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Résumé: Considérons un jeu stochastique à deux joueurs et à somme-nulle avec un espace d'état Borélien S, des espaces d'actions métriques et compacts A, B et une probabilité de transition q telle que l'intégrale sous q de toute fonction mesurable et bornée dépend mesurablement de l'état initial s et continument des actions (a,b) des joueurs. Supposons que le paiement est une fonction bornée f des histoires infinies des états et actions. Admettons enfin que f soit mesurable pour le produit des topologies Boréliennes (des espaces des cordonnées) et semi-continue inférieurement pour le produit des topologies discrètes. Alors, le jeu a une valeur et le joueur II a une stratégie optimale parfaite en sous jeux.

Abstract: Consider a two-person zero-sum stochastic game with Borel state space S, compact metric action sets A, B and law of motion q such that the integral under q of every bounded Borel measurable function depends measurably on the initial state s and continuously on the actions (a,b) of the players. Suppose the payoff is a bounded function f of the infinite history of states and actions such that f is measurable for the product of the Borel sigma-fields of the coordinate spaces and is lower semi continuous for the product of the discrete topologies on the coordinate spaces. Then the game has a value and player II has a subgame perfect optimal strategy.

Mots clés : Jeux stochastiques, perfection en sous jeux, ensembles Boréliens.

Key Words : Stochastic games, subgame perfect, Borel sets.

Classification AMS: 60G40, 91A60, 60E15, 46A55.

1 Introduction.

The stochastic games treated here are two-person and zero-sum. The state space S is a Borel subset of a Polish space. Players I and II have compact metric action sets A and B, respectively. The law of motion is a Borel transition function q from $S \times A \times B$ to S such that for each bounded Borel function g on $A \times B \times S$, the function

$$(a, b, s) \mapsto \int g(a, b, s')\, q(ds'|s, a, b)$$

is jointly continuous in a, b (for the product of the compact metric topologies on A and B) for each fixed s.

The game starts at some initial state s_1. Player I chooses an action $a_1 \in A$ and, simultaneously, player II chooses an action $b_1 \in B$. The next state s_2 has distribution $q(\cdot|s_1, a_1, b_1)$ and is announced to the players together with the actions a_1, b_1 chosen by them. This procedure is iterated so as to generate a random sequence

$$h = (s_1, (a_1, b_1, s_2), \ldots, (a_{n-1}, b_{n-1}, s_n), \ldots)$$

in the infinite product space $H = S \times (A \times B \times S)^N$ where N is the set of positive integers. The payoff from II to I is $f(h)$ where f is a bounded function from H to the real numbers. We assume that f is a Borel measurable function of h when the infinite product space H is assigned the product topology of the coordinate spaces under which S is a Borel set, and A and B are compact metric spaces. We further assume that f is lower semicontinuous when H is assigned the product of the discrete topologies on the spaces S, A, and B. In the sequel the terms "Borel" and "measurable" will always refer to the first topology on H, while the terms "open,""closed,""continuous," and "semicontinuous" will refer to the second. Thus f is assumed to be Borel, lower semicontinuous. (A simple example of such a function is the indicator of the set of histories h such that $s_n \in B$ for some n, where B is a Borel subset of S. An example of a Borel, continuous function is the familiar discounted payoff $f(h) = \sum_n \beta^{n-1} r(s_n, a_n, b_n)$ where $0 < \beta < 1$ and $r : S \times A \times B \mapsto \mathbf{R}$ is bounded Borel.) The game just described will be denoted by $\Gamma(f, s_1)$.

