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A Novel Weighted Averaging Operator of Linguistic Interval-Valued Intuitionistic Fuzzy Numbers for Cognitively Inspired Decision-Making

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Abstract

An aggregation operator of linguistic interval-valued intuitionistic fuzzy numbers (LIVIFNs) is an important tool for solving cognitively inspired decision-making problems with LIVIFNs. So far, many aggregation operators of LIVIFNs have been presented. Each of these operators works well in its specific context. But they are not always monotone because their operational rules are not always invariant and persistent. Dempster-Shafer evidence theory, a general framework for modelling epistemic uncertainty, was found to provide the capability for operational rules of fuzzy numbers to overcome these limitations. In this paper, a weighted averaging operator of LIVIFNs based on Dempster-Shafer evidence theory for cognitively inspired decision-making is proposed. Firstly, Dempster-Shafer evidence theory is introduced into linguistic interval-valued intuitionistic fuzzy environment and a definition of LIVIFNs under this theory is given. Based on this, four novel operational rules of LIVIFNs are developed and proved to be always invariant and persistent. Using the developed operational rules, a new weighted averaging operator of LIVIFNs is constructed and proved to be always monotone. Based on the constructed operator, a method for solving cognitively inspired decision-making problems with LIVIFNs is presented. The application of the presented method is illustrated via a numerical example. The effectiveness and advantage of the method are demonstrated via quantitative comparisons with several existing methods. For the numerical example, the best alternative determined by the presented method is exactly the same as that determined by other comparison methods. For some specific problems, only the presented method can generate intuitive ranking results. The demonstration results suggest that the presented method is effective in solving cognitively inspired decision-making problems with LIVIFNs. Furthermore, the method will not produce counterintuitive ranking results since its operational rules are always invariant and persistent and its aggregation operator is always monotone.

Keywords Linguistic interval-valued intuitionistic fuzzy number · Operational rule · Weighted averaging operator · Dempster-Shafer evidence theory · Aggregation operator · Cognitively inspired decision-making

List of Acronyms

AO	Aggregation operator
DSET	Dempster-Shafer evidence theory
ELECTRE	ELimination et choice translating reality
DM	Decision-making
LIVIFPWA	Linguistic interval-valued intuitionistic fuzzy prioritised weighted averaging
LIVIFPWG	Linguistic interval-valued intuitionistic fuzzy prioritised weighted geometric
LIVIFPWMM	Linguistic interval-valued intuitionistic fuzzy power weighted muirhead mean
LIVIFS	Linguistic interval-valued intuitionistic fuzzy set
LIVIFN	Linguistic interval-valued intuitionistic fuzzy number
LIVIFWA	Linguistic interval-valued intuitionistic fuzzy weighted averaging
LIVIFWG	Linguistic interval-valued intuitionistic fuzzy weighted geometric
LIVIFWMSM	Linguistic interval-valued intuitionistic fuzzy weighted maclaurin symmetric mean
MABAC	Multi-attributive border approximation area comparison
MOORA	Multi-objective optimisation by ratio analysis
OR	Operational rule

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PROMETHEE	Preference ranking organisation method for enrichment evaluation
TOPSIS	Technique for order of preference by similarity to ideal solution
WA	Weighted averaging

List of Symbols

α	An LIVIFN
α_i	The i-th ($i \in \{1, 2, ..., n\}$) LIVIFN
α_k	The k-th ($k \in \{1, 2, 3\}$) LIVIFN
α_{kij}	The LIVIFN of C_j ($j \in \{1, 2, ..., n\}$) of A_i ($i \in \{1, 2, ..., m\}$) evaluated by E_k ($k \in \{1, 2, ..., n'\}$)
α_+	An LIVIFN where $\mu_+^{\mathrm{L}} = \max\{\mu_{\alpha_i}^{\mathrm{L}}\}$, $\mu_+^{\mathrm{U}} = \max\{\mu_{\alpha_i}^{\mathrm{U}}\}$, $\nu_+^{\mathrm{L}} = \min\{\nu_{\alpha_i}^{\mathrm{L}}\}$, and $\nu_+^{\mathrm{U}} = \min\{\nu_{\alpha_i}^{\mathrm{U}}\}$
α_-	An LIVIFN where $\mu_-^{\mathrm{L}} = \min\{\mu_{\alpha_i}^{\mathrm{L}}\}$, $\mu_-^{\mathrm{U}} = \min\{\mu_{\alpha_i}^{\mathrm{U}}\}$, $\nu_-^{\mathrm{L}} = \max\{\nu_{\alpha_i}^{\mathrm{L}}\}$, and $\nu_-^{\mathrm{U}} = \max\{\nu_{\alpha_i}^{\mathrm{U}}\}$
β	An LIVIFN
β_i	The i-th ($i \in \{1, 2, ..., n\}$) LIVIFN
Θ	A frame of discernment
λ	An arbitrary positive number
λ_1	An arbitrary positive number
λ_2	An arbitrary positive number
μ^{L}	The lower bound of the linguistic membership degree of an LIVIFN
μ^{U}	The upper bound of the linguistic membership degree of an LIVIFN
ν^{L}	The lower bound of the linguistic non-membership degree of an LIVIFN
ν^{U}	The upper bound of the linguistic non-membership degree of an LIVIFN
π^{L}	The lower bound of the linguistic hesitancy degree of an LIVIFN
π^{U}	The upper bound of the linguistic hesitancy degree of an LIVIFN
A	An LIVIFS
$\mathbb{A}$	The LIVIFS A under DSET
AV( )	The accuracy value of an LIVIFN
$\mathbb{AV}_h$( )	The accuracy value of an LIVIFN under DSET with respect to h
A_i	The i-th ($i \in \{1, 2, ..., m\}$) alternative in a cognitively inspired DM problem
a	The LIVIFN α under DSET
a_i	The LIVIFN α_i under DSET or summary value of a_{ij} with respect to j ($j \in \{1, 2, ..., n\}$)
a_{ij}	The summary value of a_{kij} with respect to k ($k \in \{1, 2, ..., n'\}$)
a_k	The LIVIFN α_k under DSET
a_{kij}	The LIVIFN under DSET of C_j ($j \in \{1, 2, ..., n\}$) of A_i ($i \in \{1, 2, ..., m\}$) evaluated by E_k ($k \in \{1, 2, ..., n'\}$)
a_+	The LIVIFN α_+ under DSET
a_-	The LIVIFN α_- under DSET
BF_f( )	A belief function with respect to f
BF^{L}	The lower belief function in a linguistic lower belief interval
BF^{U}	The upper belief function in a linguistic upper belief interval
BI_f( )	A belief interval with respect to f
BI^{L}	A linguistic lower belief interval
BI^{U}	A linguistic upper belief interval
b_i	The LIVIFN β_i under DSET
C_j	The j-th ($j \in \{1, 2, ..., n\}$) criterion in a cognitively inspired DM problem
E_k	The k-th ($k \in \{1, 2, ..., n'\}$) expert in a cognitively inspired DM problem
f( )	A basic probability assignment
H_i	The i-th ($i \in \{1, 2, ..., n\}$) hypothesis
H_j	The j-th ($j \in \{1, 2, ..., n\}$) hypothesis
h	The maximum index value of all linguistic terms in a finite linguistic term set
LIVIFWA( )	The LIVIFWA operator
$\mathbb{LIVIFWA}$( )	The LIVIFWA operator under DSET
$\boldsymbol{M}_k$	The k-th ($k \in \{1, 2, ..., n'\}$) decision matrix
$\boldsymbol{N}_k$	The k-th ($k \in \{1, 2, ..., n'\}$) normalised decision matrix
P( )	The probability
PF_f( )	A plausibility function with respect to f
PF^{L}	The lower plausibility function in a linguistic lower belief interval
PF^{U}	The upper plausibility function in a linguistic upper belief interval
p^{L}	The lower bound of the linguistic membership degree of an LIVIFN in $\boldsymbol{N}_k$ ($k \in \{1, 2, ..., n'\}$)
p^{U}	The upper bound of the linguistic membership degree of an LIVIFN in $\boldsymbol{N}_k$ ($k \in \{1, 2, ..., n'\}$)
q^{L}	The lower bound of the linguistic non-membership degree of an LIVIFN in $\boldsymbol{N}_k$ ($k \in \{1, 2, ..., n'\}$)
q^{U}	The upper bound of the linguistic non-membership degree of an LIVIFN in $\boldsymbol{N}_k$ ($k \in \{1, 2, ..., n'\}$)
SV_h( )	The score value of an LIVIFN with respect to h
$\mathbb{SV}_h$( )	The score value of an LIVIFN under DSET with respect to h
2^{Θ}	The power set of Θ

w_i	The weight of α_i or a_i ($i \in \{1, 2, ..., n\}$)
w_j	The weight of C_j ($j \in \{1, 2, ..., n\}$)
w'_k	The weight of E_k ($k \in \{1, 2, ..., n'\}$)
X	A finite domain
x	An element in X

Introduction

Cognitively inspired decision-making (DM) is a cognitive process of selecting the best alternative from a certain number of alternatives based on summary values of one or multiple criteria of all alternatives, in which the values of criteria are evaluated by one or a group of domain experts [1–3]. There are two critical steps in this process. The first is to quantify the evaluation results from domain experts, while the second is to generate a ranking of all alternatives via comprehensively considering all criteria.

For the quantification of evaluation results, the main challenge is to pinpoint cognition and judgments that are close to the human brain to improve DM quality. Cognitive computation, a computing system that imitates human thought processes, has been presented to achieve computing that functions like the human brain [4]. In the era of big data, a variety of technological advancements is used in cognitive computation to handle the enormous volume of data with complicated structures [5, 6]. Since human thoughts are generally complex and vague, it is difficult to describe the data in form of crisp values. To effectively represent human cognitive process in DM, researchers proposed the use of fuzzy sets [7]. So far, over thirty different types of fuzzy sets have been presented [8]. Representative examples are fuzzy set [9], intuitionistic fuzzy set [10], interval-valued intuitionistic fuzzy set [11], linguistic intuitionistic fuzzy set [12], Pythagorean fuzzy set [13], generalised orthopair fuzzy set [14], and linguistic interval-valued intuitionistic fuzzy set (LIVIFS) [15, 16]. Using these fuzzy sets, many methods for solving cognitively inspired DM problems [17–34] have been proposed within academia.

LIVIFS, which was presented on the basis of interval-valued intuitionistic fuzzy set, linguistic term set [35, 36], and linguistic intuitionistic fuzzy set, is one of the most important types of fuzzy sets for quantifying evaluation results in cognitively inspired DM. An LIVIFS can be defined by an element and a membership degree and a non-membership degree of the element to the LIVIFS, where each degree is denoted by an interval of two linguistic terms. A pair composed of a membership degree and a non-membership degree is usually called a linguistic interval-valued intuitionistic fuzzy number (LIVIFN). Through such definition, an LIVIFS can effectively reflect the characteristics of human cognitive performance including acceptance, rejection, and hesitation. Compared with fuzzy set, intuitionistic fuzzy set, interval-valued intuitionistic fuzzy set, linguistic intuitionistic fuzzy set, Pythagorean fuzzy set, and generalised orthopair fuzzy set, LIVIFS provides stronger expressive capability and is more flexible for domain experts, since it allows them to give evaluation results using two intervals of linguistic terms (i.e. LIVIFNs). Because of these features, application of LIVIFS to express the evaluation results in cognitively inspired DM [15, 16, 27, 33, 34, 37–43] has received extensive attention and is still gaining importance and popularity.

For the generation of a ranking, there are usually two approaches. The first is to use traditional DM methods, such as analytic hierarchy process, TOPSIS method, ELECTRE method, PROMETHEE method, MABAC method, and MOORA method. The second is to adopt aggregation operators (AOs), such as weighted averaging (WA) operator, Heronian mean operator, Bonferroni mean operator, Maclaurin symmetric mean operator, and Muirhead mean operator. In general, an AO has better traceability when solving cognitively inspired DM problems than a traditional DM method, since it can produce summary values of multiple criteria and a ranking of all alternatives, while a traditional DM method can only generate a ranking [44]. To date, there have been many AOs of LIVIFNs for cognitively inspired DM. Representative examples include: a prioritised weighted averaging operator, a prioritised weighted geometric operator, a prioritised ordered weighted averaging operator, and a prioritised ordered weighted geometric operator presented by [27]; a WA operator, a weighted geometric operator, an ordered weighted averaging operator, an ordered weighted geometric operator, a hybrid average operator, and a hybrid geometric operator presented by [16]; a weighted Maclaurin symmetric mean operator presented by [37]; an Archimedean power weighted Muirhead mean operator presented by [33]; an Archimedean prioritised ‘and’ operator and an Archimedean prioritised ‘or’ operator presented by [34]; a neutrosophic Dombi hybrid weighted geometric operator presented by [42]; a partitioned weighted Hamy mean operator presented by [41]; a copula weighted Heronian mean operator presented by [43]; a Hamacher weighted averaging operator and a Hamacher weighted geometric operator presented by [40].

Most of the operators above use the operational rules (ORs) of LIVIFNs based on algebraic t-norm and t-conorm to perform their operations, while the remaining operators adopt the ORs based on other types of Archimedean t-norm and t-conorm. The ORs based on Archimedean t-norm and t-conorm make the operators general and flexible, but also make them produce counterintuitive ranking results for some cognitively inspired DM problems with LIVIFNs, because they are not always invariant and persistent and the AOs based on them are not always monotone. For example, assume a decision maker needs to select a proper additive manufacturing machine from two alternative machines M_1 and M_2 to build a part with certain material on the basis of the following conditions: the selection criteria include predicted strength (C_1) and predicted hardness (C_2) of the as-built part; the weight of C_1 (w_1) and the weight of C_2 (w_2) are respectively given as $w_1 = 0.4$ and $w_2 = 0.6$; the value of C_1 of M_1 (α_{11}), the value of C_2 of M_1 (α_{12}), the value of C_1 of M_2 (α_{21}), and the value of C_2 of M_2 (α_{22}) are all given by LIVIFNs under nine linguistic terms extremely small (0), very small (1), small (2), slightly small (3), medium (4), slightly large (5), large (6), very large (7), and extremely large (8): $\alpha_{11} = ([2,4],[1,2])$; $\alpha_{21} = ([3,4],[1,3])$; $\alpha_{12} = \alpha_{22} = ([2,4],[1,3])$. According to these conditions, it is quite intuitive for the decision maker to choose M_2 to build the part, because α_{11} is less than α_{21} according to the rules for comparing two LIVIFNs used in the operators above. However, a counterintuitive result "M_1 is better than M_2" will be obtained if any of the operators above (denoted as o) is used to solve this problem, because $o(\alpha_{11}, \alpha_{12})$ is greater than $o(\alpha_{21}, \alpha_{22})$ is obtained after using the operator and comparison rules (The reason will be explained in detail in the second comparison in Sect. 'Comparisons with Existing Methods').

Based on the analysis above, the motivations of this paper are threefold:

1. To overcome the limitation that the existing ORs of LIVIFNs are not always invariant and persistent, Dempster-Shafer evidence theory (DSET) [45, 46] is introduced to develop novel ORs of LIVIFNs. DSET, also known as theory of belief functions, is a general framework for modelling epistemic uncertainty. In this framework, there are four fundamental components: a basic probability assignment that describes the occurrence rate of criteria in basic events; a belief function that expresses the belief of a focal element; a plausibility function that expresses the uncertainty of a focal element; a belief interval that consists of a belief function and a plausibility function. As demonstrated in [47–51], fuzzy numbers can be converted into belief intervals without loss of information, and the ORs of fuzzy numbers under DSET are always invariant and persistent [52–58]. Because of these characteristics, the developed novel ORs of LIVIFNs are always invariant and persistent;
2. To address the issue that the existing AOs of LIVIFNs are not always monotone and could generate counterintuitive ranking results, the developed ORs are applied to construct a new WA operator of LIVIFNs. Benefiting from the advantages of the ORs, the constructed AO is always monotone and therefore will not produce counterintuitive ranking results. For example, an intuitive result "M_2 is better than M_1" will be obtained if the constructed AO is used to solve the problem above;
3. To solve cognitively inspired DM problems with LIVIFNs, a DM method based on the constructed AO is presented. This method has the advantages of the developed ORs and constructed AO.

The novelties of the paper lie in the following aspects:

1. The developed novel ORs. The existing ORs of LIVIFNs are based on Archimedean t-norm and t-conorm. They are general and flexible for performing operations. But they are found not to be always invariant and persistent. The developed novel ORs are based on DSET, which are proved to be always invariant and persistent. To the best of the knowledge, they are the first set of ORs of LIVIFNs based on DSET;
2. The constructed new WA operator. The existing AOs of LIVIFNs use the ORs of LIVIFNs based on Archimedean t-norm and t-conorm to perform operations. Each of them can work well under specific conditions. However, they are found to generate counterintuitive ranking results sometimes because they are not always monotone. The constructed new WA operator uses the ORs of LIVIFNs based on DSET to perform operations. It is proved to be always monotone. To the best of the knowledge, it is the first AO of LIVIFNs under DSET;
3. The proposed new DM method. The existing methods for solving DM problems with LIVIFNs are based on the existing ORs and AOs of LIVIFNs. These methods could produce counterintuitive DM results since they inherit the limitations mentioned in the first and second aspects. The proposed new DM method is based on the presented new ORs and AO. It will not generate counterintuitive DM results because the presented ORs and AO are free of the limitations.

