MDS multi-twisted Reed-Solomon codes with small dimensional hull

Harshdeep Singh and Kapish Chand Meena¹

Abstract

In this paper, we find a necessary and sufficient condition for multi-twisted Reed-Solomon codes to be MDS. In particular, we introduce a new class of MDS double-twisted Reed-Solomon codes $C_{\alpha,t,h,\eta}$ with twists $\mathbf{t} = (1,2)$ and hooks $\mathbf{h} = (0,1)$ over the finite field \mathbb{F}_q , providing a non-trivial example over \mathbb{F}_{16} and enumeration over the finite fields of size up to 17. Moreover, we obtain necessary conditions for the existence of multi-twisted Reed-Solomon codes with small dimensional hull. Consequently, we derive conditions for the existence of MDS multi-twisted Reed-Solomon codes with small dimensional hull.

Keywords: Reed-Solomon codes, MDS codes, LCD codes, one-dimensional hull, twisted Reed-Solomon codes

MSC Classification: 94B05, 11T71

1 Introduction

A linear [n, k, d] code is said to be maximum distance separable (MDS) code if its parameters achieve the Singleton bound [18], i.e. d = n - k + 1. Reed-Solomon (RS) and generalized Reed-Solomon (GRS) codes [20] are famous examples of MDS codes. For other families of MDS codes, we refer to [18], [21] and [22]. In [3], Beelen at el. introduced MDS single-twisted RS codes as a generalization of RS codes. The idea was to choose special type polynomials of degree k with twist t = 1 and hook h = 0, k - 1 such that those polynomials could have at most k-1 roots among the evaluation points. Further, in [1], Beelen at al. extended the notion of single-twisted RS codes to multi-twisted RS codes in code-based cryptography are studied by many authors (see: [1], [11]). Hence, it is significant to explore algebraic criteria of multi-twisted RS codes to be MDS.

We know that dual \mathcal{C}^{\perp} of a linear code \mathcal{C} is also a linear code over the same finite field. The hull of \mathcal{C} is denoted by $Hull(\mathcal{C})$ and defined as $Hull(\mathcal{C}) = \mathcal{C} \cap \mathcal{C}^{\perp}$. It is clear that $Hull(\mathcal{C})$ is also a linear code. There are many applications of the hull, e.g. determining the complexity of algorithms for checking equivalence of linear codes, computing the automorphism group of a linear code. It is also useful in side-channel attacks and fault injection attacks (see: [5, 12, 13, 17, 23, 24, 25]). In [19], Massey introduced *linear codes with complementary dual* (LCD), i.e. the codes having zero-dimensional hull. For other constructions of LCD codes we refer to [29, 2, 6]. Moreover, Liu and Liu studied MDS LCD single-twisted RS codes and LCD multi-twisted RS codes (under certain assumptions) in [15]. Linear codes with one-dimensional hull were studied in [14]. In [28], Wu studied single-twisted RS codes with one-dimensional hull.

Motivated by their work, in this paper, we study the multi-twisted RS codes with general twists and hooks having zero and one-dimensional hull. First, we give necessary and sufficient condition for multi-twisted RS code to be MDS. Then we introduce new k-dimensional MDS double-twisted RS codes $C_{\alpha,t,h,\eta}$ by choosing suitable η in twisted polynomials of degree k + 1 with twists t = (1, 2) and hooks h = (0, 1) such that those polynomials could have at most k - 1 roots among the evaluation points E_{α} . Moreover, we obtain necessary conditions for the existence of multi-twisted Reed-Solomon codes with small dimensional hull. Then as a consequence, we derive conditions for the existence of MDS multi-twisted RS codes with small dimensional hull.

This paper is organized as follows: in Section 2, we recall some basic terminologies and results which are needed for this paper. In Section 3, we describe a necessary and sufficient condition for a multi-twisted RS code to be MDS. In Section 4, we explicitly describe the algebraic properties of MDS double-twisted RS codes

¹Corresponding Author E-mail: meenakapishchand@gmail.com

over the finite fields \mathbb{F}_q , providing non-trivial examples over \mathbb{F}_{16} and enumeration over the finite fields of order up to 17. In Section 5, we study multi-twisted RS codes with small dimensional hull and provide examples. Also, we obtain necessary and sufficient conditions for the existence of MDS multi-twisted RS codes with small dimensional hull. Finally, in Section 6, we conclude with some future directions and compare these results with the existing results.

2 Preliminaries

In this section, we recall some fundamental concepts from literature which are needed for this paper.

Definition 2.1. [18] Let $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{F}_q$ be distinct elements and $0 \le k < n$. Then RS code is defined as

$$\mathcal{C}_{n,k}^{RS} = \{ (f(\alpha_1), f(\alpha_2), \dots, f(\alpha_n)) \},$$

$$(2.1)$$

where $f(x) \in \mathbb{F}_q[x]$ with $\deg(f(x)) < k$.

Here $\alpha_1, \alpha_2, \ldots, \alpha_n$ are called evaluation points for the code. Clearly, $\mathcal{C}_{n,k}^{RS}$ is an \mathbb{F}_q -linear subspace of \mathbb{F}_q^n having dimension k, i.e. $\mathcal{C}_{n,k}^{RS}$ is a linear code. Since the polynomial f(x) has at most k-1 roots in the field, RS codes achieve the equality in the Singleton bound.

In [3], Beelen et al. introduced the notion of twisted polynomials and using such polynomials they introduced single-twisted RS code. Let $0 \le h < k \le q - 1$ and η be a non-zero element of a finite field \mathbb{F}_q . Then the set of (k, t, h, η) -twisted polynomials over \mathbb{F}_q , is defined as

$$\mathcal{V}_{k,t,h,\eta} \coloneqq \left\{ f = \sum_{i=0}^{k-1} a_i x^i + \eta a_h x^{k-1+t} : a_i \in \mathbb{F}_q \text{ for each } i \right\},\$$

where $k, t, h \in \mathbb{N}$.

Definition 2.2. [3] Let $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{F}_q$ be distinct elements and $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. Let k < n and $0 < t \le n - k$. Then the single-twisted RS code is defined as

$$\mathcal{C}_k(\boldsymbol{\alpha}, t, h, \eta) = \{ (f(\alpha_1), f(\alpha_2), \dots, f(\alpha_n)) \},$$
(2.2)

where $f(x) \in \mathcal{V}_{k,t,h,\eta}$ with $\deg(f(x)) \leq k - 1 + t < n$.

Note that, when $\eta = 0$, then single-twisted RS codes become RS codes. Since $\deg(f(x)) \notin k$, single-twisted RS codes are not MDS in general. However, in [3], Beelen et al. obtained a necessary and sufficient condition for single-twisted RS codes to be MDS. Using SageMath implementations, they concluded that many long MDS single-twisted RS codes can be obtained using twisted polynomials for particular values of q, n, k, h and t; but not for general values. In addition, they claimed that in order to find explicit constructions of such long MDS single-twisted RS codes is difficult; however, they gave following result in favour to the existence.

Lemma 2.3. ([3], Theorem 17) Let $\mathbb{F}_s \subset \mathbb{F}_q$ be a proper subfield and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{F}_s$. If $\eta \in \mathbb{F}_q \setminus \mathbb{F}_s$, then the single-twisted code $\mathcal{C}_k(\alpha, t, h, \eta)$ is MDS.

Further, in [1], Beelen at el. introduced multi-twisted RS codes as a generalization of single-twisted RS codes. They extended the twists and hooks: let $0 \le h_i < h_{i+1} < k \le q$ and $1 \le t_i < t_{i+1} \le n - k$ for each *i*. Then, for $\mathbf{t} = (t_1, t_2, \ldots, t_\ell)$, $\mathbf{h} = (h_1, h_2, \ldots, h_\ell)$ and $\boldsymbol{\eta} = (\eta_1, \eta_2, \ldots, \eta_\ell) \in (\mathbb{F}_q^*)^\ell$, set of $(k, \mathbf{t}, \mathbf{h}, \boldsymbol{\eta})$ -twisted polynomials over \mathbb{F}_q , is defined as

$$\mathcal{V}_{k,\boldsymbol{t},\boldsymbol{h},\boldsymbol{\eta}} \coloneqq \left\{ f = \sum_{i=0}^{k-1} a_i x^i + \sum_{j=1}^{\ell} \eta_j a_{h_j} x^{k-1+t_j} : a_i \in \mathbb{F}_q \text{ for each } i \right\}.$$

Definition 2.4. [1] Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be distinct elements of a finite field \mathbb{F}_q and k < n. Then multi-twisted RS code is defined as

$$\mathcal{C}_{k}(\boldsymbol{\alpha}, \boldsymbol{t}, \boldsymbol{h}, \boldsymbol{\eta}) \coloneqq \{ (f(\alpha_{1}), f(\alpha_{2}), \dots, f(\alpha_{n})) \},$$
(2.3)

where $f(x) \in \mathcal{V}_{k,t,h,\eta}$ with $\deg(f(x)) \leq k - 1 + t_{\ell} < n$.