Denote by S^* the disjoint union of the sets S, $S \times (A \times B \times S)$, $, \ldots, S \times (A \times B \times S)^n, \ldots$, that is,

$$S^* = \bigcup_{n \geq 0} [S \times (A \times B \times S)^n].$$

The elements of S^* are called *partial histories.*

Let $P(A)$ and $P(B)$ be the sets of probability measures defined on the Borel subsets of A and B, respectively. A *strategy* σ for player I assigns to each $p = (s_1, (a_1, b_1, s_2), \ldots, (a_{n-1}, b_{n-1}, s_n)) \in S^*$ the conditional distribution $\sigma(p) \in P(A)$ for a_n given p. Formally, a strategy σ for player I is a Borel function from S^* into $P(A)$. A strategy τ for player II is defined similarly with $P(B)$ in place of $P(A)$. An initial state s_1 and strategies σ and τ for the players determine the distribution $P_{\sigma,\tau}$ of the sequence $h = (s_1, (a_1, b_1, s_2), \ldots, (a_{n-1}, b_{n-1}, s_n), \ldots)$. The expected payoff from II to I in the game $\Gamma(f, s_1)$ is then $E_{\sigma,\tau} f = \int f(h)\, P_{\sigma,\tau}(dh)$.

Let $p = (s_1, (a_1, b_1, s_2), \ldots, (a_{n-1}, b_{n-1}, s_n)) \in S \times (A \times B \times S)^{n-1}$ be a partial history. The *length* of p, $lh(p)$, is defined to be $n - 1$. (Thus $lh(s) = 0$ for $s \in S$.)

Denote by $l(p)$ the last state s_n of p. The *subgame* $\Gamma(f,p)$ is the game with initial state $s_n = l(p)$ and payoff function fp defined for sequences

$$h' = (s_n, (a'_1, b'_1, s'_2), (a'_2, b'_2, s'_3), \ldots)$$

to be

$$fp(h') = f(s_1, (a_1, b_1, s_2), \ldots, (a_{n-1}, b_{n-1}, s_n), (a'_1, b'_1, s'_2), (a'_2, b'_2, s'_3), \ldots).$$

Thus fp is the section of f at p. (Notice that the game $\Gamma(f,s)$ is itself the subgame $\Gamma(f,p)$ for which $p = s$.) For strategies σ and τ, the *conditional strategies* given the partial history p are written $\sigma[p]$ and $\tau[p]$, respectively. A strategy τ for player II, say, is said to be *subgame perfect* if $\tau[p]$ is optimal for player II in the subgame $\Gamma(f,p)$ for every $p \in S^*$.

Here is our main result and a corollary.

Theorem 1.1. *Suppose that f is a bounded, Borel, lower semicontinuous function on H. Then the game $\Gamma(f,p)$ has a value $V(p)$ for each $p \in S^*$ and the value function V is Borel measurable. For each $\epsilon > 0$ and $p \in S^*$, player I has an ϵ-optimal strategy for the game $\Gamma(f,p)$. Player II has a subgame perfect strategy.*

Corollary 1.2. *If f is bounded, Borel, and continuous, then both players have subgame perfect strategies.*

These results generalize those of Sengupta (1975) who treated the case where the state space S is compact metric, the action sets A and B are finite, and the payoff function is lower semicontinuous on H when S is given its compact metric topology.

The key to the proof of Theorem 1.1 is a result in the next section on the structure of Borel, lower semicontinuous functions. Sections 3 and 4 treat one-day and n-day games, respectively. The proof of Theorem 1.1 is completed in section 5. The final section has some additional remarks.

2 Borel, lower semicontinuous functions

Let X be a Borel subset of a Polish space, and let $\tilde{X}$ be the product space X^N. As in section 1, the term "Borel" will refer to the product of the topologies for which X is a Borel set and the term "lower semicontinuous" will refer to the product of discrete topologies. We abbreviate "lower semicontinuous" by "l.s.c" below.

