Long Time Behavior of First Order Mean Field Games on Euclidean Space

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September 25, 2018

Abstract

The aim of this paper is to study the long time behavior of solutions to deterministic mean field games systems on Euclidean space. This problem was addressed on the torus \mathbb{T}^n in [P. Cardaliaguet, *Long time average of first order mean field games and weak KAM theory*, Dyn. Games Appl. 3 (2013), 473–488], where solutions are shown to converge to the solution of a certain ergodic mean field games system on \mathbb{T}^n . By adapting the approach in [A. Fathi, E. Maderna, *Weak KAM theorem on non compact manifolds*, NoDEA Nonlinear Differential Equations Appl. 14 (2007), 1–27], we identify structural conditions on the Lagrangian, under which the corresponding ergodic system can be solved in \mathbb{R}^n . Then we show that time dependent solutions converge to the solution of such a stationary system on all compact subsets of the whole space.

Keywords: Mean field games; weak KAM theory; long time behavior

2010 Mathematics Subject Classification: 35A01; 35B40; 35F21

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1 Introduction

In this paper we study the relationship between solutions of the first order mean field games (hereinafter referred to as MFG) system with finite horizon

$$\begin{cases} -\partial_t u^T + H(x, Du^T) = F(x, m^T(t)) & \text{in } (0, T) \times \mathbb{R}^n, \\ \partial_t m^T - \operatorname{div} \left(m^T D_p H(x, Du^T) \right) = 0 & \text{in } (0, T) \times \mathbb{R}^n, \\ m^T(0) = m_0, \quad u^T(T, x) = u^f(x), \quad x \in \mathbb{R}^n, \end{cases}$$
(1.1)

and solutions of the ergodic first order MFG system

$$\begin{cases} H(x, D\bar{u}) = F(x, \bar{m}) + \bar{\lambda} & \text{in} \quad \mathbb{R}^n, \\ \operatorname{div} \left(\bar{m} D_p H(x, D\bar{u}) \right) = 0 & \text{in} \quad \mathbb{R}^n, \\ \int_{\mathbb{R}^n} \bar{m}(dx) = 1, \end{cases}$$
(1.2)

where $0 < T < +\infty$ and H is a reversible strict Tonelli Hamiltonian on $\mathbb{R}^n \times \mathbb{R}^n$. More precisely, we will study the long time behavior of the solution of system (1.1) by showing that it converges to a solution of system (1.2) in some weak sense.

MFG theory was introduced independently by Lasry-Lions [18], [19], [20] and Huang, Malhamé, and Caines [17], [16] in order to study large population deterministic and stochastic differential games. In system (1.1), the function u^T can be understood as the value function for a typical small player of a finite horizon optimal control problem in which the density m^T of the other players enters as a datum. Moreover, the players density evolves in time, according to the second equation of the system, following the vector field given by the optimal feedback of each agent.

Our analysis is partially based on tools from weak KAM theory for Lagrangians defined on the tangent bundle of \mathbb{R}^n . Fathi [13] proved the existence of solutions for stationary Hamilton-Jacobi equations, for Lagrangians defined on the tangent bundle of a compact smooth manifold, generalizing the existence result due to Lions, Papanicolaou and Varadhan [21]. Later, Fathi and Maderna [15] extended this existence result to noncompact manifolds. Moreover, they showed that backward weak KAM solutions coincide with viscosity solutions.

When the state space is the flat torus \mathbb{T}^n , the asymptotic behavior as $T \to +\infty$ of solutions to the MFG system (1.1) was studied by Cardaliaguet [8]. In this paper, we remove such a compactness assumption and address the convergence problem as $T \to +\infty$ for solutions of (1.1) on the whole space \mathbb{R}^n . The first step of our analysis is to prove the existence of solutions of system (1.2) as well as the uniqueness of the corresponding critical value (Theorem 3.4 below). A key point, here, is the regularity of viscosity solutions of the first equation of system (1.2) on the projected Mather set. Since such a set, for noncompact state spaces, might be empty (see, for instance, [11]), we need to impose a certain structural assumption ((F4) below) on the mean field Lagrangian.

Our second main result (Theorem 4.13 below) describes the behavior of the solution (u^T, m^T) of system (1.1), as $T \to +\infty$, on compact subsets of \mathbb{R}^n . More precisely, let $(\bar{\lambda}, \bar{u}, \bar{m})$ be a solution of (1.2), where \bar{m} is a projected Mather measure and $\bar{\lambda}$ denotes the Mañé critical value of $H(x, p) - F(x, \bar{m})$. See Definition 3.2 and Definition 2.8 below for definitions of projected Mather measures and Mañé's critical value for Tonelli Lagrangian systems, respectively. Our first main result Theorem 3.4 below guarantees the existence of such solutions $(\bar{\lambda}, \bar{u}, \bar{m})$ and the uniqueness of Mañé's critical value $\bar{\lambda}$. We show that for every $R > R_1$ (see the definition of $R_1 > 0$ in Proposition 4.8) there exists a constant C(R) > 0, such that for any $T \ge 1$ the unique solution (u^T, m^T) of (1.1) satisfies:

$$\sup_{t\in[0,T]} \left\| \frac{u^T(t,\cdot) - \bar{u}(\cdot)}{T} + \bar{\lambda} \left(1 - \frac{t}{T} \right) \right\|_{\infty,\overline{B}_R} \leq \frac{C(R)}{T^{\frac{1}{n+2}}},$$
$$\frac{1}{T} \int_0^T \left\| F(\cdot, m^T(s)) - F(\cdot, \bar{m}) \right\|_{\infty,\overline{B}_R} ds \leq \frac{C(R)}{T^{\frac{1}{n+2}}}.$$

This paper is organized as follows: In Section 2, we fix the notation and recall preliminaries on measure theory and weak KAM theory. In Section 3, we prove the existence of solutions to the ergodic system (1.2) and we give a uniqueness criterion under a monotonicity assumption on F. In Section 4, after proving some preliminary lemmas, we obtain the main convergence result. The Appendix contains the proof of (*ii*) of Theorem 3.4 and a technical result which is used in the proof of Theorem 4.13.

2 Preliminaries

In this section, we recall definitions and preliminary results from measure theory and weak KAM theory, which will be used later in this paper.

