{k-1+t_{2}} & \eta_{2}\left(\beta \alpha_{2}\right)^{k-1+t_{2}} & \cdots & \eta_{2}\left(\beta \alpha_{k}\right)^{k-1+t_{2}} \\
\vdots & \vdots & \vdots & \vdots \\
\eta_{\ell}\left(\beta \alpha_{1}\right)^{k-1+t_{\ell}} & \eta_{\ell}\left(\beta \alpha_{2}\right)^{k-1+t_{\ell}} & \cdots & \eta_{\ell}\left(\beta \alpha_{k}\right)^{k-1+t_{\ell}} \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0
\end{array}\right) \leftarrow\left(h_{1}+1\right)^{t h} \text { row }
$$

we get $B_{\beta} B_{\beta}^{T}$ as the following matrix:

$$
\left(\begin{array}{ccccccc}
0 & \cdots & 0 & \cdots & \cdots & \cdots & 0  \tag{5.3}\\
& & & \vdots & \cdots 0 \\
0 & \cdots & \eta_{1}^{2} \sum_{i=1}^{k}\left(\beta \alpha_{i}\right)^{2 k-2+t_{1}+t_{1}} & \cdots & \eta_{1} \eta_{2} \sum_{i=1}^{k}\left(\beta \alpha_{i}\right)^{2 k-2+t_{1}+t_{2}} & \cdots & \eta_{1} \eta_{\ell} \sum_{i=1}^{k}\left(\beta \alpha_{i}\right)^{2 k-2+t_{1}+t_{\ell}} \\
\vdots & & \cdots & & \cdots 0 \\
0 & \cdots & \eta_{2} \eta_{1} \sum_{i=1}^{k}\left(\beta \alpha_{i}\right)^{2 k-2+t_{2}+t_{1}} & \cdots & \eta_{2}^{2} \sum_{i=1}^{k}\left(\beta \alpha_{i}\right)^{2 k-2+t_{2}+t_{2}} & \cdots & \eta_{2} \eta_{\ell} \sum_{i=1}^{k}\left(\beta \alpha_{i}\right)^{2 k-2+t_{2}+t_{\ell}} \\
& & & \cdots & & \cdots 0 \\
0 & \cdots & \eta_{\ell} \eta_{1} \sum_{i=1}^{k}\left(\beta \alpha_{i}\right)^{2 k-2+t_{\ell}+t_{1}} & \cdots & \eta_{\ell} \eta_{2} \sum_{i=1}^{k}\left(\beta \alpha_{i}\right)^{2 k-2+t_{\ell}+t_{2}} & \cdots & \eta_{\ell}^{2} \sum_{i=1}^{k}\left(\beta \alpha_{i}\right)^{2 k-2+t_{\ell}+t_{\ell}} \\
& & \vdots & \cdots & \cdots & \cdots 0 \\
0 & \cdots & 0 & \cdots & \cdots & 0 & \\
& & \cdots & \cdots
\end{array}\right)
$$

Notice that, by using (5.1), the elements in $\left(h_{i}+1\right)^{\text {th }}$ row (for $\left.1 \leq i \leq \ell\right)$ of the above matrix are all zero except at most one element. Additionally, since $h_{1}>0$, the first row is all-zero. Also, the above matrix is symmetric as well. Next, we compute $A_{\beta} B_{\beta}^{T}+B_{\beta} A_{\beta}^{T}$ and get the matrix:
where $w_{i}=k-1+t_{i}$ for $1 \leq i \leq \ell$. Again, notice that, by using 5.1), the elements in $\left(h_{i}+1\right)^{t h}$ row (for $\left.1 \leq i \leq \ell\right)$ in the above matrix are all zero except at most one element. Also, the above matrix is symmetric, with diagonal entries all-zero because $\left(h_{i}+1, h_{i}+1\right)^{t h}$ entry is multiple of 2 , for each $i$. In addition, since $t_{1}>1$, the first row is all-zero. Then, the generator matrix of multi-twisted RS code $\mathcal{C}_{k}(\boldsymbol{\alpha}, \boldsymbol{t}, \boldsymbol{h}, \boldsymbol{\eta})$, given in (2.4), can be written as $G=\left[A_{1}: A_{\gamma}\right]+\left[B_{1}: B_{\gamma}\right]$. Therefore $G \cdot G^{T}$ is given by