The remainder of the paper is organised as follows. Section 'Preliminaries' gives a brief introduction of some prerequisites. Three limitations of the existing ORs and WA operator of LIVIFNs are discussed in Sect. 'Discussion of Limitations'. Section 'New ORs, WA Operator, and DM Method' presents four ORs of LIVIFNs under DSET, a WA operator of LIVIFNs under DSET, and a method to solve cognitively inspired DM problems with LIVIFNs. The application, effectiveness, and advantage of the presented method are demonstrated in Sect. 'Application and Comparisons'. Section 'Conclusion' ends the paper with a conclusion.

Preliminaries

Brief Introduction of LIVIFS

An LIVIFS needs to be defined on a continuous linguistic term set, which is defined below:

Definition 1 [35, 36] Let h be some fixed natural number. Then $\{0, 1, ..., h\}$ is called a finite linguistic term set (which consists of $h+1$ linguistic terms denoted by 0, 1, ..., h) and $\{i \in \mathbb{R} \mid 0 \leq i \leq h\}$ is called a continuous linguistic term set (if $i \in \{0, 1, ..., h\}$, then i is called an original linguistic term; otherwise, i is called a virtual linguistic term).

For the sake of clarity and convenience of description, the following conventions will be adopted in the whole paper: Every h has the same meaning, that is, the maximum index value of all linguistic terms in a finite linguistic term set $\{0, 1, ..., h\}$; All LIVIFSs and all LIVIFNs are defined on a continuous linguistic term set $\{i \in \mathbb{R} \mid 0 \leq i \leq h\}$.

A formal definition of LIVIFS is given below:

Definition 2 [16] An LIVIFS A over a finite universal set X is $A = \left\{\left\langle x, \left[\mu^{\mathrm{L}}(x), \mu^{\mathrm{U}}(x)\right], \left[\nu^{\mathrm{L}}(x), \nu^{\mathrm{U}}(x)\right]\right\rangle \mid x \in X\right\}$, where $\mu^{\mathrm{L}}(x), \mu^{\mathrm{U}}(x), \nu^{\mathrm{L}}(x), \nu^{\mathrm{U}}(x) \in [0, h]$, $\mu^{\mathrm{L}}(x) \leq \mu^{\mathrm{U}}(x)$, $\nu^{\mathrm{L}}(x) \leq \nu^{\mathrm{U}}(x)$, and $\mu^{\mathrm{U}}(x) + \nu^{\mathrm{U}}(x) \leq h$ for any $x \in X$. $\left[\mu^{\mathrm{L}}(x), \mu^{\mathrm{U}}(x)\right]$ is the linguistic membership degree of x to A. $\left[\nu^{\mathrm{L}}(x), \nu^{\mathrm{U}}(x)\right]$ is the linguistic non-membership degree of x to A.

In an LIVIFS A, for some x in a finite universal set X, $\left(\left[\mu^{\mathrm{L}}(x), \mu^{\mathrm{U}}(x)\right], \left[\nu^{\mathrm{L}}(x), \nu^{\mathrm{U}}(x)\right]\right)$ is called an LIVIFN. An LIVIFN α is generally denoted as $\alpha = \left(\left[\mu_{\alpha}^{\mathrm{L}}, \mu_{\alpha}^{\mathrm{U}}\right], \left[\nu_{\alpha}^{\mathrm{L}}, \nu_{\alpha}^{\mathrm{U}}\right]\right)$. To compare two LIVIFNs, the score and accuracy values of them are required. Two functions for respectively calculating these values are defined as follows:

Definition 3 [16] Let α be an arbitrary LIVIFN. The score value of α with respect to h and the accuracy value of α are respectively calculated by the following two equations:

$$\mathrm{SV}_h(\alpha) = (2h + \mu_{\alpha}^{\mathrm{L}} + \mu_{\alpha}^{\mathrm{U}} - \nu_{\alpha}^{\mathrm{L}} - \nu_{\alpha}^{\mathrm{U}})/4 \tag{1}$$

$$\mathrm{AV}(\alpha) = (\mu_{\alpha}^{\mathrm{L}} + \mu_{\alpha}^{\mathrm{U}} + \nu_{\alpha}^{\mathrm{L}} + \nu_{\alpha}^{\mathrm{U}})/2 \tag{2}$$

Based on the score and accuracy functions, rules for comparing two LIVIFNs are defined below:

Definition 4 [16] Let α_1 and α_2 be two arbitrary LIVIFNs. The following notations are used to express five relationships between two LIVIFNs: $\alpha_1 \circleddash\!\!< \alpha_2$ represents α_1 is less than α_2; $\alpha_1 \circleddash\!\!> \alpha_2$ represents α_1 is greater than α_2; $\alpha_1 \ominus \alpha_2$ represents α_1 is equal to α_2 ($\ominus$ is an ordering, i.e. it is reflexive, symmetric, and transitive); $\alpha_1 \circleddash\!\!\leq \alpha_2$ represents α_1 is less than or equal to α_2; $\alpha_1 \circleddash\!\!\geq \alpha_2$ represents α_1 is greater than or equal to α_2. The comparison rules are: If $\mathrm{SV}_h(\alpha_1) < \mathrm{SV}_h(\alpha_2)$, then $\alpha_1 \circleddash\!\!< \alpha_2$; If $\mathrm{SV}_h(\alpha_1) = \mathrm{SV}_h(\alpha_2)$ and $\mathrm{AV}(\alpha_1) < \mathrm{AV}(\alpha_2)$, then $\alpha_1 \circleddash\!\!< \alpha_2$; If $\mathrm{SV}_h(\alpha_1) = \mathrm{SV}_h(\alpha_2)$ and $\mathrm{AV}(\alpha_1) = \mathrm{AV}(\alpha_2)$, then $\alpha_1 \ominus \alpha_2$.

Existing ORs of LIVIFNs

Operations related to LIVIFNs can be performed using certain ORs of LIVIFNs. There are currently several sets of ORs of LIVIFNs. The most used one is based on algebraic t-norm and t-conorm, which is defined as follows:

Definition 5 [16] Let α, α_1, and α_2 be three arbitrary LIVIFNs and λ be an arbitrary positive number. The following notations are used to express two operations between α_1 and α_2 and two operations between λ and α: $\alpha_1 \oplus \alpha_2$ represents α_1 plus α_2; $\alpha_1 \otimes \alpha_2$ represents α_1 times α_2; $\lambda\alpha$ represents λ times α; α^{λ} represents the λ power of α. These operations can be performed using the following rules:

$$\begin{aligned}\alpha_1 \oplus \alpha_2 =& \Bigg(\left[\mu_{\alpha_1}^{\mathrm{L}} + \mu_{\alpha_2}^{\mathrm{L}} - \mu_{\alpha_1}^{\mathrm{L}}\mu_{\alpha_2}^{\mathrm{L}}/h, \mu_{\alpha_1}^{\mathrm{U}} + \mu_{\alpha_2}^{\mathrm{U}} - \mu_{\alpha_1}^{\mathrm{U}}\mu_{\alpha_2}^{\mathrm{U}}/h\right],\\ &\left[\nu_{\alpha_1}^{\mathrm{L}}\nu_{\alpha_2}^{\mathrm{L}}/h, \nu_{\alpha_1}^{\mathrm{U}}\nu_{\alpha_2}^{\mathrm{U}}/h\right]\Bigg)\end{aligned} \tag{3}$$

$$\begin{aligned}\alpha_1 \otimes \alpha_2 =& \Bigg(\left[\mu_{\alpha_1}^{\mathrm{L}}\mu_{\alpha_2}^{\mathrm{L}}/h, \mu_{\alpha_1}^{\mathrm{U}}\mu_{\alpha_2}^{\mathrm{U}}/h\right],\\ &\left[\nu_{\alpha_1}^{\mathrm{L}} + \nu_{\alpha_2}^{\mathrm{L}} - \nu_{\alpha_1}^{\mathrm{L}}\nu_{\alpha_2}^{\mathrm{L}}/h, \nu_{\alpha_1}^{\mathrm{U}} + \nu_{\alpha_2}^{\mathrm{U}} - \nu_{\alpha_1}^{\mathrm{U}}\nu_{\alpha_2}^{\mathrm{U}}/h\right]\Bigg)\end{aligned} \tag{4}$$

$$\begin{aligned}\lambda\alpha =& \Bigg(\left[h - h(1 - \mu_{\alpha}^{\mathrm{L}}/h)^{\lambda}, h - h(1 - \mu_{\alpha}^{\mathrm{U}}/h)^{\lambda}\right],\\ &\left[h(\nu_{\alpha}^{\mathrm{L}}/h)^{\lambda}, h(\nu_{\alpha}^{\mathrm{U}}/h)^{\lambda}\right]\Bigg)\end{aligned} \tag{5}$$

$$\begin{aligned}\alpha^{\lambda} =& \Bigg(\left[h(\mu_{\alpha}^{\mathrm{L}}/h)^{\lambda}, h(\mu_{\alpha}^{\mathrm{U}}/h)^{\lambda}\right],\\ &\left[h - h(1 - \nu_{\alpha}^{\mathrm{L}}/h)^{\lambda}, h - h(1 - \nu_{\alpha}^{\mathrm{U}}/h)^{\lambda}\right]\Bigg)\end{aligned} \tag{6}$$

It is worth noting that $\lambda\alpha$ will be equal to ([0, 0], [h, h]) and α^{λ} will be equal to ([h, h], [0, 0]) if $\lambda = 0$. The four operations in the ORs above satisfy the following algebraic laws [16]:

$$\alpha_1 \oplus \alpha_2 = \alpha_2 \oplus \alpha_1 \tag{7}$$

$$\lambda(\alpha_1 \oplus \alpha_2) = \lambda\alpha_1 \oplus \lambda\alpha_2 \tag{8}$$

$$\lambda_1\alpha \oplus \lambda_2\alpha = (\lambda_1 + \lambda_2)\alpha \tag{9}$$

$$\alpha_1 \otimes \alpha_2 = \alpha_2 \otimes \alpha_1 \tag{10}$$

$$(\alpha_1 \otimes \alpha_2)^{\lambda} = \alpha_1^{\lambda} \otimes \alpha_2^{\lambda} \tag{11}$$

$$\alpha^{\lambda_1} \otimes \alpha^{\lambda_2} = \alpha^{\lambda_1+\lambda_2} \tag{12}$$

where λ_1 and λ_2 are two arbitrary positive numbers. Further, there is yet no evidence that the four operations satisfy other algebraic laws, such as associativity of $\oplus$: $(\alpha_1 \oplus \alpha_2) \oplus \alpha_3 = \alpha_1 \oplus (\alpha_2 \oplus \alpha_3)$, associativity of $\otimes$: $(\alpha_1 \otimes \alpha_2) \otimes \alpha_3 = \alpha_1 \otimes (\alpha_2 \otimes \alpha_3)$, and distributive law of $\otimes$: $(\alpha_1 \oplus \alpha_2) \otimes \beta = (\alpha_1 \otimes \beta) \oplus (\alpha_2 \otimes \beta)$, where α_1, α_2, α_3, and β are four arbitrary LIVIFNs.

Existing WA Operator of LIVIFNs

An AO is a function for grouping together two or more values to achieve a summary value. The most common AO for solving cognitively inspired DM problems is the WA operator. A WA operator of LIVIFNs based on the ORs of LIVIFNs in Definition 5 is defined below:

Definition 6 [16] Let $\alpha_i\,(i \in \{1, 2, ..., n\})$ be n arbitrary LIVIFNs, and w_i be the weight of α_i such that $0 \le w_i \le 1$ and $\Sigma_{i=1}^{n} w_i = 1$. The aggregation function

$$\begin{aligned}\mathrm{LIVIFWA}(\alpha_1, \alpha_2, ..., \alpha_n) &= \oplus_{i=1}^{n}(w_i \alpha_i)\\ = \Bigg(\Bigg[h - h\textstyle\prod_{i=1}^{n}(1-\mu_{\alpha_i}^{\mathrm{L}}/h)^{w_i}, h - h\prod_{i=1}^{n}(1-\mu_{\alpha_i}^{\mathrm{U}}/h)^{w_i}\Bigg],\\ \Bigg[h\textstyle\prod_{i=1}^{n}(\nu_{\alpha_i}^{\mathrm{L}}/h)^{w_i}, h\prod_{i=1}^{n}(\nu_{\alpha_i}^{\mathrm{U}}/h)^{w_i}\Bigg]\Bigg)\end{aligned} \tag{13}$$

is called the linguistic interval-valued intuitionistic fuzzy weighted averaging (LIVIFWA) operator.

The LIVIFWA operator above has the following properties [16]:

1. Idempotency: Let α be an arbitrary LIVIFN. If $\mu_{\alpha_i}^{\mathrm{L}} = \mu_{\alpha}^{\mathrm{L}}$, $\mu_{\alpha_i}^{\mathrm{U}} = \mu_{\alpha}^{\mathrm{U}}, \nu_{\alpha_i}^{\mathrm{L}} = \nu_{\alpha}^{\mathrm{L}}$, and $\nu_{\alpha_i}^{\mathrm{U}} = \nu_{\alpha}^{\mathrm{U}}$ for all $i \in \{1, 2, ..., n\}$, then $\mathrm{LIVIFWA}(\alpha_1, \alpha_2, ..., \alpha_n) = \left(\left[\mu_{\alpha}^{\mathrm{L}}, \mu_{\alpha}^{\mathrm{U}}\right], \left[\nu_{\alpha}^{\mathrm{L}}, \nu_{\alpha}^{\mathrm{U}}\right]\right) = \alpha$;
2. Monotonicity: Let $\beta_i\,(i \in \{1, 2, ..., n\})$ be n arbitrary LIVIFNs. If $\mu_{\alpha_i}^{\mathrm{L}} \le \mu_{\beta_i}^{\mathrm{L}}$, $\mu_{\alpha_i}^{\mathrm{U}} \le \mu_{\beta_i}^{\mathrm{U}}$, $\nu_{\alpha_i}^{\mathrm{L}} \ge \nu_{\beta_i}^{\mathrm{L}}$, and $\nu_{\alpha_i}^{\mathrm{U}} \ge \nu_{\beta_i}^{\mathrm{U}}$ for all $i \in \{1, 2, ..., n\}$, then $\mathrm{LIVIFWA}(\alpha_1, \alpha_2, ..., \alpha_n) \lessapprox \mathrm{LIVIFWA}(\beta_1, \beta_2, ..., \beta_n)$;
3. Boundedness: Let $\alpha_- = \left(\left[\mu_-^{\mathrm{L}}, \mu_-^{\mathrm{U}}\right], \left[\nu_-^{\mathrm{L}}, \nu_-^{\mathrm{U}}\right]\right)$ and $\alpha_+ = \left(\left[\mu_+^{\mathrm{L}}, \mu_+^{\mathrm{U}}\right], \left[\nu_+^{\mathrm{L}}, \nu_+^{\mathrm{U}}\right]\right)$, where $\mu_-^{\mathrm{L}} = \min\{\mu_{\alpha_i}^{\mathrm{L}}\}$, $\mu_-^{\mathrm{U}} = \min\{\mu_{\alpha_i}^{\mathrm{U}}\}$, $\nu_-^{\mathrm{L}} = \max\{\nu_{\alpha_i}^{\mathrm{L}}\}$, $\nu_-^{\mathrm{U}} = \max\{\nu_{\alpha_i}^{\mathrm{U}}\}$, $\mu_+^{\mathrm{L}} = \max\{\mu_{\alpha_i}^{\mathrm{L}}\}$, $\mu_+^{\mathrm{U}} = \max\{\mu_{\alpha_i}^{\mathrm{U}}\}$, $\nu_+^{\mathrm{L}} = \min\{\nu_{\alpha_i}^{\mathrm{L}}\}$, and $\nu_+^{\mathrm{U}} = \min\{\nu_{\alpha_i}^{\mathrm{U}}\}$ for all $i \in \{1, 2, ..., n\}$. Then $\alpha_- \lessapprox \mathrm{LIVIFWA}(\alpha_1, \alpha_2, ..., \alpha_n) \lessapprox \alpha_+$.

Discussion of Limitations

Limitations of Existing ORs of LIVIFNs

The ORs of LIVIFNs in Eqs. (3) and (5) are found to generate counterintuitive ranking results for some cognitively inspired DM problems with LIVIFNs due to the following limitations:

1. The operation in the OR of LIVIFNs in Eq. (3) is not always invariant with respect to the score function in Eq. (1), the accuracy function in Eq. (2), and the comparison rules in Definition 4: For three arbitrary LIVIFNs α_1, α_2, and α_3, $\alpha_1 \lessdot \alpha_2$ cannot always imply $(\alpha_1 \oplus \alpha_3) \lessdot (\alpha_2 \oplus \alpha_3)$;
2. The operation in the OR of LIVIFNs in Eq. (5) is not always persistent with respect to the score function in Eq. (1), the accuracy function in Eq. (2), and the comparison rules in Definition 4: For two arbitrary LIVIFNs α_1 and α_2 and an arbitrary positive number λ, $\alpha_1 \lessdot \alpha_2$ cannot always imply $\lambda\alpha_1 \lessdot \lambda\alpha_2$.