2.1 Notation

We write below a list of symbols used throughout this paper.

- Denote by N the set of positive integers, by Rⁿ the n-dimensional real Euclidean space, by ⟨·, ·⟩ the Euclidean scalar product, by | · | the usual norm in Rⁿ, and by B_R the open ball with center 0 and radius R.
- π_1 denotes the projection of $\mathbb{R}^n \times \mathbb{R}^n$ onto the first factor.
- Let a, b ∈ ℝ. a ∨ b and a ∧ b are used to stand for maximum and minimum, respectively: a ∨ b = max{a, b} and a ∧ b = min{a, b}. The positive part of a real function f is defined by f⁺ = f ∨ 0.
- Let Λ be a real $n \times n$ matrix. Define the norm of Λ by

$$\|\Lambda\| = \sup_{|x|=1,x\in\mathbb{R}^n} \|\Lambda x\|.$$

Let A be a Lebesgue-measurable subset of ℝⁿ. Denote by Lⁿ(A) the n-dimensional Lebesgue measure of A. Denote by 1_A : ℝⁿ → {0,1} the characteristic function of A, i.e.,

$$\mathbf{1}_A(x) = \begin{cases} 1 & x \in A, \\ 0 & x \notin A. \end{cases}$$

• Let f be a real-valued function on \mathbb{R}^n . The set

$$D^+f(x) = \left\{ p \in \mathbb{R}^n : \limsup_{y \to x} \frac{f(y) - f(x) - \langle p, y - x \rangle}{|y - x|} \leqslant 0 \right\},$$

is called the superdifferential of f at x. Let u(t, x) be a real-valued function on $[0, T] \times \mathbb{R}^n$ for some T > 0. The symbol $\nabla^+ u(t, x)$ denotes the superdifferential of the function $x \mapsto u(t, x)$.

• Lip(A) stands for the space of Lipschitz functions on $A \subset \mathbb{R}^n$ and denote by

$$\operatorname{Lip}(f) = \sup_{\substack{x \neq y \\ x, y \in A}} \frac{|f(x) - f(y)|}{|x - y|},$$

the Lipschitz seminorm of $f \in Lip(A)$. Define

$$||Df||_{\infty,A} := \operatorname{ess\,sup}_{x \in A} |Df(x)|.$$

Let A be a Lebesgue-measurable subset of ℝⁿ. Let 1 ≤ p ≤ ∞. Denote by L^p(A) the space of Lebesgue-measurable functions f with ||f||_{p,A} < ∞, where

$$||f||_{\infty,A} := \operatorname{ess sup}_{x \in A} |f(x)|,$$

$$||f||_{p,A} := \left(\int_{A} |f|^{p} dx\right)^{\frac{1}{p}}, \quad 1 \le p < \infty.$$

Denote $||f||_{\infty,\mathbb{R}^n}$ by $||f||_{\infty}$ and $||f||_{p,\mathbb{R}^n}$ by $||f||_p$, for brevity.

- Let 1 ≤ p ≤ ∞. The function f belongs to the Sobolev space W^{1,p}(ℝⁿ) if f ∈ L^p(ℝⁿ) and for i = 1, ..., n the weak derivatives ∂f/∂x_i exist and belong to L^p(ℝⁿ). The function f belongs to W^{1,p}_{loc}(ℝⁿ) if f ∈ W^{1,p}(A) for each open set A such that Ā is compact and Ā ⊂ ℝⁿ.
- C_b(ℝⁿ) stands for the function space of bounded uniformly continuous functions on ℝⁿ. C²_b(ℝⁿ) stands for the space of bounded functions on ℝⁿ with bounded uniformly continuous first and second derivatives. C^k(ℝⁿ) (k ∈ ℕ) stands for the function space of k-times continuously differentiable functions on ℝⁿ, and C[∞](ℝⁿ) := ∩[∞]_{k=0}C^k(ℝⁿ). C[∞]_c(ℝⁿ) stands for the space of functions in C[∞](ℝⁿ) with compact support. Let a < b ∈ ℝ. AC([a, b]; ℝⁿ) denotes the space of absolutely continuous curves [a, b] → ℝⁿ.
- For $f \in C^1(\mathbb{R}^n)$, the gradient vector of f is denoted by $Df = (D_{x_1}f, ..., D_{x_n}f)$, where $D_{x_i}f = \frac{\partial f}{\partial x_i}$, $i = 1, 2, \cdots, n$. Let k be a nonnegative integer and let $\alpha = (\alpha_1, \cdots, \alpha_n)$ be a multiindex of order k, i.e., $k = |\alpha| = \alpha_1 + \cdots + \alpha_n$, where each component α_i is a nonnegative integer. For $f \in C^k(\mathbb{R}^n)$, define $D^{\alpha}f := D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}f$.

2.2 Measure theory

Denote by $\mathscr{B}(\mathbb{R}^n)$ the Borel σ -algebra on \mathbb{R}^n and by $\mathcal{P}(\mathbb{R}^n)$ the space of Borel probability measures on \mathbb{R}^n . The support of a measure $\mu \in \mathcal{P}(\mathbb{R}^n)$, denoted by $\operatorname{supp}(\mu)$, is the closed set defined by

 $\operatorname{supp}(\mu) := \Big\{ x \in \mathbb{R}^n : \mu(V_x) > 0 \text{ for each open neighborhood } V_x \text{ of } x \Big\}.$

We say that a sequence $\{\mu_k\}_{k\in\mathbb{N}} \subset \mathcal{P}(\mathbb{R}^n)$ is weakly-* convergent to $\mu \in \mathcal{P}(\mathbb{R}^n)$, denoted by $\mu_k \xrightarrow{w^*} \mu$, if

$$\lim_{n \to \infty} \int_{\mathbb{R}^n} f(x) \, d\mu_n(x) = \int_{\mathbb{R}^n} f(x) \, d\mu(x), \quad \forall f \in C_b(\mathbb{R}^n).$$

