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ORIGINAL RESEARCH

Dynamical analysis of a stochastic non-autonomous SVIR model with multiple stages of vaccination

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Abstract

In this paper, we analyze the dynamics of a new proposed stochastic non-autonomous SVIR model, with an emphasis on multiple stages of vaccination, due to the vaccine ineffectiveness. The parameters of the model are allowed to depend on time, to incorporate the seasonal variation. Furthermore, the vaccinated population is divided into three subpopulations, each one representing a different stage. For the proposed model, we prove the mathematical and biological well-posedness. That is, the existence of a unique global almost surely positive solution. Moreover, we establish conditions under which the disease vanishes or persists. Furthermore, based on stochastic stability theory and by constructing a suitable new Lyapunov function, we provide a condition under which the model admits a non-trivial periodic solution. The established theoretical results along with the performed numerical simulations exhibit the effect of the different stages of vaccination along with the stochastic Gaussian noise on the dynamics of the studied population.

Keywords Epidemic model · Extinction · Persistence in the mean · Stochastic differential equations · Periodic solution

Mathematics Subject Classification 34K50 · 65C30 · 92B05

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1 Introduction

Throughout history, researchers from different disciplines have developed scientific knowledge that played a major role in the advancement of Epidemiology. In the mathematical framework, the contribution of mathematicians consists of developing adequate models, based on a good understanding of the modeled disease, which allows to describe the evolution of the latter within the studied population, predict the worst outcome by performing virtual numerical simulations and even propose control strategies that can help reduce the severity of the situation, especially when it comes to disease outbreaks. Kermack–McKendrick theory [1] has been a cornerstone to the mathematical modeling of epidemics. The basic idea is to divide the studied population into so called compartments, based on the number of clinical states induced by the modeled disease. Then, to incorporate the transition of individuals from one clinical state to another, to each compartment, a set of parameters describing all the possible transitions are considered. Once the epidemic model is derived, it takes the form of a dynamical system, which then can be interpreted from two points of view. The first one is the deterministic point of view, which assumes that the output of the system is a time-dependent function that is entirely determined by the initial conditions and the input parameters, while the second is the stochastic point of view, which assumes that the same initial conditions and input parameters can lead to different outputs due to the random effect present in the environment. Consequently, the output, in this case, takes the form of a stochastic process. In the deterministic framework, numerous pioneering results in term of the dynamical and numerical analysis of epidemic and ecological models have been established by many authors [2–12], while several other works were done in the aim of extending the deterministic results to the stochastic case [13–20]. In further work, non-autonomous stochastic models have gained the attention of several researchers, due to their ability to incorporate the seasonal variation of diseases [21–23]. We briefly outline some of the existing literature in this sense for the stochastic case. For instance, in [24], Qi et al. analyzed an SEIS model and were able to prove that it admits a non-trivial periodic solution. Additionally, conditions under which the model admits an ergodic stationary distribution were obtained. The same results were proved by Shangguan et al. [25] for an SEIR model and by Liu et al. [26] for an SIR model. In [27], Lin et al. considered an SIR model and were able to derive a threshold characterizing the persistence and extinction of the disease. Furthermore, in the case of persistence, they proved the existence of a non-trivial periodic solution. However, to the best of the authors' knowledge, the extension of these types of results to SVIR-type models, incorporating vaccination, has not yet been done.

When it comes to stochastic epidemic models incorporating the ineffectiveness of vaccination, most of the current research works neglect the dynamics of the vaccinated population, and make use of time delays to take into consideration the duration elapsed before the effectiveness of the vaccine wears off. In this context, we mention for instance the results presented in [28, 29]. Another limitation of the aforementioned works is assuming that the immunity can be gained solely after one stage of vaccination. These assumptions can be considered in order to simplify the formulation of the model. However, for some new emerging diseases such as COVID-19 and its variants, not taking these characteristics into account in the formulation of the model can reduce the

amount of information acquired from the numerical simulation. To highlight the crucial role of the multiple stages of vaccination in the acquisition of immunity, we refer the reader to the recent studies presented in [30, 31]. Hence, the main contributions of our work is to address the previous limitations by providing a different approach, allowing to incorporate the multiple stages of vaccination as well as the ineffectiveness of the first stages. More precisely, we propose a new non-autonomous stochastic model extending the standard SVIR model [2], on one hand by considering time-varying parameters, incorporating the seasonal variation, and on the other, by dividing the vaccinated population V into three sub-populations V_1, V_2 and V_3, such that V_1 and V_2 stand for the vaccinated sub-population of individuals in the first and second stages of vaccination, respectively, and are not supposed to develop immunity against the disease. Consequently, they become infected. While V_3 stands for the vaccinated sub-population of individuals who complete the third stage of vaccination and are supposed to develop immunity against the disease, for a large period of time.

The model in question is expressed by the following system of coupled nonlinear stochastic differential equations.

$$\begin{cases} dS(t) = (\Lambda(t) - \beta_S(t)S(t)I(t) - \mu(t)S(t) - \kappa_1(t)S(t))\,dt - \sigma_1(t)S(t)I(t)dB_1(t), \\ dV_1(t) = \big(-\beta_{V_1}(t)V_1(t)I(t) + \kappa_1(t)S(t) - \mu(t)V_1(t) - \kappa_2(t)V_1(t)\big)\,dt \\ \qquad - \sigma_2(t)V_1(t)I(t)dB_2(t), \\ dV_2(t) = \big(-\beta_{V_2}(t)V_2(t)I(t) + \kappa_2(t)V_1(t) - \mu(t)V_2(t) - \kappa_3(t)V_2(t)\big)\,dt \\ \qquad - \sigma_3(t)V_2(t)I(t)dB_3(t), \\ dV_3(t) = \big(\kappa_3(t)V_2(t) - \mu(t)V_3(t) - \gamma_{V_3}(t)V_3(t)\big)\,dt, \\ dI(t) = \big(\beta_S(t)S(t) + \beta_{V_1}(t)V_1(t) + \beta_{V_2}(t)V_2(t) - \gamma(t) - \mu(t)\big)\,I(t)dt \\ \qquad + \sigma_1(t)S(t)I(t)dB_1(t) + \sigma_2(t)V_1(t)I(t)dB_2(t) + \sigma_3(t)V_2(t)I(t)dB_3(t), \\ dR(t) = \big(\gamma(t)I(t) - \mu(t)R(t) + \gamma_{V_3}(t)V_3(t)\big)\,dt, \end{cases} \tag{1}$$

equipped with the following initial conditions

$$S(0) := S_0 \geq 0,\ \ V_1(0) := V_{10} \geq 0,\ \ V_2(0) := V_{20} \geq 0,$$
$$V_3(0) := V_{30} \geq 0,\ \ I(0) := I_0 \geq 0 \quad \text{and} \quad R(0) := R_0 \geq 0,$$

where $(B_1(t))_{t\geq 0}$, $(B_2(t))_{t\geq 0}$ and $(B_3(t))_{t\geq 0}$ are mutually independent Brownian motions defined on a probabilistic space $(\Omega, \mathbb{F}, \{\mathbb{F}_t\}_{t\geq 0}, \mathbb{P})$ with a filtration $\{\mathbb{F}_t\}_{t\geq 0}$ which is increasing, right-continuous and such that $\mathbb{F}_0$ contains the null sets, while $\sigma_1(t), \sigma_2(t)$ and $\sigma_3(t)$ denote the time-dependent intensities of the environmental Gaussian noise present in the disease transmission rates (Fig. 1, 1).

In order to unify the notations, we set

$$\begin{cases} u(t) \triangleq (S(t), V_1(t), V_2(t), V_3(t), I(t), R(t))^\top, \\ u_0 \triangleq (S_0, V_{10}, V_{20}, V_{30}, I_0, R_0)^\top, \\ dB(t) \triangleq (dB_1(t), dB_2(t), dB_3(t), dB_4(t), dB_5(t), dB_6(t))^\top, \\ \theta(t) \triangleq (\Lambda(t), \beta_S(t), \mu(t), \kappa_1(t), \kappa_2(t), \kappa_3(t), \beta_{V_1}(t), \beta_{V_2}(t), \gamma(t), \gamma_{V_3}(t))^\top, \\ f(t, u(t)) \triangleq (f_1(t, u(t)), f_2(t, u(t)), f_3(t, u(t)), f_4(t, u(t)), f_5(t, u(t)), f_6(t, u(t)))^\top, \end{cases}$$

Table 1 Signification of the model parameters

Parameter	Biological signification
$\Lambda(t)$	Natural birth rate at time t
$\mu(t)$	Natural death rate at time t
$\beta_S(t)$	Rate in which a susceptible individual at time t becomes infected
$\beta_{V_i}(t)$	Rate in which an individual at time t and in the ith stage of vaccination ($i \in \{1, 2\}$) becomes infected
$\gamma(t)$	Natural recovery rate at time t
$\gamma_{V_3}(t)$	Rate in which an individual at time t and in the third stage of vaccination possesses immunity
$\kappa_i(t)$	Rate in which a susceptible individual at time t reaches the ith stage of vaccination ($i \in \{1, 2, 3\}$)

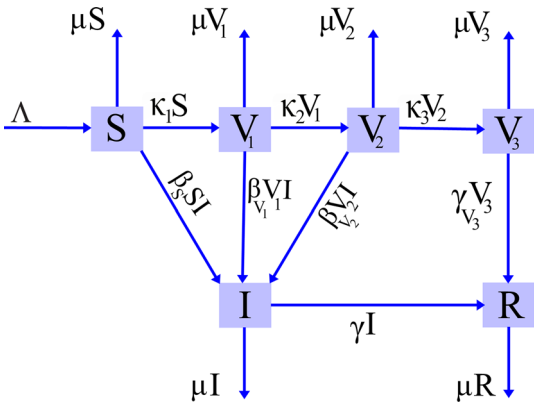

Fig. 1 Flow diagram of the model (1) in the deterministic case

where

$$\begin{cases} f_1(t, u(t)) := \Lambda(t) - \beta_S(t)S(t)I(t) - \mu(t)S(t) - \kappa_1(t)S(t), \\ f_2(t, u(t)) := -\beta_{V_1}(t)V_1(t)I(t) + \kappa_1(t)S(t) - \mu(t)V_1(t) - \kappa_2(t)V_1(t), \\ f_3(t, u(t)) := -\beta_{V_2}(t)V_2(t)I(t) + \kappa_2(t)V_1(t) - \mu(t)V_2(t) - \kappa_3(t)V_2(t), \\ f_4(t, u(t)) := \kappa_3(t)V_2(t) - \mu(t)V_3(t) - \gamma_{V_3}(t)V_3(t), \\ f_5(t, u(t)) := \beta_S(t)S(t)I(t) + \beta_{V_1}(t)V_1(t)I(t) + \beta_{V_2}(t)V_2(t)I(t) - \gamma(t)I(t) - \mu(t)I(t), \\ f_6(t, u(t)) := \gamma(t)I(t) - \mu(t)R(t) + \gamma_{V_3}(t)V_3(t), \end{cases}$$

and

$$g(t, u(t)) := \begin{pmatrix} -\sigma_1(t)S(t)I(t) & 0 & 0 & 0\ 0\ 0 \\ 0 & -\sigma_2(t)V_1(t)I(t) & 0 & 0\ 0\ 0 \\ 0 & 0 & -\sigma_3(t)V_2(t)I(t) & 0\ 0\ 0 \\ 0 & 0 & 0 & 0\ 0\ 0 \\ \sigma_1(t)S(t)I(t) & \sigma_2(t)V_1(t)I(t) & \sigma_3(t)V_2(t)I(t) & 0\ 0\ 0 \\ 0 & 0 & 0 & 0\ 0\ 0 \end{pmatrix}.$$

Then, the model (1) can be rewritten in the following abstract compact form

$$\begin{cases} du(t) = f(t, u(t))dt + g(t, u(t))dB(t), \\ u(0) = u_0 \geq 0. \end{cases} \tag{2}$$

When no confusion occurs, the value of a given function h at time $t \in (0, T)$ will occasionally be denoted h and we shall omit the explicit notation.

Given a function $V \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^6, \mathbb{R})$. The differential operator associated with (2) is defined as follows

$$\mathcal{L}V(t, u) = \frac{\partial V(t, u)}{\partial t} + \nabla_u V(t, u).f(t, u) + \frac{1}{2} tr \left(g^\top(t, u) Hess_u(V(t, u)) g(t, u) \right),$$

where $\nabla_u := \left(\frac{\partial}{\partial u_1}, \cdots, \frac{\partial}{\partial u_6} \right)$, tr denotes the trace operator, $^\top$ stands for the transpose operation, while $Hess_u$ is the Hessian matrix with respect to u.

Itô's formula [32] states that

$$dV(t, u) = \mathcal{L}V(t, u)dt + \nabla_u f(t, u(t)).g(t, u(t))dB(t).$$

We now announce some definitions and notations that will be used throughout the paper.

- For $T > 0$, denote by $C([0, T])$ the Banach space of real-valued continuous functions defined on $[0, T]$. Given $f \in C([0, T])$, we define

$$\overline{f} := \sup_{t \in [0,T]} |f(t)| \quad \text{and} \quad \underline{f} := \inf_{t \in [0,T]} |f(t)|.$$

- For an integrable function $f : (0, T) \to \mathbb{R}$, we set

$$\langle f \rangle_t := \frac{1}{t} \int_0^t f(s)ds \quad \forall t \in (0, T).$$

- Given $a, b \in \mathbb{R}$, we set $a \vee b \triangleq \sup\{a, b\}$ and $a \wedge b \triangleq \inf\{a, b\}$.
- Consider the following open bounded set

$$U := \left\{ u \in (0, +\infty)^6, \quad \sum_{i=1}^{6} u_i < \frac{\overline{\Lambda}}{\underline{\mu}} \right\}.$$

Hereafter, T is a strictly positive real number and it is assumed that

$$\theta_i, \sigma_j \in C([0, T]) \quad \text{and} \quad \underline{\theta_i}, \overline{\theta_i}, \underline{\sigma_j}, \overline{\sigma_j} > 0, \qquad \forall (i, j) \in \{1, \cdots, 10\} \times \{1, 2, 3\}.$$

The rest of this paper is organized as follows: In Sect. 2, we study the mathematical and biological well-posedness of the model (1). We devote Sect. 3 to establish

conditions under which the infected population becomes extinct or persistent in the mean. While in Sect. 4, we provide a condition under which the model (1) admits a non-trivial periodic solution. Additionally, in order to support the theoretical results, in Sect. 5, we present the outcome of the performed numerical simulations. Finally, we leave Sect. 6 to state some conclusions and future works.

2 Mathematical and biological well-posedness

We begin this section by stating a remark, which will be useful overall throughout the paper.