Theorem 2.1. *Let f be a bounded, Borel l.s.c. function on $\tilde{X}$. Then there exist uniformly bounded Borel functions $f_n, n \in N$, on $\tilde{X}$ such that $f_n \le f_{n+1}$, f_n depends only on the first n coordinates of $\tilde{X}$ and $f = \lim_n f_n$.*

Proof. For each rational r, let $B_r = [f > r]$. Then each B_r is Borel, open, and

$$r < r' \Rightarrow B_{r'} \subseteq B_r.$$

By Corollary 2.4 in Maitra et al (1990), there is for each r a Borel stopping time t_r such that $B_r = [t_r < \infty]$. Define, for each $n \ge 1$, $C_{r,n} = [t_r \le n]$. Then, for each r and n, $C_{r,n}$ is a Borel cylinder set with base contained in X^n, $C_{r,n} \subseteq C_{r,n+1}$, and

$B_r = \bigcup_{n\geq 1} C_{r,n}$. Now let $B_{r,n} = \bigcup_{r\leq r'} C_{r',n}$. Then the sets $B_{r,n}$ are cylinders with base contained in X^n, $B_{r,n} \subseteq B_{r,n+1}$, and

$$r < r' \Rightarrow B_{r',n} \subseteq B_{r,n}.$$

Moreover,

$$\bigcup_{n\geq 1} B_{r,n} = \bigcup_{n\geq 1} \bigcup_{r\leq r'} C_{r',n} = \bigcup_{r\leq r'} \bigcup_{n\geq 1} C_{r',n} = \bigcup_{r\leq r'} B_{r'} = B_r.$$

Define $f_n : \tilde{X} \mapsto \mathbf{R}$ by

$$f_n(h) = \sup\{r \mid h \in B_{r,n}\} = \sup_r [r 1_{B_{r,n}}(h)].$$

Since the sets $B_{r,n}$ depend only on the first n coordinates, so does f_n. Plainly, f_n is bounded and Borel measurable. Since $B_{r,n} \subseteq B_{r,n+1}$, it follows that $f_n \leq f_{n+1}$, so that $\lim_n f_n \leq f$. Suppose now that $\lim_n f_n(h) < f(h)$. Then there is a rational r such that $\lim_n f_n(h) < r < f(h)$. Hence, $h \in B_r$. So there is an n such that $1_{B_{r,n}}(h) = 1$. Hence, $r \leq f_n(h)$, a contradiction. Thus $f \leq \lim_n f_n$. This completes the proof. □

We now apply Theorem 2.1 to a function defined on the space $(A \times B \times S)^N$ and we write $((a_1, b_1, s_1), (a_2, b_2, s_2), \ldots)$ for a typical element of this space.

Corollary 2.2. *Suppose that f is a bounded, Borel l.s.c. function on $(A\times B\times S)^N$ and that f does not depend on the coordinates a_1, b_1. Then there exist uniformly bounded Borel functions f_n such that $f_n \leq f_{n+1}$, f_n depends only on $s_1, (a_2, b_2, s_2), \ldots, (a_n, b_n, s_n)$ and $\lim_n f_n = f$.*

Proof. By Theorem 2.1, there exist bounded Borel functions g_n such that $g_n \leq g_{n+1}$, g_n depends only on $((a_1, b_1, s_1), (a_2, b_2, s_2), \ldots, (a_n, b_n, s_n))$, and $f = \lim_n g_n$. Fix $a^* \in A$, $b^* \in B$ and define $f_n((a_1, b_1, s_1), (a_2, b_2, s_2), \ldots, (a_n, b_n, s_n), \ldots)$ to equal $g_n((a^*, b^*, s_1), (a_2, b_2, s_2), \ldots, (a_n, b_n, s_n), \ldots)$. Then the functions f_n satisfy the assertions of the corollary. □

3 Parametrized one-day games

Let Y be a Borel subset of a Polish space and let $u : Y \times A \times B \times S \mapsto \mathbf{R}$ be a bounded, Borel function. We regard Y as a space of parameters and write $u_y(a, b, s)$ for $u(y, a, b, s)$. The one-day game with initial state $s \in S$ and terminal payoff u_y is played as follows. Players I and II choose actions $a \in A$, $b \in B$ simultaneously and a new state s' with distribution $q(\cdot|s, a, b)$ results. Player II then pays player I the amount $u_y(a, b, s')$. This game will be denoted by $\Gamma_1(u_y, s)$.