As per the Definition 2.4, the generator matrix of multi-twisted RS code $C_k(\alpha, t, h, \eta)$ is

3 MDS multi-twisted RS codes

In this section, we obtain a necessary and sufficient condition for a multi-twisted RS code to be MDS. In [3], authors gave the criteria for a single-twisted ($\ell = 1$) RS code $C_k(\alpha, t, h, \eta)$ to be MDS. We have extended their study for the multi-twisted RS code ($\ell > 1$) and give a more generalized structure as follows:

Theorem 3.1. Let $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{F}_q$ be distinct and for k < n < q and $\mathbf{t} = (t_1, t_2, \ldots, t_\ell)$, $\mathbf{h} = (h_1, h_2, \ldots, h_\ell)$ and $\boldsymbol{\eta} = (\eta_1, \eta_2, \ldots, \eta_\ell)$ be as defined in previous section. Then the multi-twisted RS code $C_k(\boldsymbol{\alpha}, \mathbf{t}, \mathbf{h}, \boldsymbol{\eta})$ is MDS if and only if for each $\mathcal{I} \subset \{1, 2, \ldots, n\}$ with cardinality k correspondingly the polynomial $\prod_{i \in \mathcal{I}} (x - \alpha_i) = \sum_i \sigma_i x^i$ with $\sigma_i = 0$ for i < 0, the matrix

$$diag\left(\eta_{\ell}^{-1}, 1, \dots, 1, \eta_{\ell-1}^{-1}, 1, \dots, 1, \eta_{1}^{-1}, 1, \dots, \frac{1}{1}\right) \cdot A_{\mathcal{I}} + B_{\mathcal{I}}$$

is non-singular. Here, $A_{\mathcal{I}}$ and $B_{\mathcal{I}}$ are lower and upper-triangular $t_{\ell} \times t_{\ell}$ matrices, respectively, given by:

$$A_{\mathcal{I}} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \sigma_{k-1} & 1 & 0 & \cdots & 0 \\ \sigma_{k-2} & \sigma_{k-1} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{k-\ell_{\ell}+1} & \cdots & \cdots & \sigma_{k-1} & 1 \end{pmatrix} and$$

Proof. It suffices to show that the codeword with at least k zero positions is precisely the zero codeword. Let $f \in \mathcal{V}_{k,t,h,\eta}$ corresponds to the codeword with at least k zero positions, i.e. let f have k roots among the evaluation points. Then we can write

$$f(x) = \sigma(x)g(x), \tag{3.1}$$

where, $\sigma(x) = \prod_{i \in \mathcal{I}} (x - \alpha_i) = \sum_{i=0}^k \sigma_i x^i$ for some $\mathcal{I} \subset \{1, 2, \dots, n\}$ with cardinality k and $g(x) = \sum_{i=0}^{t_\ell - 1} g_i x^i \in \mathbb{F}_q[x]$. Note that $\sigma_k = 1$ and all the coefficients of x^k to $x^{k+t_\ell - 1}$ are zero except the coefficients of $x^{k+t_1 - 1}$, $x^{k+t_2 - 1}$, ..., $x^{k+t_\ell - 1}$, $x^{k+t_\ell - 1}$ in f. Consequently, we obtain the following system of $(t_\ell - \ell)$ equations in g_j s:

$$\sum_{j=0}^{i} \sigma_{i-j} g_j = 0, \tag{3.2}$$

where $i \in \{k, k+1, \ldots, k+t_{\ell}-2\} \setminus \{k+t_1-1, k+t_2-1, \ldots, k+t_{\ell-1}-1\}$ and $g_j = 0$ when $j \notin \{0, 1, \ldots, t_{\ell}-1\}$ and $\sigma_j = 0$ when $j \notin \{0, 1, \ldots, k\}$. Comparing the coefficients of x^{k+t_s-1} in both sides of (3.1) and substituting the value of $a_{h_s} = \sum_{j=0}^{h_s} \sigma_{h_s-j}g_j$, we obtain the following system of ℓ equations in g_j s:

$$\eta_s^{-1} \left(\sum_{i=0}^k \sigma_{k-i} g_{t_s-1+i} \right) - \sum_{j=0}^{h_s} \sigma_{h_s-j} g_j = 0,$$
(3.3)

for $s = 1, 2, ..., \ell$. From (3.2) and (3.3), we have a homogeneous system of t_{ℓ} equations in t_{ℓ} variables. The code $C_k(\alpha, t, h, \eta)$ is MDS if and only if f is zero polynomial, i.e. the above homogeneous system has only the zero vector as solution for all choices of \mathcal{I} . This completes the proof.

Example 3.2. Consider the finite field $\mathbb{F}_{2^4} = \mathbb{F}_2(\alpha)$ with $\alpha^4 + \alpha + 1 = 0$. Let n = 5, k = 3, $\alpha = (0, \alpha^2, \alpha + 1, \alpha^2 + \alpha, \alpha^3 + \alpha + 1)$, $\mathbf{t} = (1, 2)$ and $\mathbf{h} = (0, 1)$. Let $C_k(\alpha, \mathbf{t}, \mathbf{h}, \boldsymbol{\eta})$ be a multi-twisted RS code having $\ell = 2$ twists with $\boldsymbol{\eta} = (\eta_1, \eta_2)$, where $\eta_1 = \alpha^3 + \alpha^2$ and $\eta_2 \in \{1, \alpha^2 + 1, \alpha^2 + \alpha + 1, \alpha^3, \alpha^3 + \alpha^2, \alpha^3 + \alpha^2 + \alpha\}$. Now for each $\mathcal{I} \subset \{1, 2, 3, 4, 5\}$ with cardinality 3 correspondingly polynomial $\prod_{i \in \mathcal{I}} (x - \alpha_i) = \sum_i \sigma_i x^i$, the matrix

$$\begin{bmatrix} \eta_2^{-1} & 0 \\ 0 & \eta_1^{-1} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ \sigma_2 & 1 \end{bmatrix} + \begin{bmatrix} -\sigma_0 & -\sigma_1 \\ 0 & -\sigma_0 \end{bmatrix}$$

is non-singular. Then by Theorem 3.1 the code $C_k(\alpha, t, h, \eta)$ is MDS. For detailed implementation we refer to Appendix A.

Remark 3.3. By SageMath implementations, similar to the existence of MDS single-twisted RS codes one can see that many long MDS multi-twisted RS codes can be obtained using twisted polynomials for particular values

MDS multi-twisted RS codes with small dimensional hull

of q, n, k, h and t; but not for general values. Also, it seems hard to find explicit constructions of such long MDS multi-twisted RS codes. However, the existence of MDS multi-twisted RS code given by Beelen et al. [1], uses the fact that $C_k(\alpha, t, h, \eta)$ is MDS if and only if every k columns of generator matrix (2.4) are linearly independent over \mathbb{F}_q . For this, the authors assumed some constraints on the evaluation vector α and the vector η . Now, we give an alternate proof of the same result as a consequence of the Theorem 3.1.

Theorem 3.4. [1] Let $\mathbb{F}_{q_0} \not\subseteq \mathbb{F}_{q_1} \not\subseteq \mathbb{F}_{q_2} \not\subseteq \cdots \not\subseteq \mathbb{F}_{q_\ell} = \mathbb{F}_q$ be a proper chain of subfields and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{F}_{q_0}$. If $\eta_i \in \mathbb{F}_{q_i} \setminus \mathbb{F}_{q_{i-1}}$, for all $1 \leq i \leq \ell$. Then the multi-twisted RS code $\mathcal{C}_k(\alpha, t, h, \eta)$ is MDS.

Proof. Let $\mathcal{I} \subset \{1, 2, ..., n\}$ with cardinality k and $W = \operatorname{diag}\left(\eta_{\ell}^{-1}, 1, ..., 1, \eta_{\ell-1}^{-1}, 1, ..., 1, \eta_1^{-1}, 1, ..., 1\right) \cdot A_{\mathcal{I}} + B_{\mathcal{I}}$ be the corresponding matrix as described in Theorem 3.1. Since $\alpha_i \in \mathbb{F}_{q_0}$ for each i, σ_j also belongs to \mathbb{F}_{q_0} for each j. In particular, $\sigma_0 \in \mathbb{F}_{q_0}$. Further, $\eta_j^{-1} \in \mathbb{F}_{q_j} \setminus \mathbb{F}_{q_{j-1}}$ implies that $\eta_j^{-1} - \sigma_j \neq 0$. Hence, the diagonal entries in W are all non-zero and on applying elementary row operations, we can convert this matrix to lower triangular matrix with diagonal elements $\eta_{\ell}^{-1} + T_{\ell}, 1, \ldots, 1, \eta_{\ell-1}^{-1} + T_{\ell-1}, 1, \ldots, \ldots, 1, \eta_1^{-1} + T_1, 1, \ldots, 1$ for some $T_i \in \mathbb{F}_{q_{i-1}}$. Hence, corresponding matrix is non-singular. Thus by Theorem 3.1, $\mathcal{C}_k(\alpha, t, h, \eta)$ is MDS. This completes the proof.

4 MDS double-twisted RS codes

In this section, we give algebraic criteria for double-twisted RS codes having twists $\mathbf{t} = (1, 2)$ and hooks $\mathbf{h} = (0, 1)$ to be MDS with enumeration. For these double-twisted RS codes, we assume $\boldsymbol{\eta} = (\eta_1, \eta_2) \in (\mathbb{F}_q^*)^2$. Let k < n < q then we have $\mathcal{V}_{k, \mathbf{t}, \mathbf{h}, \mathbf{\eta}} := \{a_0 + a_1 x + \dots + a_{k-1} x^{k-1} + \eta_1 a_0 x^k + \eta_2 a_1 x^{k+1} : a_i \in \mathbb{F}_q \text{ for each } i\}$. Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{F}_q^n$ be an evaluation vector and $E_{\boldsymbol{\alpha}} = \{\alpha_1, \alpha_2, \dots, \alpha_n\} \subset \mathbb{F}_q$ be set of evaluation points, where $\alpha_i \neq \alpha_j$ for each $i \neq j$.