Remark 1 It can be seen that the set $\mathcal{I} := \left\{u \in \mathbb{R}_+^6, \quad \sum_{i=1}^{6} u_i \leq \frac{\overline{\Lambda}}{\underline{\mu}}\right\}$, is positively invariant for the stochastic system (1). Indeed, define the total population at time $t \in (0, T)$ by $N(t) := \sum_{i=1}^{6} u_i(t)$. Direct application of the comparison principle yields

$$N(t) \leq N(0)\exp(-\underline{\mu}t) + \frac{\overline{\Lambda}}{\underline{\mu}}\left(1 - \exp(-\underline{\mu}t)\right).$$

Then if $u_0 \in \mathcal{I}$, it follows that $u(t) \in \mathcal{I}\ \forall t \in (0, T)$. Additionally,

$$\lim_{t\to+\infty} N(t) \leq \frac{\overline{\Lambda}}{\underline{\mu}} \quad \text{almost surely.}$$

Theorem 1 *For every initial condition $u_0 \in \mathcal{I}$, the stochastic system (2) admits a unique global, almost surely positive solution.*

Proof Since the coefficients of the stochastic system (2) satisfy the local Lipschitz condition, by the standard theory of stochastic differential equations [32], there exists a unique local solution u defined up to a maximal time of existence that we denote T_{max}. In order to prove that the local solution is a global one that remains almost surely positive, let $\tilde{n} \in \mathbb{N}^*$ be sufficiently large such that $u_0 \in \left[\frac{1}{\tilde{n}}, \frac{\overline{\Lambda}}{\underline{\mu}}\right)^6$. Then, for $n \geq \tilde{n}$ define the following stopping time

$$\tau_n := \inf_{t\in[0,T_{max})} \left\{ \quad \exists i_0 \in \{1, \cdots, 6\} \quad u_{i_0}(t) \leq \frac{1}{n}\right\},$$

with the usual convention $\inf \emptyset = +\infty$, where $\emptyset$ denotes the empty set. It is clear that the sequence $(\tau_n)_{n\geq\tilde{n}}$ is increasing and $\tau_n \leq T_{max}$. Hence, there exists τ^l such that $\lim_{n\to+\infty} \tau_n = \tau^l \leq T_{max}$. Thus, it suffices to prove that $\tau^l = +\infty$. We argue by

contradiction and suppose that there exist $\epsilon \in (0,1)$, $T > 0$ and $n_0 \geq \tilde{n}$ such that

$$\forall n \geq n_0, \qquad \mathbb{P}(\tau_n \leq T) \geq \epsilon.$$

Now, consider the following function $\mathcal{F} : \quad U \longrightarrow \mathbb{R}^+$ defined by $\mathcal{F}(u) := -\sum_{i=1}^{6} \ln\left(\frac{\underline{\mu} u_i}{\overline{\Lambda}}\right)$. By Itô's formula, it holds that

$$\begin{aligned} d\mathcal{F} = \Bigg\{ & -\left[\frac{1}{S}\left(\Lambda - \beta_S SI - \mu S - \kappa_1 S\right)\right] - \left[\frac{1}{V_1}\left(-\beta_{V_1} V_1 I + \kappa_1 S - \mu V_1 - \kappa_2 V_1\right)\right] \\ & - \left[\frac{1}{V_2}\left(-\beta_{V_2} V_2 I + \kappa_2 V_1 - \mu V_2 - \kappa_3 V_2\right)\right] - \left[\frac{1}{V_3}\left(\kappa_3 V_2 - \mu V_3 - \gamma_{V_3} V_3\right)\right] \\ & - \left[\frac{1}{I}\left(\beta_S SI + \beta_{V_1} V_1 I + \beta_{V_2} V_2 I - \gamma I - \mu I\right)\right] - \left[\frac{1}{R}\left(\gamma I - \mu R + \gamma_{V_3} V_3\right)\right] \\ & + \left[\frac{1}{2}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right) I^2 + \frac{1}{2}\left(\sigma_1^2 S^2 + \sigma_2^2 V_1^2 + \sigma_3^2 V_2^2\right)\right]\Bigg\} dt \\ & + \sigma_1 (I - S) dB_1 + \sigma_2 (I - V_1) dB_2 + \sigma_3 (I - V_2) dB_3. \end{aligned}$$

Thereby, by using Remark 1, it follows that

$$d\mathcal{F} \leq C dt + \sigma_1 (I - S) dB_1 + \sigma_2 (I - V_1) dB_2 + \sigma_3 (I - V_2) dB_3, \tag{3}$$

where

$$C := \frac{\overline{\Lambda}}{\underline{\mu}}\left(\overline{\beta_S} + \overline{\beta_{V_1}} + \overline{\beta_{V_2}}\right) + 6\overline{\mu} + \overline{\kappa_1} + \overline{\kappa_2} + \overline{\kappa_3} + \overline{\gamma} + \overline{\gamma_{V_3}} + \frac{\overline{\Lambda}^2}{\underline{\mu}^2}\left(\overline{\sigma_1}^2 \vee \overline{\sigma_2}^2 \vee \overline{\sigma_3}^2\right).$$

By integrating both sides of inequality (3) from 0 to $T \wedge \tau_n$ and evaluating the expectation, we obtain

$$\mathbb{E}(\mathcal{F}(u(T \wedge \tau_n))) \leq \mathcal{F}(u(0)) + CT.$$

On the other hand, by definition of τ_n, there exists $i_0 \in \{1, \cdots, 6\}$ such that $u_{i_0}(\tau_n) \leq \frac{1}{n}$. Consequently, $-\ln\left(\frac{\underline{\mu} u_{i_0}(\tau_n)}{\overline{\Lambda}}\right) \geq -\ln\left(\frac{\underline{\mu}}{\overline{\Lambda} n}\right)$. Therefore, $\mathcal{F}(u(\tau_n)) \geq -\ln\left(\frac{\underline{\mu}}{\overline{\Lambda} n}\right)$. Hence, due to the positiveness of $\mathcal{F}$, it holds that

$$-\ln\left(\frac{\underline{\mu}}{\overline{\Lambda} n}\right) \leq \mathbb{E}(\mathcal{F}(u(\tau_n)\mathbf{1}_{\tau_n \leq T})) \leq \mathbb{E}(\mathcal{F}(u(T \wedge \tau_n))) \leq \mathcal{F}(u(0)) + CT. \tag{4}$$

where $\mathbf{1}$ stands for the indicator function.

Letting $n \to +\infty$ in inequality (4) leads to the contradiction $+\infty \leq \mathcal{F}(u(0)) + CT < +\infty$. Thus, $T_{max} = +\infty$ and the solution is global and remains almost surely positive.

□

3 Analysis of the disease extinction and persistence

In this section, we are interested in establishing conditions under which the disease vanishes or persists. To this end, we define the following parameters

$$\mathcal{R}_1^s(t) := \frac{\overline{\Lambda}}{\underline{\mu}} \frac{\beta_S(t) + \beta_{V_1}(t) + \beta_{V_2}(t)}{\mu(t) + \gamma(t)} - \frac{\overline{\Lambda}^2}{\underline{\mu}^2(\mu(t) + \gamma(t))} \left(\frac{1}{2}\sigma_1^2(t) + \frac{1}{2}\sigma_2^2(t) + \frac{1}{2}\sigma_3^2(t) \right),$$

$\forall t \in (0, T)$, and

$$\mathcal{R}_2^s := \frac{\underline{\Lambda}\, \underline{\beta_S}\, (\overline{\mu} + \overline{\kappa_2})\, (\overline{\mu} + \overline{\kappa_3}) + \underline{\Lambda}\, \underline{\kappa_1}\, \underline{\beta_{V_1}}\, (\overline{\mu} + \overline{\kappa_3}) + \underline{\Lambda}\, \underline{\kappa_1}\, \underline{\kappa_2}\, \underline{\beta_{V_2}}}{(\overline{\mu} + \overline{\kappa_1})\, (\overline{\mu} + \overline{\kappa_2})\, (\overline{\mu} + \overline{\kappa_3}) \left(\overline{\mu} + \overline{\gamma} + \frac{1}{2} \frac{\overline{\Lambda}^2}{\underline{\mu}^2} \left(\overline{\sigma_1}^2 + \overline{\sigma_2}^2 + \overline{\sigma_3}^2 \right) \right)}.$$

Theorem 2 *Let u be the solution of the system (2) with the initial value $u_0 \in \mathcal{I}$. If one of the following conditions*

1. $\limsup_{t \to +\infty} \varrho(t) < 0$, *where* $\varrho(t) := \left\langle \frac{\beta_S^2}{\sigma_1^2} \right\rangle_t + \left\langle \frac{\beta_{V_1}^2}{\sigma_2^2} \right\rangle_t + \left\langle \frac{\beta_{V_2}^2}{\sigma_3^2} \right\rangle_t - 2\left(\langle \mu \rangle_t + \langle \gamma \rangle_t \right)$,
2. $\langle \mathcal{R}_1^s \rangle_T < 1$, *and* $\forall t \in (0, T)$, $\frac{\underline{\mu}}{\overline{\Lambda}} \beta_S(t) > \sigma_1^2(t)$, $\frac{\underline{\mu}}{\overline{\Lambda}} \beta_{V_1}(t) > \sigma_2^2(t)$, $\frac{\underline{\mu}}{\overline{\Lambda}} \beta_{V_2}(t) > \sigma_3^2(t)$,

is satisfied, then the infected population goes to extinction. That is, $\limsup_{t \to +\infty} I(t) = 0$ *almost surely.*

Proof By using Itô's formula, it holds that

$$\begin{aligned} d(\ln(I)) &= \left\{ \beta_S S + \beta_{V_1} V_1 + \beta_{V_2} V_2 - \mu - \gamma - \frac{1}{2}\sigma_1^2 S^2 - \frac{1}{2}\sigma_2^2 V_1^2 - \frac{1}{2}\sigma_3^2 V_2^2 \right\} dt \\ &\quad + \sigma_1 S dB_1 + \sigma_2 V_1 dB_2 + \sigma_3 V_2 dB_3 \\ &= \left\{ -\left(\frac{\beta_S}{\sqrt{2}\sigma_1} - \frac{\sigma_1 S}{\sqrt{2}} \right)^2 - \left(\frac{\beta_{V_1}}{\sqrt{2}\sigma_2} - \frac{\sigma_2 V_1}{\sqrt{2}} \right)^2 - \left(\frac{\beta_{V_2}}{\sqrt{2}\sigma_3} - \frac{\sigma_3 V_2}{\sqrt{2}} \right)^2 \right. \\ &\quad \left. + \frac{\beta_S^2}{2\sigma_1^2} + \frac{\beta_{V_1}^2}{2\sigma_2^2} + \frac{\beta_{V_2}^2}{2\sigma_3^2} - \mu - \gamma \right\} dt + \sigma_1 S dB_1 + \sigma_2 V_1 dB_2 + \sigma_3 V_2 dB_3 \\ &\leq \left\{ \frac{\beta_S^2}{2\sigma_1^2} + \frac{\beta_{V_1}^2}{2\sigma_2^2} + \frac{\beta_{V_2}^2}{2\sigma_3^2} - \mu - \gamma \right\} dt + \sigma_1 S dB_1 + \sigma_2 V_1 dB_2 + \sigma_3 V_2 dB_3. \end{aligned} \tag{5}$$

Dividing inequality (5) by $t > 0$, then integrating from 0 to t yields

$$\frac{\ln(I(t))}{t} \leq \frac{\ln(I(0))}{t} + \frac{1}{2}\left\langle \frac{\beta_S^2}{\sigma_1^2} \right\rangle_t + \frac{1}{2}\left\langle \frac{\beta_{V_1}^2}{\sigma_2^2} \right\rangle_t + \frac{1}{2}\left\langle \frac{\beta_{V_2}^2}{\sigma_3^2} \right\rangle_t - \langle \mu \rangle_t - \langle \gamma \rangle_t + \frac{1}{t} M_1(t), \tag{6}$$

where $M_1(t)$ is a local continuous martingale satisfying $M_1(0) = 0$, and is defined by

$$M_1(t) := \int_0^t \sigma_1(s)S(s)dB_1(s) + \sigma_2(s)V_1(s)dB_2(s) + \sigma_3(s)V_2(s)dB_3(s).$$

By evaluating the supremum limit on both sides of inequality (6) and by the law of large numbers for local martingales [32], we have $\lim\limits_{t\to+\infty} \dfrac{M_1(t)}{t} = 0,$ almost surely. Consequently, we obtain

$$\limsup_{t\to+\infty} \frac{\ln(I(t))}{t} \le \limsup_{t\to+\infty} \left[\frac{1}{2}\left(\left\langle \frac{\beta_S^2}{\sigma_1^2} \right\rangle_t + \left\langle \frac{\beta_{V_1}^2}{\sigma_2^2} \right\rangle_t + \left\langle \frac{\beta_{V_2}^2}{\sigma_3^2} \right\rangle_t \right) - \langle \mu \rangle_t - \langle \gamma \rangle_t \right].$$

Hence, if condition (1) is satisfied. Then $\limsup\limits_{t\to+\infty} I(t) = 0$ almost surely. Now, we suppose that $\overset{\mu}{\underset{\Lambda}{=}} \beta_S(t) > \sigma_1^2(t)$, $\overset{\mu}{\underset{\Lambda}{=}} \beta_{V_1}(t) > \sigma_2^2(t)$ and $\overset{\mu}{\underset{\Lambda}{=}} \beta_{V_2}(t) > \sigma_3^2(t)$, $\forall t \in (0, T)$. By using Itô's formula and taking Remark 1 into account, we obtain

$$\begin{aligned}
d(\ln(I)) &= \left\{ \beta_S S - \frac{1}{2}\sigma_1^2 S^2 + \beta_{V_1} V_1 - \frac{1}{2}\sigma_2^2 V_1^2 + \beta_{V_2} V_2 - \frac{1}{2}\sigma_3^2 V_2^2 - \mu - \gamma \right\} dt \\
&\quad + \sigma_1 S dB_1 + \sigma_2 V_1 dB_2 + \sigma_3 V_2 dB_3 \\
&\le \left\{ \beta_S \frac{\overline{\Lambda}}{\underline{\mu}} - \frac{1}{2}\sigma_1^2 \frac{\overline{\Lambda}^2}{\underline{\mu}^2} + \beta_{V_1} \frac{\overline{\Lambda}}{\underline{\mu}} - \frac{1}{2}\sigma_2^2 \frac{\overline{\Lambda}^2}{\underline{\mu}^2} + \beta_{V_2} \frac{\overline{\Lambda}}{\underline{\mu}} - \frac{1}{2}\sigma_3^2 \frac{\overline{\Lambda}^2}{\underline{\mu}^2} - \mu - \gamma \right\} dt \\
&\quad + \sigma_1 S dB_1 + \sigma_2 V_1 dB_2 + \sigma_3 V_2 dB_3 \\
&= \left\{ (\mu + \gamma) \left(\frac{\overline{\Lambda}}{\underline{\mu}} \frac{\beta_S + \beta_{V_1} + \beta_{V_2}}{\mu + \gamma} - \frac{\overline{\Lambda}^2}{\underline{\mu}^2(\mu + \gamma)} \left(\frac{1}{2}\sigma_1^2 + \frac{1}{2}\sigma_2^2 + \frac{1}{2}\sigma_3^2 \right) - 1 \right) \right\} dt \\
&\quad + \sigma_1 S dB_1 + \sigma_2 V_1 dB_2 + \sigma_3 V_2 dB_3.
\end{aligned} \tag{7}$$

By dividing inequality (7) by $t > 0$ and integrating from 0 to t, we acquire that

$$\frac{\ln(I(t))}{t} \le \frac{\ln(I(0))}{t} + (\langle \mu \rangle_t + \langle \gamma \rangle_t) \left(\langle \mathcal{R}_1^s \rangle_t - 1 \right) + \frac{1}{t} M_1(t). \tag{8}$$

By applying the supremum limit on both sides of inequality (8), it follows that

$$\limsup_{t\to+\infty} \frac{\ln(I(t))}{t} \le \left(\langle \mathcal{R}_1^s \rangle_T - 1 \right) \limsup_{t\to+\infty} (\langle \mu \rangle_t + \langle \gamma \rangle_t).$$

Hence, if $\langle \mathcal{R}_1^s \rangle_T < 1$, it follows that $\limsup\limits_{t\to+\infty} I(t) = 0$ almost surely. □

We now proceed to derive the condition under which the infected population becomes persistent in the mean. Namely, under a suitable condition, we prove that: $\exists \alpha > 0, \quad \liminf\limits_{t\to+\infty} \langle I \rangle_t \ge \alpha$ almost surely.