For $p \in [1, +\infty)$, the Wasserstein space of order p is defined as

$$\mathcal{P}_p(\mathbb{R}^n) := \{ m \in \mathcal{P}(\mathbb{R}^n) : \int_{\mathbb{R}^n} |x_0 - x|^p \, dm(x) < +\infty \}$$

where $x_0 \in \mathbb{R}^n$ is arbitrary. Given any two measures m and m' in $\mathcal{P}_p(\mathbb{R}^n)$, define

$$\Pi(m,m') := \left\{ \lambda \in \mathcal{P}(\mathbb{R}^n \times \mathbb{R}^n) : \lambda(A \times \mathbb{R}^n) = m(A), \ \lambda(\mathbb{R}^n \times A) = m'(A), \ \forall A \in \mathscr{B}(\mathbb{R}^n) \right\}.$$

The Wasserstein distance of order p between m and m' is defined by

$$d_p(m,m') = \inf_{\lambda \in \Pi(m,m')} \left(\int_{\mathbb{R}^n \times \mathbb{R}^n} |x-y|^p \, d\lambda(x,y) \right)^{1/p}.$$

The distance d_1 is also commonly called the Kantorovich-Rubinstein distance and can be characterized by a useful duality formula (see, for instance, [23]) as follows

$$d_1(m,m') = \sup\Big\{\int_{\mathbb{R}^n} f(x)\,dm(x) - \int_{\mathbb{R}^n} f(x)\,dm'(x) \mid f: \mathbb{R}^n \to \mathbb{R} \text{ is 1-Lipschitz}\Big\},\$$

for all $m, m' \in \mathcal{P}_1(\mathbb{R}^n)$.

We now recall that weak-* convergence is equivalent to convergence in the metric space $(\mathcal{P}_p(\mathbb{R}^n), d_p)$ (see, for instance, [23]).

Proposition 2.1. Let $\{\mu_k\}_{k\in\mathbb{N}}$ be a sequence of measures in $\mathcal{P}_p(\mathbb{R}^n)$ and let μ be another element of $\mathcal{P}_p(\mathbb{R}^n)$. Then

- (i) if $d_p(\mu_k, \mu) \to 0$, then $\mu_k \xrightarrow{w^*} \mu$, as $k \to +\infty$;
- (ii) if supp (μ_k) is contained in a fixed compact subset of \mathbb{R}^n for all $k \in \mathbb{N}$ and $\mu_k \xrightarrow{w^*} \mu$, as $k \to +\infty$, then $d_p(\mu_k, \mu) \to 0$, as $k \to +\infty$.

Let (X_1, S_1, μ) be a measure space, (X_2, S_2) a measurable space, and $f : X_1 \to X_2$ a measurable map. The push-forward of μ through f is the measure $f \sharp \mu$ on (X_2, S_2) defined by

$$f \sharp \mu(B) := \mu\left(f^{-1}(B)\right), \quad \forall B \in S_2.$$

The push-forward has the property that a measurable map $g: X_2 \to \mathbb{R}$ is integrable with respect to $f \sharp \mu$ if and only if $g \circ f$ is integrable on X_1 with respect to μ . In this case, we have that

$$\int_{X_1} g(f(x)) \, d\mu(x) = \int_{X_2} g(y) \, df \, \sharp \mu(y).$$

2.3 Weak KAM theory on \mathbb{R}^n

Definition 2.2 (Tonelli Lagrangians). A C^2 function $L : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is called a Tonelli Lagrangian if it satisfies the following:

- (i) for each $(x, v) \in \mathbb{R}^n \times \mathbb{R}^n$, the Hessian $D_{vv}^2 L(x, v)$ is positive definite;
- (ii) for each A > 0 there exists $B(A) \in \mathbb{R}$ such that

$$L(x,v) > A|v| + B(A), \quad \forall (x,v) \in \mathbb{R}^n \times \mathbb{R}^n;$$

(iii) for each R > 0, $A(R) := \sup \{ L(x, v) : |v| \le R \} < +\infty$.

Definition 2.3 (Strict Tonelli Lagrangians). A C^2 function $L : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is called a strict Tonelli Lagrangian if there exist positive constants C_i (i = 1, 2, 3) such that, for all $(x, v) \in \mathbb{R}^n \times \mathbb{R}^n$:

(a) $\frac{I}{C_1} \leq D_{vv}^2 L(x,v) \leq C_1 I$, where I is the identity matrix;

(b)
$$||D_{vx}^2L(x,v)|| \le C_2(1+|v|);$$

(c)
$$|L(x,0)| + |D_xL(x,0)| + |D_vL(x,0)| \le C_3.$$

Remark 2.4. Let *L* be a strict Tonelli Lagrangian. It is easy to check that there are two positive constants α , β depending only on C_i (i = 1, 2, 3) in Definition 2.3, such that

(e) $|D_v L(x,v)| \le \alpha (1+|v|), \quad \forall (x,v) \in \mathbb{R}^n \times \mathbb{R}^n;$

(f)
$$|D_x L(x,v)| \le \alpha (1+|v|^2), \quad \forall (x,v) \in \mathbb{R}^n \times \mathbb{R}^n;$$

- (g) $\frac{1}{4\beta}|v|^2 \alpha \le L(x,v) \le 4\beta|v|^2 + \alpha, \quad \forall (x,v) \in \mathbb{R}^n \times \mathbb{R}^n;$
- (h) $\sup \left\{ L(x,v) : |v| \le R \right\} < +\infty, \quad \forall R \ge 0.$

In view of (a), (g), and (h), it is clear that a strict Tonelli Lagrangian is a Tonelli Lagrangian.

From now on to the end of this section, we always assume that L is a Tonelli Lagrangian on $\mathbb{R}^n \times \mathbb{R}^n$.

Define the Hamiltonian $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ associated with L by

$$H(x,p) = \sup_{v \in \mathbb{R}^n} \left\{ \left\langle p, v \right\rangle - L(x,v) \right\}, \quad \forall (x,p) \in \mathbb{R}^n \times \mathbb{R}^n.$$

It is straightforward to check that if L is a Tonelli Lagrangian (resp. a strict Tonelli Lagrangian), then H defined above also satisfies (i), (ii), and (iii) in Definition 2.1 (resp. (a), (b), and (c) in Definition 2.2). Such a function H is called a Tonelli Hamiltonian (resp. a strict Tonelli Hamiltonian). Moreover, if L is a *reversible* Lagrangian, i.e., L(x, v) = L(x, -v) for all $(x, v) \in \mathbb{R}^n \times \mathbb{R}^n$, then H(x, p) = H(x, -p) for all $(x, p) \in \mathbb{R}^n \times \mathbb{R}^n$.