Two numerical examples for respectively illustrating the limitations above are given below:

Example 1 Assume $h = 8$, $\alpha_1 = ([4, 5], [2, 3])$, $\alpha_2 = ([3, 5], [1, 2])$, and $\alpha_3 = ([2, 4], [1, 3])$. According to the score function in Eq. (1), we have $\mathrm{SV}_h(\alpha_1) = 5.0000$ and $\mathrm{SV}_h(\alpha_2) = 5.2500$. Since $\mathrm{SV}_h(\alpha_1) < \mathrm{SV}_h(\alpha_2)$, we obtain from the comparison rules in Definition 4 that $\alpha_1 \lessdot \alpha_2$. Using the OR of LIVIFNs in Eq. (3), we have

$$\alpha_1 \oplus \alpha_3 = ([5.0000, 6.5000], [0.2500, 1.1250])$$

$$\alpha_2 \oplus \alpha_3 = ([4.2500, 6.5000], [0.1250, 0.7500])$$

According to the score function in Eq. (1), we further have $\mathrm{SV}_h(\alpha_1 \oplus \alpha_3) = 6.5313$ and $\mathrm{SV}_h(\alpha_2 \oplus \alpha_3) = 6.4688$. Since $\mathrm{SV}_h(\alpha_1 \oplus \alpha_3) > \mathrm{SV}_h(\alpha_2 \oplus \alpha_3)$, we obtain from the comparison rules in Definition 4 that $(\alpha_1 \oplus \alpha_3) \gtrdot (\alpha_2 \oplus \alpha_3)$.

Example 2 Assume $h = 8$, $\alpha_1 = ([2, 4], [1, 2])$, $\alpha_2 = ([3, 4], [1, 3])$, and $\lambda = 0.6000$. According to the score function in Eq. (1) and the accuracy function in Eq. (2), we have $\mathrm{SV}_h(\alpha_1) = 4.7500$, $\mathrm{SV}_h(\alpha_2) = 4.7500$, $\mathrm{AV}(\alpha_1) = 4.5000$, and $\mathrm{AV}(\alpha_2) = 5.5000$. Since $\mathrm{SV}_h(\alpha_1) = \mathrm{SV}_h(\alpha_2)$ and $\mathrm{AV}(\alpha_1) < \mathrm{AV}(\alpha_2)$, we obtain from the comparison rules in Definition 4 that $\alpha_1 \lessdot \alpha_2$. Using the OR of LIVIFNs in Eq. (5), we have

$$\lambda\alpha_1 = ([1.2683, 2.7220], [2.2974, 3.4822])$$

$\lambda\alpha_2 = ([1.9658, 2.7220], [2.2974, 4.4413])$

According to the score function in Eq. (1), we further have $\mathrm{SV}_h(\lambda\alpha_1) = 3.5527$ and $\mathrm{SV}_h(\lambda\alpha_2) = 3.4873$. Since $\mathrm{SV}_h(\lambda\alpha_1) > \mathrm{SV}_h(\lambda\alpha_2)$, we obtain from the comparison rules in Definition 4 that $\lambda\alpha_1 \circledcirc \lambda\alpha_2$.

Limitation of Existing WA Operator of LIVIFNs

The LIVIFWA operator in Eq. (13) is found to produce counterintuitive ranking results for some cognitively inspired DM problems with LIVIFNs because of the following limitation:

The LIVIFWA operator in Eq. (13) is not always monotone with respect to the score function in Eq. (1), the accuracy function in Eq. (2), and the comparison rules in Definition 4: For three arbitrary LIVIFNs α_1, α_2, and α_3 and certain weights w_1 and w_2, $\alpha_1 \circledcirc \alpha_2$ cannot always imply $\mathrm{LIVIFWA}(\alpha_1, \alpha_3) \circledcirc \mathrm{LIVIFWA}(\alpha_2, \alpha_3)$.

A numerical example for illustrating the limitation above is given as follows:

Example 3 Assume $h = 8$, $\alpha_1 = ([2, 4], [1, 2])$, $\alpha_2 = ([3, 4], [1, 3])$, $\alpha_3 = ([2, 4], [1, 3])$, $w_1 = 0.4000$, and $w_2 = 0.6000$. According to the score function in Eq. (1) and the accuracy function in Eq. (2), we have $\mathrm{SV}_h(\alpha_1) = 4.7500$, $\mathrm{SV}_h(\alpha_2) = 4.7500$, $\mathrm{AV}(\alpha_1) = 4.5000$, and $\mathrm{AV}(\alpha_2) = 5.5000$. Since $\mathrm{SV}_h(\alpha_1) = \mathrm{SV}_h(\alpha_2)$ and $\mathrm{AV}(\alpha_1) < \mathrm{AV}(\alpha_2)$, we obtain from the comparison rules in Definition 4 that $\alpha_1 \circledcirc \alpha_2$. Using the LIVIFWA operator in Eq. (13), we have

$$\mathrm{LIVIFWA}(\alpha_1, \alpha_3) = ([2.0000, 4.0000], [1.0000, 2.5508])$$

$$\mathrm{LIVIFWA}(\alpha_2, \alpha_3) = ([2.4220, 4.0000], [1.0000, 3.0000])$$

According to the score function in Eq. (1), we further have $\mathrm{SV}_h(\mathrm{LIVIFWA}(\alpha_1, \alpha_3)) = 4.6123$ and $\mathrm{SV}_h(\mathrm{LIVIFWA}(\alpha_2, \alpha_3)) = 4.6055$. Since $\mathrm{SV}_h(\mathrm{LIVIFWA}(\alpha_1, \alpha_3)) > \mathrm{SV}_h(\mathrm{LIVIFWA}(\alpha_2, \alpha_3))$, we obtain from the comparison rules in Definition 4 that $\mathrm{LIVIFWA}(\alpha_1, \alpha_3) \circledcirc \mathrm{LIVIFWA}(\alpha_2, \alpha_3)$.

New ORs, WA Operator, and DM Method

LIVIFS Based on DSET

Five fundamental concepts in DSET are frame of discernment, basic probability assignment, belief function, plausibility function, and belief interval, which are respectively defined as follows:

Definition 7 [45, 46] Let $\Theta = \{H_1, H_2, ..., H_n\}$ be a set of n hypotheses $H_1, H_2, ..., H_n$. If the probability of every two different hypotheses in Θ being true is zero (i.e. $P(H_i \cap H_j) = 0$ for any $i, j \in \{1, 2, ..., n\}$ and $i \neq j$) and the probability of at least one hypothesis in Θ being true is one (i.e. $P(H_1 \cup H_2 \cup ... \cup H_n) = 1$), then Θ is called a frame of discernment.

Definition 8 [45, 46] Let Θ be a frame of discernment and 2^Θ be the power set of Θ. A basic probability assignment over Θ is a mapping $f : 2^\Theta \to [0, 1]$ such that $f(\emptyset) = 0$ and $\Sigma_{H \in 2^\Theta} f(H) = 1$.

Definition 9 [45, 46] Let Θ be a frame of discernment, f be a basic probability assignment over Θ and $f(\Theta) > 0$, and H be an element of the power set of Θ (i.e. $H \in 2^\Theta$). A belief function of H with respect to f is $\mathrm{BF}_f(H) = \Sigma_{H' \subseteq H} f(H')$.

Definition 10 [45, 46] Let Θ be a frame of discernment, f be a basic probability assignment over Θ, and H be an element of the power set of Θ (i.e. $H \in 2^\Theta$). A plausibility function of H with respect to f is $\mathrm{PF}_f(H) = \Sigma_{H' \cap H \neq \emptyset} f(H')$.

Definition 11 [45, 46] Let Θ be a frame of discernment, f be a basic probability assignment over Θ, H be an element of the power set of Θ (i.e. $H \in 2^\Theta$), $\mathrm{BF}_f(H)$ be a belief function of H with respect to f, and $\mathrm{PF}_f(H)$ be a plausibility function of H with respect to f. A belief interval over H with respect to f (denoted as $\mathrm{BI}_f(H)$) is an interval whose lower bound is $\mathrm{BF}_f(H)$ and upper bound is $\mathrm{PF}_f(H)$. That is, $\mathrm{BI}_f(H) = [\mathrm{BF}_f(H), \mathrm{PF}_f(H)]$.

Based on the definitions above and the definition of interval-valued intuitionistic fuzzy set under DSET [51], the definition of LIVIFS in Definition 2 can be rewritten below:

Definition 12 Let $A = \left\{\left\langle x, \left[\mu^{\mathrm{L}}(x), \mu^{\mathrm{U}}(x)\right], \left[\nu^{\mathrm{L}}(x), \nu^{\mathrm{U}}(x)\right]\right\rangle \mid x \in X\right\}$ be an LIVIFS over a finite universal set X. Then A under DSET is defined as $\mathbb{A} = \left\{\left\langle x, \left[\mathrm{BI}^{\mathrm{L}}(x), \mathrm{BI}^{\mathrm{U}}(x)\right]\right\rangle \mid x \in X\right\}$, where $\mathrm{BI}^{\mathrm{L}}(x) = \left[\mathrm{BF}^{\mathrm{L}}(x), \mathrm{PF}^{\mathrm{L}}(x)\right] = \left[\mu^{\mathrm{L}}(x)/h, 1 - \nu^{\mathrm{U}}(x)/h\right]$ is called the linguistic lower belief interval of x to $\mathbb{A}$, and $\mathrm{BI}^{\mathrm{U}}(x) = \left[\mathrm{BF}^{\mathrm{U}}(x), \mathrm{PF}^{\mathrm{U}}(x)\right] = \left[\mu^{\mathrm{U}}(x)/h, 1 - \nu^{\mathrm{L}}(x)/h\right]$ is called the linguistic upper belief interval of x to $\mathbb{A}$.

An LIVIFN α under DSET is $a = \left[\left[\mathrm{BF}_a^{\mathrm{L}}, \mathrm{PF}_a^{\mathrm{L}}\right], \left[\mathrm{BF}_a^{\mathrm{U}}, \mathrm{PF}_a^{\mathrm{U}}\right]\right] = \left[\left[\mu_\alpha^{\mathrm{L}}/h, 1 - \nu_\alpha^{\mathrm{U}}/h\right], \left[\mu_\alpha^{\mathrm{U}}/h, 1 - \nu_\alpha^{\mathrm{L}}/h\right]\right]$. To compare two LIVIFNs under DSET, the score and accuracy values of them are needed. Two functions for respectively calculating these values are defined as follows:

Definition 13 Let α be an arbitrary LIVIFN and a be α under DSET. The score and accuracy values of a with respect to h can be respectively calculated by the following two equations:

$$\mathbb{SV}_h(a) = h\left(\mathrm{BF}_a^{\mathrm{L}} + \mathrm{BF}_a^{\mathrm{U}} + \mathrm{PF}_a^{\mathrm{L}} + \mathrm{PF}_a^{\mathrm{U}}\right) = 2h + \left(\mu_\alpha^{\mathrm{L}} + \mu_\alpha^{\mathrm{U}} - \nu_\alpha^{\mathrm{L}} - \nu_\alpha^{\mathrm{U}}\right) \tag{14}$$

$$\mathbb{AV}_h(a) = h\left(\mathrm{PF}_a^{\mathrm{L}} + \mathrm{PF}_a^{\mathrm{U}} - \mathrm{BF}_a^{\mathrm{L}} - \mathrm{BF}_a^{\mathrm{U}}\right) = 2h - \left(\mu_\alpha^{\mathrm{L}} + \mu_\alpha^{\mathrm{U}} + \nu_\alpha^{\mathrm{L}} + \nu_\alpha^{\mathrm{U}}\right) \tag{15}$$

Based on the score and accuracy functions, rules for comparing two LIVIFNs under DSET are defined below:

Definition 14 Let α_1 and α_2 be two arbitrary LIVIFNs and a_1 and a_2 be respectively α_1 and α_2 under DSET. The following notations are used to express five relationships between two LIVIFNs under DSET: $a_1 \boxed{<} a_2$ represents a_1 is less than a_2; $a_1 \boxed{>} a_2$ represents a_1 is greater than a_2; $a_1 \boxed{=} a_2$ represents a_1 is equal to a_2; $a_1 \boxed{\leq} a_2$ represents a_1 is less than or equal to a_2; $a_1 \boxed{\geq} a_2$ represents a_1 is greater than or equal to a_2. The comparison rules are: If $\mathbb{SV}_h(a_1) < \mathbb{SV}_h(a_2)$, then $a_1 \boxed{<} a_2$; If $\mathbb{SV}_h(a_1) = \mathbb{SV}_h(a_2)$ and $\mathbb{AV}_h(a_1) < \mathbb{AV}_h(a_2)$, then $a_1 \boxed{>} a_2$; If $\mathbb{SV}_h(a_1) = \mathbb{SV}_h(a_2)$ and $\mathbb{AV}_h(a_1) = \mathbb{AV}_h(a_2)$, then $a_1 \boxed{=} a_2$.

ORs of LIVIFNs Based on DSET

To perform the operations related to LIVIFNs under DSET, a set of novel ORs of LIVIFNs under DSET is developed as follows:

Definition 15 Let α be an arbitrary LIVIFN, a be α under DSET, α_i $(i \in \{1, 2, ..., n\})$ be n arbitrary LIVIFNs, a_i be α_i under DSET, and λ be an arbitrary positive number. The following notations are used to express two operations between a_1 and a_2 and two operations between λ and a: $a_1 \boxplus a_2$ represents a_1 plus a_2; $a_1 \boxtimes a_2$ represents a_1 times a_2; λa represents λ times a; a^λ represents the λ power of a. These operations can be performed using the following rules:

$$\begin{aligned}\boxplus_{i=1}^n a_i &= \left[\left[\frac{1}{n}\sum_{i=1}^n \mathrm{BF}_{a_i}^{\mathrm{L}}, \frac{1}{n}\sum_{i=1}^n \mathrm{PF}_{a_i}^{\mathrm{L}}\right], \left[\frac{1}{n}\sum_{i=1}^n \mathrm{BF}_{a_i}^{\mathrm{U}}, \frac{1}{n}\sum_{i=1}^n \mathrm{PF}_{a_i}^{\mathrm{U}}\right]\right] \\ &= \left[\left[\frac{1}{n}\sum_{i=1}^n \left(\mu_{\alpha_i}^{\mathrm{L}}/h\right), \frac{1}{n}\sum_{i=1}^n \left(1 - \nu_{\alpha_i}^{\mathrm{U}}/h\right)\right], \left[\frac{1}{n}\sum_{i=1}^n \left(\mu_{\alpha_i}^{\mathrm{U}}/h\right), \right.\right. \\ &\qquad \left.\left. \frac{1}{n}\sum_{i=1}^n \left(1 - \nu_{\alpha_i}^{\mathrm{L}}/h\right)\right]\right]\end{aligned} \tag{16}$$

$$\begin{aligned}a_1 \boxtimes a_2 &= \left[\left[\mathrm{BF}_{a_1}^{\mathrm{L}}\mathrm{BF}_{a_2}^{\mathrm{L}}, \mathrm{PF}_{a_1}^{\mathrm{L}}\mathrm{PF}_{a_2}^{\mathrm{L}}\right], \left[\mathrm{BF}_{a_1}^{\mathrm{U}}\mathrm{BF}_{a_2}^{\mathrm{U}}, \mathrm{PF}_{a_1}^{\mathrm{U}}\mathrm{PF}_{a_2}^{\mathrm{U}}\right]\right] \\ &= \left[\left[\left(\mu_{\alpha_1}^{\mathrm{L}}\mu_{\alpha_2}^{\mathrm{L}}/h^2\right), \left(1 - \nu_{\alpha_1}^{\mathrm{U}}/h\right)\left(1 - \nu_{\alpha_2}^{\mathrm{U}}/h\right)\right], \right. \\ &\qquad \left. \left[\left(\mu_{\alpha_1}^{\mathrm{U}}\mu_{\alpha_2}^{\mathrm{U}}/h^2\right), \left(1 - \nu_{\alpha_1}^{\mathrm{L}}/h\right)\left(1 - \nu_{\alpha_2}^{\mathrm{L}}/h\right)\right]\right]\end{aligned} \tag{17}$$

$$\begin{aligned}\lambda a &= \left[\left[\lambda\mathrm{BF}_a^{\mathrm{L}}, \lambda\mathrm{PF}_a^{\mathrm{L}}\right], \left[\lambda\mathrm{BF}_a^{\mathrm{U}}, \lambda\mathrm{PF}_a^{\mathrm{U}}\right]\right] \\ &= \left[\left[\lambda\left(\mu_\alpha^{\mathrm{L}}/h\right), \lambda\left(1 - \nu_\alpha^{\mathrm{U}}/h\right)\right], \left[\lambda\left(\mu_\alpha^{\mathrm{U}}/h\right), \lambda\left(1 - \nu_\alpha^{\mathrm{L}}/h\right)\right]\right]\end{aligned} \tag{18}$$

$$\begin{aligned}a^\lambda &= \left[\left[\left(\mathrm{BF}_a^{\mathrm{L}}\right)^\lambda, \left(\mathrm{PF}_a^{\mathrm{L}}\right)^\lambda\right], \left[\left(\mathrm{BF}_a^{\mathrm{U}}\right)^\lambda, \left(\mathrm{PF}_a^{\mathrm{U}}\right)^\lambda\right]\right] \\ &= \left[\left[\left(\mu_\alpha^{\mathrm{L}}/h\right)^\lambda, \left(1 - \nu_\alpha^{\mathrm{U}}/h\right)^\lambda\right], \left[\left(\mu_\alpha^{\mathrm{U}}/h\right)^\lambda, \left(1 - \nu_\alpha^{\mathrm{L}}/h\right)^\lambda\right]\right]\end{aligned} \tag{19}$$

It is easy to prove that the four operations in the ORs above satisfy the following algebraic laws:

$$a_1 \boxplus a_2 = a_2 \boxplus a_1 \tag{20}$$

$$\lambda(a_1 \boxplus a_2) = \lambda a_1 \boxplus \lambda a_2 \tag{21}$$

$$\lambda_1 a \boxplus \lambda_2 a = (\lambda_1 + \lambda_2)a \tag{22}$$

$$a_1 \boxtimes a_2 = a_2 \boxtimes a_1 \tag{23}$$

$$(a_1 \boxtimes a_2)^\lambda = (a_1)^\lambda \boxtimes (a_2)^\lambda \tag{24}$$

$$(a)^{\lambda_1} \boxtimes (a)^{\lambda_2} = (a)^{\lambda_1 + \lambda_2} \tag{25}$$

where λ_1 and λ_2 are two arbitrary positive numbers.

