Cournot Maps for Intercepting Evader Evolutions by a Pursuer

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Abstract

Instead of studying evolutions governed by an evolutionary system starting at a given *initial state on a* prescribed future *time interval, finite or infinite, we tackle the problem of looking* both *for a* past interval $[T - D, T]$ *of aperture (or length, duration) D and for the* viable evolutions arriving at a prescribed terminal state *at the end of the temporal window (and thus telescoping if more than one such evolutions exist).*

Hence, given time and duration dependent evolutionary system and viability constraints, as well as time dependent departure constraints, the Cournot map associates with any terminal time T and state x the apertures $D(T, x)$ *of the intervals $[T - D(T, x), T]$, the starting (or initial) states at the beginning of the temporal window from which at least one viable evolution will reach the given terminal state x at T. Cournot maps can be used by a pursuer to intercept an evader's evolution in dynamic game theory. After providing some properties of Cournot maps are next investigated, above all, the* regulation map *piloting the viables evolutions at each time and for each duration from the beginning of the temporal window up to terminal time.*

The next question investigated is the selection of controls or regulons in the regulation map whenever several of them exist. Selection processes are either time dependent, *when the selection operates at each time, duration and state for selecting a regulon satisfying required properties (for instance, minimal norm, minimal speed), or* intertemporal. *In this case, viable evolutions are required to optimize some prescribed intertemporal functional, as in optimal control. This generates value functions, the topics of the second part of this study.*

Mathematics Subject Classification:

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1 Introduction

We attempt to translate mathematically an important concept of *uncertainty* suggested in *Exposition de la théorie des chances et des probabilités*, [27, Cournot], 1843, by *Augustin Cournot* as the meeting of two independent causal series: *"A myriad partial series can coexist in time: they can meet, so that a single event, to the production of which several events took part, come from several distinct series of generating causes."* The search for causes amounts in this case to look for "retrodictions" (so to speak) instead of predictions[4].

We suggest to combine this Cournot approach uncertainty with the Darwinian view of *contingent uncertainty* (differential inclusions) for facing *necessity*[5] (viability constraints) by introducing the concept of Cournot map.

We provide a viability characterization of Cournot maps which relates them to the concept of capture basins viable in an environment (see Chapter 8, p. 273, of *Viability Theory. New Directions*, [8, Aubin, Bayen & Saint-Pierre][6]).

More generally, Cournot maps are motivated by traffic congestion (where the duration is the travel time), by economic dynamics (where the duration of investment evolution), by population dynamics[7] (where the duration is age), by collision problems (where the duration is time until collision): Cournot maps can also be used by a pursuer to intercept an evader's evolution in dynamic game theory.

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[4]This idea probably goes back to the presocratic Greeks, according to the biologist *Antoine Danchin* in [28, 30, 31, Danchin] and his book, *La Barque de Delphes. Ce que révèle le texte des génomes*, [29, Danchin]. He denotes what we called "Cournot uncertainty" as "contingent uncertainty", whereas we use the adjective "contingent" in viability theory for translating mathematically the uncertainty encapsulated in differential inclusions, for actually capturing the concept of redundancy, not only describing the telescoping of evolutions, but the choice of adequate (for instance, viable), regulons or controls in a "contingent reservoir", which may itself evolve.

[5]In [30, Danchin], the author quotes the following sentence of *Leucippus*: *"Nothing happens in vain, but everything from reason (logos), and by necessity"*: "law" is described by a differential inclusion, "necessity" by constraints to abide to. This "Law and Necessity" statement is more in tune with viability theory than *Chance and Necessity*, title of the celebrated book [43, Monod] by *Jacques Monod*, who attributed this concept to *Democritus*. Indeed, "chance" remains to be defined (see Chapter 2 of *La valeur n'existe pas. À moins que ...*, [5, Aubin] and Chapter 8 of *La mort du devin, l'émergence du démiurge. Essai sur la contingence, la viabilité et l'inertie des systèmes*, [3, Aubin]).

[6]In this book, the evolutions are still defined on the usual future interval $[0, T]$ with prescribed finite or infinite horizon.

[7]See, among an abundant literature, [2, Anita], [9, Aubin], [38, Iannelli], [41, Keyfitz N. & Keyfitz B.], [46, Von Foerster], [47, Webb], etc.

Following the suggestion to study evolutions on (sliding) *temporal window* $[T - \Omega, T]$ on which the evolution is defined[8], *the search for the temporal window is* also *part of the solution of the problem.*

2 Cournot Maps

2.1 Definitions

Let us consider

1. a set-valued map $F : \mathbb{R} \times \mathbb{R} \times X \rightsquigarrow X$ with which we associate

   (a) the arrival map

   $$\mathcal{A}_F : \mathbb{R} \times \mathbb{R}_+ \times X \times X \rightsquigarrow \mathcal{C}(-\infty, +\infty; X)$$

   associating with any terminal pair (T, x), aperture $\Omega \geq 0$ and s the (possibly empty) set $\mathcal{A}_F(T, x)[\Omega, s]$ of evolutions $x(\cdot)$ restricted to the temporal window $[T - \Omega, T]$ governed by the *duration-structured differential inclusion*

   $$\forall\, t \in [T - \Omega, T], \ \ x'(t) \ \in \ F(t, t - (T - \Omega), x(t)) \tag{1}$$

   defined on the temporal window $[T - \Omega, T]$ starting from s at the beginning of the temporal window and arriving at $x(T) = x \in K(T)$ at its end[9]. Such an evolution linking s at time $T - \Omega$ to x at time T is called a *Cournot evolution* at (T, x);

   (b) the map $(T, x, \Omega) \rightsquigarrow \mathcal{A}_F(T, x)[\Omega] := \bigcup_{s \in C(T-\Omega)} \mathcal{A}_F(T, x)[\Omega, s]$, the set of evolutions defined on the temporal window $[T - \Omega, T]$ arriving at x at time T;

   (c) the map $(T, x) \rightsquigarrow \bigcup_{\Omega \geq 0} \mathcal{A}_F(T, x)[\Omega]$, the set of evolutions arriving at x at time T;

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[8]see Chapter 5, p. 67, of *Time and Money. How Long and How Much Money is Needed to Regulate a Viable Economy*,[4, Aubin] and *La valeur n'existe pas. À moins que ...*, [5, Aubin].