Let us recall definitions of weak KAM solutions and viscosity solutions of the Hamilton-Jacobi equation

$$H(x, Du) = c, \quad x \in \mathbb{R}^n, \tag{2.1}$$

where c is a real constant.

Definition 2.5 (Weak KAM solutions). A function $u \in C(\mathbb{R}^n)$ is called a backward (resp. forward) weak KAM solution of equation (2.1) if:

(i) for each continuous piecewise C^1 curve $\gamma : [t_1, t_2] \to \mathbb{R}^n$, we have that

$$u(\gamma(t_2)) - u(\gamma(t_1)) \le \int_{t_1}^{t_2} L(\gamma(s), \dot{\gamma}(s)) ds + c(t_2 - t_1) \le \frac{1}{2} \int_{t_1}^{t_2} L(\gamma(s), \dot{\gamma}(s)) ds + c(t_2 - t_1) \le \frac{1}{2} \int_{t_1}^{t_2} L(\gamma(s), \dot{\gamma}(s)) ds + c(t_2 - t_1) \le \frac{1}{2} \int_{t_1}^{t_2} L(\gamma(s), \dot{\gamma}(s)) ds + c(t_2 - t_1) \le \frac{1}{2} \int_{t_1}^{t_2} L(\gamma(s), \dot{\gamma}(s)) ds + c(t_2 - t_1) \le \frac{1}{2} \int_{t_1}^{t_2} L(\gamma(s), \dot{\gamma}(s)) ds + c(t_2 - t_1) \le \frac{1}{2} \int_{t_1}^{t_2} L(\gamma(s), \dot{\gamma}(s)) ds + c(t_2 - t_1) \le \frac{1}{2} \int_{t_1}^{t_2} L(\gamma(s), \dot{\gamma}(s)) ds + c(t_2 - t_1) \le \frac{1}{2} \int_{t_1}^{t_2} L(\gamma(s), \dot{\gamma}(s)) ds + c(t_2 - t_1) \le \frac{1}{2} \int_{t_1}^{t_2} L(\gamma(s), \dot{\gamma}(s)) ds + c(t_2 - t_1) \le \frac{1}{2} \int_{t_1}^{t_2} L(\gamma(s), \dot{\gamma}(s)) ds + c(t_2 - t_1) \le \frac{1}{2} \int_{t_1}^{t_2} L(\gamma(s), \dot{\gamma}(s)) ds + c(t_2 - t_1) \le \frac{1}{2} \int_{t_1}^{t_2} L(\gamma(s), \dot{\gamma}(s)) ds + c(t_2 - t_1) \le \frac{1}{2} \int_{t_1}^{t_2} L(\gamma(s), \dot{\gamma}(s)) ds + c(t_2 - t_1) \le \frac{1}{2} \int_{t_1}^{t_2} L(\gamma(s), \dot{\gamma}(s)) ds + c(t_2 - t_1) \le \frac{1}{2} \int_{t_1}^{t_2} L(\gamma(s), \dot{\gamma}(s)) ds + c(t_2 - t_1) \le \frac{1}{2} \int_{t_1}^{t_2} L(\gamma(s), \dot{\gamma}(s)) ds + c(t_2 - t_1) \le \frac{1}{2} \int_{t_1}^{t_2} L(\gamma(s), \dot{\gamma}(s)) ds + c(t_2 - t_1) \le \frac{1}{2} \int_{t_1}^{t_2} L(\gamma(s), \dot{\gamma}(s)) ds + c(t_2 - t_1) \le \frac{1}{2} \int_{t_1}^{t_2} L(\gamma(s), \dot{\gamma}(s)) ds + c(t_2 - t_1) \le \frac{1}{2} \int_{t_1}^{t_2} L(\gamma(s), \dot{\gamma}(s)) ds + c(t_2 - t_1) \le \frac{1}{2} \int_{t_1}^{t_2} L(\gamma(s), \dot{\gamma}(s)) ds + c(t_2 - t_1) + c(t_2 - t_$$

(ii) for each $x \in \mathbb{R}^n$, there exists a C^1 curve $\gamma : (-\infty, 0] \to \mathbb{R}^n$ (resp. $\gamma : [0, +\infty) \to \mathbb{R}^n$) with $\gamma(0) = x$ such that

$$u(x) - u(\gamma(t)) = \int_t^0 L(\gamma(s), \dot{\gamma}(s)) ds - ct, \quad \forall t < 0$$

(resp. $u(\gamma(t)) - u(x) = \int_0^t L(\gamma(s), \dot{\gamma}(s)) ds + ct, \quad \forall t > 0$).

Remark 2.6. A function u on \mathbb{R}^n is said to be dominated by L + c, denoted by $u \prec L + c$, if u satisfies condition (i) of Definition 2.5. A curve γ is said to be (u, L, c)-calibrated if it satisfies condition (ii) of Definition 2.5.

Definition 2.7 (Viscosity solutions). Let $V \subset \mathbb{R}^n$ be an open set.

(i) A function $u: V \to \mathbb{R}$ is called a viscosity subsolution of equation (2.1), if for every C^1 function $\varphi: V \to \mathbb{R}$ and every point $x_0 \in V$ such that $u - \varphi$ has a local maximum at x_0 , we have that

$$H(x_0, D\varphi(x_0)) \le c;$$

(ii) A function $u : V \to \mathbb{R}$ is called a viscosity supersolution of equation (2.1), if for every C^1 function $\psi : V \to \mathbb{R}$ and every point $y_0 \in V$ such that $u - \psi$ has a local minimum at y_0 , we have that

$$H(y_0, D\psi(y_0)) \ge c;$$

(iii) A function $u : V \to \mathbb{R}$ is called a viscosity solution of equation (2.1) if it is both a viscosity subsolution and a viscosity supersolution.

Definition 2.8 (Mañé critical value). *The Mañé critical value of a Tonelli Hamiltonian H is defined by*

 $c(H) := \inf \{ c \in \mathbb{R} : \text{ there exists a viscosity solution } u \in C(\mathbb{R}^n) \text{ of } H(x, Du) = c \}.$

See [15, Theorem 1.1] for the following weak KAM theorem for noncompact state spaces.