Theorem 3.1. *For each $s \in S$ and $y \in Y$ the game $\Gamma_1(u_y, s)$ has a value $Gu_y(s)$. Moreover, there exist Borel functions $\bar{\mu} : Y \times S \mapsto P(A)$ and $\bar{\nu} : Y \times S \mapsto P(B)$ such that for each y, s, $\bar{\mu}(y, s)$ and $\bar{\nu}(y, s)$ are optimal strategies for I and II, respectively, in $\Gamma_1(u_y, s)$. The function $(s, y) \mapsto Gu_y(s)$ is jointly Borel measurable in y and s.*

Proof. By Ky Fan's (1953) minimax theorem, the game $\Gamma_1(u_y, s)$ has a value, which will be denoted by $Gu_y(s)$, and both players have optimal strategies. Now define a multifunction F on $Y \times S$, having subsets of $P(A) \times P(B)$ as values, by setting $F(y, s)$

equal to the set of all pairs (μ,ν) such that μ is optimal for I and ν is optimal for II in $\Gamma_1(u_y,s)$. It is straightforward to verify that $(\mu,\nu)\in F(y,s)$ if and only if, for all $n\geq 1$

$$\int\int w_y(s,a,b)\,\mu(da)\nu(db)\geq \int w_y(s,a_n,b)\,\nu(db)$$

and

$$\int\int w_y(s,a,b)\,\mu(da)\nu(db)\leq \int w_y(s,a,b_n)\,\mu(da),$$

where

$$w_y(s,a,b)=\int u_y(a,b,s')\,q(ds'|s,a,b)$$

and $\{a_n,n\geq 1\}$ and $\{b_n,n\geq 1\}$ are dense in A and B, respectively.

Now $F(y,s)$ is a nonempty compact subset of $P(A)\times P(B)$ for each y,s, and it is easy to check that the graph of F, namely, the set

$$Gr(F)=\{(y,s,\mu,\nu)\in Y\times S\times P(A)\times P(B)\,|\,(\mu,\nu)\in F(y,s)\},$$

is a Borel subset of $Y\times S\times P(A)\times P(B)$. It therefore follows from the Kunugui-Novikov theorem (Theorem 5.7.1, Srivastava(1998)) that there is a Borel function $\phi: Y\times S\mapsto P(A)\times P(B)$ such that $\phi(y,s)\in F(y,s)$ for each y,s. Take $\bar{\mu}(y,s)=\pi_A(\phi(y,s))$ and $\bar{\nu}(y,s)=\pi_B(\phi(y,s))$, $y\in Y$, $s\in S$, where π_A and π_B are the projections of $P(A)\times P(B)$ onto $P(A)$ and $P(B)$, respectively. Clearly, the maps $\bar{\mu}$ and $\bar{\nu}$ are Borel. Also, since $\bar{\mu}(y,s)$ ($\bar{\nu}(y,s)$) is optimal for player I (player II) in $\Gamma_1(u_y,s)$, it follows that

$$Gu_y(s)=\int\int w_y(s,a,b)\,\bar{\mu}(y,s)(da)\bar{\nu}(y,s)(db),\qquad y\in Y,\ s\in S.$$

Hence, $(s,y)\mapsto Gu_y(s)$ is Borel measurable.

□

Let $u_n: Y\times A\times B\times S\mapsto \mathbf{R}$ for $n\geq 1$ be a sequence of bounded Borel functions.We write $u_{n,y}(a,b,s)$ for $u_n(y,a,b,s)$ below.

Theorem 3.2. *Suppose that $u_n\leq u_{n+1}$ for $n\geq 1$ and let $u=\lim_n u_n$. Then, for all $y\in Y$,*

a) $Gu_{n,y}\leq Gu_{n+1,y}$, and

b) $Gu_y=\lim_n Gu_{n,y}$.