Proposition 4.1. The polynomial $f(x) = a_0 + a_1x + \dots + a_{k-1}x^{k-1} + \eta_1a_0x^k + \eta_2a_1x^{k+1} \in \mathcal{V}_{k,t,h,\eta}$ has at most k-1 roots among E_{α} if any of the following conditions hold:

- (*i*) $a_0 = a_1 = 0;$
- (*ii*) $a_0 \neq 0$, $a_1 = 0$, and $\eta_1 \neq \frac{(-1)^k}{\prod_{i \in \mathcal{T}_h} \alpha_i}$;

$$(iii) \ a_0 \neq 0, \ a_1 \neq 0, \ \eta_2 \neq \frac{(-1)^k}{\prod\limits_{i \in \mathcal{J}_k} \alpha_i}, \ and \ \eta_1 \neq \frac{\left(\sum\limits_{i \in \mathcal{J}_k} \prod\limits_{j \neq i} \alpha_j\right) \left(\sum\limits_{i \in \mathcal{J}_k} \alpha_i\right) + \left(\frac{(-1)^k}{\eta_2} - \prod\limits_{i \in \mathcal{J}_k} \alpha_i\right)}{(-1)^k \left(\prod\limits_{i \in \mathcal{J}_k} \alpha_i\right) \left(\frac{(-1)^k}{\eta_2} - \prod\limits_{i \in \mathcal{J}_k} \alpha_i\right)};$$

(iv)
$$a_0 = 0, a_1 \neq 0, and \eta_2 \neq \frac{(-1)^{k-1}}{\left(\sum\limits_{j \in \mathcal{J}_{k-1}} \alpha_j\right) \left(\prod\limits_{j \in \mathcal{J}_{k-1}} \alpha_j\right)};$$

where $\mathcal{J}_s \subset \{1, 2, \ldots, n\}$ having cardinality s such that $\alpha_j \neq 0$ for each $j \in \mathcal{J}_s$.

Proof. It is immediate to observe that if (i) or (ii) holds, then f has at most k-1 roots among E_{α} . If (iii) holds then since $a_0 \neq 0$, f doesn't have 0 among its roots. Now, suppose roots of f are $\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_k}$ and β , where $\alpha_{i_j} \in E_{\alpha}$ for $1 \leq j \leq k$. Then $f(x) = \eta_2 a_1(x-\beta) \prod_{j=1}^k (x-\alpha_{i_j})$, where $\eta_2 \neq 0$. Now the product of roots of f(x) is

$$\beta \prod_{j=1}^{k} \alpha_{i_j} = \frac{(-1)^{k+1} a_0}{\eta_2 a_1} \tag{4.1}$$

and, the sum of product of k-roots of f, i.e.

$$\prod_{j=1}^{k} \alpha_{i_j} + \beta \left(\alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_{k-1}} + \alpha_{i_1} \cdots \alpha_{i_{k-2}} \alpha_{i_k} + \cdots + \alpha_{i_2} \alpha_{i_3} \cdots \alpha_{i_k} \right) = \frac{(-1)^k}{\eta_2}.$$

Now if $\alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_{k-1}} + \alpha_{i_1}\cdots\alpha_{i_{k-2}}\alpha_{i_k} + \cdots + \alpha_{i_2}\alpha_{i_3}\cdots\alpha_{i_k} = 0$, then $\eta_2 = \frac{(-1)^k}{\prod_{j=1}^k \alpha_{i_j}}$, which contradicts the assumption *(iii)*. Therefore

Ĵ

$$\beta = \frac{\left(\frac{(-1)^k}{\eta_2} - \prod_{j=1}^k \alpha_{i_j}\right)}{\left(\alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_{k-1}} + \alpha_{i_1}\cdots\alpha_{i_{k-2}}\alpha_{i_k} + \cdots + \alpha_{i_2}\alpha_{i_3}\cdots\alpha_{i_k}\right)}.$$
(4.2)

Also the sum of roots of f is

$$\sum_{i=1}^{k} \alpha_{i_j} + \beta = \frac{-\eta_1 a_0}{\eta_2 a_1}.$$
(4.3)

Using (4.1) and (4.3), we obtain

$$\sum_{j=1}^{k} \alpha_{i_j} + \beta = (-1)^k \eta_1 \left(\beta \prod_{j=1}^{k} \alpha_{i_j} \right).$$

$$(4.4)$$

Using (4.2) in (4.4), we obtain

$$\eta_{1} = \frac{\left(\alpha_{i_{1}}\alpha_{i_{2}}\cdots\alpha_{i_{k-1}} + \alpha_{i_{1}}\cdots\alpha_{i_{k-2}}\alpha_{i_{k}} + \cdots + \alpha_{i_{2}}\alpha_{i_{3}}\cdots\alpha_{i_{k}}\right)\left(\sum_{j=1}^{k}\alpha_{i_{j}}\right) + \left(\frac{(-1)^{k}}{\eta_{2}} - \prod_{j=1}^{k}\alpha_{i_{j}}\right)}{\left(-1\right)^{k}\left(\prod_{j=1}^{k}\alpha_{i_{j}}\right)\left(\frac{(-1)^{k}}{\eta_{2}} - \prod_{j=1}^{k}\alpha_{i_{j}}\right)}$$
(4.5)

but this contradicts the assumption (*iii*). Note that this contradiction is free from the choices of β . Observe that in (4.5) denominator is non-zero by (4.1) and (4.2). Hence f can have at most k - 1 roots among E_{α} . Lastly, assume (*iv*) holds then since $a_0 = 0$, one of the roots of f is definitely 0. This gives a factor of f; denoted by f/x as $a_1 + a_2x + \cdots + a_{k-1}x^{k-2} + \eta_2a_1x^k$. Therefore, we have following two cases:

• $0 \notin E_{\alpha}$: Let roots of f be $0, \alpha_{i_2}, \alpha_{i_3}, \dots, \alpha_{i_{k+1}}$, then $\sum_{j=2}^{k+1} \alpha_{i_j} = 0$ since sum of roots of f is zero. That means

$$\alpha_{i_2} = -\sum_{j=3}^{k+1} \alpha_{i_j}.$$
(4.6)

Assume $\alpha_{i_j} \in E_{\alpha}$ for each j, then f/x also has these k roots in E_{α} . Then the product of roots of f/x is

$$\frac{(-1)^k}{\eta_2} = \prod_{j=2}^{k+1} \alpha_{i_j}.$$
(4.7)

Using (4.6) in (4.7), we get $\eta_2 = \frac{(-1)^{k-1}}{\binom{k+1}{\sum \alpha_{i_j}}\binom{k+1}{\prod \alpha_{i_j}}}$. In general we obtain $\eta_2 = \frac{(-1)^{k-1}}{\binom{\sum \alpha_{i_j}}{j \in \mathcal{J}_{k-1}} \alpha_j}$ where

 $\mathcal{J}_s \subset \{1, 2, \dots, n\}$ having cardinality s such that $\alpha_j \neq 0$ for each $j \in \mathcal{J}_s$. This contradicts the assumption.

• $0 \in E_{\alpha}$: Let roots of f/x be $\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_{k-1}}, \beta$ where $\alpha_{i_j} \in E_{\alpha}$. The product and sum of roots of f/x are given by

$$\beta \prod_{j=1}^{k-1} \alpha_{i_j} = \frac{(-1)^k}{\eta_2} \text{ and } \beta + \sum_{j=1}^{k-1} \alpha_{i_j} = 0, \text{ respectively.}$$

Now eliminating β we obtain a contradiction to the condition in (*iv*). Again note that this contradiction is free from the choices of β . Hence f/x can have at most k-2 roots among E_{α} .

This completes the proof.

. . . h

Theorem 4.2. Let k < n and $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{F}_q^n$ where $\alpha_i \neq \alpha_j$ for each $i \neq j$, $\boldsymbol{t} = (1, 2)$, $\boldsymbol{h} = (0, 1)$ and $\boldsymbol{\eta} = (\eta_1, \eta_2) \in (\mathbb{F}_q^*)^2$. Then the double-twisted RS code

$$\mathcal{C}_k(\boldsymbol{\alpha}, \boldsymbol{t}, \boldsymbol{h}, \boldsymbol{\eta}) = \{ (f(\alpha_1), f(\alpha_2), \dots, f(\alpha_n)) \in \mathbb{F}_q^n : f \in \mathcal{V}_{k, \boldsymbol{t}, \boldsymbol{h}, \boldsymbol{\eta}} \}$$

is MDS if and only if the following conditions hold:

$$(i) \ \eta_{1} \neq \frac{(-1)^{k}}{\prod\limits_{i \in \mathcal{J}_{k}} \alpha_{i}} \ whenever \sum_{i \in \mathcal{J}_{k}} \prod_{j \in \mathcal{J}_{k}} \alpha_{j} = 0;$$

$$(ii) \ \eta_{1} \neq \frac{\left(\sum_{i \in \mathcal{J}_{k}} \prod_{j \in \mathcal{J}_{k}} \alpha_{j}\right) \left(\sum_{i \in \mathcal{J}_{k}} \alpha_{i}\right) + \left(\frac{(-1)^{k}}{\eta_{2}} - \prod_{i \in \mathcal{J}_{k}} \alpha_{i}\right)}{(-1)^{k} \left(\prod_{i \in \mathcal{J}_{k}} \alpha_{i}\right) \left(\frac{(-1)^{k}}{\eta_{2}} - \prod_{i \in \mathcal{J}_{k}} \alpha_{i}\right)} \ whenever \ \eta_{2} \neq \frac{(-1)^{k}}{\prod_{j \in \mathcal{J}_{k}} \alpha_{j}};$$

$$(iii) \ \eta_{2} \neq \frac{(-1)^{k-1}}{\left(\sum_{j \in \mathcal{J}_{k-1}} \alpha_{j}\right) \left(\prod_{j \in \mathcal{J}_{k-1}} \alpha_{j}\right)} \ whenever \ \prod_{i \in \mathcal{I}_{k}} \alpha_{i} = 0$$

where \mathcal{I}_s and $\mathcal{J}_s \subset \{1, 2, \dots, n\}$ having cardinality s such that $\alpha_j \neq 0$ for each $j \in \mathcal{J}_s$.