$$
G \cdot G^{T}=\left(\left[A_{1}: A_{\gamma}\right]+\left[B_{1}: B_{\gamma}\right]\right)\left(\left[\begin{array}{c}
A_{1}^{T} \\
\cdot \\
A_{\gamma}^{T}
\end{array}\right]+\left[\begin{array}{c}
B_{1}^{T} \\
\cdot \\
B_{\gamma}^{T}
\end{array}\right]\right)
$$

This implies that

$$
\begin{equation*}
G \cdot G^{T}=\left(A_{1} A_{1}^{T}+A_{\gamma} A_{\gamma}^{T}\right)+\left(B_{1} B_{1}^{T}+B_{\gamma} B_{\gamma}^{T}\right)+\left(A_{\gamma} B_{\gamma}^{T}+B_{\gamma} A_{\gamma}^{T}+A_{1} B_{1}^{T}+B_{1} A_{1}^{T}\right) \tag{5.5}
\end{equation*}
$$

By using $\beta=1+\gamma$ in the Equations: (5.2) and (5.3), we get

$$
A_{1} A_{1}^{T}+A_{\gamma} A_{\gamma}^{T}=\left(\begin{array}{ccccc}
0 & 0 & \cdots & 0 & 0  \tag{5.6}\\
0 & \cdots & \cdots & 0 & k\left(1+\gamma^{k}\right) \\
0 & \cdots & \cdots & k\left(1+\gamma^{k}\right) & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
0 & k\left(1+\gamma^{k}\right) & 0 & \cdots & 0
\end{array}\right)
$$

and $B_{1} B_{1}^{T}+B_{\gamma} B_{\gamma}^{T}=$

Similarly, the remainder terms of 5.5 can be obtained by replacing $\beta$ with $(1+\gamma)$ in 5.4 . Then, by 5.5 we deduce that

$$
G \cdot G^{T}=\left(\begin{array}{cccccc}
0 & 0 & \cdots & 0 & 0 \\
0 & \cdots & \cdots & 0 & k\left(1+\gamma^{k}\right) \\
0 & \cdots & \cdots & k\left(1+\gamma^{k}\right) & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
0 & k\left(1+\gamma^{k}\right) & 0 & \cdots & 0
\end{array}\right)+\left(\begin{array}{ccccccc}
0 & \cdots & & 0 & & & \cdots \\
\vdots & & & *_{1} & & & \\
& \ddots & & & *_{2} & & \\
0 & *_{1} & \Delta_{1} & & & & \\
\vdots & *_{2} & & \ddots & & & \Delta_{\ell} \\
& & & & & \ddots & \\
0 & & & *_{\ell} & & \Delta_{\ell} & \\
0 & & & & \ddots
\end{array}\right)_{k \times k}
$$

In the above equation, the right-most symmetric matrix has the first row all-zero, and $*_{i}$ represents the possible non-zero entry for each $i$; also, $\Delta_{i}$ is the possible non-zero entry at the $\left(h_{i}+1\right)^{t h}$ diagonal position. Clearly, $G \cdot G^{T}$ has rank at most $k-1$, and this completes the proof.