Proof. Assertion (a) is clear. For (b), fix y and s. Consider the game $\Gamma_1(u_y,s)$. Choose an optimal strategy μ for I, so that

$$\int\int u_y(a,b,s')\,q(ds'|s,a,b)\mu(da)\geq Gu_y(s)$$

for all $b\in B$. By the monotone convergence theorem

$$\int\int u_{n,y}(a,b,s')\,q(ds'|s,a,b)\mu(da)\uparrow\int\int u_y(a,b,s')\,q(ds'|s,a,b)\mu(da). \tag{3.1}$$

Now both the expression on the left and that on the right of (3.1) are continuous functions of b for fixed y and s. Since B is compact, by Dini's theorem, it follows that the convergence in (3.1) is uniform on B.

Let $\epsilon > 0$. Choose M so that for all $m \geq M$ and $b \in B$,

$$\int \int u_{m,y}(a, b, s')\, q(ds'|s, a, b)\mu(da) \geq Gu_y(s) - \epsilon.$$

Consequently, for all $m \geq M$,

$$\inf_{\nu \in P(B)} \int \int \int u_{m,y}(a, b, s')\, q(ds'|s, a, b)\mu(da)\nu(db) \geq Gu_y(s) - \epsilon.$$

It follows that

$$Gu_{m,y}(s) \geq Gu(s) - \epsilon$$

for all $m \geq M$. As ϵ is arbitrary, this proves that

$$\lim_m Gu_{m,y}(s) \geq Gu_y(s).$$

The inequality in the opposite direction is trivial.

□

4 n-day games

Here we consider games in which the payoff depends only on the first n days of play. The main result is the following Borel version of classical backward induction.

Theorem 4.1. *Suppose f is a bounded, Borel function on H which depends only on $s_1, (a_1, b_1, s_2), \ldots, (a_n, b_n, s_{n+1})$. Then players I and II have subgame perfect strategies in $\Gamma(f, \cdot)$. If, for $p \in S^*$, $V(p)$ is the value of the subgame $\Gamma(f, p)$, then V is Borel measurable on S^*. Moreover,*

$$V(p) = GV_p(l(p)), \tag{4.1}$$

where $V_p(a, b, s) = V(p(a, b, s))$.

Proof. We will construct subgame perfect optimal strategies σ^*, τ^* for I, II, respectively, by backward induction.

If $p \in S^*$ has length $lh(p) \geq n$, then fp is constant and equal to $f(ph^*)$, where h^* is a fixed, arbitrary element of $(A \times B \times S)^N$. Thus the value $V(p)$ must also be equal to $f(ph^*)$, and every strategy is optimal for the two players in the game $\Gamma(f, p)$. So we can define $\sigma^*(p)$ and $\tau^*(p)$ to equal μ^* and ν^*, for all p such that $lh(p) \geq n$, where μ^* and ν^* are arbitrary elements of $P(A)$ and $P(B)$, respectively.

Suppose now that σ^*, τ^*, and V have been defined for all p such that $lh(p) \geq l$, where $1 \leq l \leq n$. We will define σ, τ, and V for all p of length $l - 1$. So let p be of length $l - 1$. Consider the one-day parametrized game with payoff u_p, where

$$u_p(a, b, s) = V(p(a, b, s)).$$

By Theorem 3.1, the game $\Gamma_1(u_p, l(p))$ has value $Gu_p(l(p))$ and optimal strategies $\bar{\mu}(p)$, $\bar{\nu}(p)$ that are Borel measurable in $p \in S \times (A \times B \times S)^{l-1}$. We set

$$\sigma^*(p) = \bar{\mu}(p), \ \tau^*(p) = \bar{\nu}(p), \text{ and } V(p) = Gu_p(l(p)).$$

We check by backward induction that σ^*, τ^* are subgame perfect strategies in $\Gamma(f, \cdot)$ and that the value of $\Gamma(f, p)$ is $V(p)$ for all p.