The developed OR of LIVIFNs under DSET in Eq. (16) is free of the limitation of the OR of LIVIFNs in Eq. (3), as stated in the following theorem:

Theorem 1 *The operation in the OR of LIVIFNs under DSET in Eq. (16) is always invariant with respect to the score function in Eq. (14), the accuracy function in Eq. (15), and the comparison rules in Definition 14: For three arbitrary LIVIFNs under DSET a_1, a_2, and a_3, $a_1 \boxed{<} a_2$ can always imply $(a_1 \boxplus a_3) \boxed{<} (a_2 \boxplus a_3)$.*

Proof Let $a_k = \left[\left[\mathrm{BF}_{a_k}^{\mathrm{L}}, \mathrm{PF}_{a_k}^{\mathrm{L}}\right], \left[\mathrm{BF}_{a_k}^{\mathrm{U}}, \mathrm{PF}_{a_k}^{\mathrm{U}}\right]\right] = \left[\left[\mu_{\alpha_k}^{\mathrm{L}}/h, 1 - \nu_{\alpha_k}^{\mathrm{U}}/h\right], \left[\mu_{\alpha_k}^{\mathrm{U}}/h, 1 - \nu_{\alpha_k}^{\mathrm{L}}/h\right]\right] (k \in \{1, 2, 3\})$. According to the score function in Eq. (14) and the accuracy function in Eq. (15), we have

$$\begin{aligned}\mathbb{SV}_h(a_1) &= h\left(\mathrm{BF}_{a_1}^{\mathrm{L}} + \mathrm{BF}_{a_1}^{\mathrm{U}} + \mathrm{PF}_{a_1}^{\mathrm{L}} + \mathrm{PF}_{a_1}^{\mathrm{U}}\right) \\ &= 2h + \left(\mu_{\alpha_1}^{\mathrm{L}} + \mu_{\alpha_1}^{\mathrm{U}} - \nu_{\alpha_1}^{\mathrm{L}} - \nu_{\alpha_1}^{\mathrm{U}}\right)\end{aligned}$$

$$\begin{aligned}\mathbb{SV}_h(a_2) &= h\left(\mathrm{BF}_{a_2}^{\mathrm{L}} + \mathrm{BF}_{a_2}^{\mathrm{U}} + \mathrm{PF}_{a_2}^{\mathrm{L}} + \mathrm{PF}_{a_2}^{\mathrm{U}}\right) \\ &= 2h + \left(\mu_{\alpha_2}^{\mathrm{L}} + \mu_{\alpha_2}^{\mathrm{U}} - \nu_{\alpha_2}^{\mathrm{L}} - \nu_{\alpha_2}^{\mathrm{U}}\right)\end{aligned}$$

$$\begin{aligned}\mathbb{AV}_h(a_1) &= h\left(\mathrm{PF}_{a_1}^{\mathrm{L}} + \mathrm{PF}_{a_1}^{\mathrm{U}} - \mathrm{BF}_{a_1}^{\mathrm{L}} - \mathrm{BF}_{a_1}^{\mathrm{U}}\right) \\ &= 2h - \left(\mu_{\alpha_1}^{\mathrm{L}} + \mu_{\alpha_1}^{\mathrm{U}} + \nu_{\alpha_1}^{\mathrm{L}} + \nu_{\alpha_1}^{\mathrm{U}}\right)\end{aligned}$$

$$\begin{aligned}\mathbb{AV}_h(a_2) &= h\left(\mathrm{PF}_{a_2}^{\mathrm{L}} + \mathrm{PF}_{a_2}^{\mathrm{U}} - \mathrm{BF}_{a_2}^{\mathrm{L}} - \mathrm{BF}_{a_2}^{\mathrm{U}}\right) \\ &= 2h - \left(\mu_{\alpha_2}^{\mathrm{L}} + \mu_{\alpha_2}^{\mathrm{U}} + \nu_{\alpha_2}^{\mathrm{L}} + \nu_{\alpha_2}^{\mathrm{U}}\right)\end{aligned}$$

Using the OR of LIVIFNs under DSET in Eq. (16), we obtain

$$a_1 \boxplus a_3 = \left[\left[\frac{1}{2}\left(\mathrm{BF}^{\mathrm{L}}_{a_1} + \mathrm{BF}^{\mathrm{L}}_{a_3}\right), \frac{1}{2}\left(\mathrm{PF}^{\mathrm{L}}_{a_1} + \mathrm{PF}^{\mathrm{L}}_{a_3}\right)\right], \left[\frac{1}{2}\left(\mathrm{BF}^{\mathrm{U}}_{a_1} + \mathrm{BF}^{\mathrm{U}}_{a_3}\right), \frac{1}{2}\left(\mathrm{PF}^{\mathrm{U}}_{a_1} + \mathrm{PF}^{\mathrm{U}}_{a_3}\right)\right]\right]$$
$$= \left[\left[\left(\mu^{\mathrm{L}}_{\alpha_1} + \mu^{\mathrm{L}}_{\alpha_3}\right)/(2h), 1 - \left(\nu^{\mathrm{U}}_{\alpha_1} + \nu^{\mathrm{U}}_{\alpha_3}\right)/(2h)\right], \left[\left(\mu^{\mathrm{U}}_{\alpha_1} + \mu^{\mathrm{U}}_{\alpha_3}\right)/(2h), 1 - \left(\nu^{\mathrm{L}}_{\alpha_1} + \nu^{\mathrm{L}}_{\alpha_3}\right)/(2h)\right]\right]$$

$$a_2 \boxplus a_3 = \left[\left[\frac{1}{2}\left(\mathrm{BF}^{\mathrm{L}}_{a_2} + \mathrm{BF}^{\mathrm{L}}_{a_3}\right), \frac{1}{2}\left(\mathrm{PF}^{\mathrm{L}}_{a_2} + \mathrm{PF}^{\mathrm{L}}_{a_3}\right)\right], \left[\frac{1}{2}\left(\mathrm{BF}^{\mathrm{U}}_{a_2} + \mathrm{BF}^{\mathrm{U}}_{a_3}\right), \frac{1}{2}\left(\mathrm{PF}^{\mathrm{U}}_{a_2} + \mathrm{PF}^{\mathrm{U}}_{a_3}\right)\right]\right]$$
$$= \left[\left[\left(\mu^{\mathrm{L}}_{\alpha_2} + \mu^{\mathrm{L}}_{\alpha_3}\right)/(2h), 1 - \left(\nu^{\mathrm{U}}_{\alpha_2} + \nu^{\mathrm{U}}_{\alpha_3}\right)/(2h)\right], \left[\left(\mu^{\mathrm{U}}_{\alpha_2} + \mu^{\mathrm{U}}_{\alpha_3}\right)/(2h), 1 - \left(\nu^{\mathrm{L}}_{\alpha_2} + \nu^{\mathrm{L}}_{\alpha_3}\right)/(2h)\right]\right]$$

According to the score function in Eq. (14) and the accuracy function in Eq. (15), we have

$$\mathbb{SV}_h(a_1 \boxplus a_3) = 2h + \left(\mu^{\mathrm{L}}_{\alpha_1} + \mu^{\mathrm{U}}_{\alpha_1} - \nu^{\mathrm{L}}_{\alpha_1} - \nu^{\mathrm{U}}_{\alpha_1} + \mu^{\mathrm{L}}_{\alpha_3} + \mu^{\mathrm{U}}_{\alpha_3} - \nu^{\mathrm{L}}_{\alpha_3} - \nu^{\mathrm{U}}_{\alpha_3}\right)/2$$

$$\mathbb{SV}_h(a_2 \boxplus a_3) = 2h + \left(\mu^{\mathrm{L}}_{\alpha_2} + \mu^{\mathrm{U}}_{\alpha_2} - \nu^{\mathrm{L}}_{\alpha_2} - \nu^{\mathrm{U}}_{\alpha_2} + \mu^{\mathrm{L}}_{\alpha_3} + \mu^{\mathrm{U}}_{\alpha_3} - \nu^{\mathrm{L}}_{\alpha_3} - \nu^{\mathrm{U}}_{\alpha_3}\right)/2$$

$$\mathbb{AV}_h(a_1 \boxplus a_3) = 2h - \left(\mu^{\mathrm{L}}_{\alpha_1} + \mu^{\mathrm{U}}_{\alpha_1} + \nu^{\mathrm{L}}_{\alpha_1} + \nu^{\mathrm{U}}_{\alpha_1} + \mu^{\mathrm{L}}_{\alpha_3} + \mu^{\mathrm{U}}_{\alpha_3} + \nu^{\mathrm{L}}_{\alpha_3} + \nu^{\mathrm{U}}_{\alpha_3}\right)/2$$

$$\mathbb{AV}_h(a_2 \boxplus a_3) = 2h - \left(\mu^{\mathrm{L}}_{\alpha_2} + \mu^{\mathrm{U}}_{\alpha_2} + \nu^{\mathrm{L}}_{\alpha_2} + \nu^{\mathrm{U}}_{\alpha_2} + \mu^{\mathrm{L}}_{\alpha_3} + \mu^{\mathrm{U}}_{\alpha_3} + \nu^{\mathrm{L}}_{\alpha_3} + \nu^{\mathrm{U}}_{\alpha_3}\right)/2$$

There are two possible situations where $a_1 \lessdot a_2$ on the basis of the comparison rules in Definition 14:

1. $\mathbb{SV}_h(a_1) < \mathbb{SV}_h(a_2)$: According to the expressions of $\mathbb{SV}_h(a_1)$ and $\mathbb{SV}_h(a_2)$, we obtain $\mu^{\mathrm{L}}_{\alpha_1} + \mu^{\mathrm{U}}_{\alpha_1} - \nu^{\mathrm{L}}_{\alpha_1} - \nu^{\mathrm{U}}_{\alpha_1} < \mu^{\mathrm{L}}_{\alpha_2} + \mu^{\mathrm{U}}_{\alpha_2} - \nu^{\mathrm{L}}_{\alpha_2} - \nu^{\mathrm{U}}_{\alpha_2}$. Based on this, we further obtain from the expressions of $\mathbb{SV}_h(a_1 \boxplus a_3)$ and $\mathbb{SV}_h(a_2 \boxplus a_3)$ that $\mathbb{SV}_h(a_1 \boxplus a_3) < \mathbb{SV}_h(a_2 \boxplus a_3)$. Therefore, we can obtain from the comparison rules in Definition 14 that $(a_1 \boxplus a_3) \lessdot (a_2 \boxplus a_3)$;
2. $\mathbb{SV}_h(a_1) = \mathbb{SV}_h(a_2)$ and $\mathbb{AV}_h(a_1) > \mathbb{AV}_h(a_2)$: According to the expressions of $\mathbb{SV}_h(a_1)$, $\mathbb{SV}_h(a_2)$, $\mathbb{AV}_h(a_1)$, and $\mathbb{AV}_h(a_2)$, we obtain $\mu^{\mathrm{L}}_{\alpha_1} + \mu^{\mathrm{U}}_{\alpha_1} - \nu^{\mathrm{L}}_{\alpha_1} - \nu^{\mathrm{U}}_{\alpha_1} = \mu^{\mathrm{L}}_{\alpha_2} + \mu^{\mathrm{U}}_{\alpha_2} - \nu^{\mathrm{L}}_{\alpha_2} - \nu^{\mathrm{U}}_{\alpha_2}$ and $\mu^{\mathrm{L}}_{\alpha_1} + \mu^{\mathrm{U}}_{\alpha_1} + \nu^{\mathrm{L}}_{\alpha_1} + \nu^{\mathrm{U}}_{\alpha_1} < \mu^{\mathrm{L}}_{\alpha_2} + \mu^{\mathrm{U}}_{\alpha_2} + \nu^{\mathrm{L}}_{\alpha_2} + \nu^{\mathrm{U}}_{\alpha_2}$. Based on this, we further obtain from the expressions of $\mathbb{SV}_h(a_1 \boxplus a_3)$, $\mathbb{SV}_h(a_2 \boxplus a_3)$, $\mathbb{AV}_h(a_1 \boxplus a_3)$, and $\mathbb{AV}_h(a_2 \boxplus a_3)$ that $\mathbb{SV}_h(a_1 \boxplus a_3) = \mathbb{SV}_h(a_2 \boxplus a_3)$ and $\mathbb{AV}_h(a_1 \boxplus a_3) > \mathbb{AV}_h(a_2 \boxplus a_3)$. Therefore, we can obtain from the comparison rules in Definition 14 that $(a_1 \boxplus a_3) \lessdot (a_2 \boxplus a_3)$.

On the basis of the two situations above, we can conclude that $a_1 \lessdot a_2$ can always imply $(a_1 \boxplus a_3) \lessdot (a_2 \boxplus a_3)$. □

The developed OR of LIVIFNs under DSET in Eq. (18) is free of the limitation of the OR of LIVIFNs in Eq. (5), as stated in the following theorem:

Theorem 2 *The operation in the OR of LIVIFNs under DSET in Eq. (18) is always persistent with respect to the score function in Eq. (14), the accuracy function in Eq. (15), and the comparison rules in Definition 14: For two arbitrary LIVIFNs under DSET a_1 and a_2 and an arbitrary positive number λ, $a_1 \lessdot a_2$ can always imply $\lambda a_1 \lessdot \lambda a_2$.*

Proof Let $a_k = \left[\left[\mathrm{BF}^{\mathrm{L}}_{a_k}, \mathrm{PF}^{\mathrm{L}}_{a_k}\right], \left[\mathrm{BF}^{\mathrm{U}}_{a_k}, \mathrm{PF}^{\mathrm{U}}_{a_k}\right]\right] = \left[\left[\mu^{\mathrm{L}}_{\alpha_k}/h, 1 - \nu^{\mathrm{U}}_{\alpha_k}/h\right], \left[\mu^{\mathrm{U}}_{\alpha_k}/h, 1 - \nu^{\mathrm{L}}_{\alpha_k}/h\right]\right]$ $(k \in \{1, 2\})$. According to the score function in Eq. (14) and the accuracy function in Eq. (15), we have

$$\mathbb{SV}_h(a_1) = h\left(\mathrm{BF}^{\mathrm{L}}_{a_1} + \mathrm{BF}^{\mathrm{U}}_{a_1} + \mathrm{PF}^{\mathrm{L}}_{a_1} + \mathrm{PF}^{\mathrm{U}}_{a_1}\right) = 2h + \left(\mu^{\mathrm{L}}_{\alpha_1} + \mu^{\mathrm{U}}_{\alpha_1} - \nu^{\mathrm{L}}_{\alpha_1} - \nu^{\mathrm{U}}_{\alpha_1}\right)$$

$$\mathbb{SV}_h(a_2) = h\left(\mathrm{BF}^{\mathrm{L}}_{a_2} + \mathrm{BF}^{\mathrm{U}}_{a_2} + \mathrm{PF}^{\mathrm{L}}_{a_2} + \mathrm{PF}^{\mathrm{U}}_{a_2}\right) = 2h + \left(\mu^{\mathrm{L}}_{\alpha_2} + \mu^{\mathrm{U}}_{\alpha_2} - \nu^{\mathrm{L}}_{\alpha_2} - \nu^{\mathrm{U}}_{\alpha_2}\right)$$

$$\mathbb{AV}_h(a_1) = h\left(\mathrm{PF}^{\mathrm{L}}_{a_1} + \mathrm{PF}^{\mathrm{U}}_{a_1} - \mathrm{BF}^{\mathrm{L}}_{a_1} - \mathrm{BF}^{\mathrm{U}}_{a_1}\right) = 2h - \left(\mu^{\mathrm{L}}_{\alpha_1} + \mu^{\mathrm{U}}_{\alpha_1} + \nu^{\mathrm{L}}_{\alpha_1} + \nu^{\mathrm{U}}_{\alpha_1}\right)$$

$$\mathbb{AV}_h(a_2) = h\left(\mathrm{PF}^{\mathrm{L}}_{a_2} + \mathrm{PF}^{\mathrm{U}}_{a_2} - \mathrm{BF}^{\mathrm{L}}_{a_2} - \mathrm{BF}^{\mathrm{U}}_{a_2}\right) = 2h - \left(\mu^{\mathrm{L}}_{\alpha_2} + \mu^{\mathrm{U}}_{\alpha_2} + \nu^{\mathrm{L}}_{\alpha_2} + \nu^{\mathrm{U}}_{\alpha_2}\right)$$