Proof. The code $C_k(\alpha, t, h, \eta)$ is MDS if and only if every polynomial $f \in \mathcal{V}_{k,t,h,\eta}$ has at most k-1 roots among the evaluation points E_{α} . The proof of if part follows by Proposition 4.1. On the other hand, if any of the conditions (*i*)-(*iii*) do not hold then by similar approach to [3] one can find a polynomial $f \in \mathcal{V}_{k,t,h,\eta}$ having at least k roots among the evaluation points E_{α} . Thus we get required proof.

Example 4.3. Consider a finite field $\mathbb{F}_{2^4} = \frac{\mathbb{F}_2[x]}{\langle x^4 + x + 1 \rangle} = \mathbb{F}_2(\alpha)$. Let n = 5, k = 3, t = (1, 2), h = (0, 1), and $\eta = (\eta_1, \eta_2) \in \mathbb{F}_2(\alpha)^2$, then we have the set of double-twisted polynomials $\mathcal{V}_{k,t,h,\eta} = \{a_0 + a_1x + a_2x^2 + \eta_1a_0x^3 + \eta_2a_1x^4 : a_i \in \mathbb{F}_2(\alpha) \text{ for each } i\}$. Then every $f \in \mathcal{V}_{k,t,h,\eta}$ has at most 2 roots among $\{0, \alpha^3 + \alpha^2, \alpha^3 + \alpha^2 + \alpha + 1, \alpha^3 + 1, 1\} \subset \mathbb{F}_2(\alpha)$ for $\eta_1 = \alpha^2 + \alpha$ and $\eta_2 \in \{1, \alpha, \alpha^2 + \alpha, \alpha^3, \alpha^3 + \alpha, \alpha^3 + \alpha^2\}$. Thus the double-twisted RS code $\mathcal{C}_k(\alpha, t, h, \eta)$ is MDS.

Remark 4.4. If we apply the similar approach as Theorem 3.3 of [26, 27] and Theorem 3.2 of [9], we obtain that the double-twisted RS code $C_k(\alpha, t, h, \eta)$ in Theorem 4.2 is MDS if and only if $\eta = (\eta_1, \eta_2) \in (\mathbb{F}_q^*)^2$ satisfies

$$1 - \eta_1 (-1)^k \left(\prod_{i \in \mathcal{I}_k} \alpha_i\right) + \eta_2 (-1)^k \left(\left(\sum_{j \in \mathcal{I}_k} \prod_{\substack{i \neq j \\ i \in \mathcal{I}_k}} \alpha_i\right) \left(\sum_{i \in \mathcal{I}_k} \alpha_i\right) - \left(\prod_{j \in \mathcal{I}_k} \alpha_j\right) \right) + \eta_1 \eta_2 \left(\prod_{i \in \mathcal{I}_k} \alpha_i^2\right) \neq 0$$

for every $\mathcal{I}_k \subseteq \{1, 2, ..., n\}$ of cardinality k. One can easily verify that η_1 and η_2 obtained by Theorem 4.2 satisfy the above condition and vice-versa. However, Theorem 4.2 gives an approach to discard some choices for η_1 and η_2 , which is better than exhaustive search.

Note that the MDS codes described in Theorem 4.2 are double-twisted RS codes with twists t = (1, 2) and hooks h = (0, 1), which are particular multi-twisted RS codes with $\ell = 2$, and different from the MDS doubletwisted RS codes obtained in [27] and [9]. Now we give the enumeration of such classes of MDS double-twisted RS codes over finite fields of size up to 17 in Table 1. By SageMath implementations, one can see that many long MDS double-twisted RS codes can be obtained using twisted polynomials for particular values of q, n, k, h and t; but not for general values. Also, it seems hard to find explicit constructions of such long MDS double-twisted RS codes. However, if we include additional conditions defined in Theorem 3.4 then MDS double-twisted RS codes exist always.

a		k = 1	2	2	4	E	6	7	Q	0	10	11	12	12	14	15
4	2	κ – 1 «	2	3	-	3	0	/	0	,	10	- 11	12	15	14	15
3	2	39														
4	3	24	15													
_	2	116														
5	5 4	44	82 28	36												
	2	588														
-	3	870	777	704												
8	4 5	426	213	186	339											
	6	132	54	32	6	84										
	2	1099	2009													
	4	2275	1883	1764												
Ŭ	5	1694 805	1029 322	420	1281	630										
	7	224	49	0	0	63	77									
	2	1880	2012													
	4	5432	4072	4712												
9	5	4984	2592	1840	4574											
	0 7	1240	292	0	40	8	948									
	8	296	52	0	4	0	4	152								
	2	4640 12810	12525													
	4	23640	19260	22180												
11	5	30660 28560	19420 13520	16450 5190	28360 9890	24892										
	7	19140	6670	610	760	3260	15870									
	8	9060 2890	2435	20	10	60 0	565	6630 40	1915							
	10	560	120	Ő	Ő	Ő	0	0	10	240						
	2	9708	22520													
	4	77220	52538 64626	74368												
	5	129888	86082	80676	122991	140204										
13	7	152856	54708	10616	10776	34368	135396									
	8	108504	28206	1092	426	312	13086	90552	44247							
	10	22044	3738	0	0	0	40	0	462	14968						
	11 12	5820 948	942 156	0	0	0	0	0	0	48	3450	348				
	2	23955	150				0					510				
	3	105420	105885	2125.00												
	4 5	734370	290295 554265	494760	698790											
	6	1276275	762255	443560	613560	1210998	100000									
16	8	1/33160 1859715	581595	51900	1/6625	34305	365085	1652925								
	9 10	1583010	329895	5700	120	510	2220	145755	1349640	862810						
	10	562380	45795	0	0	0	0	0	0	14085	415848					
	12	227955	11085	0	0	0	0	0	0	0	4185	150300	40760			
	13	14505	225	0	0	0	0	0	0	0	0	0	0	6495		
	15	1920	15	0	0	0	0	0	0	0	0	0	0	210	345	
	2 3	31088 146960	145928													
	4	486640	429648	475184	1164604											
	5	2272816	890464 1358488	852864	1164604 1214568	2174976										
	7	3393104	1581464	460080	473896	1216256	3197848	2710014								
17	8 9	4038320 3855280	1446208 1062512	121280	46008 932	512	895712 10680	451648	3431638							
	10	2953808	638104	544	0	0	0	480	174648	2511296	1440414					
	12	872144	128704	0	0	0	0	0	0	0	13488	644224				
	13 14	325360	42832	0	0	0	0	0	0	0	0	2864	217084	51840		
	14	17776	2184	0	0	0	0	0	0	0	0	0	0	80	8680	
	16	2192	272	0	0	0	0	0	0	0	0	0	0	0	16	624

Table 1: Number of MDS double-twisted RS codes with twists t = (1, 2) and hooks h = (0, 1) respectively over finite fields of size ≤ 17

5 Multi-twisted RS codes with small dimensional hull

In this section, we study the existence of multi-twisted RS codes with small-dimensional hull. For this, we first state some useful results based on hull.

Lemma 5.1. ([14], Proposition 1) Let C be an [n,k] linear code over \mathbb{F}_q with generator matrix G. Then code C has a one-dimensional hull if and only if the rank of matrix $G \cdot G^T$ is k-1. Alternately, the code C has a one-dimensional hull if and only if the rank of matrix $H \cdot H^T$ is n-k-1.

Proposition 5.2. [14] Let C be an [n,k] linear code over \mathbb{F}_q with generator matrix G. Then code C has a \mathfrak{L} -dimensional hull if and only if the rank of matrix $G \cdot G^T$ is $k - \mathfrak{L}$, where $0 \leq \mathfrak{L} \leq \min\{k, n-k\}$.

In the remaining part of this section, we fix the following notations:

- : a primitive element of \mathbb{F}_q γ
- : a positive integer such that k|(q-1):= $\gamma^{\frac{q-1}{k}i}$ for $1 \le i \le k$ k
- α_i

It is clear that $\gamma^{\frac{q-1}{k}}$ generates a subgroup of \mathbb{F}_q^* of order k. Also, by [28], for any integer m,

$$\theta^m = \alpha_1^m + \alpha_2^m + \dots + \alpha_k^m = \begin{cases} k & \text{if } m \equiv 0 \mod k \\ 0 & \text{otherwise.} \end{cases}$$
(5.1)

Next, we give proof for the existence of multi-twisted RS codes as defined in Definition 2.4, with small dimensional hull in two scenarios when q is even and odd.

Theorem 5.3. Let q be a power of 2, and k > 1 be an integer such that k | (q-1). Then there exists a [2k, k]multi-twisted RS code $C_k(\alpha, t, h, \eta)$ over \mathbb{F}_q with \mathfrak{L} -dimensional hull $(\mathfrak{L} > 0)$ for $\alpha = (\alpha_1, \ldots, \alpha_k, \gamma \alpha_1, \ldots, \gamma \alpha_k)$, $h_1 > 0$ and $t_1 > 1$.