Theorem 3 *Let u be the solution of the system (2) with the initial value $u_0 \in \mathcal{I}$. Under the following condition*

$$\mathcal{R}_2^s > 1, \tag{9}$$

the infected population is persistent in the mean. More precisely, $\liminf_{t\to+\infty} \langle I\rangle_t \geq \dfrac{\lambda}{\lambda_0}$ almost surely, where

$$\lambda := \left(\overline{\mu} + \overline{\gamma} + \frac{1}{2}\overline{\sigma_1}^2\frac{\overline{\Lambda}^2}{\underline{\mu^2}} + \frac{1}{2}\overline{\sigma_2}^2\frac{\overline{\Lambda}^2}{\underline{\mu^2}} + \frac{1}{2}\overline{\sigma_3}^2\frac{\overline{\Lambda}^2}{\underline{\mu^2}}\right)\left(\mathcal{R}_2^s - 1\right),$$

$$\lambda_0 := \frac{\alpha_3\left(\underline{\beta_S}\overline{\beta_S}\ \overline{\Lambda}\alpha_2 + \underline{\beta_{V_1}}\overline{\beta_{V_1}}\ \overline{\Lambda}\alpha_1 + \overline{\Lambda}\ \overline{\beta_S}\ \underline{\kappa_1}\ \underline{\beta_{V_1}}\right)}{\underline{\mu}\alpha_1\alpha_2\alpha_3} + \frac{\alpha_1\left(\overline{\beta_{V_2}}\underline{\beta_{V_2}}\overline{\Lambda}\alpha_2 + \overline{\Lambda}\underline{\kappa_2}\overline{\beta_{V_1}}\underline{\beta_{V_2}}\right) + \overline{\Lambda}\ \overline{\beta_S}\underline{\kappa_1}\ \underline{\kappa_2}\ \underline{\beta_{V_2}}}{\underline{\mu}\alpha_1\alpha_2\alpha_3},$$

$$\alpha_1 := (\overline{\mu} + \overline{\kappa_1}), \quad \alpha_2 := (\overline{\mu} + \overline{\kappa_2}) \quad \textit{and} \quad \alpha_3 := (\overline{\mu} + \overline{\kappa_3}).$$

Proof By using Itô's formula, it holds that

$$\begin{aligned} d(\ln(I)) &= \left\{\beta_S S + \beta_{V_1}V_1 + \beta_{V_2}V_2 - \mu - \gamma - \frac{1}{2}\sigma_1^2 S^2 - \frac{1}{2}\sigma_2^2 V_1^2 - \frac{1}{2}\sigma_3^2 V_2^2\right\} dt \\ &\quad + \sigma_1 S dB_1 + \sigma_2 V_1 dB_2 + \sigma_3 V_2 dB_3 \\ &\geq \left\{\underline{\beta_S} S + \underline{\beta_{V_1}}V_1 + \underline{\beta_{V_2}}V_2 - \overline{\mu} - \overline{\gamma} - \frac{1}{2}\overline{\sigma_1}^2\frac{\overline{\Lambda}^2}{\underline{\mu^2}} - \frac{1}{2}\overline{\sigma_2}^2\frac{\overline{\Lambda}^2}{\underline{\mu^2}} - \frac{1}{2}\overline{\sigma_3}^2\frac{\overline{\Lambda}^2}{\underline{\mu^2}}\right\} dt \\ &\quad + \sigma_1 S dB_1 + \sigma_2 V_1 dB_2 + \sigma_3 V_2 dB_3. \end{aligned} \tag{10}$$

An integration of inequality (10) from 0 to t and a divison by $t > 0$ lead to

$$\begin{aligned} \frac{\ln(I(t)) - \ln(I(0))}{t} &\geq \underline{\beta_S}\langle S\rangle_t + \underline{\beta_{V_1}}\langle V_1\rangle_t + \underline{\beta_{V_2}}\langle V_2\rangle_t - \overline{\mu} - \overline{\gamma} - \frac{1}{2}\overline{\sigma_1}^2\frac{\overline{\Lambda}^2}{\underline{\mu^2}} - \frac{1}{2}\overline{\sigma_2}^2\frac{\overline{\Lambda}^2}{\underline{\mu^2}} \\ &\quad - \frac{1}{2}\overline{\sigma_3}^2\frac{\overline{\Lambda}^2}{\underline{\mu^2}} + \frac{M_1(t)}{t}, \end{aligned} \tag{11}$$

where $M_1(t)$ is the local continuous martingale defined in the proof of Theorem 2.

Now, by taking Remark 1 into account, an integration of the first three equations of the stochastic system (1) from 0 to t and a division by $t > 0$ yield

$$\begin{cases} \langle S\rangle_t \geq \dfrac{1}{\overline{\mu}+\overline{\kappa_1}}\left(-\overline{\beta_S}\dfrac{\overline{\Lambda}}{\underline{\mu}}\langle I\rangle_t + \underline{\Lambda} - \dfrac{S(t)-S(0)}{t} + \dfrac{M_2(t)}{t}\right), \\ \langle V_1\rangle_t \geq \dfrac{1}{\overline{\mu}+\overline{\kappa_2}}\left(-\overline{\beta_{V_1}}\dfrac{\overline{\Lambda}}{\underline{\mu}}\langle I\rangle_t + \dfrac{\underline{\kappa_1}}{\overline{\mu}+\overline{\kappa_1}}\left(\underline{\Lambda} - \overline{\beta_S}\dfrac{\overline{\Lambda}}{\underline{\mu}}\langle I\rangle_t - \dfrac{S(t)-S(0)}{t} + \dfrac{M_2(t)}{t}\right)\right. \\ \qquad \left. - \dfrac{V_1(t)-V_1(0)}{t} + \dfrac{M_3(t)}{t}\right), \\ \langle V_2\rangle_t \geq \dfrac{1}{\overline{\mu}+\overline{\kappa_3}}\left(-\overline{\beta_{V_2}}\dfrac{\overline{\Lambda}}{\underline{\mu}}\langle I\rangle_t + \dfrac{\underline{\kappa_2}}{\overline{\mu}+\overline{\kappa_2}}\left(-\overline{\beta_{V_1}}\dfrac{\overline{\Lambda}}{\underline{\mu}}\langle I\rangle_t + \dfrac{\underline{\kappa_1}}{\overline{\mu}+\overline{\kappa_1}}\right.\right. \\ \qquad \left.\times\left(\underline{\Lambda} - \overline{\beta_S}\dfrac{\overline{\Lambda}}{\underline{\mu}}\langle I\rangle_t - \dfrac{S(t)-S(0)}{t} + \dfrac{M_2(t)}{t}\right) - \dfrac{V_1(t)-V_1(0)}{t} + \dfrac{M_3(t)}{t}\right) \\ \qquad \left. - \dfrac{V_2(t)-V_2(0)}{t} + \dfrac{M_4(t)}{t}\right), \end{cases} \tag{12}$$

where $M_2(t), M_3(t)$ and $M_4(t)$ are continuous local martingales, satisfying $M_2(0) = M_3(0) = M_4(0) = 0$, and are defined by $M_2(t) := \int_0^t -\sigma_1(s)S(s)I(s)\,dB_1(s)$, $M_3(t) := \int_0^t -\sigma_2(s)V_1(s)I(s)\,dB_2(s)$, and $M_4(t) := \int_0^t -\sigma_3(s)V_2(s)I(s)\,dB_3(s)$. By injecting the inequalities of (12) into the inequality (11) and rearranging the terms, we obtain

$$\begin{aligned} \frac{\ln(I(t)) - \ln(I(0))}{t} &\geq \left(\overline{\mu} + \overline{\gamma} + \frac{1}{2}\overline{\sigma_1}^2\frac{\overline{\Lambda}^2}{\underline{\mu^2}} + \frac{1}{2}\overline{\sigma_2}^2\frac{\overline{\Lambda}^2}{\underline{\mu^2}} + \frac{1}{2}\overline{\sigma_3}^2\frac{\overline{\Lambda}^2}{\underline{\mu^2}}\right)\left(\mathcal{R}_2^s - 1\right) \\ &+ \frac{M_1(t)}{t} + \left(\frac{\underline{\beta_S}}{\overline{\mu}+\overline{\kappa_1}} + \frac{\underline{\kappa_1}\,\underline{\beta_{V_1}}}{(\overline{\mu}+\overline{\kappa_1})\,(\overline{\mu}+\overline{\kappa_2})} + \frac{\underline{\kappa_1}\,\underline{\kappa_2}\,\underline{\beta_{V_2}}}{(\overline{\mu}+\overline{\kappa_1})\,(\overline{\mu}+\overline{\kappa_2})\,(\overline{\mu}+\overline{\kappa_3})}\right) \\ &\times \frac{M_2(t)}{t} + \left(\frac{\underline{\beta_{V_1}}}{(\overline{\mu}+\overline{\kappa_2})} + \frac{\underline{\kappa_2}\,\underline{\beta_{V_2}}}{(\overline{\mu}+\overline{\kappa_2})\,(\overline{\mu}+\overline{\kappa_3})}\right)\frac{M_3(t)}{t} + \frac{\underline{\beta_{V_2}}}{(\overline{\mu}+\overline{\kappa_3})}\frac{M_4(t)}{t} \\ &- \left(\frac{\overline{\Lambda}\,\overline{\beta_S}\,\underline{\beta_S}}{\underline{\mu}\,(\overline{\mu}+\overline{\kappa_1})} + \frac{\overline{\Lambda}\,\overline{\beta_{V_1}}\,\underline{\beta_{V_1}}}{\underline{\mu}\,(\overline{\mu}+\overline{\kappa_2})} + \frac{\underline{\kappa_1}\,\underline{\beta_{V_1}}\,\overline{\beta_S}\,\overline{\Lambda}}{\underline{\mu}\,(\overline{\mu}+\overline{\kappa_1})\,(\overline{\mu}+\overline{\kappa_2})} + \frac{\underline{\kappa_2}\,\underline{\beta_{V_2}}\,\overline{\beta_{V_1}}\,\overline{\Lambda}}{\underline{\mu}\,(\overline{\mu}+\overline{\kappa_2})\,(\overline{\mu}+\overline{\kappa_3})}\right. \\ &\left.+ \frac{\overline{\Lambda}\,\overline{\beta_{V_2}}\,\underline{\beta_{V_2}}}{\underline{\mu}\,(\overline{\mu}+\overline{\kappa_3})} + \frac{\underline{\kappa_1}\,\underline{\kappa_2}\,\underline{\beta_{V_2}}\,\overline{\beta_S}\,\overline{\Lambda}}{\underline{\mu}\,(\overline{\mu}+\overline{\kappa_1})\,(\overline{\mu}+\overline{\kappa_2})\,(\overline{\mu}+\overline{\kappa_3})}\right)\langle I\rangle_t - \left(\frac{\underline{\beta_S}}{\overline{\mu}+\overline{\kappa_1}}\right. \\ &\left.+ \frac{\underline{\kappa_1}\,\underline{\beta_{V_1}}}{(\overline{\mu}+\overline{\kappa_1})\,(\overline{\mu}+\overline{\kappa_2})} + \frac{\underline{\kappa_1}\,\underline{\kappa_2}\,\underline{\beta_{V_2}}}{(\overline{\mu}+\overline{\kappa_1})\,(\overline{\mu}+\overline{\kappa_2})\,(\overline{\mu}+\overline{\kappa_3})}\right)\left(\frac{S(t)-S(0)}{t}\right) \\ &- \left(\frac{\underline{\beta_{V_1}}}{\overline{\mu}+\overline{\kappa_2}}\right) + \frac{\underline{\kappa_2}\,\underline{\beta_{V_2}}}{(\overline{\mu}+\overline{\kappa_2})\,(\overline{\mu}+\overline{\kappa_3})}\Bigg)\left(\frac{V_1(t)-V_1(0)}{t}\right) - \frac{\underline{\beta_{V_2}}}{(\overline{\mu}+\overline{\kappa_3})} \\ &\times \left(\frac{V_2(t)-V_2(0)}{t}\right). \end{aligned}$$

Consequently

$$\frac{\ln(I(t))}{t} \geq \lambda - \lambda_0 \langle I \rangle_t + \frac{H(t)}{t} \quad \text{almost surely,} \quad \forall t \geq 0, \tag{13}$$

where λ and λ_0 are as defined in Theorem 3, and

$$\begin{aligned} H(t) := {} & M_1(t) + \left(\frac{\beta_S}{\overline{\mu} + \overline{\kappa_1}} + \frac{\underline{\kappa_1}\, \underline{\beta_{V_1}}}{(\overline{\mu} + \overline{\kappa_1})\,(\overline{\mu} + \overline{\kappa_2})} + \frac{\underline{\kappa_1}\, \underline{\kappa_2}\, \underline{\beta_{V_2}}}{(\overline{\mu} + \overline{\kappa_1})\,(\overline{\mu} + \overline{\kappa_2})\,(\overline{\mu} + \overline{\kappa_3})} \right) M_2(t) \\ & + \left(\frac{\underline{\beta_{V_1}}}{(\overline{\mu} + \overline{\kappa_2})} + \frac{\underline{\kappa_2}\, \underline{\beta_{V_2}}}{(\overline{\mu} + \overline{\kappa_2})\,(\overline{\mu} + \overline{\kappa_3})} \right) M_3(t) + \frac{\underline{\beta_{V_2}}}{(\overline{\mu} + \overline{\kappa_3})} M_4(t) - \left(\frac{\beta_S}{\overline{\mu} + \overline{\kappa_1}} \right. \\ & \left. + \frac{\underline{\kappa_1}\, \underline{\beta_{V_1}}}{(\overline{\mu} + \overline{\kappa_1})\,(\overline{\mu} + \overline{\kappa_2})} + \frac{\underline{\kappa_1}\, \underline{\kappa_2}\, \underline{\beta_{V_2}}}{(\overline{\mu} + \overline{\kappa_1})\,(\overline{\mu} + \overline{\kappa_2})\,(\overline{\mu} + \overline{\kappa_3})} \right) (S(t) - S(0)) - \left(\frac{\underline{\beta_{V_1}}}{\overline{\mu} + \overline{\kappa_2}} \right. \\ & \left. + \frac{\underline{\kappa_2}\, \underline{\beta_{V_2}}}{(\overline{\mu} + \overline{\kappa_2})\,(\overline{\mu} + \overline{\kappa_3})} \right) (V_1(t) - V_1(0)) - \frac{\underline{\beta_{V_2}}}{(\overline{\mu} + \overline{\kappa_3})} (V_2(t) - V_2(0)) + ln(I(0)). \end{aligned}$$

By the law of large numbers for local martingales and by taking Remark 1 into account, it follows that $\lim_{t \to +\infty} \frac{H(t)}{t} = 0$, almost surely. The result follows by letting $t \longrightarrow +\infty$ in (13). □

Remark 2 We emphasize that in the case of non-autonomous epidemic models with Gaussian noise in the disease transmission, the characterization of the disease extinction and persistence in terms of one stochastic threshold has not been done, due to major difficulties caused by the considered type of noise as well as the time varying parameters, prohibiting to define a unified stochastic threshold. Such a characterization can be obtained for the autonomous case (see e.g. [19]).