Using the OR of LIVIFNs under DSET in Eq. (18), we obtain

$$\lambda a_1 = \left[\left[\lambda\mathrm{BF}^{\mathrm{L}}_{a_1}, \lambda\mathrm{PF}^{\mathrm{L}}_{a_1}\right], \left[\lambda\mathrm{BF}^{\mathrm{U}}_{a_1}, \lambda\mathrm{PF}^{\mathrm{U}}_{a_1}\right]\right] = \left[\left[\lambda\left(\mu^{\mathrm{L}}_{\alpha_1}/h\right), \lambda\left(1 - \nu^{\mathrm{U}}_{\alpha_1}/h\right)\right], \left[\lambda\left(\mu^{\mathrm{U}}_{\alpha_1}/h\right), \lambda\left(1 - \nu^{\mathrm{L}}_{\alpha_1}/h\right)\right]\right]$$

$$\lambda a_2 = \left[\left[\lambda \mathrm{BF}^{\mathrm{L}}_{a_2}, \lambda \mathrm{PF}^{\mathrm{L}}_{a_2}\right], \left[\lambda \mathrm{BF}^{\mathrm{U}}_{a_2}, \lambda \mathrm{PF}^{\mathrm{U}}_{a_2}\right]\right] = \left[\left[\lambda\left(\mu^{\mathrm{L}}_{\alpha_2}/h\right), \lambda\left(1-v^{\mathrm{U}}_{\alpha_2}/h\right)\right], \left[\lambda\left(\mu^{\mathrm{U}}_{\alpha_2}/h\right), \lambda\left(1-v^{\mathrm{L}}_{\alpha_2}/h\right)\right]\right]$$

According to the score function in Eq. (14) and the accuracy function in Eq. (15), we have

$$\mathbb{SV}_h(\lambda a_1) = 2\lambda h + \lambda\left(\mu^{\mathrm{L}}_{\alpha_1} + \mu^{\mathrm{U}}_{\alpha_1} - v^{\mathrm{L}}_{\alpha_1} - v^{\mathrm{U}}_{\alpha_1}\right)$$

$$\mathbb{SV}_h(\lambda a_2) = 2\lambda h + \lambda\left(\mu^{\mathrm{L}}_{\alpha_2} + \mu^{\mathrm{U}}_{\alpha_2} - v^{\mathrm{L}}_{\alpha_2} - v^{\mathrm{U}}_{\alpha_2}\right)$$

$$\mathbb{AV}_h(\lambda a_1) = 2\lambda h - \lambda\left(\mu^{\mathrm{L}}_{\alpha_1} + \mu^{\mathrm{U}}_{\alpha_1} + v^{\mathrm{L}}_{\alpha_1} + v^{\mathrm{U}}_{\alpha_1}\right)$$

$$\mathbb{AV}_h(\lambda a_2) = 2\lambda h - \lambda\left(\mu^{\mathrm{L}}_{\alpha_2} + \mu^{\mathrm{U}}_{\alpha_2} + v^{\mathrm{L}}_{\alpha_2} + v^{\mathrm{U}}_{\alpha_2}\right)$$

There are two possible situations where $a_1 \boxed{<} a_2$ on the basis of the comparison rules in Definition 14:

1. $\mathbb{SV}_h(a_1) < \mathbb{SV}_h(a_2)$: According to the expressions of $\mathbb{SV}_h(a_1)$ and $\mathbb{SV}_h(a_2)$, we obtain $\mu^{\mathrm{L}}_{\alpha_1} + \mu^{\mathrm{U}}_{\alpha_1} - v^{\mathrm{L}}_{\alpha_1} - v^{\mathrm{U}}_{\alpha_1} < \mu^{\mathrm{L}}_{\alpha_2} + \mu^{\mathrm{U}}_{\alpha_2} - v^{\mathrm{L}}_{\alpha_2} - v^{\mathrm{U}}_{\alpha_2}$. Based on this, we further obtain from the expressions of $\mathbb{SV}_h(\lambda a_1)$ and $\mathbb{SV}_h(\lambda a_2)$ that $\mathbb{SV}_h(\lambda a_1) < \mathbb{SV}_h(\lambda a_2)$. Therefore, we can obtain from the comparison rules in Definition 14 that $\lambda a_1 \boxed{<} \lambda a_2$;
2. $\mathbb{SV}_h(a_1) = \mathbb{SV}_h(a_2)$ and $\mathbb{AV}_h(a_1) > \mathbb{AV}_h(a_2)$: According to the expressions of $\mathbb{SV}_h(a_1)$, $\mathbb{SV}_h(a_2)$, $\mathbb{AV}_h(a_1)$, and $\mathbb{AV}_h(a_2)$, we obtain $\mu^{\mathrm{L}}_{\alpha_1} + \mu^{\mathrm{U}}_{\alpha_1} - v^{\mathrm{L}}_{\alpha_1} - v^{\mathrm{U}}_{\alpha_1} = \mu^{\mathrm{L}}_{\alpha_2} + \mu^{\mathrm{U}}_{\alpha_2} - v^{\mathrm{L}}_{\alpha_2} - v^{\mathrm{U}}_{\alpha_2}$ and $\mu^{\mathrm{L}}_{\alpha_1} + \mu^{\mathrm{U}}_{\alpha_1} + v^{\mathrm{L}}_{\alpha_1} + v^{\mathrm{U}}_{\alpha_1} < \mu^{\mathrm{L}}_{\alpha_2} + \mu^{\mathrm{U}}_{\alpha_2} + v^{\mathrm{L}}_{\alpha_2} + v^{\mathrm{U}}_{\alpha_2}$. Based on this, we further obtain from the expressions of $\mathbb{SV}_h(\lambda a_1)$, $\mathbb{SV}_h(\lambda a_2)$, $\mathbb{AV}_h(\lambda a_1)$, and $\mathbb{AV}_h(\lambda a_2)$ that $\mathbb{SV}_h(\lambda a_1) = \mathbb{SV}_h(\lambda a_2)$ and $\mathbb{AV}_h(\lambda a_1) > \mathbb{AV}_h(\lambda a_2)$. Therefore, we can obtain from the comparison rules in Definition 14 that $\lambda a_1 \boxed{<} \lambda a_2$.

On the basis of the two situations above, we can conclude that $a_1 \boxed{<} a_2$ can always imply $\lambda a_1 \boxed{<} \lambda a_2$. □

WA Operator of LIVIFNs Based on DSET

Based on the developed ORs of LIVIFNs under DSET, a WA operator of LIVIFNs under DSET is constructed below:

Definition 16 Let α_i $(i \in \{1, 2, ..., n\})$ be n arbitrary LIVIFNs, a_i be α_i under DSET, and w_i be the weight of a_i such that $0 \leq w_i \leq 1$ and $\Sigma_{i=1}^{n} w_i = 1$. The aggregation function

$$\begin{aligned}\mathbb{LIVIFWA}(a_1, a_2, ..., a_n) &= \boxplus_{i=1}^{n}(w_i a_i) \\ &= \left[\left[\frac{1}{n}\sum_{i=1}^{n}\left(w_i \mathrm{BF}^{\mathrm{L}}_{a_i}\right), \frac{1}{n}\sum_{i=1}^{n}\left(w_i \mathrm{PF}^{\mathrm{L}}_{a_i}\right)\right], \left[\frac{1}{n}\sum_{i=1}^{n}\left(w_i \mathrm{BF}^{\mathrm{U}}_{a_i}\right), \frac{1}{n}\sum_{i=1}^{n}\left(w_i \mathrm{PF}^{\mathrm{U}}_{a_i}\right)\right]\right] \\ &= \left[\left[\frac{1}{n}\sum_{i=1}^{n}\left(w_i \mu^{\mathrm{L}}_{\alpha_i}/h\right), \frac{1}{n}\sum_{i=1}^{n}\left(w_i - w_i v^{\mathrm{U}}_{\alpha_i}/h\right)\right], \left[\frac{1}{n}\sum_{i=1}^{n}\left(w_i \mu^{\mathrm{U}}_{\alpha_i}/h\right), \frac{1}{n}\sum_{i=1}^{n}\left(w_i - w_i v^{\mathrm{L}}_{\alpha_i}/h\right)\right]\right]\end{aligned} \tag{26}$$

is called the LIVIFWA operator under DSET.

The LIVIFWA operator under DSET above has the property of monotonicity, as stated in the following theorem:

Theorem 3 *Let β_i $(i \in \{1, 2, ..., n\})$ be n arbitrary LIVIFNs and b_i be β_i under DSET. If $\mu^{\mathrm{L}}_{\alpha_i} \leq \mu^{\mathrm{L}}_{\beta_i}$, $\mu^{\mathrm{U}}_{\alpha_i} \leq \mu^{\mathrm{U}}_{\beta_i}$, $v^{\mathrm{L}}_{\alpha_i} \geq v^{\mathrm{L}}_{\beta_i}$, and $v^{\mathrm{U}}_{\alpha_i} \geq v^{\mathrm{U}}_{\beta_i}$ for all $i \in \{1, 2, ..., n\}$, then $\mathbb{LIVIFWA}(a_1, a_2, ..., a_n) \boxed{\leqslant} \mathbb{LIVIFWA}(b_1, b_2, ..., b_n)$.*

Proof According to the LIVIFWA operator under DSET in Eq. (26), we have

$$\mathbb{LIVIFWA}(b_1, b_2, ..., b_n) = \left[\left[\frac{1}{n}\sum_{i=1}^{n}\left(w_i \mu^{\mathrm{L}}_{\beta_i}/h\right), \frac{1}{n}\sum_{i=1}^{n}\left(w_i - w_i v^{\mathrm{U}}_{\beta_i}/h\right)\right], \left[\frac{1}{n}\sum_{i=1}^{n}\left(w_i \mu^{\mathrm{U}}_{\beta_i}/h\right), \frac{1}{n}\sum_{i=1}^{n}\left(w_i - w_i v^{\mathrm{L}}_{\beta_i}/h\right)\right]\right]$$

From $\mu^{\mathrm{L}}_{\alpha_i} \leq \mu^{\mathrm{L}}_{\beta_i}$ and $\mu^{\mathrm{U}}_{\alpha_i} \leq \mu^{\mathrm{U}}_{\beta_i}$, we have $\frac{1}{n}\sum_{i=1}^{n}\left(w_i \mu^{\mathrm{L}}_{\alpha_i}/h\right) \leq \frac{1}{n}\sum_{i=1}^{n}\left(w_i \mu^{\mathrm{L}}_{\beta_i}/h\right)$ and $\frac{1}{n}\sum_{i=1}^{n}\left(w_i \mu^{\mathrm{U}}_{\alpha_i}/h\right) \leq \frac{1}{n}\sum_{i=1}^{n}\left(w_i \mu^{\mathrm{U}}_{\beta_i}/h\right)$. From $v^{\mathrm{U}}_{\alpha_i} \geq v^{\mathrm{U}}_{\beta_i}$ and $v^{\mathrm{L}}_{\alpha_i} \geq v^{\mathrm{L}}_{\beta_i}$, we have $\frac{1}{n}\sum_{i=1}^{n}\left(w_i - w_i v^{\mathrm{U}}_{\alpha_i}/h\right) \leq \frac{1}{n}\sum_{i=1}^{n}\left(w_i - w_i v^{\mathrm{U}}_{\beta_i}/h\right)$ and $\frac{1}{n}\sum_{i=1}^{n}\left(w_i - w_i v^{\mathrm{L}}_{\alpha_i}/h\right) \leq \frac{1}{n}\sum_{i=1}^{n}\left(w_i - w_i v^{\mathrm{L}}_{\beta_i}/h\right)$. Based on this, we can obtain from the score function in Eq. (14) and the accuracy function in Eq. (15) that $\mathbb{SV}_h(\mathbb{LIVIFWA}(a_1, a_2, ..., a_n)) \leq \mathbb{SV}_h(\mathbb{LIVIFWA}(b_1, b_2, ..., b_n))$ and $\mathbb{AV}_h(\mathbb{LIVIFWA}(a_1, a_2, ..., a_n)) = \mathbb{AV}_h(\mathbb{LIVIFWA}(b_1, b_2, ..., b_n))$ if and only if $\mathbb{SV}_h(\mathbb{LIVIFWA}(a_1, a_2, ..., a_n)) = \mathbb{SV}_h(\mathbb{LIVIFWA}(b_1, b_2, ..., b_n))$. According to the comparison rules in Definition 14, we have $\mathbb{LIVIFWA}(a_1, a_2, ..., a_n) \boxed{\leqslant} \mathbb{LIVIFWA}(b_1, b_2, ..., b_n)$. □

The constructed LIVIFWA operator under DSET does not have the properties of idempotency and boundedness. However, a small modification of its expression (multiplying by n, i.e. $n\mathbb{LIVIFWA}(a_1, a_2, ..., a_n)$) can generate an LIVIFWA operator under DSET having idempotency and boundedness, as respectively stated in the following two theorems:

Theorem 4 *Let α be an arbitrary LIVIFN and a be α under DSET. If* $\mathrm{BF}^{\mathrm{L}}_{a_i} = \mathrm{BF}^{\mathrm{L}}_{a}$, $\mathrm{PF}^{\mathrm{L}}_{a_i} = \mathrm{PF}^{\mathrm{L}}_{a}$, $\mathrm{BF}^{\mathrm{U}}_{a_i} = \mathrm{BF}^{\mathrm{U}}_{a}$, *and* $\mathrm{PF}^{\mathrm{U}}_{a_i} = \mathrm{PF}^{\mathrm{U}}_{a}$ *for all* $i \in \{1, 2, ..., n\}$, *then* $n\mathbb{LIVIFWA}(a_1, a_2, ..., a_n) = \left[\left[\mathrm{BF}^{\mathrm{L}}_{a}, \mathrm{PF}^{\mathrm{L}}_{a}\right], \left[\mathrm{BF}^{\mathrm{U}}_{a}, \mathrm{PF}^{\mathrm{U}}_{a}\right]\right] = a$.

Proof According to the LIVIFWA operator under DSET in Eq. (26), we have

$$
\begin{aligned}
&n\mathbb{LIVIFWA}(a_1, a_2, ..., a_n) = n \boxplus_{i=1}^{n} (w_i a_i)\\
&= \left[\left[\sum_{i=1}^{n}\left(w_i \mathrm{BF}^{\mathrm{L}}_{a_i}\right), \sum_{i=1}^{n}\left(w_i \mathrm{PF}^{\mathrm{L}}_{a_i}\right)\right],\right.\\
&\quad \left.\left[\sum_{i=1}^{n}\left(w_i \mathrm{BF}^{\mathrm{U}}_{a_i}\right), \sum_{i=1}^{n}\left(w_i \mathrm{PF}^{\mathrm{U}}_{a_i}\right)\right]\right]\\
&= \left[\left[\sum_{i=1}^{n}\left(w_i \mu^{\mathrm{L}}_{\alpha_i}/h\right), \sum_{i=1}^{n}\left(w_i - w_i \nu^{\mathrm{U}}_{\alpha_i}/h\right)\right],\right.\\
&\quad \left.\left[\sum_{i=1}^{n}\left(w_i \mu^{\mathrm{U}}_{\alpha_i}/h\right), \sum_{i=1}^{n}\left(w_i - w_i \nu^{\mathrm{L}}_{\alpha_i}/h\right)\right]\right]
\end{aligned}
$$

From $\mathrm{BF}^{\mathrm{L}}_{a_i} = \mathrm{BF}^{\mathrm{L}}_{a}$, $\mathrm{PF}^{\mathrm{L}}_{a_i} = \mathrm{PF}^{\mathrm{L}}_{a}$, $\mathrm{BF}^{\mathrm{U}}_{a_i} = \mathrm{BF}^{\mathrm{U}}_{a}$, and $\mathrm{PF}^{\mathrm{U}}_{a_i} = \mathrm{PF}^{\mathrm{U}}_{a}$ for all $i \in \{1, 2, ..., n\}$, we can obtain $\sum_{i=1}^{n}\left(w_i \mu^{\mathrm{L}}_{\alpha_i}/h\right) = \mu^{\mathrm{L}}_{\alpha}/h$, $\sum_{i=1}^{n}\left(w_i - w_i \nu^{\mathrm{U}}_{\alpha_i}/h\right) = 1 - \nu^{\mathrm{U}}_{\alpha}/h$, $\sum_{i=1}^{n}\left(w_i \mu^{\mathrm{U}}_{\alpha_i}/h\right) = \mu^{\mathrm{U}}_{\alpha}/h$, and $\sum_{i=1}^{n}\left(w_i - w_i \nu^{\mathrm{L}}_{\alpha_i}/h\right) = 1 - \nu^{\mathrm{L}}_{\alpha}/h$. Therefore, we have $n\mathbb{LIVIFWA}(a_1, a_2, ..., a_n) = \left[\left[\mu^{\mathrm{L}}_{\alpha}/h, 1 - \nu^{\mathrm{U}}_{\alpha}/h\right], \left[\mu^{\mathrm{U}}_{\alpha}/h, 1 - \nu^{\mathrm{L}}_{\alpha}/h\right]\right] = a$. □

Theorem 5 *Let* $\alpha_- = \left(\left[\mu^{\mathrm{L}}_-, \mu^{\mathrm{U}}_-\right], \left[\nu^{\mathrm{L}}_-, \nu^{\mathrm{U}}_-\right]\right)$, $\alpha_+ = \left(\left[\mu^{\mathrm{L}}_+, \mu^{\mathrm{U}}_+\right], \left[\nu^{\mathrm{L}}_+, \nu^{\mathrm{U}}_+\right]\right)$, $a_- = \left[\left[\mathrm{BF}^{\mathrm{L}}_-, \mathrm{PF}^{\mathrm{L}}_-\right], \left[\mathrm{BF}^{\mathrm{U}}_-, \mathrm{PF}^{\mathrm{U}}_-\right]\right] = \left[\left[\mu^{\mathrm{L}}_-/h, 1 - \nu^{\mathrm{U}}_-/h\right], \left[\mu^{\mathrm{U}}_-/h, 1 - \nu^{\mathrm{L}}_-/h\right]\right]$, *and* $a_+ = \left[\left[\mathrm{BF}^{\mathrm{L}}_+, \mathrm{PF}^{\mathrm{L}}_+\right], \left[\mathrm{BF}^{\mathrm{U}}_+, \mathrm{PF}^{\mathrm{U}}_+\right]\right] = \left[\left[\mu^{\mathrm{L}}_+/h, 1 - \nu^{\mathrm{U}}_+/h\right], \left[\mu^{\mathrm{U}}_+/h, 1 - \nu^{\mathrm{L}}_+/h\right]\right]$, *where* $\mu^{\mathrm{L}}_- = \min\{\mu^{\mathrm{L}}_{\alpha_i}\}$, $\mu^{\mathrm{U}}_- = \min\{\mu^{\mathrm{U}}_{\alpha_i}\}$, $\nu^{\mathrm{L}}_- = \max\{\nu^{\mathrm{L}}_{\alpha_i}\}$, $\nu^{\mathrm{U}}_- = \max\{\nu^{\mathrm{U}}_{\alpha_i}\}$, $\mu^{\mathrm{L}}_+ = \max\{\mu^{\mathrm{L}}_{\alpha_i}\}$, $\mu^{\mathrm{U}}_+ = \max\{\mu^{\mathrm{U}}_{\alpha_i}\}$, $\nu^{\mathrm{L}}_+ = \min\{\nu^{\mathrm{L}}_{\alpha_i}\}$, *and* $\nu^{\mathrm{U}}_+ = \min\{\nu^{\mathrm{U}}_{\alpha_i}\}$ *for all* $i \in \{1, 2, ..., n\}$. *Then* $a_- \lesseqqgtr n\mathbb{LIVIFWA}(a_1, a_2, ..., a_n) \lesseqqgtr a_+$.