Corollary 5.4. Let $q$ be a power of 2 , and $k>1$ be an integer such that $k \mid(q-1)$. Then there exists an MDS $[2 k, k]$ multi-twisted $R S$ code $\mathcal{C}_{k}(\boldsymbol{\alpha}, \boldsymbol{t}, \boldsymbol{h}, \boldsymbol{\eta})$ over $\mathbb{F}_{q}$ with $\mathfrak{L}$-dimensional hull $(\mathfrak{L}>0)$ for $\boldsymbol{\alpha}=\left(\alpha_{1}, \ldots, \alpha_{k}, \gamma \alpha_{1}, \ldots, \gamma \alpha_{k}\right)$, $h_{1}>0$ and $t_{1}>1$ if and only if $\boldsymbol{\eta}$ satisfies conditions of Theorem 3.1.
Example 5.5. Consider the finite field $\mathbb{F}_{2^{4}}=\frac{\mathbb{F}_{2}[x]}{\left\langle x^{4}+x+1\right\rangle}$. Let $\gamma=\alpha$ be the primitive element as defined above, $k=3, \boldsymbol{t}=(2,3), \boldsymbol{h}=(1,2)$, and $\boldsymbol{\eta}=\left(\alpha^{3}, \alpha^{3}+\alpha^{2}\right)$. Corresponding to these parameters, consider $a[6,3,4]$ double-twisted $R S$ code $\mathcal{C}_{k, \boldsymbol{t}, \boldsymbol{h}, \boldsymbol{\eta}}$, where $\boldsymbol{\alpha}=\left(\alpha^{2}+\alpha, \alpha^{2}+\alpha+1,1, \alpha^{3}+\alpha^{2}, \alpha^{3}+\alpha^{2}+\alpha, \alpha\right)$. Following to the proof of above theorem, we have

$$
\begin{gathered}
A_{1} A_{1}^{T}=\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{array}\right), \quad B_{1} B_{1}^{T}=\left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & \alpha^{3}+\alpha \\
0 & \alpha^{3}+\alpha & 0
\end{array}\right), \\
A_{\alpha} A_{\alpha}^{T}=\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 0 & \alpha^{3} \\
0 & \alpha^{3} & 0
\end{array}\right), \quad B_{\alpha} B_{\alpha}^{T}=\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 0 & \alpha^{3} \\
0 & \alpha^{3} & 0
\end{array}\right) .
\end{gathered}
$$

Therefore

$$
A_{1} A_{1}^{T}+A_{\alpha} A_{\alpha}^{T}=\left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & \alpha^{3}+1 \\
0 & \alpha^{3}+1 & 0
\end{array}\right), \quad B_{1} B_{1}^{T}+B_{\alpha} B_{\alpha}^{T}=\left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & \alpha \\
0 & \alpha & 0
\end{array}\right)
$$

and

$$
A_{\alpha} B_{\alpha}^{T}+B_{\alpha} A_{\alpha}^{T}+A_{1} B_{1}^{T}+B_{1} A_{1}^{T}=\left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{array}\right)
$$

Thus we can write

$$
G \cdot G^{T}=\left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & \alpha^{3}+1 \\
0 & \alpha^{3}+1 & 0
\end{array}\right)+\left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & \alpha+1 \\
0 & \alpha+1 & 0
\end{array}\right)
$$

which has rank 2. Thus, $\mathcal{C}_{k, \boldsymbol{t}, \boldsymbol{h}, \boldsymbol{\eta}}$ is an MDS double-twisted $R S$ code with one-dimensional hull.

Theorem 5.6. Let $q$ be a power of an odd prime, and $k>2$ be an integer such that $k \mid(q-1)$. Then there exists a $[2 k, k-1]$ multi-twisted $R S$ code $\mathcal{C}_{k}(\boldsymbol{\alpha}, \boldsymbol{t}, \boldsymbol{h}, \boldsymbol{\eta})$ over $\mathbb{F}_{q}$ with $\mathfrak{L}$-dimensional hull ( $\left.\mathfrak{L}>0\right)$ for $\boldsymbol{\alpha}=\left(\alpha_{1}, \ldots, \alpha_{k}, \gamma \alpha_{1}, \ldots, \gamma \alpha_{k}\right), h_{1}>1$ and $t_{\ell}<k$.