Proof. Let, for $\beta \in \mathbb{F}_q^*$, A_β be a $k \times k$ matrix over \mathbb{F}_q given by

$$A_{\beta} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \beta \alpha_{1} & \beta \alpha_{2} & \cdots & \beta \alpha_{k} \\ (\beta \alpha_{1})^{2} & (\beta \alpha_{2})^{2} & \cdots & (\beta \alpha_{k})^{2} \\ \vdots & \vdots & \vdots & \vdots \\ (\beta \alpha_{1})^{k-1} & (\beta \alpha_{2})^{k-1} & \cdots & (\beta \alpha_{k})^{k-1} \end{pmatrix},$$

then, by using (5.1), we have

$$A_{\beta}A_{\beta}^{T} = \begin{pmatrix} k & 0 & \cdots & 0 & 0 \\ 0 & \cdots & \cdots & 0 & k\beta^{k} \\ 0 & \cdots & \cdots & k\beta^{k} & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & k\beta^{k} & 0 & \cdots & 0 \end{pmatrix}.$$
 (5.2)

Considering B_{β} a $k \times k$ matrix over \mathbb{F}_q as

$$B_{\beta} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \eta_{1}(\beta\alpha_{1})^{k-1+t_{1}} & \eta_{1}(\beta\alpha_{2})^{k-1+t_{1}} & \cdots & \eta_{1}(\beta\alpha_{k})^{k-1+t_{1}} \\ \vdots & \vdots & \vdots & \vdots \\ \eta_{2}(\beta\alpha_{1})^{k-1+t_{2}} & \eta_{2}(\beta\alpha_{2})^{k-1+t_{2}} & \cdots & \eta_{2}(\beta\alpha_{k})^{k-1+t_{2}} \\ \vdots & \vdots & \vdots & \vdots \\ \eta_{\ell}(\beta\alpha_{1})^{k-1+t_{\ell}} & \eta_{\ell}(\beta\alpha_{2})^{k-1+t_{\ell}} & \cdots & \eta_{\ell}(\beta\alpha_{k})^{k-1+t_{\ell}} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \leftarrow (h_{\ell}+1)^{th} row$$

we get $B_{\beta}B_{\beta}^{T}$ as the following matrix:

$$\begin{pmatrix} 0 & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & 0 & \cdots & 0 \\ & & & \vdots & & & \\ 0 & \cdots & \eta_{1}^{2} \sum_{i=1}^{k} (\beta \alpha_{i})^{2k-2+t_{1}+t_{1}} & \cdots & \eta_{1} \eta_{2} \sum_{i=1}^{k} (\beta \alpha_{i})^{2k-2+t_{1}+t_{2}} & \cdots & \eta_{1} \eta_{\ell} \sum_{i=1}^{k} (\beta \alpha_{i})^{2k-2+t_{1}+t_{\ell}} & \cdots & 0 \\ & & & & \vdots & & & \\ 0 & \cdots & \eta_{2} \eta_{1} \sum_{i=1}^{k} (\beta \alpha_{i})^{2k-2+t_{2}+t_{1}} & \cdots & \eta_{2}^{2} \sum_{i=1}^{k} (\beta \alpha_{i})^{2k-2+t_{2}+t_{2}} & \cdots & \eta_{2} \eta_{\ell} \sum_{i=1}^{k} (\beta \alpha_{i})^{2k-2+t_{2}+t_{\ell}} & \cdots & 0 \\ & & & & \vdots & & & \\ 0 & \cdots & \eta_{\ell} \eta_{1} \sum_{i=1}^{k} (\beta \alpha_{i})^{2k-2+t_{\ell}+t_{1}} & \cdots & \eta_{\ell} \eta_{2} \sum_{i=1}^{k} (\beta \alpha_{i})^{2k-2+t_{\ell}+t_{2}} & \cdots & \eta_{\ell}^{2} \sum_{i=1}^{k} (\beta \alpha_{i})^{2k-2+t_{\ell}+t_{\ell}} & \cdots & 0 \\ & & & & \vdots & & \\ 0 & \cdots & 0 & \cdots & \cdots & 0 & \cdots & 0 \end{pmatrix}$$
(5.3)

Notice that, by using (5.1), the elements in $(h_i + 1)^{th}$ row (for $1 \le i \le \ell$) of the above matrix are all zero except at most one element. Additionally, since $h_1 > 0$, the first row is all-zero. Also, the above matrix is symmetric as well. Next, we compute $A_{\beta}B_{\beta}^{T} + B_{\beta}A_{\beta}^{T}$ and get the matrix:

where $w_i = k - 1 + t_i$ for $1 \le i \le \ell$. Again, notice that, by using (5.1), the elements in $(h_i + 1)^{th}$ row (for $1 \le i \le \ell$) in the above matrix are all zero except at most one element. Also, the above matrix is symmetric, with diagonal entries all-zero because $(h_i + 1, h_i + 1)^{th}$ entry is multiple of 2, for each *i*. In addition, since $t_1 > 1$, the first row is all-zero. Then, the generator matrix of multi-twisted RS code $C_k(\alpha, t, h, \eta)$, given in (2.4), can be written as $G = [A_1 : A_{\gamma}] + [B_1 : B_{\gamma}]$. Therefore $G \cdot G^T$ is given by

$$G \cdot G^{T} = \left(\begin{bmatrix} A_{1} : A_{\gamma} \end{bmatrix} + \begin{bmatrix} B_{1} : B_{\gamma} \end{bmatrix} \right) \left(\begin{bmatrix} A_{1}^{T} \\ \vdots \\ A_{\gamma}^{T} \end{bmatrix} + \begin{bmatrix} B_{1}^{T} \\ \vdots \\ B_{\gamma}^{T} \end{bmatrix} \right).$$

This implies that

$$G \cdot G^{T} = \left(A_{1}A_{1}^{T} + A_{\gamma}A_{\gamma}^{T}\right) + \left(B_{1}B_{1}^{T} + B_{\gamma}B_{\gamma}^{T}\right) + \left(A_{\gamma}B_{\gamma}^{T} + B_{\gamma}A_{\gamma}^{T} + A_{1}B_{1}^{T} + B_{1}A_{1}^{T}\right).$$
(5.5)

By using $\beta = 1 + \gamma$ in the Equations: (5.2) and (5.3), we get

$$A_{1}A_{1}^{T} + A_{\gamma}A_{\gamma}^{T} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & \cdots & \cdots & 0 & k(1+\gamma^{k}) \\ 0 & \cdots & \cdots & k(1+\gamma^{k}) & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & k(1+\gamma^{k}) & 0 & \cdots & 0 \end{pmatrix}$$
(5.6)

and
$$B_1 B_1^T + B_{\gamma} B_{\gamma}^T =$$

(0 ... 0 ... 0 ... 0 ... 0 ... 0 ... 0
 \vdots
0 ... $\eta_1^2 \sum_{i=1}^k ((1+\gamma)\alpha_i)^{2k-2+t_1+t_1} \cdots \eta_1 \eta_2 \sum_{i=1}^k ((1+\gamma)\alpha_i)^{2k-2+t_1+t_2} \cdots \eta_1 \eta_\ell \sum_{i=1}^k ((1+\gamma)\alpha_i)^{2k-2+t_1+t_\ell} \cdots 0$
 \vdots
0 ... $\eta_2 \eta_1 \sum_{i=1}^k ((1+\gamma)\alpha_i)^{2k-2+t_2+t_1} \cdots \eta_2^2 \sum_{i=1}^k ((1+\gamma)\alpha_i)^{2k-2+t_2+t_2} \cdots \eta_2 \eta_\ell \sum_{i=1}^k ((1+\gamma)\alpha_i)^{2k-2+t_2+t_\ell} \cdots 0$
 \vdots
0 ... $\eta_\ell \eta_1 \sum_{i=1}^k ((1+\gamma)\alpha_i)^{2k-2+t_\ell+t_1} \cdots \eta_\ell \eta_2 \sum_{i=1}^k ((1+\gamma)\alpha_i)^{2k-2+t_\ell+t_2} \cdots \eta_\ell^2 \sum_{i=1}^k ((1+\gamma)\alpha_i)^{2k-2+t_\ell+t_\ell} \cdots 0$
 \vdots
0 ... 0

Similarly, the remainder terms of (5.5) can be obtained by replacing β with (1 + γ) in (5.4). Then, by (5.5) we deduce that

$$G \cdot G^{T} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & \cdots & \cdots & 0 & k \left(1 + \gamma^{k} \right) \\ 0 & \cdots & \cdots & k \left(1 + \gamma^{k} \right) & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & k \left(1 + \gamma^{k} \right) & 0 & \cdots & 0 \end{pmatrix} + \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & *_{1} & & & \\ & \ddots & & *_{2} & & \\ 0 & *_{1} & \Delta_{1} & & & \\ & & & \ddots & & *_{\ell} \\ \vdots & *_{2} & & \Delta_{2} & & \\ & & & *_{\ell} & & \Delta_{\ell} \\ 0 & & & & & \ddots \end{pmatrix}_{k \times k}$$

In the above equation, the right-most symmetric matrix has the first row all-zero, and $*_i$ represents the possible non-zero entry for each i; also, Δ_i is the possible non-zero entry at the $(h_i + 1)^{th}$ diagonal position. Clearly, $G \cdot G^T$ has rank at most k - 1, and this completes the proof.

Corollary 5.4. Let q be a power of 2, and k > 1 be an integer such that k|(q-1). Then there exists an MDS [2k,k] multi-twisted RS code $C_k(\alpha, t, h, \eta)$ over \mathbb{F}_q with \mathfrak{L} -dimensional hull $(\mathfrak{L} > 0)$ for $\alpha = (\alpha_1, \ldots, \alpha_k, \gamma \alpha_1, \ldots, \gamma \alpha_k)$, $h_1 > 0$ and $t_1 > 1$ if and only if η satisfies conditions of Theorem 3.1.