Theorem 2.9 (Weak KAM theorem). Let H be a Tonelli Hamiltonian. Then, there exists a global viscosity solution of equation

$$H(x, Du) = c(H), \quad x \in \mathbb{R}^n.$$

In [15], viscosity solutions are shown to coincide with backward weak KAM solutions.

Observe that, as \mathbb{R}^n can be seen as a covering of the torus \mathbb{T}^n , Mañé's critical value can be characterized as follows ([10]):

$$c(H) = \inf_{u \in C^{\infty}(\mathbb{R}^n)} \sup_{x \in \mathbb{R}^n} H(x, Du(x)).$$
(2.2)

We conclude this section by recalling the notion of Mather set and the role such a set plays for the regularity of viscosity solutions. Let L be a Tonelli Lagrangian. As is well known, the associated Euler-Lagrange equation, i.e.,

$$\frac{d}{dt}D_v L(x,\dot{x}) = D_x L(x,\dot{x}), \qquad (2.3)$$

generates a flow of diffeomorphisms $\phi_t^L : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$, with $t \in \mathbb{R}$, defined by

$$\phi_t^L(x_0, v_0) = (x(t), \dot{x}(t)),$$

where $x : \mathbb{R} \to \mathbb{R}^n$ is the maximal solution of (2.3) with initial conditions $x(0) = x_0$, $\dot{x}(0) = v_0$. It should be noted that, for any Tonelli Lagrangian, the flow ϕ_t^L is complete ([15]).

We recall that a Borel probability measure μ on $\mathbb{R}^n \times \mathbb{R}^n$ is called ϕ_t^L -invariant, if

$$\mu(B) = \mu(\phi_t^L(B)), \quad \forall t \in \mathbb{R}, \quad \forall B \in \mathscr{B}(\mathbb{R}^n \times \mathbb{R}^n),$$

or, equivalently,

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} f(\phi_t^L(x, v)) \, d\mu(x, v) = \int_{\mathbb{R}^n \times \mathbb{R}^n} f(x, v) \, d\mu(x, v), \quad \forall f \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^n).$$

We denote by \mathcal{M}_L the class of all ϕ_t^L -invariant probability measures.

Definition 2.10 (Mather measures [22]). A probability measure $\mu \in M_L$ is called a Mather measure for L, if it satisfies

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} L(x, v) \, d\mu(x, v) = \inf_{\nu \in \mathcal{M}_L} \int_{\mathbb{R}^n \times \mathbb{R}^n} L(x, v) \, d\nu(x, v).$$

Under the assumption (F4) below, we deduce that the set of Mather measures is nonempty. Moreover, in [14], it was proved that

$$c(H) = -\inf_{\nu \in \mathcal{M}_L} \int_{\mathbb{R}^n \times \mathbb{R}^n} L(x, v) \, d\nu(x, v).$$

Denote by \mathcal{M}_L^* the set of Mather measures. Observe that, if L (resp. H) is a reversible Lagrangian (resp. reversible Hamiltonian), then

$$-c(H) = \inf_{x \in \mathbb{R}^n} L(x, 0).$$
(2.4)

The Mather set is the subset $\mathcal{M}_0 \subset \mathbb{R}^n \times \mathbb{R}^n$ defined by

$$\mathcal{M}_0 = \bigcup_{\mu \in \mathcal{M}_L^*} \operatorname{supp}(\mu).$$

We call $M_0 = \pi_1(\mathcal{M}_0) \subset \mathbb{R}^n$ the projected Mather set. See [14, Theorem 4.12.3] for the following result.

Theorem 2.11. If u is dominated by L + c(H), then it is differentiable at every point of the projected Mather set M_0 . Moreover, if $(x, v) \in \mathcal{M}_0$, then

$$Du(x) = D_v L(x, v)$$

and the map $M_0 \to \mathbb{R}^n \times \mathbb{R}^n$, defined by $x \mapsto (x, Du(x))$, is locally Lipschitz with a Lipschitz constant which is independent of u.

3 Ergodic MFG system: existence and uniqueness

In this section we prove an existence and uniqueness result for (1.2).

3.1 Assumptions

From now on, we suppose that L is a reversible strict Tonelli Lagrangian. Let $F : \mathbb{R}^n \times \mathcal{P}_1(\mathbb{R}^n) \to \mathbb{R}$ be a function, satisfying the following assumptions:

(F1) for every measure $m \in \mathcal{P}_1(\mathbb{R}^n)$ the function $x \mapsto F(x,m)$ is of class $C_b^2(\mathbb{R}^n)$ and

$$\sup_{m \in \mathcal{P}_1(\mathbb{R}^n)} \sum_{|\alpha| \le 2} \|D^{\alpha} F(\cdot, m)\|_{\infty} < +\infty,$$

where $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $D^{\alpha} = D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}$;

(F2) for every $x \in \mathbb{R}^n$ the function $m \mapsto F(x, m)$ is Lipschitz continuous and

$$\operatorname{Lip}_{2}(F) := \sup_{\substack{x \in \mathbb{R}^{n} \\ m_{1}, \ m_{2} \in \mathcal{P}_{1}(\mathbb{R}^{n}) \\ m_{1} \neq m_{2}}} \frac{|F(x, m_{1}) - F(x, m_{2})|}{d_{1}(m_{1}, m_{2})} < +\infty;$$

(F3) there is a constant $C_F > 0$ such that for every $m_1, m_2 \in \mathcal{P}_1(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \left(F(x, m_1) - F(x, m_2) \right) \, d(m_1 - m_2) \ge C_F \int_{\mathbb{R}^n} \left(F(x, m_1) - F(x, m_2) \right)^2 \, dx,$$

and for each $x \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} \left(F(x, m_1) - F(x, m_2) \right) d(m_1 - m_2) = 0 \quad \text{if and only if} \quad F(x, m_1) = F(x, m_2);$$

(F4) there exist a compact set $K_0 \subset \mathbb{R}^n$ and a constant $\delta_0 > 0$ such that, for every $m \in \mathcal{P}_1(\mathbb{R}^n)$,

$$\inf_{x \in \mathbb{R}^n \setminus K_0} \left\{ L(x,0) + F(x,m) \right\} - \min_{x \in K_0} \left\{ L(x,0) + F(x,m) \right\} \ge \delta_0$$

Now we give an example where F and L satisfy conditions (F1)-(F4).