Proof. Let, for $\beta \in \mathbb{F}_{q}^{*}, A_{\beta}$ be a $(k-1) \times k$ matrix over $\mathbb{F}_{q}$ given by

$$
A_{\beta}=\left(\begin{array}{cccc}
1 & 1 & \cdots & 1 \\
\beta \alpha_{1} & \beta \alpha_{2} & \cdots & \beta \alpha_{k} \\
\left(\beta \alpha_{1}\right)^{2} & \left(\beta \alpha_{2}\right)^{2} & \cdots & \left(\beta \alpha_{k}\right)^{2} \\
\vdots & \vdots & \vdots & \vdots \\
\left(\beta \alpha_{1}\right)^{k-2} & \left(\beta \alpha_{2}\right)^{k-2} & \cdots & \left(\beta \alpha_{k}\right)^{k-2}
\end{array}\right) \text {, }
$$

then, by using 5.1, we have a $(k-1) \times(k-1)$ matrix

$$
A_{\beta} A_{\beta}^{T}=\left(\begin{array}{ccccc}
k & 0 & \cdots & 0 & 0  \tag{5.8}\\
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & k \beta^{k} \\
\vdots & \vdots & \cdots & \therefore & \vdots \\
0 & 0 & k \beta^{k} & \cdots & 0
\end{array}\right)
$$

Considering $B_{\beta}$ a $(k-1) \times k$ matrix over $\mathbb{F}_{q}$ as

$$
B_{\beta}=\left(\begin{array}{cccc}
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots \\
\eta_{1}\left(\beta \alpha_{1}\right)^{k-1+t_{1}} & \eta_{1}\left(\beta \alpha_{2}\right)^{k-1+t_{1}} & \cdots & \eta_{1}\left(\beta \alpha_{k}\right)^{k-1+t_{1}} \\
\vdots & \vdots & \vdots & \vdots \\
\eta_{2}\left(\beta \alpha_{1}\right)^{k-1+t_{2}} & \eta_{2}\left(\beta \alpha_{2}\right)^{k-1+t_{2}} & \cdots & \eta_{2}\left(\beta \alpha_{k}\right)^{k-1+t_{2}} \\
\vdots & \vdots & \vdots & \vdots \\
\eta_{\ell}\left(\beta \alpha_{1}\right)^{k-1+t_{\ell}} & \eta_{\ell}\left(\beta \alpha_{2}\right)^{k-1+t_{\ell}} & \cdots & \eta_{\ell}\left(\beta \alpha_{k}\right)^{k-1+t_{\ell}} \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0
\end{array}\right) \leftarrow\left(h_{1}+1\right)^{t h} \text { row }
$$

we get $B_{\beta} B_{\beta}^{T}$ as the following matrix:

$$
\left(\begin{array}{ccccccc}
0 & \cdots & 0 & \cdots & \cdots & \cdots & 0  \tag{5.9}\\
0 & & \vdots & \cdots & \cdots \\
0 & \cdots & \eta_{1}^{2} \sum_{i=1}^{k}\left(\beta \alpha_{i}\right)^{2 k-2+t_{1}+t_{1}} & \cdots & \eta_{1} \eta_{2} \sum_{i=1}^{k}\left(\beta \alpha_{i}\right)^{2 k-2+t_{1}+t_{2}} & \cdots & \eta_{1} \eta_{\ell} \sum_{i=1}^{k}\left(\beta \alpha_{i}\right)^{2 k-2+t_{1}+t_{\ell}} \\
& & & \cdots & \cdots \\
0 & \cdots & \eta_{2} \eta_{1} \sum_{i=1}^{k}\left(\beta \alpha_{i}\right)^{2 k-2+t_{2}+t_{1}} & \cdots & \eta_{2}^{2} \sum_{i=1}^{k}\left(\beta \alpha_{i}\right)^{2 k-2+t_{2}+t_{2}} & \cdots & \eta_{2} \eta_{\ell} \sum_{i=1}^{k}\left(\beta \alpha_{i}\right)^{2 k-2+t_{2}+t_{\ell}} \\
& & \vdots & & \cdots 0 \\
0 & \cdots & \eta_{\ell} \eta_{1} \sum_{i=1}^{k}\left(\beta \alpha_{i}\right)^{2 k-2+t_{\ell}+t_{1}} & \cdots & \eta_{\ell} \eta_{2} \sum_{i=1}^{k}\left(\beta \alpha_{i}\right)^{2 k-2+t_{\ell}+t_{2}} & \cdots & \eta_{\ell}^{2} \sum_{i=1}^{k}\left(\beta \alpha_{i}\right)^{2 k-2+t_{\ell}+t_{\ell}} \\
& & \vdots & & \cdots & \cdots 0 \\
0 & \cdots & 0 & \cdots & \cdots & 0 & \\
& & \cdots & \cdots 0
\end{array}\right)
$$