On the other hand, for the model (1), considered in this paper, the characterization of the disease extinction and persistence is given independently, in terms of the two stochastic parameters $\mathcal{R}_1^s$ and $\mathcal{R}_2^s$. However, for the autonomous counterpart of the model, that is, when the model parameters don't depend on time, following the approach used in Theorem 3, it can be proved that when $\mathcal{R}_2^s < 1$, the infected population goes to extinction. Consequently, $\mathcal{R}_2^s$ can be seen as a stochastic threshold characterizing the disease persistence and extinction, in the stochastic case. Furthermore, in the absence of Gaussain noise, $\mathcal{R}_2^s$ coincides with the basic reproduction number corresponding to the deterministic counterpart of the model.

4 Existence of a non-trivial periodic solution

In this section, we investigate the condition under which the system (2) admits a non-trivial periodic solution. From the biological point of view, the existence of such a solution means that the susceptible, vaccinated, infected and recovered populations are persistent. Meaning that their corresponding densities remain strictly positive throughout time. Hence, for diseases with seasonal characteristics, by analyzing the existence of such solutions, one can obtain additional conditions under which, the disease per-

sists within the studied population. In order to achieve the main result of this section, we recall the definition of a periodic stochastic process.

Definition 1 (See [33]) A stochastic process $(\eta(t))_{t\in\mathbb{R}}$ is said to be periodic with period ν if for every finite sequence of numbers $t_1, t_2, \ldots, t_n$, the joint distribution of random variables $\eta(t_1+h), \eta(t_2+h), \cdots, \eta(t_n+h)$, is independent of h, where $h := k\nu$ $(k = \pm 1, \pm 2, \cdots)$.

Lemma 1 *(See [33]) Let $(X(t))_{t\geq t_0}$ be an l-dimensional stochastic process, consider the following system $dX(t) = b(t, X(t))dt + \sigma(t, X(t))dB(t)$, such that the corresponding coefficients are ν-periodic in t and satisfy the local Lipschitz condition with respect to X. If there exists a function $V \in C^{1,2}((0,+\infty)\times\mathbb{R}^l, \mathbb{R})$ such that*

1. *V is ν-periodic with respect to $t \in (0,+\infty)$.*
2. *$\inf\limits_{|x|>R} V(t,x) \longrightarrow +\infty$ as $R \longrightarrow +\infty$ $\forall t \in (0,+\infty)$.*
3. *$\mathcal{L}V(t,x) \leq -1$ outside some compact set.*

Then, there exists a solution of the above system, which is a ν-periodic Markov process.

Theorem 4 *Suppose that $(\theta_i)_{i=1}^{10}$ and $(\sigma)_{i=1}^{3}$ are periodic functions and denote by $\nu > 0$ their corresponding period. Moreover, let u be the solution of the system (2) with the initial value $u_0 \in \mathcal{I}$. Define the following parameters*

$$\mathscr{R}_1 := \frac{\left(\langle(\Lambda\beta_S)^{\frac{1}{2}}\rangle_\nu\right)^2}{\left(\langle\mu+\gamma\rangle_\nu + \frac{1}{2}\left(\langle\sigma_1^2+\sigma_2^2+\sigma_3^2\rangle_\nu\right)\frac{\overline{\Lambda}^2}{\underline{\mu}^2}\right)\left(\langle\mu+\kappa_1\rangle_\nu + \frac{1}{2}\langle\sigma_1^2\rangle_\nu\frac{\overline{\Lambda}^2}{\underline{\mu}^2}\right)},$$

$$\mathscr{R}_2 := \frac{\left(\langle\left(\Lambda\kappa_1\beta_{V_1}\right)^{\frac{1}{3}}\rangle_\nu\right)^3}{\left(\langle\mu+\gamma\rangle_\nu + \frac{1}{2}\left(\langle\sigma_1^2+\sigma_2^2+\sigma_3^2\rangle_\nu\right)\frac{\overline{\Lambda}^2}{\underline{\mu}^2}\right)\left(\langle\mu+\kappa_2\rangle_\nu + \frac{1}{2}\langle\sigma_2^2\rangle_\nu\frac{\overline{\Lambda}^2}{\underline{\mu}^2}\right)} \times \frac{1}{\left(\langle\mu+\kappa_1\rangle_\nu + \frac{1}{2}\langle\sigma_1^2\rangle_\nu\frac{\overline{\Lambda}^2}{\underline{\mu}^2}\right)},$$

and

$$\mathscr{R}_3 := \frac{\left(\langle\left(\Lambda\kappa_1\kappa_2\beta_{V_2}\right)^{\frac{1}{4}}\rangle_\nu\right)^4}{\left(\langle\mu+\gamma\rangle_\nu + \frac{1}{2}\left(\langle\sigma_1^2+\sigma_2^2+\sigma_3^2\rangle_\nu\right)\frac{\overline{\Lambda}^2}{\underline{\mu}^2}\right)\left(\langle\mu+\kappa_3\rangle_\nu + \frac{1}{2}\langle\sigma_3^2\rangle_\nu\frac{\overline{\Lambda}^2}{\underline{\mu}^2}\right)} \times \frac{1}{\left(\langle\mu+\kappa_2\rangle_\nu + \frac{1}{2}\langle\sigma_2^2\rangle_\nu\frac{\overline{\Lambda}^2}{\underline{\mu}^2}\right)}.$$

Set $\mathscr{R} := \mathscr{R}_1 + \mathscr{R}_2 + \mathscr{R}_3$. *If the following condition*

$$\mathscr{R} > 1, \tag{14}$$

is satisfied, then the stochastic system (2) *admits a* ν*-periodic solution.*

Proof We consider the following function $\mathbf{V} : (0, +\infty) \times U \longrightarrow \mathbb{R}$ defined by

$$\begin{aligned}\mathbf{V}(t,u) := M\Big(&-(b_1+b_2+b_3)\ \ln\left(\frac{\mu}{\overline{\overline{\Lambda}}}S\right) - (b_4+b_5)\ln\left(\frac{\mu}{\overline{\overline{\Lambda}}}V_1\right) - b_6\ln\left(\frac{\mu}{\overline{\overline{\Lambda}}}V_2\right)\\ &- \ln\left(\frac{\mu}{\overline{\overline{\Lambda}}}I\right) + \omega(t)\Big) + S + V_1 + V_2 + V_3 + I + R - \ln\left(\frac{\mu}{\overline{\overline{\Lambda}}}S\right) - \ln\left(\frac{\mu}{\overline{\overline{\Lambda}}}V_1\right)\\ &- \ln\left(\frac{\mu}{\overline{\overline{\Lambda}}}V_2\right) - \ln\left(\frac{\mu}{\overline{\overline{\Lambda}}}V_3\right) - \ln\left(\frac{\mu}{\overline{\overline{\Lambda}}}R\right),\end{aligned}$$

such that

$$b_1 := \frac{\left(\langle(\Lambda\beta_S)^{\frac{1}{2}}\rangle_\nu\right)^2}{\left(\langle\mu+\kappa_1\rangle_\nu + \frac{1}{2}\langle\sigma_1^2\rangle_\nu\frac{\overline{\Lambda}^2}{\underline{\mu^2}}\right)^2},$$

$$b_2 := \frac{\left(\langle(\Lambda\kappa_1\beta_{V_1})^{\frac{1}{3}}\rangle_\nu\right)^3}{\left(\langle\mu+\kappa_2\rangle_\nu + \frac{1}{2}\langle\sigma_2^2\rangle_\nu\frac{\overline{\Lambda}^2}{\underline{\mu^2}}\right)\left(\langle\mu+\kappa_1\rangle_\nu + \frac{1}{2}\langle\sigma_1^2\rangle_\nu\frac{\overline{\Lambda}^2}{\underline{\mu^2}}\right)^2},$$

$$b_3 := \frac{\left(\langle(\Lambda\kappa_1\kappa_2\beta_{V_2})^{\frac{1}{4}}\rangle_\nu\right)^4}{\left(\langle\mu+\kappa_3\rangle_\nu + \frac{1}{2}\langle\sigma_3^2\rangle_\nu\frac{\overline{\Lambda}^2}{\underline{\mu^2}}\right)\left(\langle\mu+\kappa_2\rangle_\nu + \frac{1}{2}\langle\sigma_2^2\rangle_\nu\frac{\overline{\Lambda}^2}{\underline{\mu^2}}\right)\left(\langle\mu+\kappa_1\rangle_\nu + \frac{1}{2}\langle\sigma_1^2\rangle_\nu\frac{\overline{\Lambda}^2}{\underline{\mu^2}}\right)^2},$$

$$b_4 := \frac{\left(\langle(\Lambda\kappa_1\beta_{V_1})^{\frac{1}{3}}\rangle_\nu\right)^3}{\left(\langle\mu+\kappa_2\rangle_\nu + \frac{1}{2}\langle\sigma_2^2\rangle_\nu\frac{\overline{\Lambda}^2}{\underline{\mu^2}}\right)^2\left(\langle\mu+\kappa_1\rangle_\nu + \frac{1}{2}\langle\sigma_1^2\rangle_\nu\frac{\overline{\Lambda}^2}{\underline{\mu^2}}\right)},$$

$$b_5 := \frac{\left(\langle(\Lambda\kappa_1\kappa_2\beta_{V_2})^{\frac{1}{4}}\rangle_\nu\right)^4}{\left(\langle\mu+\kappa_3\rangle_\nu + \frac{1}{2}\langle\sigma_3^2\rangle_\nu\frac{\overline{\Lambda}^2}{\underline{\mu^2}}\right)\left(\langle\mu+\kappa_2\rangle_\nu + \frac{1}{2}\langle\sigma_2^2\rangle_\nu\frac{\overline{\Lambda}^2}{\underline{\mu^2}}\right)^2\left(\langle\mu+\kappa_1\rangle_\nu + \frac{1}{2}\langle\sigma_1^2\rangle_\nu\frac{\overline{\Lambda}^2}{\underline{\mu^2}}\right)},$$

and

$$b_6 := \frac{\left(\langle(\Lambda\kappa_1\kappa_2\beta_{V_2})^{\frac{1}{4}}\rangle_\nu\right)^4}{\left(\langle\mu+\kappa_3\rangle_\nu + \frac{1}{2}\langle\sigma_3^2\rangle_\nu\frac{\overline{\Lambda}^2}{\underline{\mu^2}}\right)^2\left(\langle\mu+\kappa_2\rangle_\nu + \frac{1}{2}\langle\sigma_2^2\rangle_\nu\frac{\overline{\Lambda}^2}{\underline{\mu^2}}\right)\left(\langle\mu+\kappa_1\rangle_\nu + \frac{1}{2}\langle\sigma_1^2\rangle_\nu\frac{\overline{\Lambda}^2}{\underline{\mu^2}}\right)},$$

while $M \in \mathbb{R}_+^*$ and $\omega : (0, T) \longrightarrow \mathbb{R}$ is a periodic function, which will both be chosen thereafter accordingly. For simplicity, set

$$\mathbf{V}_1(u) := -(b_1 + b_2 + b_3)\ \ln\left(\frac{\mu}{\overline{\overline{\Lambda}}} S\right) - (b_4 + b_5) \ln\left(\frac{\mu}{\overline{\overline{\Lambda}}} V_1\right) - b_6 \ln\left(\frac{\mu}{\overline{\overline{\Lambda}}} V_2\right) - \ln\left(\frac{\mu}{\overline{\overline{\Lambda}}} I\right).$$

By applying Itô's formula and rearranging the terms, we obtain

$$\begin{aligned}
\mathcal{L}(\mathbf{V}_1(u)) = &-b_1 \frac{\Lambda}{S} - \beta_S S + b_1 \left(\mu + \kappa_1 + \frac{1}{2}\sigma_1^2 I^2\right) - b_2 \frac{\Lambda}{S} - \beta_{V_1} V_1 - b_4 \kappa_1 \frac{S}{V_1} \\
&+ b_2 \left(\mu + \kappa_1 + \frac{1}{2}\sigma_1^2 I^2\right) + b_4 \left(\mu + \kappa_2 + \frac{1}{2}\sigma_2^2 I^2\right) - b_3 \frac{\Lambda}{S} - b_5 \kappa_1 \frac{S}{V_1} - b_6 \kappa_2 \frac{V_1}{V_2} - \beta_{V_2} V_2 \\
&+ b_3 \left(\mu + \kappa_1 + \frac{1}{2}\sigma_1 I^2\right) + b_5 \left(\mu + \kappa_2 + \frac{1}{2}\sigma_2^2 I^2\right) + b_6 \left(\mu + \kappa_3 + \frac{1}{2}\sigma_3^2 I^2\right) \\
&+ \left((b_1 + b_2 + b_3)\, \beta_S + (b_4 + b_5)\, \beta_{V_1} + b_6 \beta_{V_2}\right) I + \mu + \gamma + \frac{1}{2}\left(\sigma_1^2 S^2 + \sigma_2^2 V_1^2 + \sigma_3^2 V_2^2\right).
\end{aligned}$$