Example 3.1. Let L(x, v) = L(v) be a reversible strict Tonelli Lagrangian. Let

$$F(x,m) = f(x)g(m),$$

where

- $f: \mathbb{R}^n \to \mathbb{R}$ satisfies
 - (i) $f \in C_b^2(\mathbb{R}^n)$, and $\int_{\mathbb{R}^n} |f|^2 dx < +\infty$;
 - (*ii*) $\operatorname{argmin}_{x \in \mathbb{R}^n} f(x)$ is nonempty and bounded.
- $g(m) = G(\int_{\mathbb{R}^n} f(x) \, dm)$ for $m \in \mathcal{P}_1(\mathbb{R}^n)$, with $G \in C^1(\mathbb{R})$ satisfying the following:

- (*iii*) $G \ge \delta_1$, where δ_1 is a positive constant;
- (iv) for each R > 0, there is $\nu(R) > 0$ such that for any $s \in [-R, R]$,

$$\nu(R) \le G'(s) \le \frac{1}{\nu(R)}.$$

Let $K_0 \subset \mathbb{R}^n$ be a compact set such that

int
$$K_0 \supset \operatorname*{argmin}_{x \in \mathbb{R}^n} f(x)$$
,

where int K_0 denotes the interior of K_0 .

Then, we claim that that assumptions (F1)-(F4) are fulfilled. Indeed, (F1) and (F2) follow, immediately, from (i), (iv), and the differentiability of G. In order to check that F satisfies (F3), fix $m_1, m_2 \in \mathcal{P}_1(\mathbb{R}^n)$ and observe that, since m_1, m_2 are probability measures,

$$\left|\int_{\mathbb{R}^n} f \, dm_i\right| \le \|f\|_{\infty},$$

for i = 1, 2. Moreover, by definition, we have that

$$\int_{\mathbb{R}^n} (F(x,m_1) - F(x,m_2))^2 dx$$
$$= \left(G\left(\int_{\mathbb{R}^n} f \, dm_1 \right) - G\left(\int_{\mathbb{R}^n} f \, dm_2 \right) \right)^2 \|f\|_2^2,$$

and

$$\int_{\mathbb{R}^n} (F(x,m_1) - F(x,m_2)) d(m_1 - m_2)$$
$$= \left(G\left(\int_{\mathbb{R}^n} f \, dm_1 \right) - G\left(\int_{\mathbb{R}^n} f \, dm_2 \right) \right) \int_{\mathbb{R}^n} f \, d(m_1 - m_2).$$

If $g(m_1) = g(m_2)$, then the inequality in (F3) is obvious. Suppose $g(m_1) \neq g(m_2)$. Then, $\int_{\mathbb{R}^n} f d(m_1 - m_2) \neq 0$ and we have that

$$\frac{\int_{\mathbb{R}^{n}} (F(x,m_{1}) - F(x,m_{2}))^{2} dx}{\int_{\mathbb{R}^{n}} (F(x,m_{1}) - F(x,m_{2})) d(m_{1} - m_{2})} \\
\leq \|f\|_{2}^{2} \left| \frac{\left(G\left(\int_{\mathbb{R}^{n}} f dm_{1}\right) - G\left(\int_{\mathbb{R}^{n}} f dm_{2}\right)\right)}{\int_{\mathbb{R}^{n}} f d(m_{1} - m_{2})} \right| \\
\leq \frac{\|f\|_{2}^{2}}{\nu(\|f\|_{\infty})}.$$

So far, we have checked that the inequality in (F3) holds true. For the necessary and sufficient condition in (F3), we only need to prove that $0 = \int_{\mathbb{R}^n} (F(x, m_1) - F(x, m_2)) d(m_1 - m_2)$ implies that $F(x, m_1) = F(x, m_2)$ for all $x \in \mathbb{R}^n$. Note that

$$0 = \int_{\mathbb{R}^n} \left(F(x, m_1) - F(x, m_2) \right) d(m_1 - m_2)$$

= $\left(G\left(\int_{\mathbb{R}^n} f \, dm_1 \right) - G\left(\int_{\mathbb{R}^n} f \, dm_2 \right) \right) \int_{\mathbb{R}^n} f \, d(m_1 - m_2)$
 $\geq \nu(\|f\|_{\infty}) \left(\int_{\mathbb{R}^n} f \, d(m_1 - m_2) \right)^2.$

Hence,

$$\int_{\mathbb{R}^n} f \, dm_1 = \int_{\mathbb{R}^n} f \, dm_2$$

and thus for every $x \in \mathbb{R}^n$, we have that $F(x, m_1) = F(x, m_2)$.

Finally, we prove that L and F satisfy (F4). Fix $m \in \mathcal{P}_1(\mathbb{R}^n)$. Since K_0 is a compact neighborhood of $\operatorname{argmin}_{x \in \mathbb{R}^n} f(x)$ it follows that

$$\inf_{x \in \mathbb{R}^n \setminus K_0} f(x) > \min_{x \in K_0} f(x)$$

Then, there is a constant $\delta_2 > 0$ such that

$$\inf_{x \in \mathbb{R}^n \setminus K_0} f(x) - \min_{x \in K_0} f(x) \ge \delta_2.$$

Therefore, since $g(m) \ge \delta_1$, we obtain that

$$\inf_{x \in \mathbb{R}^n \setminus K_0} \left\{ L(0) + F(x,m) \right\} - \min_{x \in K_0} \left\{ L(0) + F(x,m) \right\} \ge \delta_1 \delta_2 := \delta_0. \qquad \Box$$

Let *H* be the reversible strict Tonelli Hamiltonian associated with *L*. For any $m \in \mathcal{P}_1(\mathbb{R}^n)$, define the mean field Lagrangian and Hamiltonian associated with *m* by

$$L_m(x,v) := L(x,v) + F(x,m), \quad (x,v) \in \mathbb{R}^n \times \mathbb{R}^n,$$
(3.1)

$$H_m(x,p) := H(x,p) - F(x,m), \quad (x,p) \in \mathbb{R}^n \times \mathbb{R}^n.$$
(3.2)

By assumptions (F1) and (F2), it is clear that for any given $m \in \mathcal{P}_1(\mathbb{R}^n)$, L_m (resp. H_m) is a strict Tonelli Lagrangian (resp. Hamiltonian).