Proof On the basis of the proof of Theorem 3, it is easy to prove that $n\mathbb{LIVIFWA}$ has the property of monotonicity. From $\mu^{\mathrm{L}}_- = \min\{\mu^{\mathrm{L}}_{\alpha_i}\}$, $\mu^{\mathrm{U}}_- = \min\{\mu^{\mathrm{U}}_{\alpha_i}\}$, $\nu^{\mathrm{L}}_- = \max\{\nu^{\mathrm{L}}_{\alpha_i}\}$, $\nu^{\mathrm{U}}_- = \max\{\nu^{\mathrm{U}}_{\alpha_i}\}$, $\mu^{\mathrm{L}}_+ = \max\{\mu^{\mathrm{L}}_{\alpha_i}\}$, $\mu^{\mathrm{U}}_+ = \max\{\mu^{\mathrm{U}}_{\alpha_i}\}$, $\nu^{\mathrm{L}}_+ = \min\{\nu^{\mathrm{L}}_{\alpha_i}\}$, and $\nu^{\mathrm{U}}_+ = \min\{\nu^{\mathrm{U}}_{\alpha_i}\}$, we can obtain $\mu^{\mathrm{L}}_- \le \mu^{\mathrm{L}}_{\alpha_i} \le \mu^{\mathrm{L}}_+$, $\mu^{\mathrm{U}}_- \le \mu^{\mathrm{U}}_{\alpha_i} \le \mu^{\mathrm{U}}_+$, $\nu^{\mathrm{L}}_- \ge \nu^{\mathrm{L}}_{\alpha_i} \ge \nu^{\mathrm{L}}_+$, and $\nu^{\mathrm{U}}_- \ge \nu^{\mathrm{U}}_{\alpha_i} \ge \nu^{\mathrm{U}}_+$. Based on this, we have $n\mathbb{LIVIFWA}(a_-, a_-, ..., a_-) \lesseqqgtr n\mathbb{LIVIFWA}(a_1, a_2, ..., a_n) \lesseqqgtr n\mathbb{LIVIFWA}(a_+, a_+, ..., a_+)$. According to Theorem 4, we can obtain $n\mathbb{LIVIFWA}(a_-, a_-, ..., a_-) = a_-$ and $n\mathbb{LIVIFWA}(a_+, a_+, ..., a_+) = a_+$. Therefore, we have $a_- \lesseqqgtr n\mathbb{LIVIFWA}(a_1, a_2, ..., a_n) \lesseqqgtr a_+$. □

The constructed LIVIFWA operator under DSET in Eq. (26) is free of the limitation of the LIVIFWA operator in Eq. (13), as stated in the following theorem:

Theorem 6 *The LIVIFWA operator under DSET in Eq. (26) is always monotone with respect to the score function in Eq. (14), the accuracy function in Eq. (15), and the comparison rules in Definition 14: For three arbitrary LIVIFNs under DSET a_1, a_2, and a_3 and certain weights w_1 and w_2, $a_1 \lesseqqgtr a_2$ can always imply* $\mathbb{LIVIFWA}(a_1, a_3) \lesseqqgtr \mathbb{LIVIFWA}(a_2, a_3)$.

Proof Let $a_k = \left[\left[\mathrm{BF}^{\mathrm{L}}_{a_k}, \mathrm{PF}^{\mathrm{L}}_{a_k}\right], \left[\mathrm{BF}^{\mathrm{U}}_{a_k}, \mathrm{PF}^{\mathrm{U}}_{a_k}\right]\right] = \left[\left[\mu^{\mathrm{L}}_{\alpha_k}/h, 1 - \nu^{\mathrm{U}}_{\alpha_k}/h\right], \left[\mu^{\mathrm{U}}_{\alpha_k}/h, 1 - \nu^{\mathrm{L}}_{\alpha_k}/h\right]\right] (k \in \{1, 2, 3\})$. According to the score function in Eq. (14) and the accuracy function in Eq. (15), we have

$$
\begin{aligned}
\mathbb{SV}_h(a_1) &= h\left(\mathrm{BF}^{\mathrm{L}}_{a_1} + \mathrm{BF}^{\mathrm{U}}_{a_1} + \mathrm{PF}^{\mathrm{L}}_{a_1} + \mathrm{PF}^{\mathrm{U}}_{a_1}\right)\\
&= 2h + \left(\mu^{\mathrm{L}}_{\alpha_1} + \mu^{\mathrm{U}}_{\alpha_1} - \nu^{\mathrm{L}}_{\alpha_1} - \nu^{\mathrm{U}}_{\alpha_1}\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{SV}_h(a_2) &= h\left(\mathrm{BF}^{\mathrm{L}}_{a_2} + \mathrm{BF}^{\mathrm{U}}_{a_2} + \mathrm{PF}^{\mathrm{L}}_{a_2} + \mathrm{PF}^{\mathrm{U}}_{a_2}\right)\\
&= 2h + \left(\mu^{\mathrm{L}}_{\alpha_2} + \mu^{\mathrm{U}}_{\alpha_2} - \nu^{\mathrm{L}}_{\alpha_2} - \nu^{\mathrm{U}}_{\alpha_2}\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{AV}_h(a_1) &= h\left(\mathrm{PF}^{\mathrm{L}}_{a_1} + \mathrm{PF}^{\mathrm{U}}_{a_1} - \mathrm{BF}^{\mathrm{L}}_{a_1} - \mathrm{BF}^{\mathrm{U}}_{a_1}\right)\\
&= 2h - \left(\mu^{\mathrm{L}}_{\alpha_1} + \mu^{\mathrm{U}}_{\alpha_1} + \nu^{\mathrm{L}}_{\alpha_1} + \nu^{\mathrm{U}}_{\alpha_1}\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{AV}_h(a_2) &= h\left(\mathrm{PF}^{\mathrm{L}}_{a_2} + \mathrm{PF}^{\mathrm{U}}_{a_2} - \mathrm{BF}^{\mathrm{L}}_{a_2} - \mathrm{BF}^{\mathrm{U}}_{a_2}\right)\\
&= 2h - \left(\mu^{\mathrm{L}}_{\alpha_2} + \mu^{\mathrm{U}}_{\alpha_2} + \nu^{\mathrm{L}}_{\alpha_2} + \nu^{\mathrm{U}}_{\alpha_2}\right)
\end{aligned}
$$

Using the LIVIFWA operator under DSET in Eq. (26), we obtain

$$
\begin{aligned}
\mathbb{LIVIFWA}(a_1, a_3) &= \left[\left[\frac{1}{2}\left(w_1 \mathrm{BF}^{\mathrm{L}}_{a_1} + w_2 \mathrm{BF}^{\mathrm{L}}_{a_3}\right), \frac{1}{2}\left(w_1 \mathrm{PF}^{\mathrm{L}}_{a_1} + w_2 \mathrm{PF}^{\mathrm{L}}_{a_3}\right)\right],\right.\\
&\quad \left.\left[\frac{1}{2}\left(w_1 \mathrm{BF}^{\mathrm{U}}_{a_1} + w_2 \mathrm{BF}^{\mathrm{U}}_{a_3}\right), \frac{1}{2}\left(w_1 \mathrm{PF}^{\mathrm{U}}_{a_1} + w_2 \mathrm{PF}^{\mathrm{U}}_{a_3}\right)\right]\right]\\
&= \left[\left[\frac{1}{2h}\left(w_1 \mu^{\mathrm{L}}_{\alpha_1} + w_2 \mu^{\mathrm{L}}_{\alpha_3}\right), \frac{1}{2h}\left(h - w_1 \nu^{\mathrm{U}}_{\alpha_1} - w_2 \nu^{\mathrm{U}}_{\alpha_3}\right)\right],\right.\\
&\quad \left.\left[\frac{1}{2h}\left(w_1 \mu^{\mathrm{U}}_{\alpha_1} + w_2 \mu^{\mathrm{U}}_{\alpha_3}\right), \frac{1}{2h}\left(h - w_1 \nu^{\mathrm{L}}_{\alpha_1} - w_2 \nu^{\mathrm{L}}_{\alpha_3}\right)\right]\right]
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{LIVIFWA}(a_2, a_3) &= \left[\left[\frac{1}{2}\left(w_1 \mathrm{BF}^{\mathrm{L}}_{a_2} + w_2 \mathrm{BF}^{\mathrm{L}}_{a_3}\right), \frac{1}{2}\left(w_1 \mathrm{PF}^{\mathrm{L}}_{a_2} + w_2 \mathrm{PF}^{\mathrm{L}}_{a_3}\right)\right],\right.\\
&\quad \left.\left[\frac{1}{2}\left(w_1 \mathrm{BF}^{\mathrm{U}}_{a_2} + w_2 \mathrm{BF}^{\mathrm{U}}_{a_3}\right), \frac{1}{2}\left(w_1 \mathrm{PF}^{\mathrm{U}}_{a_2} + w_2 \mathrm{PF}^{\mathrm{U}}_{a_3}\right)\right]\right]\\
&= \left[\left[\frac{1}{2h}\left(w_1 \mu^{\mathrm{L}}_{\alpha_2} + w_2 \mu^{\mathrm{L}}_{\alpha_3}\right), \frac{1}{2h}\left(h - w_1 \nu^{\mathrm{U}}_{\alpha_2} - w_2 \nu^{\mathrm{U}}_{\alpha_3}\right)\right],\right.\\
&\quad \left.\left[\frac{1}{2h}\left(w_1 \mu^{\mathrm{U}}_{\alpha_2} + w_2 \mu^{\mathrm{U}}_{\alpha_3}\right), \frac{1}{2h}\left(h - w_1 \nu^{\mathrm{L}}_{\alpha_2} - w_2 \nu^{\mathrm{L}}_{\alpha_3}\right)\right]\right]
\end{aligned}
$$

According to the score function in Eq. (14) and the accuracy function in Eq. (15), we have

$$
\begin{aligned}
\mathbb{SV}_h(\mathbb{LIVIFWA}(a_1, a_3)) = h &+ \frac{w_1}{2}\left(\mu^{\mathrm{L}}_{\alpha_1} + \mu^{\mathrm{U}}_{\alpha_1} - \nu^{\mathrm{L}}_{\alpha_1} - \nu^{\mathrm{U}}_{\alpha_1}\right)\\
&+ \frac{w_2}{2}\left(\mu^{\mathrm{L}}_{\alpha_3} + \mu^{\mathrm{U}}_{\alpha_3} - \nu^{\mathrm{L}}_{\alpha_3} - \nu^{\mathrm{U}}_{\alpha_3}\right)
\end{aligned}
$$

$$\mathbb{SV}_h(\mathbb{LIVIFWA}(a_2,a_3)) = h + \frac{w_1}{2}\left(\mu^{\mathrm{L}}_{\alpha_2} + \mu^{\mathrm{U}}_{\alpha_2} - \nu^{\mathrm{L}}_{\alpha_2} - \nu^{\mathrm{U}}_{\alpha_2}\right) + \frac{w_2}{2}\left(\mu^{\mathrm{L}}_{\alpha_3} + \mu^{\mathrm{U}}_{\alpha_3} - \nu^{\mathrm{L}}_{\alpha_3} - \nu^{\mathrm{U}}_{\alpha_3}\right)$$

$$\mathbb{AV}_h(\mathbb{LIVIFWA}(a_1,a_3)) = h - \frac{w_1}{2}\left(\mu^{\mathrm{L}}_{\alpha_1} + \mu^{\mathrm{U}}_{\alpha_1} + \nu^{\mathrm{L}}_{\alpha_1} + \nu^{\mathrm{U}}_{\alpha_1}\right) - \frac{w_2}{2}\left(\mu^{\mathrm{L}}_{\alpha_3} + \mu^{\mathrm{U}}_{\alpha_3} + \nu^{\mathrm{L}}_{\alpha_3} + \nu^{\mathrm{U}}_{\alpha_3}\right)$$

$$\mathbb{AV}_h(\mathbb{LIVIFWA}(a_2,a_3)) = h - \frac{w_1}{2}\left(\mu^{\mathrm{L}}_{\alpha_2} + \mu^{\mathrm{U}}_{\alpha_2} + \nu^{\mathrm{L}}_{\alpha_2} + \nu^{\mathrm{U}}_{\alpha_2}\right) - \frac{w_2}{2}\left(\mu^{\mathrm{L}}_{\alpha_3} + \mu^{\mathrm{U}}_{\alpha_3} + \nu^{\mathrm{L}}_{\alpha_3} + \nu^{\mathrm{U}}_{\alpha_3}\right)$$

There are two possible situations where $a_1 \lessdot a_2$ on the basis of the comparison rules in Definition 14:

1. $\mathbb{SV}_h(a_1) < \mathbb{SV}_h(a_2)$: According to the expressions of $\mathbb{SV}_h(a_1)$ and $\mathbb{SV}_h(a_2)$, we obtain $\mu^{\mathrm{L}}_{\alpha_1} + \mu^{\mathrm{U}}_{\alpha_1} - \nu^{\mathrm{L}}_{\alpha_1} - \nu^{\mathrm{U}}_{\alpha_1} < \mu^{\mathrm{L}}_{\alpha_2} + \mu^{\mathrm{U}}_{\alpha_2} - \nu^{\mathrm{L}}_{\alpha_2} - \nu^{\mathrm{U}}_{\alpha_2}$. Based on this, we further obtain from the expressions of $\mathbb{SV}_h(\mathbb{LIVIFWA}(a_1,a_3))$ and $\mathbb{SV}_h(\mathbb{LIVIFWA}(a_2,a_3))$ that $\mathbb{SV}_h(\mathbb{LIVIFWA}(a_1,a_3)) < \mathbb{SV}_h(\mathbb{LIVIFWA}(a_2,a_3))$. Therefore, we can obtain from the comparison rules in Definition 14 that $\mathbb{LIVIFWA}(a_1,a_3) \lessdot \mathbb{LIVIFWA}(a_2,a_3)$;
2. $\mathbb{SV}_h(a_1) = \mathbb{SV}_h(a_2)$ and $\mathbb{AV}_h(a_1) > \mathbb{AV}_h(a_2)$: According to the expressions of $\mathbb{SV}_h(a_1)$, $\mathbb{SV}_h(a_2)$, $\mathbb{AV}_h(a_1)$, and $\mathbb{AV}_h(a_2)$, we obtain $\mu^{\mathrm{L}}_{\alpha_1} + \mu^{\mathrm{U}}_{\alpha_1} - \nu^{\mathrm{L}}_{\alpha_1} - \nu^{\mathrm{U}}_{\alpha_1} = \mu^{\mathrm{L}}_{\alpha_2} + \mu^{\mathrm{U}}_{\alpha_2} - \nu^{\mathrm{L}}_{\alpha_2} - \nu^{\mathrm{U}}_{\alpha_2}$ and $\mu^{\mathrm{L}}_{\alpha_1} + \mu^{\mathrm{U}}_{\alpha_1} + \nu^{\mathrm{L}}_{\alpha_1} + \nu^{\mathrm{U}}_{\alpha_1} < \mu^{\mathrm{L}}_{\alpha_2} + \mu^{\mathrm{U}}_{\alpha_2} + \nu^{\mathrm{L}}_{\alpha_2} + \nu^{\mathrm{U}}_{\alpha_2}$. Based on this, we further obtain from the expressions of $\mathbb{SV}_h(\mathbb{LIVIFWA}(a_1,a_3))$, $\mathbb{SV}_h(\mathbb{LIVIFWA}(a_2,a_3))$, $\mathbb{AV}_h(\mathbb{LIVIFWA}(a_1,a_3))$, and $\mathbb{AV}_h(\mathbb{LIVIFWA}(a_2,a_3))$ that $\mathbb{SV}_h(\mathbb{LIVIFWA}(a_1,a_3)) = \mathbb{SV}_h(\mathbb{LIVIFWA}(a_2,a_3))$ and $\mathbb{AV}_h(\mathbb{LIVIFWA}(a_1,a_3)) > \mathbb{AV}_h(\mathbb{LIVIFWA}(a_2,a_3))$. Therefore, we can obtain from the comparison rules in Definition 14 that $\mathbb{LIVIFWA}(a_1,a_3) \lessdot \mathbb{LIVIFWA}(a_2,a_3)$.