Let τ be a strategy for II. Then

$$\begin{aligned} E_{\sigma^*[p],\tau[p]} fp &= \int\int\int E_{\sigma^*[p(a,b,s)],\tau[p(a,b,s)]} fp(a, b, s)\, q(ds|l(p), a, b)\, \bar{\mu}(p)(da)\, \tau(p)(db) \\ &\geq \int\int\int V(p(a, b, s)\, q(ds|l(p), a, b)\, \bar{\mu}(p)(da)\, \tau(p)(db) \\ &\geq Gu_p(l(p)) \\ &= V(p). \end{aligned}$$

The first inequality is by the inductive assumption; the second holds because $\bar{\mu}(p)$ is optimal in $\Gamma_1(u_p, l(p))$.

Similarly, for any strategy σ of I, $E_{\sigma[p],\tau^*[p]} fp \leq V(p)$.

□

Theorem 4.2. *Suppose f_k, $k \geq 1$, is a sequence of uniformly bounded, Borel functions on H such that each f_k depends only on $s_1, (a_1, b_1, s_2), \ldots, (a_n, b_n, s_{n+1})$. Assume also that $f_k \leq f_{k+1}$ and $f = \lim_k f_k$. Denote by $V(\cdot)$, and $V_k(\cdot)$ the value functions for the games $\Gamma(f, \cdot)$ and $\Gamma(f_k, \cdot)$, $k \geq 1$. Then $V_k \leq V_{k+1}$ and $V = \lim_k V_k$.*

Proof. The theorem follows by backward induction, Theorem 3.2, and Theorem 4.1.

□

5 Lower semicontinuous games

We are now ready for the proof of Theorem 1.1:

Proof. By Corollary 2.2, there exist uniformly bounded Borel functions f_n, $n \geq 1$, such that each f_n depends only on the coordinates $s_1, (a_1, b_1, s_2), \ldots, (a_n, b_n, s_{n+1}), f_n \leq f_{n+1}$, and $f = \lim_n f_n$.

Let $V_n(\cdot)$ be the value function for $\Gamma(f_n, \cdot)$. Then $V_n \leq V_{n+1}$. Set $V = \lim_n V_n$. We claim that $V(\cdot)$ is the value function for $\Gamma(f, \cdot)$. To see this, let $\epsilon > 0$ and fix $p \in S^*$. Choose k such that $V_k(p) > V(p) - \epsilon$. According to Theorem 4.1, there is a subgame perfect strategy σ_k for player I in $\Gamma(f_k, \cdot)$. Plainly then, if τ is any strategy for player II,

$$E_{\sigma_k[p],\tau[p]}(fp) \geq E_{\sigma_k[p],\tau[p]}(f_k p) \geq V_k(p) > V(p) - \epsilon.$$

Thus the lower value of $\Gamma(f, p)$ is at least $V(p)$.

Now let $V_p(a, b, s) = V(p(a, b, s))$ and $V_{k,p}(a, b, s) = V_k(p(a, b, s))$. Then, by Theorems 3.2 and 4.1,

$$GV_p(l(p)) = \lim_k GV_{k,p}(l(p)) = \lim_k V_k(p) = V(p).$$

By Theorem 3.1, there is a Borel function $\tau^* : S^* \mapsto P(B)$ such that $\tau^*(p)$ is optimal for player II in $\Gamma_1(V_p, l(p))$. To complete the proof, it suffices to show that, for all strategies σ for I and all p, that

$$E_{\sigma[p],\tau^*[p]}(fp) \leq V(p), \qquad p \in S^*.$$

For this will establish that the upper value of $\Gamma(f,p)$ is at most $V(p)$, and also that τ^* is subgame perfect.