[9]This is a *"retrodiction" evolutionary system* in the sense that, for any evolving present time T, we study the evolution on the past (or historical) temporal window $[T - \Omega, T]$.

2. a (duration dependent) *environmental tube* $\mathbb{K} : (t, d) \in \mathbb{R} \times \mathbb{R}_+ \rightsquigarrow K(t, d) \subset X$ in which evolutions are required to be viable;

3. a *departure tube*[10] $\mathbb{C} : t \in \mathbb{R} \rightsquigarrow C(t) \subset K(t, 0)$, which associates with any (departure) date $t \in \mathbb{R}$ the subset of state $s \in C(t)$ from which evolutions start at time t.

The question arises whether we can find the subset of those initial states.

Definition 2.1 [*Cournot Map*] *The* Cournot map $\mathrm{Cour}_F(\mathbb{K}, \mathbb{C}) : \mathrm{Graph}(K) \rightsquigarrow \mathbb{R}_+ \times X$ *of the departure tube* $\mathbb{C}$ *viable in the environmental tube* $\mathbb{K}$ *under the differential inclusion (1), p. 3, is the set-valued map associating with any terminal pair* $(T, x) \in \mathrm{Graph}(\mathbb{K})$ *the subset* $\mathrm{Cour}_F(\mathbb{K}, \mathbb{C})(T, x)$ *of pairs* $(\Omega, s) \in \mathbb{R}_+ \times X$ *such that*

1. $s \in C(T - \Omega)$;

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[10]The departure sets $C(d)$ can be empty for some departure dates d. If, for instance, the beginning d_{fix} of the temporal window is prescribed and not computed, then the departure sets $C(d) = \emptyset$ are empty for all $d \neq d_{fix}$. If we want that all departure dates are later than a date d_{min}, we assume that the departure sets $C(d) = \emptyset$ are empty for all $d < d_{min}$. The use of departure maps cover many different situations. In ethology, departure maps could translate mathematically the *Konrad Lorenz* imprinting (actually discovered by the 19th-century biologist *Douglas Spalding*, rediscovered by *Oskar Heinroth*, Lorenz' mentor), associating with given dates of cognitive development the perception of the environment triggering imprinted behaviors (such as recognition of the mother, etc.).

2. *there exists at least one Cournot evolution $x(\cdot) \in \mathcal{A}_F(T, x)[\Omega, s]$ starting from $x(T - \Omega) = s$ at time $T - \Omega$, arriving at $x(T) = x$ at time T and viable in the environmental tube $x(\cdot)$ on the temporal window $[T - \Omega, T]$ of aperture $\Omega \geq 0$ in the sense that*

$$\forall\, t \in [T - \Omega, T], \quad x(t) \in K(t, t - (T - \Omega)) \tag{2}$$

When there is no environmental constraint, we simply set $\mathrm{Cour}_F(\mathbb{C})$.
We stress the fact that we look for both *a temporal window* $[T - \Omega, T]$ ***and*** *a* viable evolution *$x(\cdot)$ governed by $x'(t) \in F(t, t - (T - \Omega), x(t))$ on this temporal window.*

The concept of Cournot map encapsulates several features. The first one is the concept of Cournot (or minimal aperture):

Definition 2.2 *[**Cournot Aperture**] The* Cournot aperture *function $\Omega_F(\mathbb{K}, \mathbb{C})$ associates with any time T and at arrival state x the smallest aperture*

$$\Omega_F(\mathbb{K}, \mathbb{C})(T, x) \ := \ \inf_{(\Omega, s)\, \in\, \mathrm{Cour}_F(\mathbb{K},\mathbb{C})(T - \Omega_F(\mathbb{K},\mathbb{C})(T,x), x)} \Omega \tag{3}$$

of the Cournot temporal window $[T - \Omega_F(\mathbb{K}, \mathbb{C})(T, x), T]$. Its inverse $\dfrac{1}{\Omega_F(\mathbb{K}, \mathbb{C})(T, x)}$ is called the Cournot liquidity *in economics.*
When there is ambiguity, we set $\Omega(T, x) := \Omega_F(\mathbb{K}, \mathbb{C})(T, x)$

Next, we extract from the knowledge of the Cournot map the arrival tube of arrival dates and states which can be reached:

Definition 2.3 *[**Cournot Arrival Tubes**] The Cournot map $\mathrm{Cour}_F(\mathbb{K}, \mathbb{C})$ generates the* arrival tube $\mathrm{Aval}_F(\mathbb{K}, \mathbb{C}) \subset \mathbb{K}$ *defined by*

$$\mathrm{Graph}(\mathrm{Aval}_F(\mathbb{K}, \mathbb{C})) \ := \ \mathrm{Dom}(\mathrm{Cour}_F(\mathbb{K}, \mathbb{C})) \tag{4}$$

which associates with any $T \in \mathbb{R}$ the (possibly empty) subset $\mathrm{Aval}_F(\mathbb{K}, \mathbb{C})(T)$ of arrival states $x \in K(T)$ at which at least one viable *Cournot evolution starting from some $s \in C(T - \Omega)$ at $T - \Omega$ for some aperture $\Omega \geq 0$ arrives at x at time T.*

In other words, the set-valued map F generates the map

$$\mathbb{C} \mapsto \mathrm{Aval}_F(\mathbb{K}, \mathbb{C}) \ \subset \ \mathbb{K} \tag{5}$$

mapping departure tubes $\mathbb{C}$ to arrival tubes $\mathrm{Aval}_F(\mathbb{K}, \mathbb{C})$.
The Cournot map generates in turn the Cournot starting tube contained in the departure tube:

Definition 2.4 [***Cournot Starting Tubes***] *The Cournot map* $\mathrm{Cour}_F(\mathbb{K}, \mathbb{C})$ *generates the* starting map $\mathrm{Start}_F(\mathbb{K}, \mathbb{C}) \subset \mathbb{C}$ *defined by*

$$(T, x) \rightsquigarrow \mathrm{Start}_F(\mathbb{K}, \mathbb{C})(T, x) := \{(T - \Omega, s)\}_{(\Omega, s) \in \mathrm{Cour}_F(\mathbb{K}, \mathbb{C})(T, x)} \tag{6}$$

We denote by $\mathrm{Start}_F(\mathbb{K}, \mathbb{C})(T, x)[\Omega] := \mathrm{Cour}_F(\mathbb{C})(T, x)[\Omega]$ *the subset defined by*

$$\mathrm{Start}_F(\mathbb{K}, \mathbb{C})(T, x)[\Omega] := \{s \ \ such \ that \ (\Omega, s) \ \in \ \mathrm{Cour}_F(\mathbb{C})(T, x)\} \tag{7}$$

providing the starting states $s \in C(T - \Omega)$. *In particular, we single out the* Cournot earliest starting map

$$(T, x) \rightsquigarrow \mathrm{Cour}_F(\mathbb{C})(T, x)[\Omega_F(\mathbb{K}, \mathbb{C}(T, x)] \ \subset \ C(T - \Omega_F(\mathbb{K}, \mathbb{C})(T, x)) \tag{8}$$

associating the set of starting states $s \in C(T - \Omega_F(\mathbb{K}, \mathbb{C})(T, x))$ *from which starts a viable Cournot evolution* $x(\cdot) \in \mathcal{A}_F(T, x)[\Omega_F(\mathbb{K}, \mathbb{C})(T, x), s]$ *reaching* x *at time* T *with minimal duration.*
The tube $\mathrm{Start}_F(\mathbb{K}, \mathbb{C})(\mathrm{Aval}_F(\mathbb{K}, \mathbb{C})) \subset \mathbb{C}$ *is the* starting tube *of the Cournot map, the subset of starting times and states* (d, s) *from which at least one evolution arrives at some* $(T, x) \in \mathrm{Aval}_F(\mathbb{K}, \mathbb{C})$ *in the arrival tube.*

2.2 Cournot Tube of a Pursuer for Intercepting an Evader Evolution

Cournot tubes could be of some use in the context of pursuer-evader dynamical games. We consider the problem from the point of view of the pursuer, who has computed its Cournot map $\mathrm{Cour}_F(\mathbb{K}, \mathbb{C})$.

Assume that at time t_0, the pursuer observes an evolution[11] $\xi_0(\cdot) : [t_0, +\infty[\mapsto X$.

Can the pursuer intercept the evolution $\xi_0(\cdot)$, and, if the answer is positive, when and how? Cournot maps can be used for answering these questions[12].

Theorem 2.5 [***Capturability of the Evader Evolution***] *Let us assume that the arrival tube* $\mathrm{Aval}_F(\mathbb{K}, \mathbb{C})$ *of the Cournot map of the pursuer is closed. We associate with it and with the evader evolution* $\xi_0(\cdot)$ *the capturability state* $(T_{\xi_0}^\flat, x_{\xi_0}^\flat)$ *defined by*

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[11]This evolution can be extrapolated from the knowledge of the evolution on an adequate interval $[t_0 - \Omega_0, t_0]$ or, knowing the dynamics of the evader, an evolution starting at $\xi_0(t_0)$ governed by the evader's dynamics.

[12]The literature on differential games from *Differential games*, [39, Isaacs] is so abundant that it is impossible to quote all the contributions, which figure, for instance, in the recent proceedings, *Advances in Dynamic Games: Theory, Applications, and Numerical Methods for Differential and Stochastic Games*, [18, Cardaliaguet & Cressman]. However, viability techniques have been introduced in Chapter 14 of *Viability Theory*, [6, Aubin], [19, Cardaliaguet & Plaskacz], [21, 20, Cardaliaguet, Quincampoix & Saint-Pierre] among many other articles.

$$\begin{cases} (i) & T^\flat_{\xi_0} := \inf_{\{t \ge t_0 \ such \ that \ (t,\xi_0(t)) \in \mathrm{Graph}(\mathrm{Aval}_F(\mathbb{K},\mathbb{C}))\}} t \\ (ii) & x^\flat_{\xi_0} := \xi_0(T^\flat_{\xi_0}) \end{cases} \quad (9)$$

If $T^\flat_{\xi_0} < +\infty$ is finite and if the duration $\Omega_0 := \Omega(T^\flat_{\xi_0}, x^\flat_{\xi_0}) \le T^\flat_{\xi_0} - t_0$ is smaller or equal to the duration $T^\flat_{\xi_0} - t_0$, then the evader evolution $\xi_0(\cdot)$ is captured by a viable Cournot evolution $x_0(\cdot) \in \mathcal{A}_F(T^\flat_{\xi_0}, x^\flat_{\xi_0})[\Omega_0, s_0]$ where $s_0 = x_0(T^\flat_{\xi_0} - \Omega_0)$.