Definition 3.2 (Mean field ergodic solutions). We say that a triple $(\bar{\lambda}, \bar{u}, \bar{m}) \in \mathbb{R} \times C(\mathbb{R}^n) \times \mathcal{P}_1(\mathbb{R}^n)$ is a solution of system (1.2) if

- (i) \bar{u} is a Lipschitz continuous viscosity solution of the first equation of system (1.2);
- (ii) $D\bar{u}$ exists for $\bar{m} a.e. \ x \in \mathbb{R}^n$;
- (iii) \bar{m} is a projected Mather measure, i.e., there is a Mather measure $\eta_{\bar{m}}$ for $L_{\bar{m}}$ such that $\bar{m} = \pi_1 \sharp \eta_{\bar{m}}$;
- (iv) \bar{m} satisfies the second equation of system (1.2) in the sense of distributions, that is,

$$\int_{\mathbb{R}^n} \left\langle Df(x), D_p H\left(x, D\bar{u}(x)\right) \right\rangle d\bar{m}(x) = 0, \quad \forall f \in C_c^{\infty}(\mathbb{R}^n)$$

We denote by S the set of solutions of system (1.2).

Define the function $\lambda : \mathcal{P}_1(\mathbb{R}^n) \to \mathbb{R}$ by

$$\lambda(m) := c(H_m).$$

Lemma 3.3. The function $m \mapsto \lambda(m)$ is Lipschitz continuous on $\mathcal{P}_1(\mathbb{R}^n)$ with respect to the *metric* d_1 .

Proof. For any $m \in \mathcal{P}_1(\mathbb{R}^n)$, since L_m is a strict Tonelli Lagrangian, by (2.2) we have that

$$\lambda(m) = \inf_{u \in C^{\infty}(\mathbb{R}^n)} \sup_{x \in \mathbb{R}^n} H_m(x, Du(x)).$$
(3.3)

So, the conclusion follows noting that (3.3) and (F2) yield

$$\left|\lambda(m_1) - \lambda(m_2)\right| \le \inf_{u \in C^{\infty}(\mathbb{R}^n)} \sup_{x \in \mathbb{R}^n} \left|F(x, m_1) - F(x, m_2)\right| \le \operatorname{Lip}_2(F) d_1(m_1, m_2)$$

for any $m_1, m_2 \in \mathcal{P}_1(\mathbb{R}^n)$.

3.2 Main result 1

We are now in a position to state and prove our first major result.

Theorem 3.4 (Existence of solutions of (1.2)). Assume (F1), (F2), and (F4).

- (i) There exists at least one solution $(c(H_{\bar{m}}), \bar{u}, \bar{m})$ of system (1.2), i.e., $S \neq \emptyset$.
- (*ii*) Assume, in addition, (F3). Let $(c(H_{\bar{m}_1}), \bar{u}_1, \bar{m}_1), (c(H_{\bar{m}_2}), \bar{u}_2, \bar{m}_2) \in S$. Then,

$$F(x, \bar{m}_1) = F(x, \bar{m}_2), \quad \forall x \in \mathbb{R}^n \text{ and } c(H_{\bar{m}_1}) = c(H_{\bar{m}_2}).$$

Remark 3.5. By (*ii*) in Theorem 3.4, it is clear that each element of S has the form $(\lambda, \bar{u}, \bar{m})$, where \bar{m} is a projected Mather measure and $\bar{\lambda}$ denotes the common Mañé critical value of $H_{\bar{m}}$.

Proof of Theorem 3.4. (i) For any measure $m \in \mathcal{P}_1(\mathbb{R}^n)$, recall that $\phi_t^{L_m}$ denotes the Euler-Lagrange flow of L_m , where L_m is defined in (3.1). We divide the proof of (i) in two steps.

STEP 1: we show the existence of Mather measures for L_m for each $m \in \mathcal{P}_1(\mathbb{R}^n)$.

From assumption (F4), for any $m \in \mathcal{P}_1(\mathbb{R}^n)$ there exists $x_m \in K_0$ such that

$$\inf_{x \in \mathbb{R}^n} L_m(x,0) = \min_{x \in K_0} L_m(x,0) = L_m(x_m,0),$$

where K_0 is the compact set as in (F4). Note that the constant curve $t \mapsto x_m$ for $t \in \mathbb{R}$ is a solution of

$$\frac{d}{dt}D_v L_m(x,\dot{x}) = D_x L_m(x,\dot{x}),$$

i.e., $\phi_t^{L_m}(x_m, 0) = (x_m, 0)$ for all $t \in \mathbb{R}$. Thus, the atomic measure $\delta_{(x_m,0)}$, supported on $(x_m, 0)$, is a $\phi_t^{L_m}$ -invariant probability measure. Recalling the definition of Mather measures and x_m , it follows that $\delta_{(x_m,0)}$ is a Mather measure for L_m (see also [14, Proposition 4.14.3]).

Consequently, for any $m \in \mathcal{P}_1(\mathbb{R}^n)$ we have that

$$\mathcal{M}_0^m = \left\{ (x_m, 0) : L_m(x_m, 0) = \min_{x \in K_0} L_m(x, 0) \right\},\$$

where \mathcal{M}_0^m denotes the Mather set associated with L_m . So, for each $m \in \mathcal{P}_1(\mathbb{R}^n)$, all Mather measures associated with L_m are supported in $K_0 \times \{0\}$.

STEP 2 : we show the existence of solutions of (1.2).

From Step 1, for any $m \in \mathcal{P}_1(\mathbb{R}^n)$, there is a Mather measure η_m associated with L_m , i.e., $\eta_m \in \mathcal{M}_{L_m}^*$. Consider the set-valued map

$$\Psi: \mathcal{P}(K_0) \rightrightarrows \mathcal{P}(K_0), \quad m \mapsto \Psi(m),$$

where

$$\Psi(m) := \left\{ \pi_1 \sharp \eta_m : \eta_m \in \mathcal{M}_{L_m}^* \right\}$$

As is customary in MFG theory, we will apply Kakutani's theorem (see, for instance, [5]) to show that there exist a fixed point \overline{m} of Ψ .