Example 5.5. Consider the finite field $\mathbb{F}_{2^4} = \frac{\mathbb{F}_2[x]}{\langle x^4 + x + 1 \rangle}$. Let $\gamma = \alpha$ be the primitive element as defined above, $k = 3, t = (2,3), h = (1,2), and \eta = (\alpha^3, \alpha^3 + \alpha^2)$. Corresponding to these parameters, consider a [6,3,4] double-twisted RS code $C_{k,t,h,\eta}$, where $\alpha = (\alpha^2 + \alpha, \alpha^2 + \alpha + 1, 1, \alpha^3 + \alpha^2, \alpha^3 + \alpha^2 + \alpha, \alpha)$. Following to the proof of above theorem, we have

$$A_{1}A_{1}^{T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad B_{1}B_{1}^{T} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha^{3} + \alpha \\ 0 & \alpha^{3} + \alpha & 0 \end{pmatrix},$$
$$A_{\alpha}A_{\alpha}^{T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \alpha^{3} \\ 0 & \alpha^{3} & 0 \end{pmatrix}, \quad B_{\alpha}B_{\alpha}^{T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \alpha^{3} \\ 0 & \alpha^{3} & 0 \end{pmatrix}.$$

Therefore

$$A_1 A_1^T + A_\alpha A_\alpha^T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha^3 + 1 \\ 0 & \alpha^3 + 1 & 0 \end{pmatrix}, \quad B_1 B_1^T + B_\alpha B_\alpha^T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha \\ 0 & \alpha & 0 \end{pmatrix},$$

and

$$A_{\alpha}B_{\alpha}^{T} + B_{\alpha}A_{\alpha}^{T} + A_{1}B_{1}^{T} + B_{1}A_{1}^{T} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Thus we can write

$$G \cdot G^{T} = \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & \alpha^{3} + 1 \\ 0 & \alpha^{3} + 1 & 0 \end{array}\right) + \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & \alpha + 1 \\ 0 & \alpha + 1 & 0 \end{array}\right)$$

which has rank 2. Thus, $C_{k,t,h,\eta}$ is an MDS double-twisted RS code with one-dimensional hull.

Theorem 5.6. Let q be a power of an odd prime, and k > 2 be an integer such that k|(q-1). Then there exists a [2k, k-1] multi-twisted RS code $C_k(\alpha, t, h, \eta)$ over \mathbb{F}_q with \mathfrak{L} -dimensional hull ($\mathfrak{L} > 0$) for $\alpha = (\alpha_1, \ldots, \alpha_k, \gamma \alpha_1, \ldots, \gamma \alpha_k), h_1 > 1$ and $t_\ell < k$.

Proof. Let, for $\beta \in \mathbb{F}_q^*$, A_β be a $(k-1) \times k$ matrix over \mathbb{F}_q given by

$$A_{\beta} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \beta \alpha_1 & \beta \alpha_2 & \cdots & \beta \alpha_k \\ (\beta \alpha_1)^2 & (\beta \alpha_2)^2 & \cdots & (\beta \alpha_k)^2 \\ \vdots & \vdots & \vdots & \vdots \\ (\beta \alpha_1)^{k-2} & (\beta \alpha_2)^{k-2} & \cdots & (\beta \alpha_k)^{k-2} \end{pmatrix},$$

then, by using (5.1), we have a $(k-1) \times (k-1)$ matrix

$$A_{\beta}A_{\beta}^{T} = \begin{pmatrix} k & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & k\beta^{k} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & k\beta^{k} & \cdots & 0 \end{pmatrix}.$$
 (5.8)

Considering B_{β} a $(k-1) \times k$ matrix over \mathbb{F}_q as

$$B_{\beta} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \eta_{1}(\beta\alpha_{1})^{k-1+t_{1}} & \eta_{1}(\beta\alpha_{2})^{k-1+t_{1}} & \cdots & \eta_{1}(\beta\alpha_{k})^{k-1+t_{1}} \\ \vdots & \vdots & \vdots & \vdots \\ \eta_{2}(\beta\alpha_{1})^{k-1+t_{2}} & \eta_{2}(\beta\alpha_{2})^{k-1+t_{2}} & \cdots & \eta_{2}(\beta\alpha_{k})^{k-1+t_{2}} \\ \vdots & \vdots & \vdots & \vdots \\ \eta_{\ell}(\beta\alpha_{1})^{k-1+t_{\ell}} & \eta_{\ell}(\beta\alpha_{2})^{k-1+t_{\ell}} & \cdots & \eta_{\ell}(\beta\alpha_{k})^{k-1+t_{\ell}} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \leftarrow (h_{\ell}+1)^{th} row$$

we get $B_{\beta}B_{\beta}^{T}$ as the following matrix:

$$\begin{pmatrix} 0 & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & 0 & \cdots \\ & & \vdots & & & \\ 0 & \cdots & \eta_{1}^{2} \sum_{i=1}^{k} (\beta \alpha_{i})^{2k-2+t_{1}+t_{1}} & \cdots & \eta_{1} \eta_{2} \sum_{i=1}^{k} (\beta \alpha_{i})^{2k-2+t_{1}+t_{2}} & \cdots & \eta_{1} \eta_{\ell} \sum_{i=1}^{k} (\beta \alpha_{i})^{2k-2+t_{1}+t_{\ell}} & \cdots \\ & & \vdots & & \\ 0 & \cdots & \eta_{2} \eta_{1} \sum_{i=1}^{k} (\beta \alpha_{i})^{2k-2+t_{2}+t_{1}} & \cdots & \eta_{2}^{2} \sum_{i=1}^{k} (\beta \alpha_{i})^{2k-2+t_{2}+t_{2}} & \cdots & \eta_{2} \eta_{\ell} \sum_{i=1}^{k} (\beta \alpha_{i})^{2k-2+t_{2}+t_{\ell}} & \cdots \\ & & \vdots & & \\ 0 & \cdots & \eta_{\ell} \eta_{1} \sum_{i=1}^{k} (\beta \alpha_{i})^{2k-2+t_{\ell}+t_{1}} & \cdots & \eta_{\ell} \eta_{2} \sum_{i=1}^{k} (\beta \alpha_{i})^{2k-2+t_{\ell}+t_{2}} & \cdots & \eta_{\ell}^{2} \sum_{i=1}^{k} (\beta \alpha_{i})^{2k-2+t_{\ell}+t_{\ell}} & \cdots \\ & & \vdots & & \\ 0 & \cdots & 0 & \cdots & \cdots & 0 & \cdots \\ \end{pmatrix}$$
(5.9)

Notice that, by using (5.1), the elements in $(h_i + 1)^{th}$ row (for $1 \le i \le \ell$) of the above matrix are all zero except at most one element. Additionally, since $h_1 > 1$, the first and second rows are all-zero. Also, the above matrix is symmetric as well. Next, we compute $A_{\beta}B_{\beta}^T + B_{\beta}A_{\beta}^T$ and get the matrix:

where $w_i = k - 1 + t_i$ for $1 \le i \le \ell$. Again, notice that, by using (5.1), the elements in $(h_i + 1)^{th}$ row (for $1 \le i \le \ell$) in the above matrix are all zero except at most one element. Also, the above matrix is symmetric. In addition, since $t_{\ell} < k$, the second row is all-zero. Then, the generator matrix of multi-twisted RS code $C_k(\alpha, t, h, \eta)$, given in (2.4), can be written as $G = [A_1 : A_{\gamma}] + [B_1 : B_{\gamma}]$. Similar to the computations as in Theorem 5.3, the $(k-1) \times (k-1)$ matrix $G \cdot G^T$ is given by the following expression:

$$\begin{pmatrix} 2k & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ & & & & & \\ 0 & 0 & \cdots & \cdots & 0 & k(1+\gamma^{k}) \\ & & & & & \\ 0 & 0 & \cdots & \cdots & k(1+\gamma^{k}) & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & & \\ & & & & & & \\ 0 & 0 & k(1+\gamma^{k}) & 0 & \cdots & 0 \end{pmatrix} + \begin{pmatrix} 0 & *_{1} & & & & \\ 0 & 0 & \cdots & & 0 \\ \vdots & \ddots & & *_{2} & & \\ *_{1} & & \Delta_{1} & & & \\ & & & \ddots & & & *_{\ell} \\ 0 & *_{2} & & \Delta_{2} & & \\ & & & & \ddots & & \\ \vdots & & *_{\ell} & & \Delta_{\ell} & \\ 0 & & & & & \ddots \end{pmatrix}$$

In the above expression, the right-most symmetric matrix has the second row all-zero, and $*_i$ represents the possible non-zero entry for each i; also, since $h_1 > 1$, Δ_1 cannot occur in first and second rows. Clearly, $G \cdot G^T$ has rank at most k - 2, and this completes the proof.

Corollary 5.7. Let q be a power of an odd prime, and k > 2 be an integer such that k|(q-1). Then there exists an MDS [2k, k-1] multi-twisted RS code $C_k(\alpha, t, h, \eta)$ over \mathbb{F}_q with \mathfrak{L} -dimensional hull ($\mathfrak{L} > 0$) for $\alpha = (\alpha_1, \ldots, \alpha_k, \gamma \alpha_1, \ldots, \gamma \alpha_k), h_1 > 1, t_\ell < k$ if and only if η satisfies the conditions of Theorem 3.1.