Notice that, by using (5.1), the elements in $\left(h_{i}+1\right)^{t h}$ row (for $\left.1 \leq i \leq \ell\right)$ of the above matrix are all zero except at most one element. Additionally, since $h_{1}>1$, the first and second rows are all-zero. Also, the above matrix is symmetric as well. Next, we compute $A_{\beta} B_{\beta}^{T}+B_{\beta} A_{\beta}^{T}$ and get the matrix:
where $w_{i}=k-1+t_{i}$ for $1 \leq i \leq \ell$. Again, notice that, by using (5.1), the elements in $\left(h_{i}+1\right)^{t h}$ row (for $\left.1 \leq i \leq \ell\right)$ in the above matrix are all zero except at most one element. Also, the above matrix is symmetric. In addition, since $t_{\ell}<k$, the second row is all-zero. Then, the generator matrix of multi-twisted RS code $\mathcal{C}_{k}(\boldsymbol{\alpha}, \boldsymbol{t}, \boldsymbol{h}, \boldsymbol{\eta})$, given in (2.4), can be written as $G=\left[A_{1}: A_{\gamma}\right]+\left[B_{1}: B_{\gamma}\right]$. Similar to the computations as in Theorem 5.3, the $(k-1) \times(k-1)$ matrix $G \cdot G^{T}$ is given by the following expression:

$$
\left(\begin{array}{cccccc}
2 k & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 \\
& & & & & \\
0 & 0 & \cdots & \cdots & 0 & k\left(1+\gamma^{k}\right) \\
& & & & & 0 \\
0 & 0 & \cdots & \cdots & k\left(1+\gamma^{k}\right) & 0 \\
\vdots & \vdots & \vdots & \therefore & \vdots & \\
& & & & & 0
\end{array}\right)+\left(\begin{array}{cccccccc} 
& 0 & & *_{1} & & & & \\
0 & 0 & \cdots & & & & \cdots & 0 \\
& \vdots & \ddots & & & *_{2} & & \\
*_{1} & & & \Delta_{1} & & & & \\
& & & & \ddots & & & *_{\ell} \\
& 0 & *_{2} & & & \Delta_{2} & & \\
& & & & & \ddots & & \\
& \vdots & & *_{\ell} & & & \Delta_{\ell} & \\
0 & & & & & & \ddots
\end{array}\right)
$$

In the above expression, the right-most symmetric matrix has the second row all-zero, and $*_{i}$ represents the possible non-zero entry for each $i$; also, since $h_{1}>1, \Delta_{1}$ cannot occur in first and second rows. Clearly, $G \cdot G^{T}$ has rank at most $k-2$, and this completes the proof.

Corollary 5.7. Let $q$ be a power of an odd prime, and $k>2$ be an integer such that $k \mid(q-1)$. Then there exists an MDS $[2 k, k-1]$ multi-twisted $R S$ code $\mathcal{C}_{k}(\boldsymbol{\alpha}, \boldsymbol{t}, \boldsymbol{h}, \boldsymbol{\eta})$ over $\mathbb{F}_{q}$ with $\mathfrak{L}$-dimensional hull ( $\left.\mathfrak{L}>0\right)$ for $\boldsymbol{\alpha}=\left(\alpha_{1}, \ldots, \alpha_{k}, \gamma \alpha_{1}, \ldots, \gamma \alpha_{k}\right), h_{1}>1, t_{\ell}<k$ if and only if $\boldsymbol{\eta}$ satisfies the conditions of Theorem 3.1.