Proof — The case when $T^\flat_{\xi_0} = +\infty$ means that the evader evolution is not capturable by the pursuer. Otherwise, the pair $(T^\flat_{\xi_0}, x^\flat_{\xi_0})$ belongs to the graph $\mathrm{Graph}(\mathrm{Aval}_F(\mathbb{K},\mathbb{C}))$ of the arrival tube of the pursuer. Therefore, there exist one Cournot aperture Ω_0, one starting state $s_0 \in C(T^\flat_{\xi_0} - \Omega_0)$ and one viable Cournot evolution $x_0(\cdot)$ linking s_0 at time $T^\flat_{\xi_0} - \Omega_0$ to $x^\flat_{\xi_0} := \xi_0(T^\flat_{\xi_0})$ at time $T^\flat_{\xi_0}$, and thus intercepting the evader at time $T^\flat_{\xi_0}$ since $t_0 \le T^\flat_{\xi_0} - \Omega_0$ by assumption. ∎

Naturally, the assumption that the observed evolution $\xi_0(\cdot)$ at t_0 is known is too strong, since predictions are most of the time doomed to fail. Another observation may have to be made at a future time $t_1 \in [T^\flat_{\xi_0} - \Omega_0, T^\flat_{\xi_0}]$. It may happen that at time t_1 starts another evolution $\xi_1(\cdot)$.

In this case, at that time t_1, the state of the pursuer evolution $x_0(\cdot)$ is no longer viable in the departure tube $\mathbb{C}$. This departure tube has to replaced by the tube reduced to $\{x_0(\cdot)\}$ from which the evolution a possible correction must be made.

We thus compute the Cournot map $\mathrm{Cour}_F(\mathbb{K}, \{x_0(\cdot)\})$ so that, for any $t \in [T^\flat_{\xi_0} - \Omega_0, T^\flat_{\xi_0}]$, $x_0(t) \in \mathrm{Cour}_F(\mathbb{K}, \{x_0(\cdot)\})(T^\flat_{\xi_0}, x^\flat_{\xi_0})[T^\flat_{\xi_0} - t]$.

We next introduce the pair

$$\begin{cases} (i) & T^\flat_{\xi_1} := \inf_{\{t \in [t_1, T^\flat_0] \ such \ that \ (t,\xi_1(t)) \in \mathrm{Graph}(\mathrm{Aval}_F(\mathbb{K},\{x_0(\cdot)\}))\}} t \\ (ii) & x^\flat_{\xi_1} := \xi_1(T^\flat_{\xi_1}) \end{cases} \quad (10)$$

The case when $T^\flat_{\xi_1} = +\infty$ means that the evader evolution is not capturable by the pursuer before $T^\flat_{\xi_0}$. Otherwise $T^\flat_{\xi_1} \le T^\flat_{\xi_0}$. We set $\Omega_1 := \Omega(T^\flat_{\xi_1}, x^\flat_{\xi_1})$, we take $s_1 \in \mathrm{Cour}_F(\mathbb{K}, \{x_0(\cdot)\}(T^\flat_{\xi_1}, x^\flat_{\xi_1}))[\Omega_1]$ and a viable Cournot evolution $x_1(\cdot) \in \mathcal{A}_F(T^\flat_{\xi_1}, x^\flat_{\xi_1})[\Omega_1, s_1]$ linking s_1 at time $T^\flat_{\xi_1} - \Omega_1$ to $\xi_1(T^\flat_{\xi_1})$ at time $T^\flat_{\xi_1}$.

If $\Omega_1 \le T^\flat_{\xi_1} - t_1$, the new evader evolution $\xi_1(\cdot)$ can be intercepted by a Cournot evolution $x_1(\cdot) \in \mathcal{A}_F(T^\flat_{\xi_1}, x^\flat_{\xi_1})[\Omega_1, s_1]$ at time $T^\flat_{\xi_1}$ since $t_1 \le T^\flat_{\xi_s} - \Omega_1$.

We can reiterate this process until interception happens when the last prediction ξ_j at time $t_j \in [T^\flat_{\xi_{j-1}} - \Omega_j, T^\flat_{\xi_{j-1}}]$ is true.

Since Cournot maps can be characterized in terms of viable capture basins, they inherit their properties, among them, the ability of computing them thanks to the capture basin algorithm. They can be used in the field of pursuer-evader dynamical games.

2.3   Properties of Cournot Maps

We observe how the Cournot map $\text{Cour}_F(\mathbb{K}, \mathbb{C})(t, x(t))[t - (T - \Omega)]$ evolves along a viable Cournot evolution on the temporal window $[T - \Omega, T]$ reaching x at time T, an obvious consequence of the Bilateral Fixed Point of Capture Basins (see Theorem 10.2.5, p. 379, of *Viability Theory. New Directions*, [8, Aubin, Bayen & Saint-Pierre]):

Proposition 2.6 [*Evolution of Cournot Maps*] *Let us consider* $(T, x) \in \text{Aval}_F(\mathbb{K}, \mathbb{C})$.

1. *For any* $(\Omega, s) \in \text{Cour}_F(\mathbb{K}, \mathbb{C})(T, x)$ *and any viable Cournot evolution* $x(\cdot) \in \mathcal{A}_{(F,\mathbb{K})}(T, x)[\Omega, s]$ *linking* s *to* x, *then,*

$$\forall \, t \, \in \, [T - \Omega, T], \; \; (T - \Omega, s) \, \in \, \text{Cour}_F(\mathbb{K}, \mathbb{C})(t, x(t))[t - (T - \Omega)] \qquad (11)$$

2. *For any* $(\Omega_1, s_1) \in \text{Cour}_F(\mathbb{K}, \mathbb{C})(T - \Omega, s)$ *and any viable Cournot evolution* $x_1(\cdot) \in \mathcal{A}_{(F,\mathbb{K})}(T, x)[\Omega_1, s_1]$ *linking* s_1 *to* s, *then the concatenation* $(x_1 \lozenge x)(\cdot)$ *is a viable Cournot evolution linking* s_1 *to* x, *and,* $\forall \, t \, \in \, [T - (\Omega + \Omega_1), T]$,

$$(T - (\Omega + \Omega_1), s_1) \, \in \, \text{Cour}_F(\mathbb{K}, \mathbb{C})(t, (x_1 \lozenge x)(t))[t - (T - (\Omega + \Omega_1))] \qquad (12)$$

Consequently,

$$\forall \, t \in [T - \Omega, T], \; \; t \rightsquigarrow \in \text{Cour}_F(\mathbb{K}, \mathbb{C})(t, x(t)) \, \in \, \text{Cour}_F(\mathbb{K}, \mathbb{C})(T, x) \text{ and is increasing} \quad (13)$$

Cournot evolutions $x(\cdot) \in \mathcal{A}_F(T, x)[\Omega, s]]$ are not only in the in the departure tube (after $\Omega(T, x)$, at least), so that for we replace it by the Cournot evolution $\{x(\cdot)\}$ it self.