On the basis of the two situations above, we can conclude that $a_1 \lessdot a_2$ can always imply $\mathbb{LIVIFWA}(a_1,a_3) \lessdot \mathbb{LIVIFWA}(a_2,a_3)$. □

DM Method Based on the New WA Operator

A cognitively inspired DM problem with LIVIFNs is generally described by m alternatives A_i $(i \in \{1,2,...,m\})$, n criteria C_j $(j \in \{1,2,...,n\})$, a vector of weights of criteria $(w_1, w_2, ..., w_n)$ such that $0 \le w_j \le 1$ is the weight of C_j and $\Sigma_{j=1}^{n} w_j = 1$, n' experts (an expert refers to a person with special knowledge, experience, or skills in the domain to which the DM problem belongs who provides evaluation values of criteria) E_k $(k \in \{1,2,...,n'\})$, a vector of weights of experts $(w'_1, w'_2, ..., w'_{n'})$ such that $0 \le w'_k \le 1$ is the weight of E_k and $\Sigma_{k=1}^{n'} w'_k = 1$, $h+1$ linguistic terms $0, 1, ..., h$, and n' decision matrices $\boldsymbol{M}_k = [\alpha_{kij}] = \left[\left(\left[\mu^{\mathrm{L}}_{\alpha_{kij}}, \mu^{\mathrm{U}}_{\alpha_{kij}}\right], \left[\nu^{\mathrm{L}}_{\alpha_{kij}}, \nu^{\mathrm{U}}_{\alpha_{kij}}\right]\right)\right]$ such that each α_{kij} is an LIVIFN which represents the value of C_j of A_i evaluated by E_k. The aim of solving such a problem is to determine the best alternative from A_i on the basis of $\boldsymbol{M}_k$, $(w'_1, w'_2, ..., w'_{n'})$, and $(w_1, w_2, ..., w_n)$. Using a DM method based on the constructed WA operator of LIVIFNs under DSET, the problem can be solved via the following steps:

1. Normalise the decision matrices $\boldsymbol{M}_k$. There are two types of criteria in multi-criterion decision-making, which are benefit and cost criteria. A benefit criterion is a criterion that has positive effect on the decision-making result (the larger its value, the more favourable the decision-making result), while a cost criterion is a criterion that affects the decision-making result adversely (the smaller its value, the more favourable the decision-making result). For example, total area and price belong to a benefit criterion and a cost criterion in selection of a house to buy, respectively, since the larger the total area and the lower the price, the more favourable the decision-making result. A DM problem may contain only benefit criteria, both benefit and cost criteria, or only cost criteria. When it contains cost criteria, specific rules are generally applied to normalise the values of cost criteria to obtain normalised decision matrices. For the studied DM problem with LIVIFNs, the normalisation rules and normalised decision matrices are expressed as follows:

$$\boldsymbol{N}_k = \left[\left(\left[p^{\mathrm{L}}_{\alpha_{kij}}, p^{\mathrm{U}}_{\alpha_{kij}}\right], \left[q^{\mathrm{L}}_{\alpha_{kij}}, q^{\mathrm{U}}_{\alpha_{kij}}\right]\right)\right] = \begin{cases} \left[\left(\left[\mu^{\mathrm{L}}_{\alpha_{kij}}, \mu^{\mathrm{U}}_{\alpha_{kij}}\right], \left[\nu^{\mathrm{L}}_{\alpha_{kij}}, \nu^{\mathrm{U}}_{\alpha_{kij}}\right]\right)\right] & \text{if } C_j \text{ is a benefit criterion} \\ \left[\left(\left[\nu^{\mathrm{L}}_{\alpha_{kij}}, \nu^{\mathrm{U}}_{\alpha_{kij}}\right], \left[\mu^{\mathrm{L}}_{\alpha_{kij}}, \mu^{\mathrm{U}}_{\alpha_{kij}}\right]\right)\right] & \text{if } C_j \text{ is a cost criterion} \end{cases} \tag{27}$$

2. Convert the LIVIFNs in the normalised decision matrices $\boldsymbol{N}_k$ into LIVIFNs under DSET to obtain the following matrices:

$$\boldsymbol{N'}_k = \left[a_{kij}\right] = \left[\left[\left[p^{\mathrm{L}}_{\alpha_{kij}}/h, 1 - q^{\mathrm{U}}_{\alpha_{kij}}/h\right], \left[p^{\mathrm{U}}_{\alpha_{kij}}/h, 1 - q^{\mathrm{L}}_{\alpha_{kij}}/h\right]\right]\right] \tag{28}$$

3. Calculate the summary values of a_{kij} using the LIVIFWA operator under DSET in Eq. (26) with the weight vector $(w'_1, w'_2, ..., w'_{n'})$:

$$\left[a_{ij}\right] = \left[\mathbb{LIVIFWA}(a_{1ij}, a_{2ij}, ..., a_{n'ij})\right] \tag{29}$$

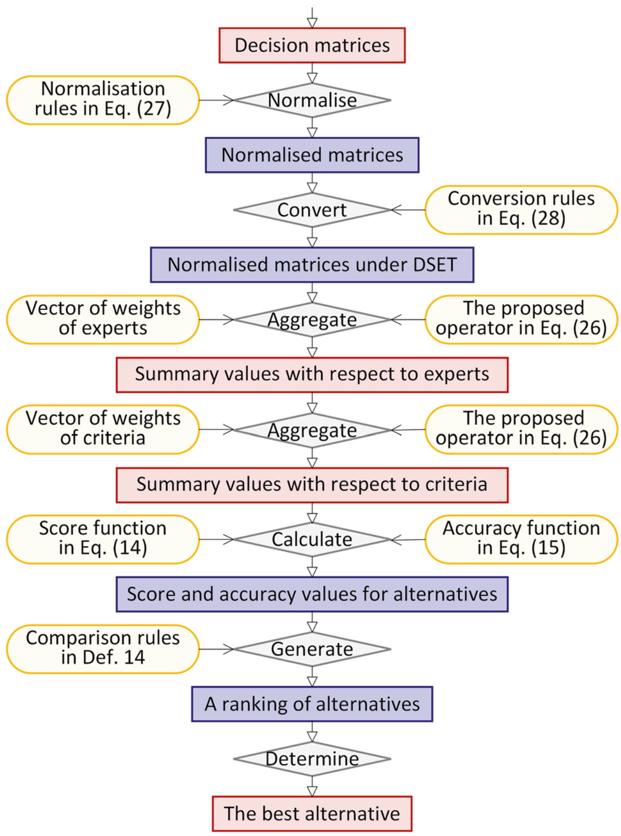

Fig. 1 General flow of the proposed DM method

4. Calculate the summary values of a_{ij} using the LIVIFWA operator under DSET in Eq. (26) with the weight vector $(w_1, w_2, ..., w_n)$:

$$[a_i] = \left[\mathbb{LIVIFWA}(a_{i1}, a_{i2}, ..., a_{in})\right] \tag{30}$$

5. Calculate the score values of a_i using the score function in Eq. (14) and the accuracy values of a_i using the accuracy function in Eq. (15).
6. Rank a_i according to the comparison rules in Definition 14 and determine the best alternative based on the ranking results.

According to the steps above, the general flow of the proposed DM method is depicted in Fig. 1.

Application and Comparisons

Application of the New DM Method

Additive manufacturing, commonly known as three-dimensional printing, is an emerging manufacturing technology that builds three-dimensional objects via adding material layer by layer. Compared to traditional manufacturing technologies, this technology has advantages in providing maximum design freedom, manufacturing objects with complex geometries at no additional cost, generating less waste material, avoiding a lot of assembly, and producing customised products. Because of these advantages, the research and application of additive manufacturing technology are gaining importance and popularity. To date, more than 1,700 (data from Senvol Database) machines that are based on the technology have been identified in the market. For a practical application, how to select a proper additive manufacturing machine from several alternatives is of great importance since this will directly affect the quality of the final product. To assist selection of additive manufacturing machines, several different types of methods have been presented during the past two decades. A representative type of methods is multi-criterion decision-making method. This type of method determines a proper additive manufacturing machine from a

Table 1 Evaluation results of the three domain experts

Matrix	Expert	Machine	C_1	C_2	C_3	C_4
$\boldsymbol{M}_1$	E_1	M_1	([2, 4], [3, 4])	([6, 6], [1, 2])	([5, 6], [1, 1])	([6, 6], [1, 1])
		M_2	([3, 5], [2, 3])	([5, 6], [1, 2])	([4, 5], [1, 1])	([4, 6], [1, 2])
		M_3	([1, 1], [6, 7])	([6, 6], [1, 2])	([3, 4], [3, 3])	([5, 6], [1, 2])
		M_4	([1, 1], [7, 7])	([3, 4], [2, 3])	([3, 5], [2, 3])	([2, 3], [3, 4])
$\boldsymbol{M}_2$	E_2	M_1	([3, 4], [3, 4])	([5, 6], [1, 2])	([6, 6], [1, 2])	([5, 6], [1, 2])
		M_2	([3, 5], [1, 3])	([6, 6], [1, 1])	([5, 6], [1, 2])	([3, 5], [1, 2])
		M_3	([1, 2], [6, 6])	([5, 6], [1, 1])	([3, 5], [3, 3])	([5, 6], [1, 2])
		M_4	([1, 2], [6, 6])	([3, 3], [3, 3])	([3, 4], [2, 3])	([3, 4], [3, 4])
$\boldsymbol{M}_3$	E_3	M_1	([3, 3], [3, 4])	([6, 6], [2, 2])	([5, 6], [1, 1])	([6, 7], [1, 1])
		M_2	([3, 4], [2, 3])	([6, 7], [1, 1])	([5, 5], [1, 3])	([5, 5], [1, 2])
		M_3	([2, 3], [5, 5])	([4, 5], [1, 2])	([3, 4], [3, 4])	([4, 5], [2, 2])
		M_4	([1, 1], [6, 7])	([3, 4], [2, 3])	([3, 5], [1, 2])	([4, 5], [3, 3])

Table 2 Elements of the normalised decision matrices

Matrix	Expert	Machine	C_1	C_2	C_3	C_4
$\boldsymbol{N}_1$	E_1	M_1	([3, 4], [2, 4])	([6, 6], [1, 2])	([5, 6], [1, 1])	([6, 6], [1, 1])
		M_2	([2, 3], [3, 5])	([5, 6], [1, 2])	([4, 5], [1, 1])	([4, 6], [1, 2])
		M_3	([6, 7], [1, 1])	([6, 6], [1, 2])	([3, 4], [3, 3])	([5, 6], [1, 2])
		M_4	([7, 7], [1, 1])	([3, 4], [2, 3])	([3, 5], [2, 3])	([2, 3], [3, 4])
$\boldsymbol{N}_2$	E_2	M_1	([3, 4], [3, 4])	([5, 6], [1, 2])	([6, 6], [1, 2])	([5, 6], [1, 2])
		M_2	([1, 3], [3, 5])	([6, 6], [1, 1])	([5, 6], [1, 2])	([3, 5], [1, 2])
		M_3	([6, 6], [1, 2])	([5, 6], [1, 1])	([3, 5], [3, 3])	([5, 6], [1, 2])
		M_4	([6, 6], [1, 2])	([3, 3], [3, 3])	([3, 4], [2, 3])	([3, 4], [3, 4])
$\boldsymbol{N}_3$	E_3	M_1	([3, 4], [3, 3])	([6, 6], [2, 2])	([5, 6], [1, 1])	([6, 7], [1, 1])
		M_2	([2, 3], [3, 4])	([6, 7], [1, 1])	([5, 5], [1, 3])	([5, 5], [1, 2])
		M_3	([5, 5], [2, 3])	([4, 5], [1, 2])	([3, 4], [3, 4])	([4, 5], [2, 2])
		M_4	([6, 7], [1, 1])	([3, 4], [2, 3])	([3, 5], [1, 2])	([4, 5], [3, 3])

certain number of alternatives via comprehensively considering multiple criteria of the alternatives. The following is an additive manufacturing machine selection example [33] for illustrating the application of the proposed DM method.

In this example, a decision maker needs to select a proper additive manufacturing machine from four alternative machines M_1, M_2, M_3, and M_4 to build a part with certain material. The decision maker invited three domain experts E_1, E_2, and E_3 to evaluate the four alternative machines. The evaluation criteria include the predicted surface roughness (C_1), predicted strength (C_2), predicted elongation (C_3), and predicted hardness (C_4) of the as-built part. The weights of the three experts are respectively 0.4, 0.3, and 0.3. The weights of the four criteria are respectively 0.1, 0.3, 0.3, and 0.3. The three experts were asked to use LIVIFNs to express their evaluation results. There are nine available linguistic terms, which are extremely small (0), very small (1), small (2), slightly small (3), medium (4), slightly large (5), large (6), very large (7), and extremely large (8). The evaluation results are listed in Table 1.

Using the proposed DM method, the problem above is solved through the following steps:

1. Since predicted surface roughness (C_1) is a cost criterion and predicted strength (C_2), predicted elongation (C_3), and predicted hardness (C_4) are three benefit criteria, according to Eq. (27), the decision matrices $\boldsymbol{M}_k$ ($k \in \{1, 2, 3\}$), whose elements are listed in Table 1, are normalised as $\boldsymbol{N}_k$, whose elements are listed in Table 2.
2. According to Eq. (28), the LIVIFNs in the normalised decision matrices $\boldsymbol{N}_k$ are converted into LIVIFNs under DSET to obtain three matrices $\boldsymbol{N'}_k$, whose elements are listed in Table 3.
3. According to Eq. (29) and the weight vector (0.4, 0.3, 0.3), the three matrices $\boldsymbol{N'}_k$ are aggregated into a single matrix $\boldsymbol{N'}$, whose elements are listed in Table 4.
4. According to Eq. (30) and the weight vector (0.1, 0.3, 0.3, 0.3), the elements in each row of the matrix $\boldsymbol{N'}$ are aggregated into a single LIVIFN under DSET:

Table 3 Elements of the converted decision matrices

Matrix	Expert	Machine	C_1	C_2	C_3	C_4
$\boldsymbol{N'}_1$	E_1	M_1	$[[3/8, 4/8], [4/8, 6/8]]$	$[[6/8, 6/8], [6/8, 7/8]]$	$[[5/8, 7/8], [6/8, 7/8]]$	$[[6/8, 7/8], [6/8, 7/8]]$
		M_2	$[[2/8, 3/8], [3/8, 5/8]]$	$[[5/8, 6/8], [6/8, 7/8]]$	$[[4/8, 7/8], [5/8, 7/8]]$	$[[4/8, 6/8], [6/8, 7/8]]$
		M_3	$[[6/8, 7/8], [7/8, 7/8]]$	$[[6/8, 6/8], [6/8, 7/8]]$	$[[3/8, 5/8], [4/8, 5/8]]$	$[[5/8, 6/8], [6/8, 7/8]]$
		M_4	$[[7/8, 7/8], [7/8, 7/8]]$	$[[3/8, 5/8], [4/8, 6/8]]$	$[[3/8, 5/8], [5/8, 6/8]]$	$[[2/8, 4/8], [3/8, 5/8]]$
$\boldsymbol{N'}_2$	E_2	M_1	$[[3/8, 4/8], [4/8, 5/8]]$	$[[5/8, 6/8], [6/8, 7/8]]$	$[[6/8, 6/8], [6/8, 7/8]]$	$[[5/8, 6/8], [6/8, 7/8]]$
		M_2	$[[1/8, 3/8], [3/8, 5/8]]$	$[[6/8, 7/8], [6/8, 7/8]]$	$[[5/8, 6/8], [6/8, 7/8]]$	$[[3/8, 6/8], [5/8, 7/8]]$
		M_3	$[[6/8, 6/8], [6/8, 7/8]]$	$[[5/8, 7/8], [6/8, 7/8]]$	$[[3/8, 5/8], [5/8, 5/8]]$	$[[5/8, 6/8], [6/8, 7/8]]$
		M_4	$[[6/8, 6/8], [6/8, 7/8]]$	$[[3/8, 5/8], [3/8, 5/8]]$	$[[3/8, 5/8], [4/8, 6/8]]$	$[[3/8, 4/8], [4/8, 5/8]]$
$\boldsymbol{N'}_3$	E_3	M_1	$[[3/8, 5/8], [4/8, 5/8]]$	$[[6/8, 6/8], [6/8, 6/8]]$	$[[5/8, 7/8], [6/8, 7/8]]$	$[[6/8, 7/8], [7/8, 7/8]]$
		M_2	$[[2/8, 4/8], [3/8, 5/8]]$	$[[6/8, 7/8], [7/8, 7/8]]$	$[[5/8, 5/8], [5/8, 7/8]]$	$[[5/8, 6/8], [5/8, 7/8]]$
		M_3	$[[5/8, 5/8], [5/8, 6/8]]$	$[[4/8, 6/8], [5/8, 7/8]]$	$[[3/8, 4/8], [4/8, 5/8]]$	$[[4/8, 6/8], [5/8, 6/8]]$
		M_4	$[[6/8, 7/8], [7/8, 7/8]]$	$[[3/8, 5/8], [4/8, 6/8]]$	$[[3/8, 6/8], [5/8, 7/8]]$	$[[4/8, 5/8], [5/8, 5/8]]$