Example 5.8. Consider the finite field $\mathbb{F}_{3^{4}}=\frac{\mathbb{F}_{2}[x]}{\left\langle x^{4}+2 x^{3}+2\right\rangle}$. Let $\gamma=\alpha$ be the primitive element as defined above, $k=5, \boldsymbol{t}=(1,2), \boldsymbol{h}=(2,3)$, and $\boldsymbol{\eta}=\left(\alpha^{3}+\alpha^{2}, \alpha\right)$. Corresponding to these parameters, consider the $[10,4,7]$ double-twisted RS code $\mathcal{C}_{k, \boldsymbol{t}, \boldsymbol{h}, \boldsymbol{\eta}}$, where $\boldsymbol{\alpha}=\left(2 \alpha^{2}+\alpha+2,2 \alpha^{3}+\alpha+2,2 \alpha^{2}+2 \alpha+1, \alpha^{3}+2 \alpha^{2}+2 \alpha, 1,2 \alpha^{3}+\alpha^{2}+\right.$ $\left.2 \alpha, 2 \alpha^{3}+\alpha^{2}+2 \alpha+2,2 \alpha^{3}+2 \alpha^{2}+\alpha, 2 \alpha^{2}+1, \alpha\right)$. Following to the proof of above theorem, we have

$$
\begin{gathered}
A_{1} A_{1}^{T}=\left(\begin{array}{cccc}
2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 \\
0 & 0 & 2 & 0
\end{array}\right), \quad B_{1} B_{1}^{T}=\left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 \\
0 & 0 & 2 \alpha^{3}+2 \alpha^{2}+2 \\
0 & 0 & 0
\end{array}\right), \\
A_{\alpha} A_{\alpha}^{T}=\left(\begin{array}{cccc}
2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 \alpha^{3}+2 \alpha+2 \\
0 & 0 & 2 \alpha^{3}+2 \alpha+2 & 0
\end{array}\right), \quad B_{\alpha} B_{\alpha}^{T}=\left(\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & \alpha^{3}+\alpha^{2} & 0 \\
0 & 0 & 0 & 0
\end{array}\right) .
\end{gathered}
$$

Therefore

$$
A_{1} A_{1}^{T}+A_{\alpha} A_{\alpha}^{T}=\left(\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 \alpha^{3}+2 \alpha+1 \\
0 & 0 & 2 \alpha^{3}+2 \alpha+1 & 0
\end{array}\right), \quad B_{1} B_{1}^{T}+B_{\alpha} B_{\alpha}^{T}=\left(\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0
\end{array}\right)
$$

and

$$
A_{\alpha} B_{\alpha}^{T}+B_{\alpha} A_{\alpha}^{T}+A_{1} B_{1}^{T}+B_{1} A_{1}^{T}=\left(\begin{array}{cccc}
0 & 0 & \alpha & 0 \\
0 & 0 & 0 & 0 \\
\alpha & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right)
$$

Thus we can write

$$
G \cdot G^{T}=\left(\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 \alpha^{3}+2 \alpha+1 \\
0 & 0 & 2 \alpha^{3}+2 \alpha+1 & 0
\end{array}\right)+\left(\begin{array}{cccc}
0 & 0 & \alpha & 0 \\
0 & 0 & 0 & 0 \\
\alpha & 0 & 2 & 0 \\
0 & 0 & 0 & 0
\end{array}\right)
$$

which have rank 3. Thus, $\mathcal{C}_{k, \boldsymbol{t}, \boldsymbol{h}, \boldsymbol{\eta}}$ is an MDS double-twisted $R S$ code with one-dimensional hull.

## 6 Conclusion

The construction of MDS codes is one of the active and hot research topics in the area of algebraic coding theory due to their maximum error-correction property. We obtained necessary and sufficient condition for multi-twisted RS codes to be MDS. Particularly, we focused MDS double-twisted RS codes. In future, one can study the existence of the error-correcting pair 10 for these double-twisted RS codes. We also studied multi-twisted RS codes with small dimensional hull. Further, we obtained necessary and sufficient conditions for such MDS multi-twisted RS codes to have small dimensional hull. As a future task, one can study such (MDS) multi-twisted RS codes with certain dimensional Hermitian hull [7] and Galois hull [4, 8, 16 with some possible applications. For comparison of these studies with existing studies, we refer to the Table 2.