Owing to the inequality of arithmetic and geometric means, we acquire that

$$\begin{aligned}
\mathcal{L}(\mathbf{V}_1(u)) \le &-2\,(\Lambda \beta_S b_1)^{\frac{1}{2}} + b_1 \left(\mu + \kappa_1 + \frac{1}{2}\sigma_1^2 \frac{\overline{\Lambda}^2}{\underline{\mu}^2}\right) - 3\left(\Lambda \kappa_1 \beta_{V_1} b_2 b_4\right)^{\frac{1}{3}} \\
&+ b_2 \left(\mu + \kappa_1 + \frac{1}{2}\sigma_1^2 \frac{\overline{\Lambda}^2}{\underline{\mu}^2}\right) + b_4 \left(\mu + \kappa_2 + \frac{1}{2}\sigma_2^2 \frac{\overline{\Lambda}^2}{\underline{\mu}^2}\right) - 4\left(\Lambda \kappa_1 \kappa_2 \beta_{V_2} b_3 b_5 b_6\right)^{\frac{1}{4}} \\
&+ b_3 \left(\mu + \kappa_1 + \frac{1}{2}\sigma_1^2 \frac{\overline{\Lambda}^2}{\underline{\mu}^2}\right) + b_6 \left(\mu + \kappa_3 + \frac{1}{2}\sigma_3^2 \frac{\overline{\Lambda}^2}{\underline{\mu}^2}\right) + b_5 \left(\mu + \kappa_2 + \frac{1}{2}\sigma_2^2 I^2\right) \\
&+ \left((b_1 + b_2 + b_3)\, \beta_S + (b_4 + b_5)\, \beta_{V_1} + b_6 \beta_{V_2}\right) I + \mu + \gamma + \frac{1}{2}\frac{\overline{\Lambda}^2}{\underline{\mu}^2}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right).
\end{aligned}$$

For $t \in (0, T)$, set

$$\begin{aligned}
\zeta(t) := &-2\,(\Lambda(t) \beta_S(t) b_1)^{\frac{1}{2}} + b_1 \left(\mu(t) + \kappa_1(t) + \frac{1}{2}\sigma_1^2(t) \frac{\overline{\Lambda}^2}{\underline{\mu}^2}\right) \\
&- 3\left(\Lambda(t) \kappa_1(t) \beta_{V_1}(t) b_2 b_4\right)^{\frac{1}{3}} \\
&+ b_2 \left(\mu(t) + \kappa_1(t) + \frac{1}{2}\sigma_1^2(t) \frac{\overline{\Lambda}^2}{\underline{\mu}^2}\right) + b_4 \left(\mu(t) + \kappa_2(t) + \frac{1}{2}\sigma_2^2(t) \frac{\overline{\Lambda}^2}{\underline{\mu}^2}\right) \\
&+ b_5 \left(\mu(t) + \kappa_2(t) + \frac{1}{2}\sigma_2^2(t) \frac{\overline{\Lambda}^2}{\underline{\mu}^2}\right) - 4\left(\Lambda(t) \kappa_1(t) \kappa_2(t) \beta_{V_2}(t) b_3 b_5 b_6\right)^{\frac{1}{4}} \\
&+ b_3 \left(\mu(t) + \kappa_1(t) + \frac{1}{2}\sigma_1^2(t) \frac{\overline{\Lambda}^2}{\underline{\mu}^2}\right) + b_6 \left(\mu(t) + \kappa_3(t) + \frac{1}{2}\sigma_3^2(t) \frac{\overline{\Lambda}^2}{\underline{\mu}^2}\right) \\
&+ \mu(t) + \gamma(t) + \frac{1}{2}\frac{\overline{\Lambda}^2}{\underline{\mu}^2}\left(\sigma_1^2(t) + \sigma_2^2(t) + \sigma_3^2(t)\right),
\end{aligned}$$

and

$$\xi := (b_1 + b_2 + b_3)\,\overline{\beta_S} + (b_4 + b_5)\,\overline{\beta_{V_1}} + b_6\overline{\beta_{V_2}},$$

so that

$$\mathcal{L}(\mathbf{V}_1(u(t))) \leq \zeta(t) + \xi I.$$

Let ω satisfy

$$\begin{cases} \omega'(t) = \langle\zeta\rangle_\nu - \zeta(t), \\ \omega(0) = 0. \end{cases}$$

Then, clearly the ν-periodicity of ζ implies that of ω. Indeed, taking into account that

$$\omega(\nu) = \nu\langle\zeta\rangle_\nu - \int_0^\nu \zeta(s)ds = \nu\left(\langle\zeta\rangle_\nu - \frac{1}{\nu}\int_0^\nu \zeta(s)ds\right) = \nu(\langle\zeta\rangle_\nu - \langle\zeta\rangle_\nu) = 0,$$

we obtain

$$\begin{aligned} \omega(t+\nu) &= \omega(\nu) + \int_\nu^{t+\nu} \langle\zeta\rangle_\nu - \zeta(s)ds = \int_0^t \langle\zeta\rangle_\nu - \zeta(u+\nu)du \\ &= \int_0^t \langle\zeta\rangle_\nu - \zeta(u)du = \omega(t). \end{aligned}$$

Thus

$$\mathcal{L}(\mathbf{V}_1(u) + \omega(t)) \leq \langle\zeta\rangle_\nu + \xi I.$$

Since

$$-2\langle(\Lambda\beta_S b_1)^{\frac{1}{2}}\rangle_\nu = -\frac{2\left(\langle(\Lambda\beta_S)^{\frac{1}{2}}\rangle_\nu\right)^2}{\left(\langle\mu+\kappa_1\rangle_\nu + \frac{1}{2}\langle\sigma_1^2\rangle_\nu\frac{\overline{\Lambda}^2}{\underline{\mu^2}}\right)},$$

$$-3\langle\left(\Lambda\kappa_1\beta_{V_1}b_2b_4\right)^{\frac{1}{3}}\rangle_\nu = -\frac{3\left(\langle\left(\Lambda\kappa_1\beta_{V_1}\right)^{\frac{1}{3}}\rangle_\nu\right)^3}{\left(\langle\mu\rangle_\nu + \frac{1}{2}\langle\sigma_2^2\rangle_\nu\frac{\overline{\Lambda}^2}{\underline{\mu^2}}\right)\left(\langle\mu+\kappa_1\rangle_\nu + \frac{1}{2}\langle\sigma_1^2\rangle_\nu\frac{\overline{\Lambda}^2}{\underline{\mu^2}}\right)},$$

and

$$
\begin{aligned}
&-4\langle\left(\Lambda\kappa_1\kappa_2\beta_{V_2}b_3b_5b_6\right)^{\frac{1}{4}}\rangle_\nu \\
&= -\frac{4\left(\langle\left(\Lambda\kappa_1\kappa_2\beta_{V_2}\right)^{\frac{1}{4}}\rangle_\nu\right)^4}{\left(\langle\mu+\kappa_3\rangle_\nu+\frac{1}{2}\langle\sigma_3^2\rangle_\nu\frac{\overline{\Lambda}^2}{\underline{\mu}^2}\right)\left(\langle\mu+\kappa_2\rangle_\nu+\frac{1}{2}\langle\sigma_2^2\rangle_\nu\frac{\overline{\Lambda}^2}{\underline{\mu}^2}\right)} \\
&\times\frac{1}{\left(\langle\mu+\kappa_1\rangle_\nu+\frac{1}{2}\langle\sigma_1^2\rangle_\nu\frac{\overline{\Lambda}^2}{\underline{\mu}^2}\right)},
\end{aligned}
$$

it follows that

$$
\mathcal{L}(\mathbf{V}_1(u)+\omega(t))\le-\left(\langle\mu\rangle_\nu+\langle\gamma\rangle_\nu+\frac{1}{2}\frac{\overline{\Lambda}}{\underline{\mu}}\left(\langle\sigma_1^2+\sigma_2^2+\sigma_3^2\rangle_\nu\right)\right)(\mathscr{R}-1)+\xi I.
$$

On the other hand, by Itô's formula, one can obtain

$$
\begin{aligned}
&\mathcal{L}\left(-\ln\left(\frac{\underline{\mu}}{\overline{\Lambda}}S\right)\right)\le-\frac{\underline{\Lambda}}{S}+\overline{\beta_S}I+\overline{\mu}+\overline{\kappa_1}+\frac{1}{2}\overline{\sigma_1}^2\frac{\overline{\Lambda}^2}{\underline{\mu}^2}, \\
&\mathcal{L}\left(-\ln\left(\frac{\underline{\mu}}{\overline{\Lambda}}V_1\right)\right)\le\overline{\beta_{V_1}}I-\underline{\kappa_1}\frac{S}{V_1}+\overline{\mu}+\overline{\kappa_2}+\frac{1}{2}\overline{\sigma_2}^2\frac{\overline{\Lambda}^2}{\underline{\mu}^2}, \\
&\mathcal{L}\left(-\ln\left(\frac{\underline{\mu}}{\overline{\Lambda}}V_2\right)\right)\le\overline{\beta_{V_2}}I-\underline{\kappa_2}\frac{V_1}{V_2}+\overline{\mu}+\overline{\kappa_3}+\frac{1}{2}\overline{\sigma_3}^2\frac{\overline{\Lambda}^2}{\underline{\mu}^2}, \\
&\mathcal{L}\left(-\ln\left(\frac{\underline{\mu}}{\overline{\Lambda}}V_3\right)\right)\le-\underline{\kappa_3}\frac{V_2}{V_3}+\overline{\mu}+\overline{\gamma_{V_3}}, \\
&\mathcal{L}\left(-\ln\left(\frac{\underline{\mu}}{\overline{\Lambda}}R\right)\right)\le\overline{\mu}-\underline{\gamma_{V_3}}\frac{V_3}{R},
\end{aligned}
$$

and $\mathcal{L}\left(S+V_1+V_2+V_3+I+R\right)\le\overline{\Lambda}-\underline{\mu}\left(S+V_1+V_2+V_3+I+R\right)$. Thereby,

$$
\begin{aligned}
\mathcal{L}(\mathbf{V}(t,u))\le{}& M\left(-\left(\langle\mu\rangle_\nu+\langle\gamma\rangle_\nu+\frac{1}{2}\frac{\overline{\Lambda}}{\underline{\mu}}\left(\sigma_1^2+\sigma_2^2+\sigma_3^2\right)\right)(\mathscr{R}-1)+\xi I\right) \\
&+\overline{\Lambda}-\underline{\mu}\left(S+V_1+V_2+V_3+I+R\right)-\frac{\underline{\Lambda}}{S}+\left(\overline{\beta_S}+\overline{\beta_{V_1}}+\overline{\beta_{V_2}}\right)I+\overline{\kappa_1} \\
&+\overline{\kappa_2}+\overline{\kappa_3}-\underline{\kappa_1}\frac{S}{V_1}-\underline{\kappa_2}\frac{V_1}{V_2}-\underline{\kappa_3}\frac{V_2}{V_3}+5\overline{\mu}+\overline{\gamma_{V_3}}-\underline{\gamma_{V_3}}\frac{V_3}{R}+\frac{1}{2}\frac{\overline{\Lambda}^2}{\underline{\mu}^2}\left(\overline{\sigma_1}^2+\overline{\sigma_2}^2+\overline{\sigma_3}^2\right).
\end{aligned}
$$

Now, consider the following compact set

$$K := \left\{ u \in U, \quad \epsilon_i \leq u_i \leq \frac{1}{\epsilon_i} \quad \forall i \in \{1, \cdots, 6\} \right\},$$

such that $\epsilon_1, \cdots, \epsilon_6 > 0$ will be chosen later.

Let $u \in U \setminus K$. To verify that $\mathcal{L}(\mathbf{V}(t,.)) \leq -1$ in $U \setminus K$, it suffices to investigate the following distinguished seven cases

Case 1 : $u \in \{u \in U, \quad I < \epsilon_5\}$.
Case 2 : $u \in \{u \in U, \quad I \geq \epsilon_5, \ S < \epsilon_1\}$.
Case 3 : $u \in \{u \in U, \quad I \geq \epsilon_5, \ S \geq \epsilon_1, V_1 < \epsilon_2\}$.
Case 4 : $u \in \{u \in U, \quad I \geq \epsilon_5, \ S \geq \epsilon_1, \ V_1 \geq \epsilon_2, \ V_2 < \epsilon_3\}$.
Case 5 : $u \in \{u \in U, \quad I \geq \epsilon_5, \ S \geq \epsilon_1, \ V_1 \geq \epsilon_2, \ V_2 \geq \epsilon_3, \ V_3 < \epsilon_4\}$.
Case 6 : $u \in \{u \in U, \quad I \geq \epsilon_5, \ S \geq \epsilon_1, \ V_1 \geq \epsilon_2, \ V_2 \geq \epsilon_3, \ V_3 \geq \epsilon_4 \ R < \epsilon_6\}$.
Case 7 : $u \in \left\{ u \in U, \quad \exists j \in \{1, \cdots, 6\}, \quad u_j > \frac{1}{\epsilon_j} \right\}$.

For case 1, we obtain

$$\begin{aligned}
\mathcal{L}(\mathbf{V}(t,u)) \leq M &\left(- \left(\langle \mu \rangle_\nu + \langle \gamma \rangle_\nu + \frac{1}{2} \frac{\overline{\Lambda}}{\underline{\mu}} \langle \sigma_1^2 + \sigma_2^2 + \sigma_3^2 \rangle_\nu \right) (\mathscr{R} - 1) + \xi \epsilon_5 \right) \\
&+ \overline{\Lambda} + \left(\overline{\beta_S} + \overline{\beta_{V_1}} + \overline{\beta_{V_2}} \right) \epsilon_5 + \overline{\kappa_1} + \overline{\kappa_2} + \overline{\kappa_3} \\
&+ 5\overline{\mu} + \overline{\gamma_{V_3}} + \frac{1}{2} \frac{\overline{\Lambda}^2}{\underline{\mu}^2} \left(\overline{\sigma_1}^2 + \overline{\sigma_2}^2 + \overline{\sigma_3}^2 \right).
\end{aligned}$$

For case 2, we obtain

$$\begin{aligned}
\mathcal{L}(\mathbf{V}(t,u)) \leq M &\left(- \left(\langle \mu \rangle_\nu + \langle \gamma \rangle_\nu + \frac{1}{2} \frac{\overline{\Lambda}}{\underline{\mu}} \langle \sigma_1^2 + \sigma_2^2 + \sigma_3^2 \rangle_\nu \right) (\mathscr{R} - 1) + \xi \frac{\overline{\Lambda}}{\underline{\mu}} \right) \\
&+ \overline{\Lambda} - \frac{\underline{\Lambda}}{\epsilon_1} + \left(\overline{\beta_S} + \overline{\beta_{V_1}} + \overline{\beta_{V_2}} \right) \frac{\overline{\Lambda}}{\underline{\mu}} + \overline{\kappa_1} + \overline{\kappa_2} + \overline{\kappa_3} + 5\overline{\mu} + \overline{\gamma_{V_3}} \\
&+ \frac{1}{2} \frac{\overline{\Lambda}^2}{\underline{\mu}^2} \left(\overline{\sigma_1}^2 + \overline{\sigma_2}^2 + \overline{\sigma_3}^2 \right).
\end{aligned}$$