Example 5.8. Consider the finite field $\mathbb{F}_{3^4} = \frac{\mathbb{F}_2[x]}{(x^4+2x^3+2)}$. Let $\gamma = \alpha$ be the primitive element as defined above, k = 5, t = (1, 2), h = (2, 3), and $\eta = (\alpha^3 + \alpha^2, \alpha)$. Corresponding to these parameters, consider the [10, 4, 7] double-twisted RS code $C_{k,t,h,\eta}$, where $\alpha = (2\alpha^2 + \alpha + 2, 2\alpha^3 + \alpha + 2, 2\alpha^2 + 2\alpha + 1, \alpha^3 + 2\alpha^2 + 2\alpha, 1, 2\alpha^3 + \alpha^2 + 2\alpha, 2\alpha^3 + \alpha^2 + 2\alpha + 2, 2\alpha^3 + \alpha^2 + 2\alpha, 2\alpha^2 + 1, \alpha)$. Following to the proof of above theorem, we have

Therefore

and

$$A_{\alpha}B_{\alpha}^{T} + B_{\alpha}A_{\alpha}^{T} + A_{1}B_{1}^{T} + B_{1}A_{1}^{T} = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus we can write

$$G \cdot G^{T} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2\alpha^{3} + 2\alpha + 1 \\ 0 & 0 & 2\alpha^{3} + 2\alpha + 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 \\ \alpha & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which have rank 3. Thus, $C_{k,t,h,\eta}$ is an MDS double-twisted RS code with one-dimensional hull.

6 Conclusion

The construction of MDS codes is one of the active and hot research topics in the area of algebraic coding theory due to their maximum error-correction property. We obtained necessary and sufficient condition for multi-twisted RS codes to be MDS. Particularly, we focused MDS double-twisted RS codes. In future, one can study the existence of the error-correcting pair [10] for these double-twisted RS codes. We also studied multi-twisted RS codes with small dimensional hull. Further, we obtained necessary and sufficient conditions for such MDS multi-twisted RS codes to have small dimensional hull. As a future task, one can study such (MDS) multi-twisted RS codes with certain dimensional Hermitian hull [7] and Galois hull [4, 8, 16] with some possible applications. For comparison of these studies with existing studies, we refer to the Table 2.

RS codes (always MDS)	$\eta = 0$	[20]
Single-twisted RS codes	$\eta \neq 0, \ 1 \leq t \leq n-k, \ 0 \leq h \leq k-1$	[3]
Multi-twisted RS codes	$0 \le h_i < h_{i+1} \le k - 1, \ 1 \le t_i < t_{i+1} \le n - k,$	[1]
	$t = (t_1, t_2, \dots, t_\ell), h = (h_1, h_2, \dots, h_\ell),$	
	$\boldsymbol{\eta} = (\eta_1, \eta_2, \dots, \eta_\ell) \in (\mathbb{F}_q^*)^\ell$	
Necessary and sufficient condition for	$\eta \neq 0, \ 1 \leq t \leq n-k, \ 0 \leq h \leq k-1$	[3, 26]
single-twisted RS codes to be MDS		
Necessary and sufficient condition for	$\eta = (\eta_1, \eta_2) \in (\mathbb{F}_q^*)^2, t = (1, 2),$	[27]
double-twisted RS codes to be MDS	h = (k - 1, k - 2)	
Necessary and sufficient condition for	$\eta = (\eta_1, \eta_2) \in (\mathbb{F}_q^*)^2, t = (1, 2), h = (0, 1)$	In this paper
double-twisted RS codes to be MDS		
Necessary and sufficient condition for	$\boldsymbol{t} = (1, 2, \dots, \ell),$	[9]
Multi-twisted RS codes to be MDS	$\boldsymbol{h} = (k - \ell, k - \ell + 1, \dots, k - 1),$	
	$\boldsymbol{\eta} = (\eta_1, \eta_2, \dots, \eta_\ell) \in (\mathbb{F}_q^*)^c,$	
	$\ell < \min\{k, n-k\}$	T
Necessary and sufficient condition for	$0 \le h_i < h_{i+1} \le k - 1, \ 1 \le t_i < t_{i+1} \le n - k,$	In this paper
Multi-twisted RS codes to be MDS	$t = (t_1, t_2, \dots, t_\ell), h = (h_1, h_2, \dots, h_\ell),$	
	$\boldsymbol{\eta} = (\eta_1, \eta_2, \dots, \eta_\ell) \in (\mathbb{F}_q^n)^c$	
Explicit construction for MDS single-	$\eta \neq 0, t = 1, h = 0, k - 1$	[3]
twisted RS codes		[]
Explicit construction for MDS single-	$\eta \neq 0, t = 2, h = 1$	[26]
twisted RS codes		T . 1 .
Explicit construction for MDS double-	$\eta = (\eta_1, \eta_2) \in (\mathbb{F}_q^*)^2, t = (1, 2), h = (0, 1)$	In this paper
twisted RS codes		
Single-twisted RS codes with one-	$\eta \neq 0; \ 1 \le t \le n-k; \ 0 \le h \le k-1$	[28]
dimensional hull		
MDS Single-twisted RS codes with		[1]
1	$\eta \neq 0, t = 1, h = k - 1$	[10]
zero-dimensional hull	$\eta \neq 0, \ t = 1, \ h = k - 1$	[10]
Zero-dimensional hull Multi-twisted RS codes with zero-	$\eta \neq 0, \ t = 1, \ h = k - 1$ $t = (1, 2, \dots, \ell), \ h = (0, 1, \dots, \ell - 1),$	[15]
zero-dimensional hull Multi-twisted RS codes with zero- dimensional hull	$\eta \neq 0, \ t = 1, \ h = k - 1$ $t = (1, 2, \dots, \ell), \ h = (0, 1, \dots, \ell - 1),$ $\eta = (\eta_1, \eta_2, \dots, \eta_\ell) \in (\mathbb{F}_q^*)^\ell, \ \ell \le n/2$	[15]
zero-dimensional hull Multi-twisted RS codes with zero- dimensional hull Multi-twisted RS codes with small di-	$\eta \neq 0, \ t = 1, \ h = k - 1$ $t = (1, 2, \dots, \ell), \ h = (0, 1, \dots, \ell - 1),$ $\eta = (\eta_1, \eta_2, \dots, \eta_\ell) \in (\mathbb{F}_q^*)^\ell, \ \ell \le n/2$ $0 \le h_i < h_{i+1} \le k - 1, \ 1 \le t_i < t_{i+1} \le n - k,$	[15] [15] In this paper
zero-dimensional hull Multi-twisted RS codes with zero- dimensional hull Multi-twisted RS codes with small di- mensional hull	$\eta \neq 0, t = 1, h = k - 1$ $t = (1, 2,, \ell), h = (0, 1,, \ell - 1),$ $\eta = (\eta_1, \eta_2,, \eta_\ell) \in (\mathbb{F}_q^*)^\ell, \ell \le n/2$ $0 \le h_i < h_{i+1} \le k - 1, 1 \le t_i < t_{i+1} \le n - k,$ $t = (t_1, t_2,, t_\ell), h = (h_1, h_2,, h_\ell),$ $(T_i = (t_i) \le t_i < t_i \le t_$	[15] [15] In this paper
zero-dimensional hull Multi-twisted RS codes with zero- dimensional hull Multi-twisted RS codes with small di- mensional hull	$\eta \neq 0, t = 1, h = k - 1$ $t = (1, 2,, \ell), h = (0, 1,, \ell - 1),$ $\eta = (\eta_1, \eta_2,, \eta_\ell) \in (\mathbb{F}_q^*)^\ell, \ \ell \le n/2$ $0 \le h_i < h_{i+1} \le k - 1, \ 1 \le t_i < t_{i+1} \le n - k,$ $t = (t_1, t_2,, t_\ell), h = (h_1, h_2,, h_\ell),$ $\eta = (\eta_1, \eta_2,, \eta_\ell) \in (\mathbb{F}_q^*)^\ell$	[15] [15] In this paper
zero-dimensional hull Multi-twisted RS codes with zero- dimensional hull Multi-twisted RS codes with small di- mensional hull Necessary and sufficient conditions for	$\eta \neq 0, t = 1, h = k - 1$ $t = (1, 2,, \ell), h = (0, 1,, \ell - 1),$ $\eta = (\eta_1, \eta_2,, \eta_\ell) \in (\mathbb{F}_q^*)^\ell, \ell \le n/2$ $0 \le h_i < h_{i+1} \le k - 1, 1 \le t_i < t_{i+1} \le n - k,$ $t = (t_1, t_2,, t_\ell), h = (h_1, h_2,, h_\ell),$ $\eta = (\eta_1, \eta_2,, \eta_\ell) \in (\mathbb{F}_q^*)^\ell$ $1 < h_i < h_{i+1} \le k - 1, 1 < t_i < t_{i+1} \le n - k - 2,$	[15] [15] In this paper In this paper
zero-dimensional hull Multi-twisted RS codes with zero- dimensional hull Multi-twisted RS codes with small di- mensional hull Necessary and sufficient conditions for MDS multi-twisted RS codes with small	$\eta \neq 0, t = 1, h = k - 1$ $t = (1, 2,, \ell), h = (0, 1,, \ell - 1),$ $\eta = (\eta_1, \eta_2,, \eta_\ell) \in (\mathbb{F}_q^*)^\ell, \ell \le n/2$ $0 \le h_i < h_{i+1} \le k - 1, 1 \le t_i < t_{i+1} \le n - k,$ $t = (t_1, t_2,, t_\ell), h = (h_1, h_2,, h_\ell),$ $\eta = (\eta_1, \eta_2,, \eta_\ell) \in (\mathbb{F}_q^*)^\ell$ $1 < h_i < h_{i+1} \le k - 1, 1 < t_i < t_{i+1} \le n - k - 2,$ $t = (t_1, t_2,, t_\ell), h = (h_1, h_2,, h_\ell),$ $(\Pi^*)^{\ell}$	[15] [15] In this paper In this paper

Table 2: Comparison with existing studies

Statements and Declarations

- Acknowledgment: We are extremely thankful and appreciate the referee(s) for his/her comments which led improvements towards the quality of the paper. In addition, we are grateful to Dr. S. K. Tiwari for his fruitful discussions and suggestions towards improvement of the paper.
- Ethical approval: The submitted work is original and not submitted to more than one journal for simultaneous consideration.
- Competing interest: None of the authors have any relevant financial or non-financial competing inter-

ests.