Proposition 2.7 [*Viability Property of Cournot Evolutions*] *For any* $(\Omega, s) \in \text{Cour}_F(\mathbb{K}, \mathbb{C})(T, x)$ *and any viable Cournot evolution* $x(\cdot) \in \mathcal{A}_{(F,\mathbb{K})}(T, x)[\Omega, s]$ *linking* s *to* x,

$$\forall \, t \in [T - \Omega, T], \; \; x(t) \, \in \, \text{Cour}_F(\mathbb{K}, \{x(\cdot)\})(T, x)[T - t] \qquad (14)$$

Next, we adapt the *dilation property* of Cournot maps stating in essence that the Cournot map of the union of departure tubes is the union of the Cournot maps of these departure tubes. This morphism property plays a crucial role in computational issues since it allows a parallelization of the computation of the Cournot maps.

We recall that a hypermap $\mathbb{V}$ is a *dilation* if $\mathbb{V} \left(\bigcup_{i \in \mathbb{I}} K_i \right) = \bigcup_{i \in \mathbb{I}} \mathbb{V}(K_i)$ and that any dilation is increasing.

Proposition 2.8 [***Morphism Property of Cournot Maps***] *The map* $(F, \mathbb{C}) \leadsto$ $\mathrm{Cour}_F(\mathbb{K}, \mathbb{C})$ *is a dilation:*

$$\mathrm{Cour}_{\bigcup_{p \in \mathbb{P}} F_p} \left(\mathbb{K}, \bigcup_{i \in \mathbb{I}} \mathbb{C}_i \right) = \bigcup_{p \in \mathbb{P}} \bigcup_{i \in \mathbb{I}} \mathrm{Cour}_{F_p}(\mathbb{K}, \mathbb{C}_i) \tag{15}$$

and thus, the map $(F, \mathbb{C}) \leadsto \mathrm{Cour}_F(\mathbb{K}, \mathbb{C})$ *is increasing.*

2.4 Viability Characterization of Cournot Maps

Our first task is to provide a viability characterization of Cournot maps which allows us to transfer the properties of viable capture basins to Cournot maps.

Theorem 2.9 [***Viability Characterization of Cournot Maps***] *Let us associate with the differential inclusion (1), p. 3, the system*

$$\begin{cases} (i) & \overleftarrow{\tau}'(t) = -1 \\ (ii) & \overleftarrow{\omega}'(t) = -1 \\ (iii) & \overleftarrow{x}'(t) \in -F(\overleftarrow{\tau}(t), \overleftarrow{\omega}(t), \overleftarrow{x}(t)) \\ (iv) & \overleftarrow{\sigma}'(t) = 0 \end{cases} \tag{16}$$

We introduce the auxiliary environment $\mathcal{K} := \mathrm{Graph}(\mathbb{K}) \times X$ *and the auxiliary target* $\mathcal{C} \subset \mathcal{K}$ *defined by*

$$(t, d, x, s) \in \mathcal{C} \text{ if and only if } x \in C(t), \ d = 0 \text{ and } s = x \tag{17}$$

Then the graph of the Cournot map $(T, x) \leadsto \mathrm{Cour}_F(\mathbb{K}, \mathbb{C})(T, x)$ *is equal to subset of elements* (T, x, Ω, s) *such that*

$$(T, \Omega, x, s) \in \mathrm{Capt}_{(16)}(\mathcal{K}, \mathcal{C})$$

Therefore, the Cournot map inherits all the properties of viable capture basins.

    Proof — To say $(T, \Omega, x, s) \in \mathrm{Capt}_{(16)}(\mathcal{K}, \mathcal{C})$ belongs to the capture basin amounts to saying that there exist $t^\star \geq 0$ and one evolution $(\overleftarrow{\tau}(\cdot), \overleftarrow{\omega}(t), \overleftarrow{x}(\cdot), \overleftarrow{\sigma}(t))$ where

$$\overleftarrow{\tau}(t) = T - t, \ \overleftarrow{\omega}(t) = \Omega - t, \ \overleftarrow{x}(t), \overleftarrow{\sigma}(t) = s \tag{18}$$

governed by differential inclusion (16), p. 9, starting at (T, Ω, x, s) such that

$$\begin{cases} (i) & (\overleftarrow{\tau}(t^\star), \overleftarrow{\omega}(t^\star), \overleftarrow{\xi}(t^\star), \overleftarrow{\sigma}(t^\star)) \in \mathcal{C} \\ & \text{or } t^\star = \Omega, \ \overleftarrow{x}(\Omega) \in C(\overleftarrow{\tau}(\Omega)) \subset K(\Omega, 0) \text{ and } s = \overleftarrow{\xi}(\Omega) \\ (ii) & \forall t \in [0, \Omega], \ (\overleftarrow{\tau}(t), \overleftarrow{\omega}(t), \overleftarrow{\xi}(t), \overleftarrow{\sigma}(t)) \in \mathcal{K} \text{ or } \overleftarrow{\xi}(t) \in K(\overleftarrow{\tau}(t), \overleftarrow{\omega}(t)) \end{cases} \tag{19}$$

Let us make the change of variable $t \mapsto T - t$ and, setting $x(t) := \overleftarrow{\xi}(T - t)$, we infer that

$$
\begin{cases}
(i) & x(T - \Omega) = s \in C(T - \Omega) \text{ and } x(T) = x \\
(ii) & \forall\, t \in [T - \Omega, T], \quad x'(t) \in F(t, t - (T - \Omega), x(t)) \\
(iii) & \forall\, t \in [T - \Omega, T], \quad x(t) \in K(t, t - (T - \Omega))
\end{cases}
\tag{20}
$$

This means that (T, x, Ω, s) belongs to the graph of the Cournot map $\mathrm{Cour}_F(\mathbb{K}, \mathbb{C})$. ■

2.5   Regulation of Viable Evolutions

Denote by $T_K^{\star\star}(x)$ the closed convex hull (or the bipolar) of the tangent cone $T_K(x)$ to K at $x \in K$.