For case 3, we obtain

$$\begin{aligned}
\mathcal{L}(\mathbf{V}(t,u)) \leq M &\left(- \left(\langle \mu \rangle_\nu + \langle \gamma \rangle_\nu + \frac{1}{2} \frac{\overline{\Lambda}}{\underline{\mu}} \langle \sigma_1^2 + \sigma_2^2 + \sigma_3^2 \rangle_\nu \right) (\mathscr{R} - 1) + \xi \frac{\overline{\Lambda}}{\underline{\mu}} \right) \\
&+ \overline{\Lambda} + \left(\overline{\beta_S} + \overline{\beta_{V_1}} + \overline{\beta_{V_2}} \right) \frac{\overline{\Lambda}}{\underline{\mu}} + \overline{\kappa_1} + \overline{\kappa_2} + \overline{\kappa_3} - \underline{\kappa_1} \frac{\epsilon_1}{\epsilon_2} + 5\overline{\mu} + \overline{\gamma_{V_3}} \\
&+ \frac{1}{2} \frac{\overline{\Lambda}^2}{\underline{\mu}^2} \left(\overline{\sigma_1}^2 + \overline{\sigma_2}^2 + \overline{\sigma_3}^2 \right).
\end{aligned}$$

For case 4, we obtain

$$\begin{aligned}\mathcal{L}(\mathbf{V}(t,u)) \leq M &\left(-\left(\langle\mu\rangle_\nu + \langle\gamma\rangle_\nu + \frac{1}{2}\frac{\overline{\Lambda}}{\underline{\mu}}\langle\sigma_1^2+\sigma_2^2+\sigma_3^2\rangle_\nu\right)(\mathscr{R}-1) + \xi\frac{\overline{\Lambda}}{\underline{\mu}}\right)\\ &+ \overline{\Lambda} + \left(\overline{\beta_S} + \overline{\beta_{V_1}} + \overline{\beta_{V_2}}\right)\frac{\overline{\Lambda}}{\underline{\mu}} + \overline{\kappa_1} + \overline{\kappa_2} + \overline{\kappa_3} - \underline{\kappa_2}\frac{\epsilon_2}{\epsilon_3} + 5\overline{\mu} + \overline{\gamma_{V_3}}\\ &+ \frac{1}{2}\frac{\overline{\Lambda}^2}{\underline{\mu}^2}\left(\overline{\sigma_1}^2 + \overline{\sigma_2}^2 + \overline{\sigma_3}^2\right).\end{aligned}$$

For case 5, we obtain

$$\begin{aligned}\mathcal{L}(\mathbf{V}(t,u)) \leq M &\left(-\left(\langle\mu\rangle_\nu + \langle\gamma\rangle_\nu + \frac{1}{2}\frac{\overline{\Lambda}}{\underline{\mu}}\langle\sigma_1^2+\sigma_2^2+\sigma_3^2\rangle_\nu\right)(\mathscr{R}-1) + \xi\frac{\overline{\Lambda}}{\underline{\mu}}\right)\\ &+ \overline{\Lambda} + \left(\overline{\beta_S} + \overline{\beta_{V_1}} + \overline{\beta_{V_2}}\right)\frac{\overline{\Lambda}}{\underline{\mu}} + \overline{\kappa_1} + \overline{\kappa_2} + \overline{\kappa_3} - \underline{\kappa_3}\frac{\epsilon_3}{\epsilon_4} + 5\overline{\mu} + \overline{\gamma_{V_3}}\\ &+ \frac{1}{2}\frac{\overline{\Lambda}^2}{\underline{\mu}^2}\left(\overline{\sigma_1}^2 + \overline{\sigma_2}^2 + \overline{\sigma_3}^2\right).\end{aligned}$$

For case 6, we obtain

$$\begin{aligned}\mathcal{L}(\mathbf{V}(t,u)) \leq M &\left(-\left(\langle\mu\rangle_\nu + \langle\gamma\rangle_\nu + \frac{1}{2}\frac{\overline{\Lambda}}{\underline{\mu}}\langle\sigma_1^2+\sigma_2^2+\sigma_3^2\rangle_\nu\right)(\mathscr{R}-1) + \xi\frac{\overline{\Lambda}}{\underline{\mu}}\right)\\ &+ \overline{\Lambda} + \left(\overline{\beta_S} + \overline{\beta_{V_1}} + \overline{\beta_{V_2}}\right)\frac{\overline{\Lambda}}{\underline{\mu}} + \overline{\kappa_1} + \overline{\kappa_2} + \overline{\kappa_3} - \underline{\gamma_{V_3}}\frac{\epsilon_4}{\epsilon_6} + 5\overline{\mu} + \overline{\gamma_{V_3}}\\ &+ \frac{1}{2}\frac{\overline{\Lambda}^2}{\underline{\mu}^2}\left(\overline{\sigma_1}^2 + \overline{\sigma_2}^2 + \overline{\sigma_3}^2\right).\end{aligned}$$

For case 7, we obtain

$$\begin{aligned}\mathcal{L}(\mathbf{V}(t,u)) \leq M &\left(-\left(\langle\mu\rangle_\nu + \langle\gamma\rangle_\nu + \frac{1}{2}\frac{\overline{\Lambda}}{\underline{\mu}}\langle\sigma_1^2+\sigma_2^2+\sigma_3^2\rangle_\nu\right)(\mathscr{R}-1) + \xi\frac{\overline{\Lambda}}{\underline{\mu}}\right)\\ &+ \overline{\Lambda} + \left(\overline{\beta_S} + \overline{\beta_{V_1}} + \overline{\beta_{V_2}}\right)\frac{\overline{\Lambda}}{\underline{\mu}} + \overline{\kappa_1} + \overline{\kappa_2} + \overline{\kappa_3} - \underline{\mu}\frac{1}{\epsilon_j} + 5\overline{\mu} + \overline{\gamma_{V_3}}\\ &+ \frac{1}{2}\frac{\overline{\Lambda}^2}{\underline{\mu}^2}\left(\overline{\sigma_1}^2 + \overline{\sigma_2}^2 + \overline{\sigma_3}^2\right).\end{aligned}$$

Now, set

$$M \geq \frac{2+\overline{\Lambda}+\left(\overline{\beta_S}+\overline{\beta_{V_1}}+\overline{\beta_{V_2}}\right)\dfrac{\overline{\Lambda}}{\underline{\mu}}+\overline{\kappa_1}+\overline{\kappa_2}+\overline{\kappa_3}+5\overline{\mu}+\overline{\gamma_{V_3}}+\dfrac{1}{2}\dfrac{\overline{\Lambda}^2}{\underline{\mu}^2}\left(\overline{\sigma_1}^2+\overline{\sigma_2}^2+\overline{\sigma_3}^2\right)}{\left(\langle\mu\rangle_\nu+\langle\gamma\rangle_\nu+\dfrac{1}{2}\dfrac{\overline{\Lambda}}{\underline{\mu}}\langle\sigma_1^2+\sigma_2^2+\sigma_3^2\rangle_\nu\right)(\mathscr{R}-1)}.$$

Then, for case 1 and $j=5$ in case 7, choose

$$\epsilon_5 := \min\left\{1, \frac{1}{M\xi+\overline{\beta_S}+\overline{\beta_{V_1}}+\overline{\beta_{V_2}}}, \frac{\underline{\mu}^2}{\overline{\Lambda}M\xi}\right\}.$$

For case 2, case 3, case 4, case 5, case 6 and $j \neq 5$ in case 7, choose

$$\epsilon_1 = \min\left\{1, \frac{\underline{\mu}\,\underline{\Lambda}}{M\xi\overline{\Lambda}}, \frac{\underline{\mu}\,\underline{\kappa_1}}{M\xi\overline{\Lambda}}, \frac{\underline{\kappa_2}\,\underline{\mu}}{M\overline{\Lambda}\xi}, \frac{\underline{\kappa_3}\,\underline{\mu}}{M\overline{\Lambda}\xi}, \frac{\underline{\gamma_{V_3}}\,\underline{\mu}}{M\overline{\Lambda}\xi}, \frac{\underline{\mu}^2}{\overline{\lambda}M\xi}, \right.$$
$$\left.\left(\frac{\underline{\mu}^2}{\overline{\lambda}M\xi}\right)^{\frac{1}{2}}, \left(\frac{\underline{\mu}^2}{\overline{\lambda}M\xi}\right)^{\frac{1}{3}}, \left(\frac{\underline{\mu}^2}{\overline{\lambda}M\xi}\right)^{\frac{1}{4}}, \left(\frac{\underline{\mu}^2}{\overline{\lambda}M\xi}\right)^{\frac{1}{5}}\right\},$$

$$\epsilon_2 = \epsilon_1^2,\ \epsilon_3 = \epsilon_1^3,\ \epsilon_4 = \epsilon_1^4 \quad \text{and } \epsilon_6 = \epsilon_1^5.$$

Consequently, $\mathcal{L}(\mathbf{V}(t,.)) \leq -1$ in $U\backslash K$. On the other hand, since $\inf_{|u|>R}\mathbf{V}(t,u) \longrightarrow +\infty$ as $R \longrightarrow +\infty$ $\forall t \in (0,+\infty)$, and taking into consideration the ν-periodicity of $\mathbf{V}$ with respect to t, the assumptions of Lemma 1 are verified. Conclusively, the stochastic system (2) admits a ν-periodic solution. □

Remark 3 For the deterministic autonomous counterpart of the model (1), that is $\sigma_1(t)=\sigma_2(t)=\sigma_3(t)=0$ and $\theta(t)=\theta \in (0,+\infty)^{10}$ $\forall t \in (0,T)$. One can use the Next Generation Method [34] to compute the basic reproduction number $\mathcal{R}_0$, which yields that $\mathcal{R}_0 = \dfrac{\Lambda\beta_S(\mu+\kappa_2)(\mu+\kappa_3)+\Lambda\kappa_1\beta_{V_1}(\mu+\kappa_3)+\Lambda\kappa_1\kappa_2\beta_{V_2}}{(\mu+\kappa_1)(\mu+\kappa_2)(\mu+\kappa_3)(\mu+\gamma)}$. Hence, the value of $\mathcal{R}_0$ coincides with that of $\mathcal{R}_2^s$ and $\mathscr{R}$ stated in Sects. 3 and 4, respectively.

5 Numerical simulations

We consider the time horizon $(0,200)$ and we choose the following initial condition $u_0 = (0.8, 0.1, 0.01, 0.04, 0.03, 0.02)$. We simulate the model (1) numerically by relying on Matlab software [35] to develop a script implementing the Milstein method presented in [36], which was chosen due to its accuracy. The resulting numerical scheme of the model (1) is the same one presented in [37, 38] and hence is omitted

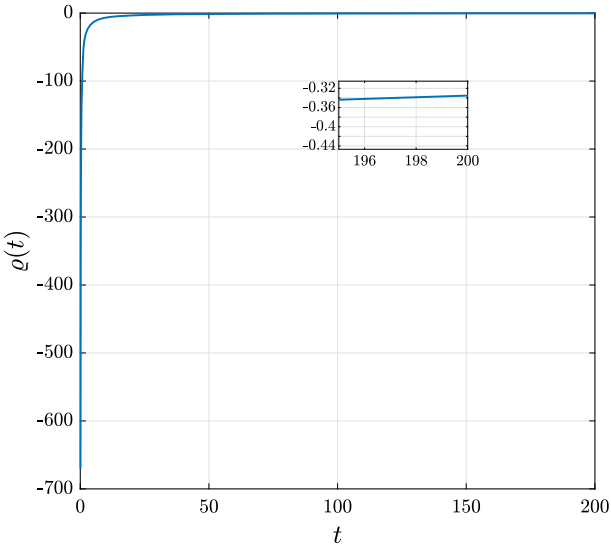

Fig. 2 Verification of the first disease extinction condition

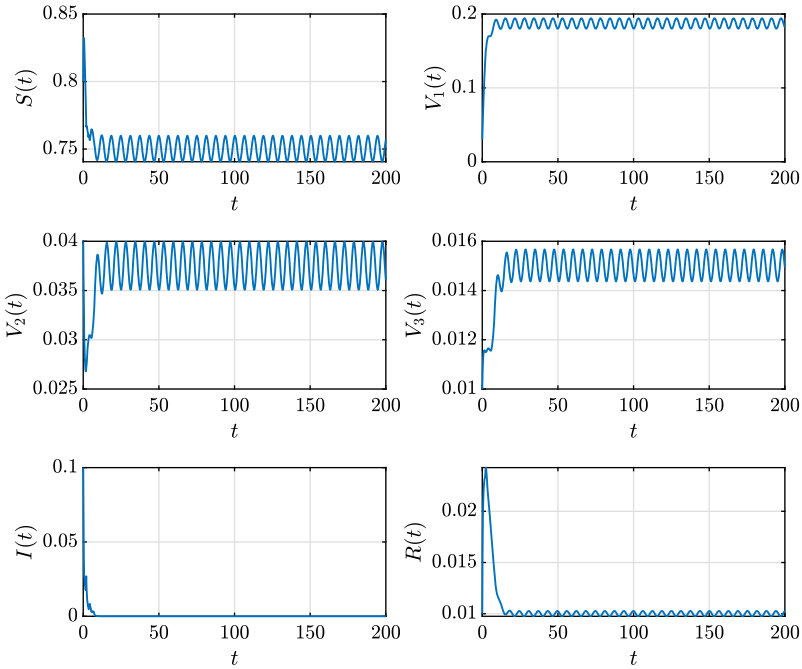

Fig. 3 Paths of S, V_1, V_2, V_3, I and R when the first condition of the disease extinction holds

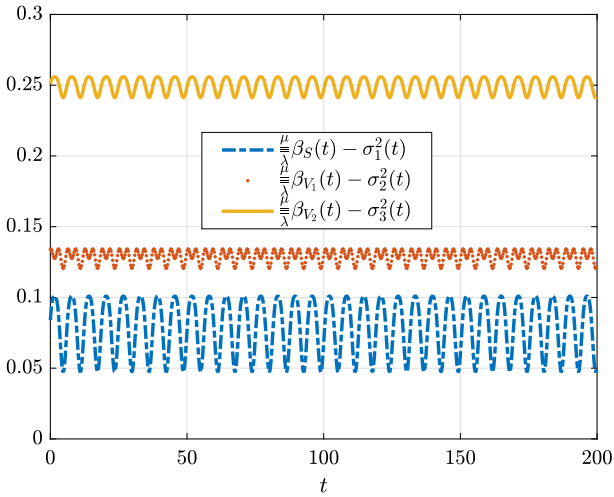

Fig. 4 Verification of the second disease extinction condition

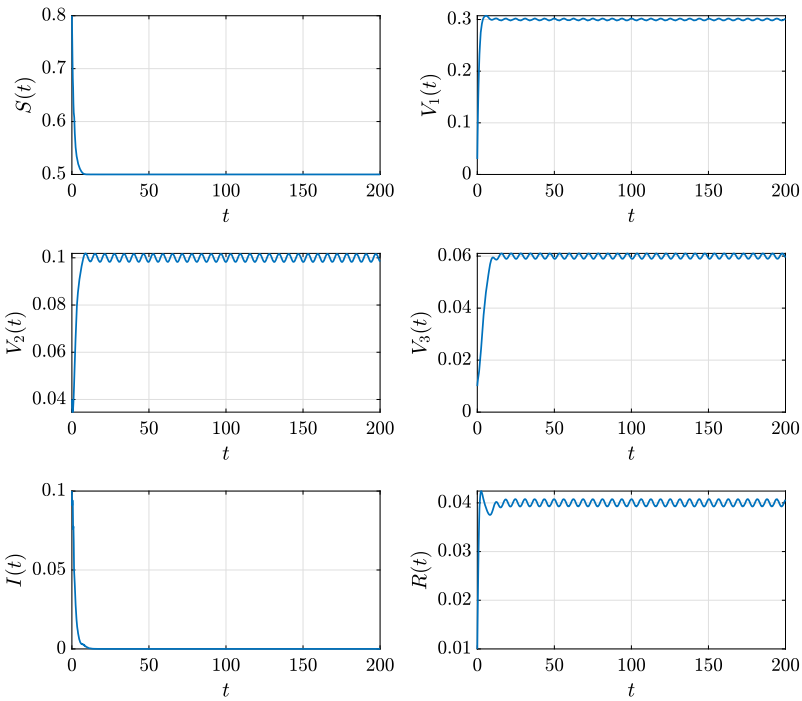

Fig. 5 Paths of S, V_1, V_2, V_3, I and R when the second condition of the disease extinction holds