- Author's contributions: The conceptualization, methodology, investigation, writing-original draft preparation, review and revision-editing have been performed by both the authors equally.
- Funding: This work has no financial supported.
- Availability of data and materials: This manuscript has no associated data.

References

- P. Beelen, M. Bossert, S. Puchinger, and J. Rosenkilde, Structural properties of twisted Reed-Solomon codes with applications to cryptography, *IEEE Int. Symp. Inform. Theory (ISIT)*, 2018.
- P. Beelen and L. Jin, Explicit MDS codes with complementary duals, *IEEE Trans. Inform. Theory*, 64(11), 7188-7193, 2018.
- [3] P. Beelen, S. Puchinger and J. R. Nielsen, Twisted Reed-Solomon codes, IEEE Int. Symp. Inform. Theory (ISIT), 2017.
- [4] M. Cao, MDS codes with Galois hulls of arbitrary dimensions and the related entanglement-assisted quantum error correction, *IEEE Trans. Inform. Theory*, 67(12), 7964-7984, 2021.
- [5] C. Carlet and S. Guilley, Complementary dual codes for counter-measures to side-channel attacks, *Coding Theory Appl.*, 97-105, Springer International Publishing, 2015.
- [6] C. Carlet, S. Mesnager, C. Tang, Y. Qi and R. Pellikaan, Linear codes over \mathbb{F}_q are equivalent to LCD codes for q > 3, *IEEE Trans. Inform. Theory*, 64(4), 3010-3017, 2018.
- [7] W. Fang, F. Fu, L. Li and S. Zhu, Euclidean and Hermitian hulls of MDS codes and their applications to EAQECCs, *IEEE Trans. Inform. Theory*, 66(6), 3527-3537, 2020.
- [8] X. Fang, R. Jin, J. Luo and W. Ma, New Galois hulls of GRS codes and application to EAQECCs, Cryptogr. Commun., 14(1), 145-159, 2022.
- [9] H. Gu and J. Zhang, On twisted generalized Reed-Solomon codes with ℓ twists, *Preprint*, arXiv:2211.06066v1 [cs.IT]
- [10] B. He and Q. Liao, The error-correcting pair for TGRS codes, *Discrete Math.*, 346(9), 1-13, 2023.
- [11] J. Lavauzelle and J. Renner, Cryptanalysis of a system based on twisted Reed-Solomon codes, Des. Codes Cryptogr., 88(7), 1285-1300, 2020.
- [12] J. S. Leon, Computing automorphism groups of error-correcting codes, IEEE Trans. Inform. Theory, 28(3), 496-511, 1982.
- [13] J. S. Leon, Permutation group algorithms based on partitions, i: Theory and algorithms. J. Symb. Comput., 12(4-5), 533-583, 1991.
- [14] C. Li and P. Zeng, Constructions of linear codes with one-dimensional hull, *IEEE Trans. Inform. Theory*, 65(3), 1668-1676, 2019.
- [15] H. Liu and S. Liu, Construction of MDS twisted Reed-Solomon codes and LCD MDS codes, Des. Codes Cryptogr., 89(9), 2051-2065, 2021.

- [16] H. Liu and X. Pan, Galois hulls of linear codes over finite fields, Des. Codes Cryptogr., 88(2), 241-255, 2020.
- [17] G. Luo, X. Cao and X. Chen, MDS codes with hulls of arbitrary dimensions and their quantum error correction, *IEEE Trans. Inform. Theory*, 65(5), 2944-2952, 2019.
- [18] F. J. MacWilliams and N. J. A. Sloane, The theory of error correcting codes, Vol. 16, Elsevier, 1977.
- [19] J. L. Massey, Linear codes with complementary duals, Discrete Math., 106-107, 337-342, 1992.
- [20] I. S. Reed and G. Solomon, Polynomial codes over certain finite fields, J. Society Indust. Applied Math., 8(2), 300-304, 1960.
- [21] R. M. Roth and A. Lempel, A construction of non-Reed-Solomon type MDS codes, *IEEE Trans. Inform. Theory*, 35(3), 655-657, 1989.
- [22] R. M. Roth and G. Seroussi, On generator matrices of MDS codes (corresp.), IEEE Trans. Inform. Theory, 31(6), 826-830, 1985.
- [23] N. Sendrier, On the dimension of the hull, SIAM J. Discrete Math., 10(2), 282-293, 1997.
- [24] N. Sendrier, Finding the permutation between equivalent linear codes: the support splitting algorithm, IEEE Trans. Inform. Theory, 46(4), 1193-1203, 2000.
- [25] N. Sendrier and G. Skersys, On the computation of the automorphism group of a linear code, In Proceedings. IEEE Int. Symp. Inform. Theory (IEEE Cat. No.01CH37252).
- [26] J. Sui, X. Zhu and X. Shi, MDS and near-MDS codes via twisted Reed-Solomon codes, Des. Codes Cryptogr., 90(8), 1937-1958, 2022.
- [27] J. Sui, Q. Yue, X. Li and D. Huang, MDS, near-MDS or 2-MDS self-dual codes via twisted generalized Reed-Solomon codes, *IEEE Trans. Inform. Theory*, 68(12), 7832-7841, 2022.
- [28] Y. Wu, Twisted Reed-Solomon codes with one-dimensional hull, IEEE Comm. Letters, 25(2), 383-386, 2021.
- [29] X. Yang and J. L. Massey, The condition for a cyclic code to have a complementary dual, Discrete Math., 126(1-3), 391-393, 1994.
- [30] C. Zhu and Q. Liao, The [1,0]-twisted generalized Reed-Solomon code, Preprint, arXiv:2211.05119v2 [cs.IT]

Scientific Analysis Group
Metcalfe House Complex
Defence Research and Development Organisation
Delhi - 110054, India.
E-mails: harshdeep.sag@gov.in {H. Singh}
meenakapishchand@gmail.com {K. C. Meena}

A Implementation of Example 3.2

For checking the non-singularity of each matrix in Example 3.2, we attach the following implementation:

```
1 F.<a> = GF(16)

2 alpha_vec = [0, a^2, a + 1, a^2 + a, a^3 + a + 1]

etal = a^3 + a^2

4 eta2_list = [1, a^2 + 1, a^2 + a + 1, a^3, a^3 + a^2, a^3 + a^2 + a]

8.<x> = PolynomialRing(F)

6 for eta2 in eta2_list:

7 print('\n eta2 : ',eta2)

8 det_matrix = []

9 for I in Combinations([0,1,2,3,4],3):

10 temp_poly = R(1)

11 for i in I:

12 temp_poly = temp_poly*(x-alpha_vec[i])

13 sigma_coeff = temp_poly.coefficients(sparse = False)

14 A_I = matrix(F,[[1,0],[sigma_coeff[2],1]])

15 B_I = matrix(F,[[-1*sigma_coeff[0], -1*sigma_coeff[0]]))

17 det_matrix.append((D*A_I + B_I).determinant())

19 print(det_matrix)
```

The output of this code is described below:

1	eta2 : 1
2	[a^3 + a^2 + 1, a^3 + a^2 + a, a^3 + a + 1, a^2, a^3 + a^2 + a + 1, a^3 + 1, a, a^3 + a, a^3, a^2]
3	
4	eta2 : a^2 + 1
5	$[a^2 + 1, a^2 + a, a + 1, a^3 + a^2, a^2 + a + 1, 1, a^3 + a^2, a^3 + 1, a^2 + 1, a^3 + a + 1]$
6	
7	eta2 : a^2 + a + 1
8	[a^3 + a^2 + a, a^3 + a^2 + 1, a^3, a^2 + a + 1, a^3 + a^2, a^3 + a, a^3 + a^2 + 1, a^3 + a^2, 1, a^3 + 1]
9	
10	eta2 : a^3
11	$[a^3 + a + 1, a^3, a^3 + a^2 + 1, a, a^3 + 1, a^3 + a^2 + a + 1, a^3 + a^2 + a + 1, a^2 + a, a^3 + 1, a^3 + a^2 + 1]$
12	
13	eta2 : a^3 + a^2
14	[a^3 + a^2 + a + 1, a^3 + a^2, a^3 + 1, a^2 + a, a^3 + a^2 + 1, a^3 + a + 1, a^3, a^3 + a^2 + a, a^2 + a, a + 1]
15	
16	eta2 : a^3 + a^2 + a
17	[a^3 + a, a^3 + 1, a^3 + a^2, a + 1, a^3, a^3 + a^2 + a, a^3 + a, a^2, a^3 + a^2 + a, a^2 + a + 1]
18	

18

B Implementation for counting of double-twisted **RS** codes

The following SageMath implementation provides number of double-twisted RS codes of length n and dimension k over the finite field \mathbb{F}_q .

```
1 def number (q,n,k):

2 f.<a> = GF(q)

3 A = Combinations(list(F),n)

4 main_count = 0

5 for C in A:

6 count = 0

7 for eta in F<sup>2</sup>:

8 if eta[0]**ta[1] !=0:

9 Ik = Combinations(C, k)

10 i = 0

11 while i < len(Ik):

2 sum_prod = 0

13 for Iki in Combinations(Ik[i], k-1):

4 sum_prod = sum_prod + prod(Ik[i])

14 flag = 1 - eta[0]*((-1)<sup>*</sup>k)*prod(Ik[i]) + eta[1]*((-1)<sup>*</sup>k)*((sum_prod)*(sum(Ik[i])) - prod(Ik[i])) + eta[0]*eta[1]*(prod(Ik[i])<sup>*</sup>2)

16 i = i+1

17 if flag=0:

18 break

19 if flag!=0:

20 count = count + 1

21 main_count = main_count
```