Table 4 Elements of the aggregated decision matrix

Machine	C_1	C_2
M_1	$[[0.1250, 0.1792], [0.1667, 0.2250]]$	$[[0.2375, 0.2500], [0.2500, 0.2792]]$
M_2	$[[0.0708, 0.1375], [0.1250, 0.2083]]$	$[[0.2333, 0.2750], [0.2625, 0.2917]]$
M_3	$[[0.2375, 0.2542], [0.2542, 0.2792]]$	$[[0.2125, 0.2625], [0.2375, 0.2917]]$
M_4	$[[0.2667, 0.2792], [0.2792, 0.2917]]$	$[[0.1250, 0.2083], [0.1542, 0.2375]]$
Machine	C_3	C_4
M_1	$[[0.2208, 0.2792], [0.2500, 0.2917]]$	$[[0.2375, 0.2792], [0.2625, 0.2917]]$
M_2	$[[0.1917, 0.2542], [0.2208, 0.2917]]$	$[[0.1667, 0.2500], [0.2250, 0.2917]]$
M_3	$[[0.1250, 0.1958], [0.1792, 0.2083]]$	$[[0.1958, 0.2500], [0.2375, 0.2792]]$
M_4	$[[0.1250, 0.2208], [0.1958, 0.2625]]$	$[[0.1208, 0.1792], [0.1625, 0.2083]]$

$$a_1 = [[0.0069, 0.2412], [0.0081, 0.2423]]$$

$$a_2 = [[0.0058, 0.2411], [0.0077, 0.2430]]$$

$$a_3 = [[0.0057, 0.2418], [0.0074, 0.2431]]$$

$$a_4 = [[0.0043, 0.2424], [0.0066, 0.2443]]$$

5. Using the score function in Eq. (14), the score values of $a_i\,(i \in \{1, 2, 3, 4\})$ are calculated as follows:

$$\mathbb{SV}_h(a_1) = 3.9887$$

$$\mathbb{SV}_h(a_2) = 3.9809$$

$$\mathbb{SV}_h(a_3) = 3.9846$$

$$\mathbb{SV}_h(a_4) = 3.9812$$

Using the accuracy function in Eq. (15), the accuracy values of a_i are calculated as follows:

$$\mathbb{AV}_h(a_1) = 3.7479$$

$$\mathbb{AV}_h(a_2) = 3.7649$$

$$\mathbb{AV}_h(a_3) = 3.7737$$

$$\mathbb{AV}_h(a_4) = 3.8071$$

6. According to the comparison rules in Definition 14, a_i are ranked as $a_1 \boxed{>} a_3 \boxed{>} a_4 \boxed{>} a_2$. Therefore, the best additive manufacturing machine is M_1.

Comparisons with Existing Methods

To verify the effectiveness of the proposed DM method, a comparison of the ranking results of the method and the DM methods with LIVIFNs presented by [16, 27, 37], and [33] is carried out. In this comparison, the problem in Sect. 5.1 is taken as a benchmark. The linguistic interval-valued intuitionistic fuzzy prioritised weighted averaging

Table 5 Details and results of the first comparison

DM method	Used aggregation operators	SV of M_1	SV of M_2	SV of M_3	SV of M_4	Generated ranking
[27]	LIVIFPWA, LIVIFPWA	6.1654	5.8171	5.5985	4.8181	$M_1 \succ M_2 \succ M_3 \succ M_4$
[27]	LIVIFPWG, LIVIFPWG	5.9921	5.5132	5.3441	4.5370	$M_1 \succ M_2 \succ M_3 \succ M_4$
[16]	LIVIFWA, LIVIFWA	6.1654	5.8171	5.5985	4.8181	$M_1 \succ M_2 \succ M_3 \succ M_4$
[16]	LIVIFWG, LIVIFWG	5.9921	5.5132	5.3441	4.5370	$M_1 \succ M_2 \succ M_3 \succ M_4$
[37]	LIVIFWMSM, LIVIFWMSM	5.4523	5.0127	5.2577	4.7343	$M_1 \succ M_3 \succ M_2 \succ M_4$
[33]	LIVIFPWMM, LIVIFPWMM	5.7674	5.4217	5.3606	4.7043	$M_1 \succ M_2 \succ M_3 \succ M_4$
The proposed method	$\mathbb{LIVIFWA}$, $\mathbb{LIVIFWA}$	3.9887	3.9809	3.9846	3.9812	$M_1 \succ M_3 \succ M_4 \succ M_2$

SV stands for score value; $\succ$ stands for 'is followed by'; For the method of [37], $k = 1$ in the first aggregation by the LIVIFWMSM operator and $k = 3$ in the second aggregation by the LIVIFWMSM operator; For the method of [33], $Q = (1, 0, 0)$ in the first aggregation by the LIVIFPWMM operator and $Q = (1, 2, 3, 0)$ in the second aggregation by the LIVIFPWMM operator; The LIVIFPWA and LIVIFPWG operators respectively reduce to the LIVIFWA and LIVIFWG operators, since the weights given in the DM problem are directly taken as their priority weights

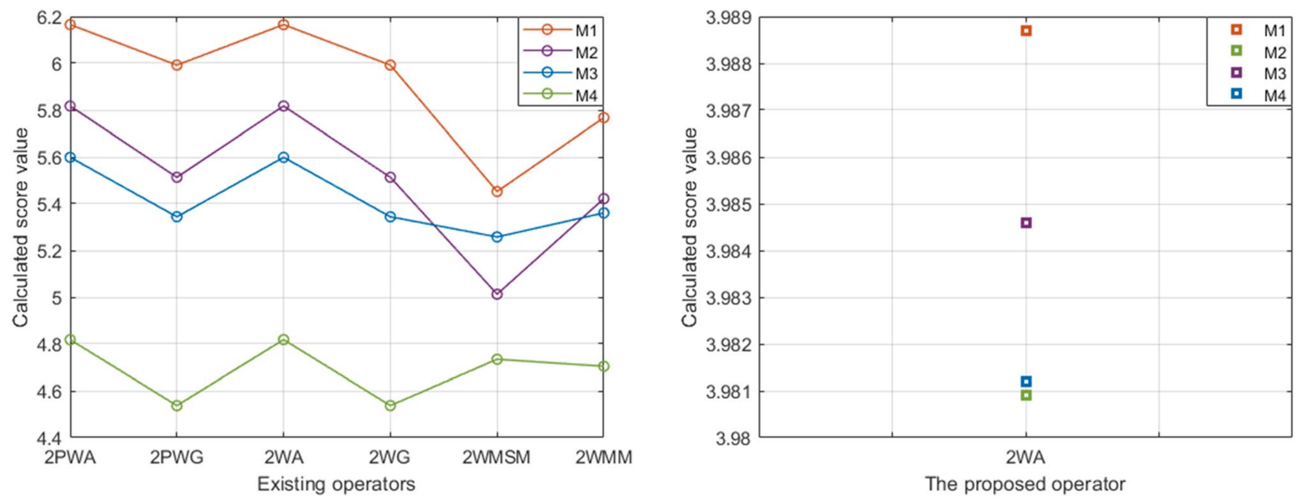

Fig. 2 Graphical presentation of the results of the first comparison

(LIVIFPWA) operator and the linguistic interval-valued intuitionistic fuzzy prioritised weighted geometric (LIVIFPWG) operator are respectively used in the method of [27]. The LIVIFWA operator and the linguistic interval-valued intuitionistic fuzzy weighted geometric (LIVIFWG) operator are respectively used in the method of [16]. The linguistic interval-valued intuitionistic fuzzy weighted Maclaurin symmetric mean (LIVIFWMSM) operator is used in the method of [37]. The linguistic interval-valued intuitionistic fuzzy power weighted Muirhead mean (LIVIFPWMM) operator is used in the method of [33]. To facilitate the comparison, the methods of [16, 27, 37], and [33] use the score function in Eq. (1), the accuracy function in Eq. (2), and the comparison rules in Definition 4 uniformly to generate the ranking results. The details and results of the comparison are listed in Table 5 and depicted in Fig. 2. It can be seen from Table 5 and Fig. 2 that the best additive manufacturing machine determined by the proposed method is exactly the same as that determined by all other methods. This demonstrates the effectiveness of the proposed method in solving practical cognitively inspired DM problems with LIVIFNs.

The advantage of the proposed method is that the developed ORs of LIVIFNs under DSET are always invariant and persistent and the presented LIVIFWA operator under DSET is always monotone. This advantage has been proved in Theorem 1, Theorem 2, and Theorem 6. To show the advantage more intuitively and how it affects the DM results, another comparison of the ranking results of the DM methods in Table 5 is carried out. In this comparison, Example 3 in Sect. 3.2 is taken as a benchmark. The AOs used in each method are the same as that in the first comparison. Further, the methods of [16, 27, 37], and [33] also use the score function in Eq. (1), the accuracy function in Eq. (2), and the comparison rules in Definition 4 uniformly to generate the ranking results. The results of the comparison are listed in Table 6 and depicted in Fig. 3. As can be seen from Table 6 and Fig. 3, the proposed method can generate an intuitive ranking for Example 3, while other comparison methods

Table 6 Details and results of the second comparison

DM method	Used operator	$\mathrm{SV}_h(o(\alpha_1,\alpha_3))$ $/\mathbb{SV}_h(o(a_1,a_3))$	$\mathrm{SV}_h(o(\alpha_2,\alpha_3))$ $/\mathbb{SV}_h(o(a_2,a_3))$	$\mathrm{AV}(o(\alpha_1,\alpha_3))$ $/\mathbb{AV}_h(o(a_1,a_3))$	$\mathrm{AV}(o(\alpha_2,\alpha_3))$ $/\mathbb{AV}_h(o(a_2,a_3))$	Generated ranking
[27]	LIVIFPWA	4.6123	4.6055	4.7754	5.2110	$o(\alpha_1,\alpha_3) \circledgtr o(\alpha_2,\alpha_3)$
[27]	LIVIFPWG	4.5946	4.5880	4.8109	5.1761	$o(\alpha_1,\alpha_3) \circledgtr o(\alpha_2,\alpha_3)$
[16]	LIVIFWA	4.6123	4.6055	4.7754	5.2110	$o(\alpha_1,\alpha_3) \circledgtr o(\alpha_2,\alpha_3)$
[16]	LIVIFWG	4.5946	4.5880	4.8109	5.1761	$o(\alpha_1,\alpha_3) \circledgtr o(\alpha_2,\alpha_3)$
[37]	LIVIFWMSM	4.5570	4.5369	4.7681	5.2683	$o(\alpha_1,\alpha_3) \circledgtr o(\alpha_2,\alpha_3)$
[33]	LIVIFPWMM	4.5664	4.5477	4.7742	5.2621	$o(\alpha_1,\alpha_3) \circledgtr o(\alpha_2,\alpha_3)$
The proposed method	$\mathbb{LIVIFWA}$	9.2000	9.2000	3.2000	2.8000	$o(a_1,a_3) \boxed{<} o(a_2,a_3)$

o stands for the used operator; a_1, a_2, and a_3 respectively stand for α_1, α_2, and α_3 under DSET; For the method of [37], $k=2$ in the aggregation by the LIVIFWMSM operator; For the method of [33], $Q=(1,2)$ in the aggregation by the LIVIFPWMM operator; The LIVIFPWA and LIVIFPWG operators respectively reduce to the LIVIFWA and LIVIFWG operators, since the weights given in Example 3 are directly taken as their priority weights

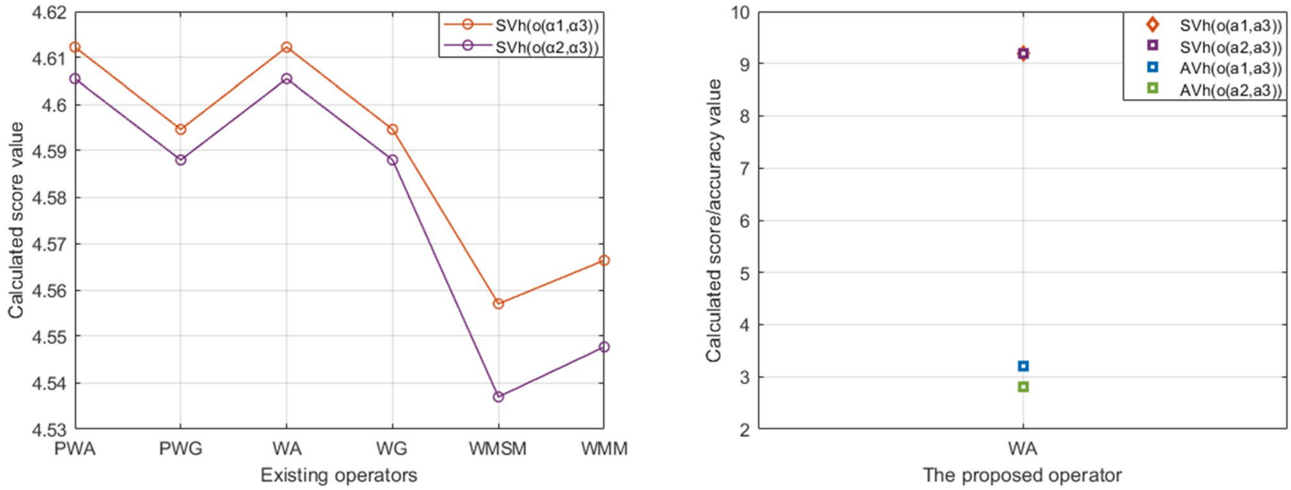

Fig. 3 Graphical presentation of the results of the second comparison

produce a counterintuitive ranking. This is because the proposed method uses the ORs of LIVIFNs under DSET that are always invariant and persistent, while all of other comparison methods use the ORs of LIVIFNs in Definition 5 that do not have these properties.

Conclusion

In this paper, a WA operator of LIVIFNs under DSET is presented to solve cognitively inspired DM problems with LIVIFNs. Firstly, an interpretation of LIVIFS under DSET is given. Based on this interpretation, four novel ORs of LIVIFNs are then developed. The characteristics of these ORs are highlighted and proved. After that, a new WA operator of LIVIFNs, i.e. the LIVIFWA operator under DSET, is constructed using the developed ORs. The properties of this operator is explored and its advantage is highlighted and proved. Finally, a method for solving cognitively inspired DM problems with LIVIFNs based on the constructed operator is proposed. The paper also introduces a numerical example to illustrate the application of the proposed method and documents quantitative comparisons with several existing methods to demonstrate the effectiveness and advantage of the method.

The main contributions of the paper are threefold:

1. Four ORs of LIVIFNs based on DSET are developed. Compared to the existing most used ORs of LIVIFNs, the developed ones are always invariant and persistent with respect to the score function, accuracy function, and comparison rules of LIVIFNs under DSET;
2. A WA operator of LIVIFNs based on DSET is constructed. Compared to the existing WA operator of LIVIFNs, the constructed one is always monotone with respect to the score function, accuracy function, and comparison rules of LIVIFNs under DSET;
3. A new method to solve cognitively inspired DM problems with LIVIFNs is proposed. This method has the advantages of the developed ORs and constructed AO.

Future work will aim especially at improving the proposed method to be capable to solve the cognitively inspired DM problems with LIVIFNs where the weights of criteria are expressed by LIVIFNs. In some cognitively inspired DM problems with LIVIFNs, decision makers may use LIVIFNs to describe the degrees of importance of the considered criteria. The proposed method is applicable for the problems where the degrees of importance of criteria are expressed as decimals. It cannot be applied to the problems where the weights of criteria are in the form of LIVIFNs. To address this limitation, two rules for the power and division operations between two LIVIFNs under DSET would be developed and a new AO would be constructed using these rules. Further, it would be interesting to combine the developed novel ORs with the power average operator, Bonferroni mean operator, Maclaurin symmetric mean operator, and Muirhead mean operator under linguistic interval-valued intuitionistic fuzzy environment to construct some more powerful AOs of LIVIFNs. Last but not least, applications of the constructed AOs to solve more cognitively inspired DM problems would also be studied.

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Data Availability The implementation code of the proposed method and related data have been deposited in the GitHub repository (https://github.com/YuchuChingQin/LivifwaDSetOperator).

Declarations

Conflict of Interest The authors declare no competing interests.

Ethical Approval This article does not contain any studies with human participants or animals performed by any of the authors.

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