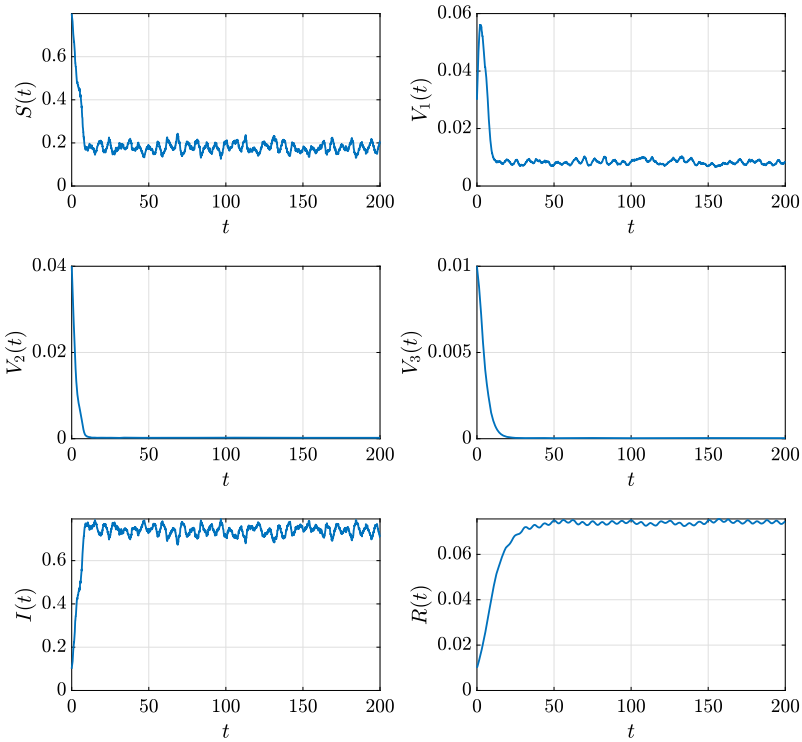

Fig. 6 Paths of S, V_1, V_2, V_3, I and R when the condition of the disease persistence in the mean holds

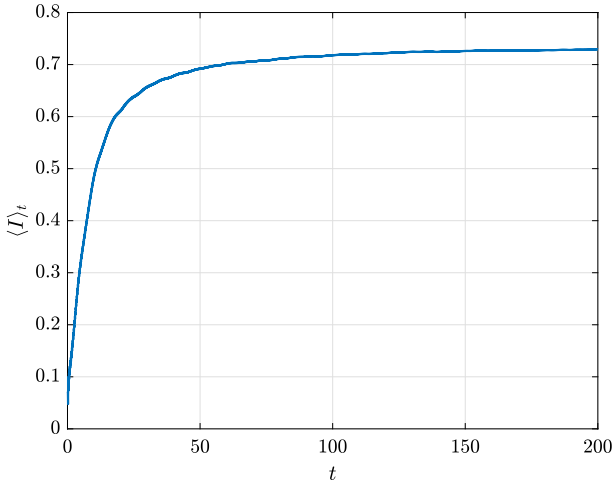

Fig. 7 Persistence in the mean of the infected population

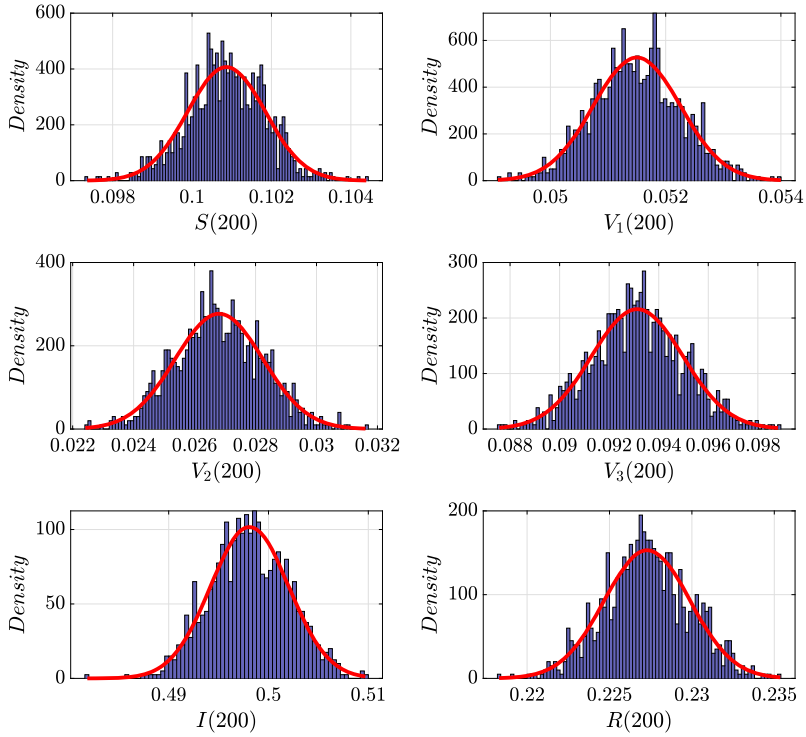

Fig. 8 Probability density functions of S, V_1, V_2, V_3, I and R at time $t = 200$ when the condition of the disease persistence in the mean holds

here for brevity. To support all the established theoretical results, five cases are numerically simulated. In the first and second cases, the parameters are chosen such that the conditions (1) and (2) stated in Theorem 2 are verified, respectively. In the third case, the parameters are chosen such that the condition (9) of Theorem 3 is verified. While the fourth and fifth cases correspond to choices of parameters in which the condition (14) of Theorem 4 is verified in both deterministic and stochastic non autonomous cases. For the first case, Fig. 2 shows that the condition (1) of Theorem 2 is satisfied. For the second case, by using Simpson's method, we have $\langle \mathcal{R}_1^s \rangle_{200} \approx 0.7812 < 1$ (Fig. 3). Moreover, from Fig. 4, it can be deduced that the condition (2) of Theorem 2 is satisfied. The numerical outcomes are shown in Figs. 3 and 5 and exhibit in both cases that the disease goes to extinction. For the third case, by calculation, we have $\mathcal{R}_2^s = 1.2192 > 1$, thereby, the condition (4) of Theorem 3 holds (Fig. 6). Consequently, $\liminf_{t \to +\infty} \langle I \rangle_t \geq 0.0029$, which is illustrated by Fig. 7. Figures 6 and 8 show the obtained solution. For the fourth case, by using Simpson's method, we have $\mathscr{R} \approx 1.4697 > 1$. Similarly, for the fifth case, we have $\mathscr{R} \approx 1.3340 > 1$. Consequently, the condition (14) of Theorem 4 is verified. Figures 9 and 10 illustrate the deterministic and stochastic periodicity of the obtained solution (Fig. 11).

The assigned values to the parameters in each case are as follows

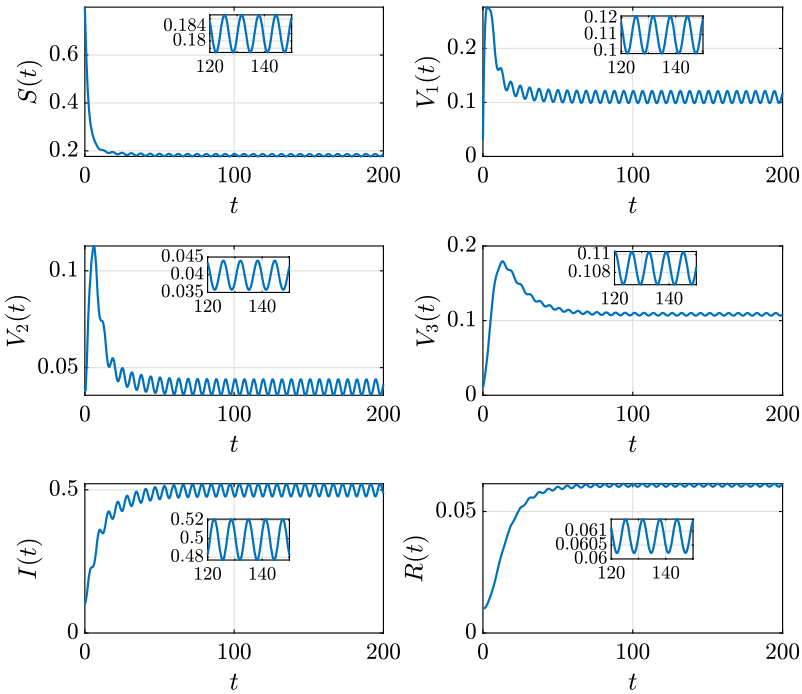

Fig. 9 Paths of S, V_1, V_2, V_3, I and R in the deterministic non-autonomous case and under the condition $\mathscr{R} > 1$

Case 1 : $\forall t \in [0, 200]$,

$$\Lambda(t) = 0.3 + 0.02\sin(t), \quad \beta_S(t) = 0.2 + 0.06\sin(t), \quad \beta_{V_1}(t) = 0.1 + 0.02\sin(t),$$
$$\beta_{V_2}(t) = 0.1 + 0.05\sin(t), \quad \gamma(t) = 0.3 + 0.001\sin(t), \quad \gamma_{V_3}(t) = 0.2 + 0.001\sin(t),$$
$$\kappa_1(t) = 0.1 + 0.02\sin(t), \quad \kappa_2(t) = 0.1 + 0.02\sin(t), \quad \kappa_3(t) = 0.2 + 0.02\sin(t),$$
$$\mu(t) = 0.3 + 0.02\sin(t).$$

Case 2 : $\forall t \in [0, 200]$

$$\sigma_1(t) = 0.3 + 0.1\sin(t), \quad \sigma_2(t) = 0.2 + 0.1\sin(t), \quad \sigma_3(t) = 0.1 + 0.05\sin(t),$$
$$\Lambda(t) = 0.3 + 0.02\sin(t), \quad \beta_S(t) = 0.2 + 0.1\sin(t), \quad \beta_{V_1}(t) = 0.2 + 0.05\sin(t),$$
$$\beta_{V_2}(t) = 0.3 + 0.02\sin(t), \quad \gamma(t) = 0.3 + 0.001\sin(t), \quad \gamma_{V_3}(t) = 0.2 + 0.001\sin(t),$$
$$\kappa_1(t) = 0.3 + 0.02\sin(t), \quad \kappa_2(t) = 0.2 + 0.02\sin(t), \quad \kappa_3(t) = 0.3 + 0.02\sin(t),$$
$$\mu(t) = 0.3 + 0.02\sin(t).$$

Case 3 : $\forall t \in [0, 200]$

$$\sigma_1(t) = 0.1 + 0.01\sin(t), \quad \sigma_2(t) = 0.05 + 0.01\sin(t), \quad \sigma_3(t) = 0.04 + 0.01\sin(t),$$
$$\Lambda(t) = 0.1 + 0.02\sin(t), \quad \beta_S(t) = 0.6 + 0.3\sin(t), \quad \beta_{V_1}(t) = 0.7 + 0.2\sin(t),$$
$$\beta_{V_2}(t) = 0.8 + 0.4\sin(t), \quad \gamma(t) = 0.01 + 0.001\sin(t), \quad \gamma_{V_3}(t) = 0.2 + 0.001\sin(t),$$
$$\kappa_1(t) = 0.03 + 0.01\sin(t), \quad \kappa_2(t) = 0.02 + 0.01\sin(t), \quad \kappa_3(t) = 0.05 + 0.02\sin(t),$$
$$\mu(t) = 0.1 + 0.02\sin(t).$$

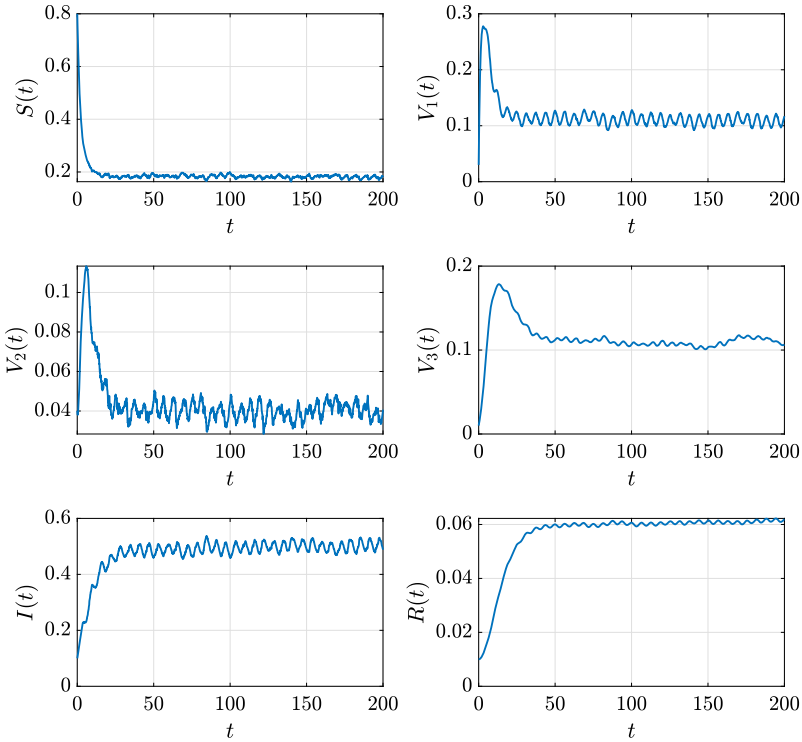

Fig. 10 Paths of S, V_1, V_2, V_3, I and R in the stochastic non-autonomous case and under the condition $\mathscr{R} > 1$