| RS codes (always MDS) | $\eta=0$ | [20] |
| :---: | :---: | :---: |
| Single-twisted RS codes | $\eta \neq 0,1 \leq t \leq n-k, 0 \leq h \leq k-1$ | [3] |
| Multi-twisted RS codes | $\begin{gathered} 0 \leq h_{i}<h_{i+1} \leq k-1,1 \leq t_{i}<t_{i+1} \leq n-k, \\ \boldsymbol{t}=\left(t_{1}, t_{2}, \ldots, t_{\ell}\right), \boldsymbol{h}=\left(h_{1}, h_{2}, \ldots, h_{\ell}\right), \\ \boldsymbol{\eta}=\left(\eta_{1}, \eta_{2}, \ldots, \eta_{\ell}\right) \in\left(\mathbb{F}_{q}^{*}\right) \end{gathered}$ | [1] |
| Necessary and sufficient condition for single-twisted RS codes to be MDS | $\eta \neq 0,1 \leq t \leq n-k, 0 \leq h \leq k-1$ | [3, 26] |
| Necessary and sufficient condition for double-twisted RS codes to be MDS | $\begin{gathered} \boldsymbol{\eta}=\left(\eta_{1}, \eta_{2}\right) \in\left(\mathbb{F}_{q}^{*}\right)^{2}, \boldsymbol{t}=(1,2), \\ \boldsymbol{h}=(k-1, k-2) \end{gathered}$ | [27] |
| Necessary and sufficient condition for double-twisted RS codes to be MDS | $\boldsymbol{\eta}=\left(\eta_{1}, \eta_{2}\right) \in\left(\mathbb{F}_{q}^{*}\right)^{2}, \boldsymbol{t}=(1,2), \boldsymbol{h}=(0,1)$ | In this paper |
| Necessary and sufficient condition for Multi-twisted RS codes to be MDS | $\begin{gathered} \boldsymbol{t}=(1,2, \ldots, \ell), \\ \boldsymbol{h}=(k-\ell, k-\ell+1, \ldots, k-1), \\ \boldsymbol{\eta}=\left(\eta_{1}, \eta_{2}, \ldots, \eta_{\ell}\right) \in\left(\mathbb{F}_{q}^{*}\right)^{\ell}, \\ \ell<\min \{k, n-k\} \end{gathered}$ | [9] |
| Necessary and sufficient condition for Multi-twisted RS codes to be MDS | $\begin{gathered} 0 \leq h_{i}<h_{i+1} \leq k-1,1 \leq t_{i}<t_{i+1} \leq n-k, \\ \boldsymbol{t}=\left(t_{1}, t_{2}, \ldots, t_{\ell}\right), \boldsymbol{h}=\left(h_{1}, h_{2}, \ldots, h_{\ell}\right), \\ \boldsymbol{\eta}=\left(\eta_{1}, \eta_{2}, \ldots, \eta_{\ell}\right) \in\left(\mathbb{F}_{q}^{*}\right)^{\ell} \\ \hline \hline \end{gathered}$ | In this paper |
| Explicit construction for MDS singletwisted RS codes | $\eta \neq 0, t=1, h=0, k-1$ | [3] |
| Explicit construction for MDS singletwisted RS codes | $\eta \neq 0, t=2, h=1$ | [26] |
| Explicit construction for MDS doubletwisted RS codes | $\boldsymbol{\eta}=\left(\eta_{1}, \eta_{2}\right) \in\left(\mathbb{F}_{q}^{*}\right)^{2}, \boldsymbol{t}=(1,2), \boldsymbol{h}=(0,1)$ | In this paper |
| Single-twisted RS codes with onedimensional hull | $\eta \neq 0 ; 1 \leq t \leq n-k ; 0 \leq h \leq k-1$ | [28] |
| MDS Single-twisted RS codes with zero-dimensional hull | $\eta \neq 0, t=1, h=k-1$ | [15] |
| Multi-twisted RS codes with zerodimensional hull | $\begin{gathered} \boldsymbol{t}=(1,2, \ldots, \ell), \boldsymbol{h}=(0,1, \ldots, \ell-1), \\ \boldsymbol{\eta}=\left(\eta_{1}, \eta_{2}, \ldots, \eta_{\ell}\right) \in\left(\mathbb{F}_{q}^{*}\right)^{\ell}, \ell \leq n / 2 \end{gathered}$ | [15] |
| Multi-twisted RS codes with small dimensional hull | $\begin{gathered} 0 \leq h_{i}<h_{i+1} \leq k-1,1 \leq t_{i}<t_{i+1} \leq n-k, \\ \boldsymbol{t}=\left(t_{1}, t_{2}, \ldots, t_{\ell}\right), \boldsymbol{h}=\left(h_{1}, h_{2}, \ldots, h_{\ell}\right), \\ \boldsymbol{\eta}=\left(\eta_{1}, \eta_{2}, \ldots, \eta_{\ell}\right) \in\left(\mathbb{F}_{q}^{*}\right)^{\ell} \end{gathered}$ | In this paper |
| Necessary and sufficient conditions for MDS multi-twisted RS codes with small dimensional hull | $\begin{gathered} 1<h_{i}<h_{i+1} \leq k-1,1<t_{i}<t_{i+1} \leq n-k-2, \\ \boldsymbol{t}=\left(t_{1}, t_{2}, \ldots, t_{\ell}\right), \boldsymbol{h}=\left(h_{1}, h_{2}, \ldots, h_{\ell}\right), \\ \boldsymbol{\eta}=\left(\eta_{1}, \eta_{2}, \ldots, \eta_{\ell}\right) \in\left(\mathbb{F}_{q}^{*}\right)^{\ell} \end{gathered}$ | In this paper |