Recall[13] that the *(forward) convexified derivative* $D^{\star\star}\mathbb{V}(t, x)$ of a tube $\mathbb{V}$ is defined by

$$
\mathrm{Graph}(D^{\star\star}\mathbb{V}(t, x)) := T_{\mathrm{Graph}(\mathbb{V})}^{\star\star}(t, x)
\tag{21}
$$

Hence the graph of the convexified forward derivative is a closed convex cone (therefore, a set-valued map analogue of a linear operator, called a closed convex process in *Convex analysis*, [44, Rockafellar]).

We introduce the concept of regulation map:

Definition 2.10 [***Regulation Map***] *Let us consider a set-valued map* $F : (t, d, x) \in \mathrm{Graph}(\mathbb{K}) \rightsquigarrow F(t, d, x) \subset X$ *and a tube* $\mathbb{V} : (t, x) \in \mathrm{Graph}(\mathbb{K}) \rightsquigarrow \mathbb{V}(t, x)$. *The* regulation map $R_{(F,\mathbb{V})} : (t, d, x, s) \rightsquigarrow R_{(F,\mathbb{V})}(t, d, x, s)$ *is defined by*

$$
R_{(F,\mathbb{V})}(t, d, x, s) := \{u \in F(t, d, x) \text{ such that } 0 \in D^{\star\star}\mathbb{V}(t, x, d, s)(1, u, 1)\}
\tag{22}
$$

Remark —   Observe that if $\mathbb{V}$ is a single-value differentiable map, then $R_{(F,\mathbb{V})}(t, d, x, s)$ is the subset of directions $u \in F(t, d, x)$ such that

$$
0 = \frac{\partial \mathbb{V}(t, d, x, s)}{\partial t} + \frac{\partial \mathbb{V}(t, d, x, s)}{\partial d} + \left\langle \frac{\partial \mathbb{V}(t, d, x, s)}{\partial x}, u \right\rangle
$$

which is a McKendrik partial differential equation of age-structure problems (see [9, Aubin]). ■

Therefore, one can reformulate the Viability Theorem in this framework:

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[13]See *Set-valued analysis*, [12, Aubin & Frankowska], *Variational Analysis*, [45, Rockafellar & Wets] and Chapter 18, p. 713, of *Viability Theory. New Directions*, [8, Aubin, Bayen & Saint-Pierre].

Theorem 2.11 [**The Viability Theorem**] *Let us assume that F is Marchaud and that the departure and environmental tubes are closed. Denote by $\mathbb{V} := \mathrm{Cour}_F(\mathbb{K}, \mathbb{C})$ the Cournot map. Then its graph is closed. The viable evolutions $x(\cdot) \in \mathcal{A}_F(T, x)[\Omega, s]$ starting from s at time $T - \Omega$ and arriving at x at time T are governed by the following differential inclusion involving the regulation map:*

$$\forall\, t \in [T - \Omega(T, x), T], \quad x'(t) \in R_{(F, \mathbb{V})}(t, t - (T - \Omega), x(t), s) \tag{23}$$

Proof — The Viability Theorem states that whenever the map F is Marchaud, the capture basin is the largest set of elements (T, Ω, x, s) between $\mathcal{C}$ and $\mathcal{K}$ which is closed and locally viable. Therefore, the backward velocities $\overleftarrow{u} \in F(t, d, x)$ such that $(-1, -1, -\overleftarrow{u}, 0) \in -(\{1\} \times \{1\} \times F(t, d, x) \times \{0\})$ belongs to convexified tangent cone to the viable capture basin $\mathrm{Capt}_{(16)}(\mathcal{K}, \mathcal{C})$. By Theorem 2.9, p. 9, this means that $(-1, -\overleftarrow{u}, -1, 0)$ belongs to the convexified tangent cone $T^{\star\star}_{\mathrm{Graph}(\mathbb{V})}(\tau, \xi, \omega, \sigma)$ to the graph of $\mathbb{V}$. Recalling that $T^{\star\star}_{\mathrm{Graph}(\mathbb{V})}(\tau, \xi, \omega, \sigma) = \mathrm{Graph}(D^{\star\star}\mathbb{V}(\tau, \xi, \omega, \sigma))$, we infer that $0 \in -F(\tau, \omega, \xi) \cap D^{\star\star}\mathbb{V}(\tau, \xi, \omega, \sigma)(-1, -\overleftarrow{u}, -1)$. Therefore, the forward directions $u := -\overleftarrow{u}$ belongs to $R_{(F, \mathbb{V})}(t, d, x, s)$, so that the forward velocities $x'(t)$ which regulate the forward viable evolutions $x(\cdot) \in \mathcal{A}_F(\Omega, T, x)$ starting at s are the ones which belong to $F(t, t - (T - \Omega), x(t))$ and satisfy $0 \in D^{\star\star}\mathbb{V}(t, x(t), t - (T - \Omega), s)(1, x'(t), 1)$, i.e., which belong to the regulation map $R_{(F, \mathbb{V})}(t, t - (T - \Omega), x(t), s)$. ∎

3 Hamilton-Jacobi-Cournot-McKendrik Optimization Problem

The Cournot map $\mathrm{Cour}_F(\mathbb{K}, \mathbb{C})$ may contain *more than one viable evolution* arriving at $(T, x) \in \mathrm{Graph}(\mathbb{K})$, starting, for example, from the Cournot beginning $T - \Omega_F(\mathbb{K}, \mathbb{C})(T, x)$ and arriving at x at arrival time T. Hence the question of selecting viable evolutions arises.