Let p have length 0. The proof for p of positive length is similar. Suppose $l(p) = s_1$. Fix a strategy σ for I. Suppose that the random sequence generated when I plays σ and II plays τ^* is

$$s_1, Y_1, Y_2, \ldots, Y_n, \ldots.$$

(Thus $Y_n = (a_n, b_n, s_{n+1})$, $n \geq 1$.) By the choice of τ^*, the process

$$V(s_1), V(s_1, Y_1), \ldots, V(s_1, Y_1, \ldots, Y_n), \ldots$$

is a supermartingale under P_{σ,τ^*}. Indeed,

$$E_{\sigma,\tau^*}[V(s_1, Y_1, \ldots, Y_{n+1}) | (s_1, Y_1, \ldots, Y_n) = p] = E_{\sigma(p),\tau^*(p)} V_p \leq GV_p(l(p)) = V(p).$$

Hence

$$\begin{aligned}
E_{\sigma,\tau^*}(f(s_1, Y_1, \ldots, Y_n, \ldots)) &= \lim_n E_{\sigma,\tau^*}(f_n(s_1, Y_1, \ldots, Y_n, \ldots)) \\
&= \lim_n E_{\sigma,\tau^*} V_n((s_1, Y_1, \ldots, Y_n)) \\
&\leq \lim_n E_{\sigma,\tau^*} V((s_1, Y_1, \ldots, Y_n)) \\
&\leq V(s_1).
\end{aligned}$$

Here the first line is by the monotone convergence theorem, the second holds because f_n depends only on $s_1, Y_1, Y_2, \ldots, Y_n$, the third holds because $V_n \leq V$, and the final line is by the supermartingale property.

□

Corollary 1.2 follows from Theorem 1.1 since, by reversing the roles of the players, we see that I also has a subgame perfect strategy.

The final result of this section generalizes an approximation theorem of Orkin (1972) for Blackwell games.

Theorem 5.1. *Suppose f_n, $n \geq 1$, are uniformly bounded Borel, l.s.c. functions on H such that $f_n \leq f_{n+1}$. Let $f = \lim_n f_n$, and denote by V, V_n the value functions of the games $\Gamma(f, \cdot), \Gamma(f_n, \cdot)$, respectively. Then $V = \lim_n V_n$.*

Proof. For each n, choose, by Theorem 2.1, bounded Borel functions $g_{n,k}$ such that

$$g_{n,k} \leq g_{n,k+1}, \quad \lim_k g_{n,k} = f_n,$$

and $g_{n,k}$ depends only on the coordinates $s_1, (a_1, b_1, s_2), \ldots, (a_k, b_k, s_{k+1})$. Let

$$f_{n,k} = \max\{g_{1,k}, g_{2,k}, \ldots, g_{n,k}\}.$$

Then $f_{n,k}$ also depends only on $s_1, (a_1, b_1, s_2), \ldots, (a_k, b_k, s_{k+1})$. Furthermore, $f_{n,k} \leq f_{n+1,k}$, and $f_{n,k} \leq f_{n,k+1}$.

Now define $g_k = \lim_n f_{n,k}$. We claim that (i) $g_k \leq g_{k+1}$ and (ii) $\lim_k g_k = f$. Claim (i) is trivial. For (ii), note that

$$\begin{aligned}
\lim_k g_k &= \lim_k \lim_n f_{n,k} \\
&= \lim_n \lim_k f_{n,k} \\
&\geq \lim_n \lim_k g_{n,k} \\
&= \lim_n f_n \\
&= f.
\end{aligned}$$