Table 2: Comparison with existing studies

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## A Implementation of Example 3.2

For checking the non-singularity of each matrix in Example 3.2 we attach the following implementation:

```
F.<a> = GF (16)
alpha_vec = [0, a^2, a + 1, a^2 + a, a^3 + a + 1]
ta1 = a^3 + a^2
* (1, a^2 + 1, a^2 + a + 1, a^3, a^3 + a^2, a^3 + a^2 + a]
R.<x> = PolynomialRing(F)
for eta2 in eta2 list
print('\n eta2,: ,eta2)
    det_matrix = []
    for I in Combinations([0,1,2,3,4],3):
    temp-poly = R(1)
    for i in I:
    temp_poly = temp_poly*(x-alpha_vec[i])
    sigma_coeff = temp_poly.coefficients(sparse= False)
    A_I = matrix(F,[[[,0],[sigma_coeff[2],1]])
    -1*sigma_coeff[1]],[0,-1*sigma_coeff[0]]])
    D= diagonal_matrix(F,[eta2^(-1), eta1^(-1)])
    print(det_matrix)
```

The output of this code is described below:


```
eta2 : a - 2 + 1
```



```
eta2 : a^2 + a + 1
eta2 : a^3
```



```
ota2: : a^3 + a`_
```



```
eta2: : a^3+ + - 2 +a
```



## B Implementation for counting of double-twisted RS codes

The following SageMath implementation provides number of double-twisted RS codes of length $n$ and dimension $k$ over the finite field $\mathbb{F}_{q}$.

```
def number (q,n,k)
    F.<a> =GF(q)
    A = Combinations(list(F),n)
    main_count
    count = 0
        for eta in F^2:
            if eta[0]*eta[1] !=0:
            Ik = Combinations(C, k)
            i = 0 i < len(Ik):
                sum_prod = 0
                for Ik1 in Combinations(Ik[i], k-1):
                flag = 1 - eta[0]*((-1)*k)*prod(Ik[i]) + eta[1]*((-1)*k)*((sum_prod)*(sum(Ik[i])) - prod(Ik[i])) + eta[0]*eta[1]*(prod(Ik[i]) -2)
                flag = 1
                if flag==0:
                if flag== 
            if flag!=0:
                count = count + 1
        main_count = main_count + count
    return main_count
```


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