There are two classes of selection procedures for reducing this set of evolutions. The first one is to operate at each time at the level of the *regulation map* by selection one control in $R_{(F, \mathbb{V})} : (t, d, x, s) \rightsquigarrow R_{(F, \mathbb{V})}(t, d, x, s)$ (see for instance Section 11.3.1, p. 453, of *Viability Theory. New Directions*, [8, Aubin, Bayen & Saint-Pierre]).

The other class of selection procedures in an intertemporal one, which consists in using an intertemporal cost functional on evolutions $x(\cdot) \in \mathcal{A}_F(T, x)$ (depending for instance on departure cost functions and velocity dependent cost functions) and looking for the viable evolutions which minimize this intertemporal criterion.

Definition 3.1 [**Departure Cost Functions and Lagrangian**] *We consider two "cost functions" **c** and **l**:*

1. *a* departure cost condition *function* $(d, s) \mapsto \mathbf{c}(d, s) \in \mathbb{R} \cup \{+\infty\}$;

2. *an* intertemporal cost functional, *called in short a* Lagrangian, $\mathbf{l} : (t, d, x, u) \mapsto \mathbf{l}(t, d, x, u) \in \mathbb{R} \cup \{+\infty\}$ $u \mapsto \mathbf{l}(t, d, x, u)$ *depending on time, duration, state and velocity;*

with which we associate

1. *the departure tube* $C : \mathbb{R} \rightsquigarrow X$ *defined by*

$$C(d) := \{s \in X \ \ such \ that \ \mathbf{c}(d, s) \ < \ +\infty\}$$

2. *the set-valued map* $F_{\mathbf{l}} : \mathbb{R} \times \mathbb{R}_+ \times X \rightsquigarrow X$ *defined by*

$$F_{\mathbf{l}}(t, d, x) := \{u \in X \ \ such \ that \ \mathbf{l}(t, d, x, u) \ < \ +\infty\} \tag{24}$$

and the arrival map $\mathcal{A}_{\mathbf{l}}(T, x)[\Omega, s]$ *associating with any final pair* (T, x) *the set of evolutions* $x(\cdot)$ *governed by the* differential inclusion

$$\forall \, t \in [T - \Omega, T], \ \ x'(t) \ \in \ F_{\mathbf{l}}(t, t - (T - \Omega), x(t)) \tag{25}$$

starting at $s \in C(T - \Omega)$ *and arriving at the terminal condition* $x(T) \ = \ x$ *at time* T.

We have to define the intertemporal cost functional. We begin by the simpler case when no constraint function is taken into account.

Definition 3.2 *[**The Hamilton-Jacobi-Cournot-McKendrik Valuation Function**] We associate with the data defined in Definition 3.1, p.11 the* Hamilton-Jacobi-Cournot-McKendrik valuation function $V_l(\mathbf{c})$ *defined by*

$$V_l(\mathbf{c})(T, x) := \inf_{\Omega \geq 0} \inf_{x(\cdot) \in \mathcal{A}_{\mathbf{l}}(T,x)} \left(\mathbf{c}(T - \Omega, x(T - \Omega)) + \int_{T-\Omega}^{T} \mathbf{l}(t, T - (T - \Omega), x(t))dt \right) \tag{26}$$

The question arises to know wether the infimum $V_l(\mathbf{c})(T, x)$ is achieved and what are the standard properties of the valuation function, and, in particular, what is the Hamilton-Jacobi-McKendrik to which it is a solution.

The way to achieve this program is to observe that the epigraph of the valuation function is the Cournot map of an auxiliary problem we now define (the vertical arrows symbolize this property and the fact that these Cournot maps are related to intertemporal minimization problems).

We introduce the

1. auxiliary target

$$\mathcal{C}_\uparrow := (t, d, x, y, s) \text{ such that } \mathbf{c}(t, s) < +\infty, \ d = 0 \text{ and } s = x$$

2. the right hand side

$$\mathcal{F}_\uparrow(t, d, x, y, s) = \{\{1\} \times \{1\} \times \mathcal{E}p(\mathbf{l}) \times \{0\}\} \tag{27}$$

of the differential inclusion

$$t' = 1, \ d' = 1, \ x' = u, \ y' \leq \mathbf{l}(t, d, x, u), \ s' = 0 \tag{28}$$

3. the Cournot map $(T, x, y) \rightsquigarrow \mathrm{Cour}_{\mathcal{F}_\uparrow}(\mathbb{K}_\uparrow, \mathbb{C}_\uparrow)(T, x, y)$.

Definition 3.3 [*The Viability Solution to the Hamilton-Jacobi-Cournot-McKendrik Optimization Problem*] *We consider the extended Cournot map*

$$(T, x, y) \rightsquigarrow \mathrm{Cour}_{\mathcal{F}_\uparrow}(\mathbb{K}_\uparrow, \mathbb{C}_\uparrow)(T, x, y) \tag{29}$$

associating with elements (T, x, y) the set of apertures $\Omega \geq 0$ and initial values $s = x(T - \Omega)$ of evolutions $x(\cdot) \in \mathcal{A}_\mathbf{l}(T, x)[\Omega, s]$ starting from $s \in \mathcal{C}(T - \Omega)$.
The Hamilton-Jacobi-Cournot-McKendrik viability solution $W_l(\mathbf{c})$ is defined by

$$W_l(\mathbf{c})(T, x) := \inf_{\substack{(T, x, y) \in \mathrm{Dom}(\mathrm{Cour}_{\mathcal{F}_\uparrow}(\mathbb{K}_\uparrow, \mathbb{C}_\uparrow))}} y \tag{30}$$

As expected, these two functions coincide.