Now $f_{n,k} \leq f_n$. So we also have

$$\lim_k g_k = \lim_k \lim_n f_{n,k} \leq \lim_n f_n = f.$$

For a function g, let $V_g(\cdot)$ denote the value function of $\Gamma(g, \cdot)$. By inspecting the proof of Theorem 1.1 above, we see that $V = \lim_k V_{g_k}$. But, by Theorem 4.2, for each k, $\lim_n V_{f_{n,k}} = V_{g_k}$. Now $V_{f_{n,k}} \leq V_n$ since $f_{n,k} \leq f_n$; so $\lim_n V_{f_{n,k}} \leq \lim_n V_n$. Hence, for each k, $V_{g_k} \leq \lim_n V_n$. Consequently, $V = \lim_k V_{g_k} \leq \lim_n V_n$. □

6 Further remarks

1. It is possible to generalize Theorems 1.1 and 5.1 to a situation where the actions available to the players depend on the state. More precisely, suppose that $A(s)$ and $B(s)$ are the actions available to players I and II, respectively, at state s, $s \in S$. Assume A and B are Borel subsets of Polish spaces and that, for every s, $A(s)$ and $B(s)$ are compact, nonempty subsets of A and B, respectively. Suppose further that the sets $\{(s, a) : a \in A(s)\}$ and $\{(s, b) : b \in B(s)\}$ are Borel subsets of $S \times A$ and $S \times B$, respectively. Then Theorems 1.1 and 5.1 remain true. The proofs go through without major changes.

2. The proof of Theorem 1.1 goes through if we replace the assumption that f is bounded by the assumptions that (a) f is bounded below and (b) for every $p \in S^*$, there is a strategy τ for player II such that

$$\sup_\sigma E_{\sigma[p],\tau[p]} fp < \infty.$$

Thus positive stochastic games are subsumed by the games treated here.

3. The games considered in this article form a subclass of the "leavable games" of Maitra and Sudderth (1993). To see this, one need only redefine the state space in this reference to be the set of partial histories. The formulation of leavable games is further complicated by allowing player I to use stop rules. In this article we have done away with stop rules. The '93 paper also required the use of universally measurable strategies and upper analytic functions. Here we have been able to restrict attention to Borel measurable strategies and functions.

4. The conditions on A and q in Theorems 1.1 and 5.1 can be relaxed as follows:

(a) A is a Borel (rather than compact) subset of a Polish space;

(b) q is a Borel measurable transition function such that for every bounded Borel function g on $A \times B \times S$, the function

$$(a, b, s) \mapsto \int g(a, b, s')\, q(ds|s, a, b)$$

is continuous in b (rather than a, b) for fixed s and a.

Then Theorem 1.1 holds with the following changes:

(i) For every p, player I has a universally measurable (rather than Borel) ϵ- optimal strategy in $\Gamma(f, p)$;

(ii) Player II has a subgame perfect universally measurable (rather than Borel) strategy in $\Gamma(f, \cdot)$;

(iii) The value function V is upper analytic (rather than Borel) on S^*.

Under these weaker conditions Theorem 3.1 has to be replaced by a weaker statement, similar to Lemmas 3.1 and 3.2 of Maitra and Sudderth (1993), whose proof relies on Theorem 5.1 in Nowak (1985). Other changes in the proof of Theorem 1.1 are of a minor nature.

Theorem 5.1 of this paper also remains true under these weaker conditions.

5. Under a continuity assumption on the payoff functions similar to that of Corollary 1.2, Mertens and Parthasarathy (2003) proved the existence of subgame perfect Nash equilibria for n-person games. However, their proof also required an assumption that the law of motion be a continuous function of the actions in the norm topology for measures on the state space. We do not know whether subgame perfect Nash equilibria always exist for n-person games when the payoffs are bounded, Borel, and continuous, the action sets are compact metric, and the law of motion $q(\cdot|s, a_1, \ldots, a_n)$ has the property that the function

$$(a_1, \ldots, a_n, s) \mapsto \int g(a_1, \ldots, a_n, s')\, q(ds'|s, a_1, \ldots, a_n)$$

is jointly continuous in the actions $(a_1, \ldots, a_n)$ for each fixed s and every bounded Borel function g of $s, a_1, \ldots, a_n$.

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