Case 4 : $\forall t \in [0, 200]$

$$\sigma_1(t) = 0, \quad \sigma_2(t) = 0, \quad \sigma_3(t) = 0,$$
$$\Lambda(t) = 0.1 + 0.02\sin(t), \quad \beta_S(t) = 0.3 + 0.2\sin(t), \quad \beta_{V_1}(t) = 0.4 + 0.2\sin(t),$$
$$\beta_{V_2}(t) = 0.3 + 0.2\sin(t), \quad \gamma(t) = 0.01 + 0.001\sin(t), \quad \gamma_{V_3}(t) = 0.01 + 0.001\sin(t),$$
$$\kappa_1(t) = 0.3 + 0.02\sin(t), \quad \kappa_2(t) = 0.2 + 0.02\sin(t), \quad \kappa_3(t) = 0.3 + 0.02\sin(t),$$
$$\mu(t) = 0.1 + 0.02\sin(t).$$

Case 5 : $\forall t \in [0, 200]$

$$\sigma_1(t) = 0.06 + 0.02\sin(t), \quad \sigma_2(t) = 0.03 + 0.02\sin(t), \quad \sigma_3(t) = 0.05 + 0.02\sin(t),$$
$$\Lambda(t) = 0.1 + 0.02\sin(t), \quad \beta_S(t) = 0.3 + 0.2\sin(t), \quad \beta_{V_1}(t) = 0.4 + 0.2\sin(t),$$
$$\beta_{V_2}(t) = 0.3 + 0.2\sin(t), \quad \gamma(t) = 0.01 + 0.001\sin(t), \quad \gamma_{V_3}(t) = 0.01 + 0.001\sin(t),$$
$$\kappa_1(t) = 0.3 + 0.02\sin(t), \quad \kappa_2(t) = 0.2 + 0.02\sin(t), \quad \kappa_3(t) = 0.3 + 0.02\sin(t),$$
$$\mu(t) = 0.1 + 0.02\sin(t).$$

6 Conclusions and future work

The appearance of new emerging diseases requires the enhancement of existing epidemic models, in order to have a more pertinent interpretation of reality [39–43]. In

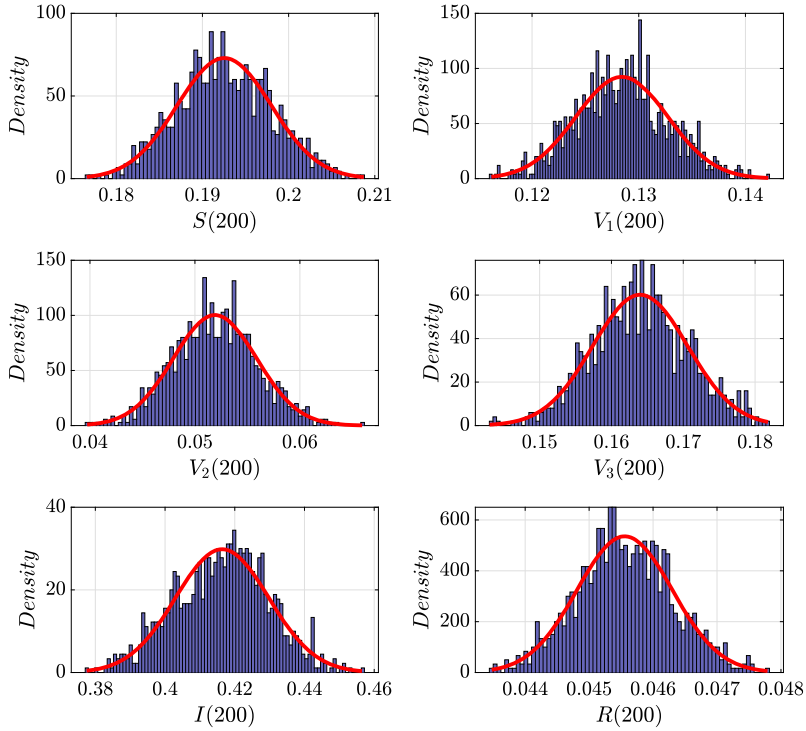

Fig. 11 Probability density functions of S, V_1, V_2, V_3, I and R at time $t = 200$ in the stochastic non-autonomous case and under the condition $\mathscr{R} > 1$

this context, inspired by the characteristics of new emerging diseases such as COVID-19, in this paper, we have conducted a dynamical study of a new-proposed stochastic SVIR model, in the aim of studying the effect of the multiple stages of vaccination, required to gain immunity, along with the environmental noise on the dynamics of the studied population. Our results are briefly outlined as follows.

- For a large values of the Gaussian noise intensities, the infected population goes to extinction if $\limsup_{t\to+\infty} \varrho(t) < 0$. For sufficiently small values of the Gaussian noise intensities, a sufficient condition guaranteeing that the infected population goes to extinction is $\langle \mathcal{R}_1^s \rangle_T < 1$, and $\forall t \in (0, T)$, $\frac{\mu}{\Lambda} \beta_S(t) > \sigma_1^2(t)$, $\frac{\mu}{\Lambda} \beta_{V_1}(t) > \sigma_2^2(t)$, and $\frac{\mu}{\Lambda} \beta_{V_2}(t) > \sigma_3^2(t)$.
- Under the condition $\mathcal{R}_2^s > 1$, the infected population becomes persistent in the mean.
- For diseases with seasonal patterns, under the condition $\mathscr{R} > 1$, the susceptible, infected, vaccinated and recovered subpopulations become persistent.

It is worth mentioning that while our primal focus in this work resided in the dynamical analysis, this paper brings about other interesting questions that need to be investigated. Case in point, we can think of dealing with the identification problem for

COVID-19 in Morocco, due to the availability of the data [44], which will permit us to identify the stochastic thresholds characterizing the disease extinction and persistence and then test the effectiveness of the vaccination strategy adopted by the authorities. On the other hand, the model (1) can be further generalized. For instance, taking into account that a certain amount of time is necessary between each stage of vaccination as well as the mean time in which the effectiveness of each stage wears off, we can add delay variables to the model and analyze the changes induced in the dynamics. Finally, by taking into account that the population may suffer from sudden environmental shocks. Precisely, ones exhibited by socio-cultural changes such as anti-vaccination movements, adding Lévy jumps to the model can increase its pertinence. All these questions will be the subjects of future work.

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Code availability The Matlab code used for the numerical simulations is available from the corresponding author upon request.

Declarations

Conflicts of interest The authors declare that they have no conflicts of interest.

References

1. Kermack, W.O., McKendrick, A.G.: A contribution to the mathematical theory of epidemics. Proc. R. Soc. Lond. Ser. A Contain Papers Math. Phys. Charact. **115**, 700–721 (1927). https://doi.org/10.1098/rspa.1927.0118
2. Liu, X., Takeuchi, Y., Iwami, S.: SVIR epidemic models with vaccination strategies. J. Theor. Biol. **253**(1), 1–11 (2008)
3. Kumar, S., Ahmadian, A., Kumar, R., Kumar, D., Singh, J., Baleanu, D., Salimi, M.: An efficient numerical method for fractional sir epidemic model of infectious disease by using Bernstein wavelets. Mathematics **8**(4), 558 (2020)
4. Khan, Z.A., Lamrani Alaoui, A., Zeb, A., Tilioua, M., Djilali, S.: Global dynamics of a SEI epidemic model with immigration and generalized nonlinear incidence functional. Results Phys. **27**, 104477 (2021)
5. Kumar, S., Kumar, A., Samet, B., Gómez-Aguilar, J., Osman, M.: A chaos study of tumor and effector cells in fractional tumor-immune model for cancer treatment. Chaos Solitons Fractals **141**, 110321 (2020)
6. Kumar, S., Kumar, R., Osman, M., Samet, B.: A wavelet based numerical scheme for fractional order SEIR epidemic of measles by using Genocchi polynomials. Numer. Methods Partial Differ. Equ. **37**(2), 1250–1268 (2021)
7. Mohammadi, H., Kumar, S., Rezapour, S., Etemad, S.: A theoretical study of the Caputo-Fabrizio fractional modeling for hearing loss due to mumps virus with optimal control. Chaos Solitons Fractals **144**, 110668 (2021)
8. Ghanbari, B., Kumar, S., Kumar, R.: A study of behaviour for immune and tumor cells in immunogenetic tumour model with non-singular fractional derivative. Chaos Solitons Fractals **133**, 109619 (2020)
9. Mehdaoui, M., Alaoui, A.L., Tilioua, M.: Optimal control for a multi-group reaction-diffusion SIR model with heterogeneous incidence rates. Int. J. Dyn. Control (2022). https://doi.org/10.1007/s40435-022-01030-3
10. Mezouaghi, A., Djilali, S., Bentout, S., Biroud, K.: Bifurcation analysis of a diffusive predator-prey model with prey social behavior and predator harvesting. Math. Methods Appl. Sci. **45**(2), 718–731 (2022)

11. Djilali, S., Bentout, S.: Pattern formations of a delayed diffusive predator-prey model with predator harvesting and prey social behavior. Math. Methods Appl. Sci. **44**(11), 9128–9142 (2021)
12. Djilali, S., Bentout, S.: Spatiotemporal patterns in a diffusive predator-prey model with prey social behavior. Acta Appl. Math. **169**(1), 125–143 (2020)
13. Kiouach, D., Sabbar, Y.: Ergodic stationary distribution of a stochastic hepatitis B epidemic model with interval-valued parameters and compensated poisson process. Comput. Math. Methods Med. **2020** (2020)
14. Kiouach, D., Sabbar, Y.: Global dynamics analysis of a stochastic SIRS epidemic model with vertical transmission and different periods of immunity. Int. J. Dyn. Syst. Differ. Equ. **10**(5), 468–491 (2020)
15. Kiouach, D., Sabbar, Y.: Dynamic characterization of a stochastic SIR infectious disease model with dual perturbation. Int. J. Biomath. **14**(04), 2150016 (2021)
16. Liu, Q., Jiang, D., Hayat, T., Alsaedi, A., Ahmad, B.: Dynamical behavior of a higher order stochastically perturbed SIRI epidemic model with relapse and media coverage. Chaos Solit. Fractals **139**, 110013 (2020)
17. Kiouach, D., Sabbar, Y., El Azami El-idrissi, S.: New results on the asymptotic behavior of an SIS epidemiological model with quarantine strategy, stochastic transmission, and Lévy disturbance. Math. Methods Appl. Sci. **44**(17), 13468–13492 (2021)
18. Shangguan, D., Liu, Z., Wang, L., Tan, R.: A stochastic epidemic model with infectivity in incubation period and homestead-isolation on the susceptible. J. Appl. Math. Comput. **67**(1), 785–805 (2021)
19. Jiang, D., Yu, J., Ji, C., Shi, N.: Asymptotic behavior of global positive solution to a stochastic SIR model. Math. Comput. Model. **54**(1–2), 221–232 (2011)
20. Sabbar, Y., Khan, A., Din, A., Kiouach, D., Rajasekar, S.: Determining the global threshold of an epidemic model with general interference function and high-order perturbation. AIMS Math. **7**(11), 19865–19890 (2022)
21. Keeling, M.J., Rohani, P., Grenfell, B.T.: Seasonally forced disease dynamics explored as switching between attractors. Phys. D Nonlinear Phenom. **148**(3–4), 317–335 (2001)
22. Weber, A., Weber, M., Milligan, P.: Modeling epidemics caused by respiratory syncytial virus (RSV). Math. Biosci. **172**(2), 95–113 (2001)
23. Greenhalgh, D., Moneim, I.A.: SIRS epidemic model and simulations using different types of seasonal contact rate. Syst. Anal. Model. Simul. **43**(5), 573–600 (2003)
24. Qi, H., Leng, X., Meng, X., Zhang, T.: Periodic solution and ergodic stationary distribution of SEIS dynamical systems with active and latent patients. Qual. Theory Dyn. Syst. **18**(2), 347–369 (2019)
25. Shangguan, D., Liu, Z., Wang, L., Tan, R.: Periodicity and stationary distribution of two novel stochastic epidemic models with infectivity in the latent period and household quarantine. J. Appl. Math. Comput. 1–20 (2021)
26. Liu, Q., Jiang, D., Hayat, T., Ahmad, B.: Periodic solution and stationary distribution of stochastic SIR epidemic models with higher order perturbation. Phys. A Stat. Mech. Appl. **482**, 209–217 (2017)
27. Lin, Y., Jiang, D., Liu, T.: Nontrivial periodic solution of a stochastic epidemic model with seasonal variation. Appl. Math. Lett. **45**, 103–107 (2015)
28. El Fatini, M., Pettersson, R., Sekkak, I., Taki, R.: A stochastic analysis for a triple delayed SIQR epidemic model with vaccination and elimination strategies. J. Appl. Math. Comput. **64**(1), 781–805 (2020)
29. Zhang, X., Liu, M.: Dynamical analysis of a stochastic delayed sir epidemic model with vertical transmission and vaccination. Adv. Contin. Discrete Models **2022**(1), 1–18 (2022)
30. Burki, T.K.: Omicron variant and booster COVID-19 vaccines. Lancet Respir. Med. **10**(2), 17 (2022)
31. Mattiuzzi, C., Lippi, G.: Primary COVID-19 vaccine cycle and booster doses efficacy: analysis of Italian nationwide vaccination campaign. Eur. J. Public Health (2022)
32. Mao, X.: Stochastic Differential Equations and Applications. Elsevier, New York (2007)
33. Khasminskii, R.: Stochastic Stability of Differential Equations, vol. 66. Springer, Berlin (2011)
34. Van den Driessche, P., Watmough, J.: Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission. Math. Biosci. **180**(1–2), 29–48 (2002)
35. MATLAB: Version 9.4.0 (R2018a). The MathWorks Inc., Natick, Massachusetts (2010)
36. Higham, D.J.: An algorithmic introduction to numerical simulation of stochastic differential equations. SIAM Rev. **43**(3), 525–546 (2001)
37. Bao, K., Zhang, Q.: Stationary distribution and extinction of a stochastic sirs epidemic model with information intervention. Adv. Differ. Equ. **2017**(1), 1–19 (2017)

38. Rao, F.: Dynamics analysis of a stochastic sir epidemic model. In: Abstract and Applied Analysis, vol. 2014 (2014). Hindawi
39. Soufiane, B., Touaoula, T.M.: Global analysis of an infection age model with a class of nonlinear incidence rates. J. Math. Anal. Appl. **434**(2), 1211–1239 (2016)
40. Bentout, S., Tridane, A., Djilali, S., Touaoula, T.M.: Age-structured modeling of Covid-19 epidemic in the USA, UAE and Algeria. Alex. Eng. J. **60**(1), 401–411 (2021)
41. Bentout, S., Chekroun, A., Kuniya, T.: Parameter estimation and prediction for coronavirus disease outbreak 2019 (Covid-19) in Algeria. AIMS Public Health **7**(2), 306 (2020)
42. Soufiane, B., Touaoula, T.M.: Global analysis of an infection age model with a class of nonlinear incidence rates. J. Math. Anal. Appl. **434**(2), 1211–1239 (2016)
43. Bentout, S., Chen, Y., Djilali, S.: Global dynamics of an SEIR model with two age structures and a nonlinear incidence. Acta Appl. Math. **171**(1), 1–27 (2021)
44. The Moroccan Ministry of Public Health: COVID-19 Platform. http://www.covidmaroc.ma/pages/Accueilfr.aspx (2022)