Theorem 3.4 [*The Hamilton-Jacobi-Cournot-McKendrik Valuation Function and Viability Solution Coincide*]

$$\forall \ (T, x), \ V_l(\mathbf{c})(T, x) = W_l(\mathbf{c})(T, x) \tag{31}$$

Therefore, the valuation function inherits the properties of Cournot maps.

Proof — Let $(T, x, y) \in \mathrm{Dom}(\mathrm{Cour}_{\mathcal{F}_\uparrow}(\mathbb{K}_\uparrow, \mathbb{C}_\uparrow))$ and $(\Omega, x(T - \Omega))$ belong to $\mathrm{Cour}_{\mathcal{F}_\uparrow}(\mathbb{K}_\uparrow, \mathbb{C}_\uparrow)(T, x, y)$. Then there exist $s \in C(T - \Omega)$ and $x(\cdot) \in \mathcal{A}_\mathbf{l}(T, x)[\Omega, s]$ such that $s = x(T - \Omega)$ and $y(T - \Omega) \geq \mathbf{c}(T - \Omega, x(T - \Omega))$. Since

$$y(T - \Omega) \leq y - \int_{T - \Omega}^{T} \mathbf{l}(t, t - (T - \Omega), x(t), u(t)) dt \tag{32}$$

because $y'(t) \leq \mathbf{l}(t, t - (T - \Omega), x(t), u(t))$ and $y(T) = y$, we infer that

$$\mathbf{c}(T - \Omega, x(T - \Omega)) + \int_{T-\Omega}^{T} \mathbf{l}(t, t - (T - \Omega), x(t), u(t))dt \ \leq \ y \qquad (33)$$

By taking the infimum over $\Omega \geq 0$ and, next, over the $x(\cdot) \in \mathcal{A}_{\mathbf{l}}(T, x)[\Omega]$, we deduce that the valuation function $V_l(\mathbf{c})(T, x) \leq y$. By taking the infimum over the set of y satisfying $(T, x, y) \in \mathrm{Dom}(\mathrm{Cour}_{\mathcal{F}_{\uparrow}}(\mathbb{K}_{\uparrow}, \mathbb{C}_{\uparrow}))$, we obtain inequality $V_l(\mathbf{c})(T, x) \leq W_l(\mathbf{c})(T, x)$.

For proving the opposite inequality, let us fix $\varepsilon > 0$ and choose $\Omega_{\varepsilon} \geq 0$ and an evolution $x_{\varepsilon}(\cdot) \in \mathcal{A}_{\mathbf{l}}(T, x)[\Omega_{\varepsilon}]$ such that

$$\mathbf{c}(T - \Omega_{\varepsilon}, x_{\varepsilon}(T - \Omega_{\varepsilon})) + \int_{T-\Omega_{\varepsilon}}^{T} \mathbf{l}(t, T - (T - \Omega_{\varepsilon}), x_{\varepsilon}(t))dt \ \leq \ V_l(\mathbf{c})(T, x) + \varepsilon \qquad (34)$$

Let us set

$$y_{\varepsilon}(t) \ := \ V_l(\mathbf{c})(T, x) + \varepsilon - \int_{t}^{T} \mathbf{l}(\tau, \tau - (T - \Omega_{\varepsilon}), x_{\varepsilon}(\tau), u_{\varepsilon}(\tau))dt \qquad (35)$$

We thus observe that $(x_{\varepsilon}(\cdot), y_{\varepsilon}(\cdot))$ is a solution to the differential inclusion $(x'_{\varepsilon}(\cdot), y'_{\varepsilon}(\cdot)) \in \mathcal{E}p(\mathbf{l})$, that $x_{\varepsilon}(T) = x$ and that $y_{\varepsilon}(T) = V_l(\mathbf{c})(T, x) + \varepsilon$ and $y(T - \Omega) = V_l(\mathbf{c})(T, x) + \varepsilon - \int_{T-\Omega}^{T} \mathbf{l}(\tau, \tau - (T - \Omega_{\varepsilon}), x_{\varepsilon}(\tau), u_{\varepsilon}(\tau))dt \ \leq \ V_l(\mathbf{c})(T, x) + \varepsilon$. Hence $(T, x, V_l(\mathbf{c})(T, x) + \varepsilon)$ belongs to the domain of the auxiliary Cournot map. This implies that $W_l(\mathbf{c})(T, x) \leq V_l(\mathbf{c})(T, x) + \varepsilon$. Letting $\varepsilon \to 0+$ implies $W_l(\mathbf{c})(T, x) \leq V_l(\mathbf{c})(T, x)$ and thus the equality we were looking for. ∎

We recall that the viability solution, when it is differentiable, is a solution to the Hamilton-Jacobi equation satisfying the trajectory conditions. Otherwise, when it is not differentiable, but only lower semicontinuous, we can give a meaning to a solution as a solution in the Barron-Jensen/Frankoska sense, using for that purpose subdifferential of lower semicontinuous functions defined in non-smooth analysis (*Set-valued analysis*, [12, Aubin & Frankowska], [7, 8, Aubin, Bayen, Saint-Pierre]). This is not that important for two reasons: all other properties of viability solutions that are proven in this paper are derived directly from the properties of capture basins without using the concept of derivatives, usual or generalized. In particular, the fact that the viability solution is a solution to the Hamilton-Jacobi equation derives from the tangential conditions characterization of viable-capture basins provided by the Viability Theorem.

The adaptation of the results of Chapters 13 and 17 of *Viability Theory. New Directions*, [8, Aubin, Bayen & Saint-Pierre] is straightforward.

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