Observe that the metric space $(\mathcal{P}(K_0), d_1)$ is convex and compact due to Prokhorov's theorem (see, for instance, [4]). Since Ψ has nonempty convex values, the only hypothesis of Kakutani's theorem we need to check is that Ψ has closed graph: for any pair of sequences $\{m_j\}_{j\in\mathbb{N}} \subset \mathcal{P}(K_0), \{\mu_j\}_{j\in\mathbb{N}} \subset \mathcal{P}(K_0)$ such that

$$m_j \xrightarrow{w^*} m, \quad \mu_j \xrightarrow{w^*} \mu, \text{ as } j \to +\infty \quad \text{and} \quad \mu_j \in \Psi(m_j) \quad \text{for all } j \in \mathbb{N},$$

we must prove that $\mu \in \Psi(m)$. Since $\mu_j \in \Psi(m_j)$ and $\mu_j \xrightarrow{w^*} \mu$ as $j \to +\infty$, there are Mather measures η_{m_j} and a measure $\eta \in \mathcal{P}_1(\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$\mu_j = \pi_1 \sharp \eta_{m_j}, \quad \eta_{m_j} \xrightarrow{w^*} \eta, \text{ as } j \to +\infty \quad \text{and} \quad \mu = \pi_1 \sharp \eta.$$
 (3.4)

So, it suffices to show that η is a Mather measure for L_m . For this purpose, let us consider the sequence of Mañé's critical values $\{\lambda(m_j)\}_{j\in\mathbb{N}}$. By (2.4) and the definition of Mather measure, we get that

$$\lambda(m_j) = -\int_{\mathbb{R}^n \times \mathbb{R}^n} L_{m_j}(x, v) \, d\eta_{m_j}.$$
(3.5)

By (3.4) and (3.5), we deduce that $\lambda(m_j)$ converges to some $\tilde{\lambda} \in \mathbb{R}$ and

$$\tilde{\lambda} = -\int_{\mathbb{R}^n \times \mathbb{R}^n} L_m(x, v) \, d\eta$$

By Lemma 3.3, we have that

$$\tilde{\lambda} = \lim_{j \to \infty} \lambda(m_j) = \lambda(\lim_{j \to \infty} m_j) = \lambda(m).$$

Therefore, $\tilde{\lambda}$ is the Mañé critical value of H_m and η is a Mather measure for L_m . This shows that Ψ has closed graph. So, by Kakutani's theorem, there exists $\bar{m} \in \mathcal{P}(K_0)$ such that $\bar{m} \in \Psi(\bar{m})$.

Then, by Theorem 2.9, there is a global viscosity solution \bar{u} of $H_{\bar{m}}(x, Du) = c(H_{\bar{m}})$, where $H_{\bar{m}}$ is defined in (3.2). Moreover, by Theorem 2.11, \bar{u} is differentiable \bar{m} -a.e because \bar{m} is supported on a subset of the projected Mather set of $H_{\bar{m}}$. Again by Theorem 2.11 we deduce that the map $\pi_1 : \operatorname{supp}(\eta_{\bar{m}}) \to \operatorname{supp}(\bar{m})$ is one-to-one and its inverse is given by $x \mapsto (x, D_p H(x, D\bar{u}(x)))$ on $\operatorname{supp}(\bar{m})$.

For any $x \in \operatorname{supp}(\bar{m})$, let $\gamma_t(x) = \pi_1 \circ \phi_t^{L_{\bar{m}}}(x, D_p H_{\bar{m}}(x, D\bar{u}(x)))$. Then, we have that

$$\frac{d}{dt}\gamma_t(x) = D_p H_{\bar{m}}\left(\gamma_t(x), D\bar{u}(\gamma_t(x))\right).$$

Since $\eta_{\bar{m}}$ is $\phi_t^{L_{\bar{m}}}$ -invariant and \bar{m} is γ_t -invariant, for any function $f \in C^{\infty}(\mathbb{R}^n)$ we get that

$$0 = \frac{d}{dt} \int_{\mathbb{R}^n} f(\gamma_t(x)) \, d\bar{m}(x) = \int_{\mathbb{R}^n} \left\langle Df(\gamma_t(x)), D_p H_{\bar{m}}(\gamma_t(x), D\bar{u}(\gamma_t(x))) \right\rangle \, d\bar{m}(x)$$
$$= \int_{\mathbb{R}^n} \left\langle Df(x), D_p H_{\bar{m}}(x, D\bar{u}(x)) \right\rangle \, d\bar{m}(x).$$

Hence, \bar{m} satisfies the second equation of system (1.2) in the sense of distributions. This completes the proof of (*i*).

(*ii*) The proof of uniqueness, which is similar to the one in [8], is given in the Appendix. \Box

4 MFG system with finite horizon

This section is devoted to the second main result of this paper—the convergence result. Let us recall the MFG system with finite horizon (1.1), i.e.,

$$\begin{cases} -\partial_t u^T + H(x, Du^T) = F(x, m^T(t)) & \text{in} \quad (0, T) \times \mathbb{R}^n, \\ \partial_t m^T - \operatorname{div} \left(m^T D_p H(x, Du^T) \right) = 0 & \text{in} \quad (0, T) \times \mathbb{R}^n, \\ m^T(0) = m_0, \quad u^T(T, x) = u^f(x), \qquad x \in \mathbb{R}^n. \end{cases}$$

In this section, we will assume (F1), (F2), (F3), (F4), and the following additional conditions.

- (U) $u^f \in C^1(\mathbb{R}^n) \cap \operatorname{Lip}(\mathbb{R}^n)$ satisfies $u^f(x) \ge -c_0$ for all $x \in \mathbb{R}^n$ and some constant $c_0 \ge 0$.
- (M) m_0 is an absolutely continuous measure with respect to the Lebesgue measure and has compact support contained in K_0 , where K_0 is as in (F4). Denote by m_0 the density function of the measure m_0 , i.e., $dm_0 = m_0 dx$.
- (F5) $\bigcap_{m \in \mathcal{P}_1(\mathbb{R}^n)} \operatorname{argmin}_{x \in K_0} L_m(x, 0) \neq \emptyset$, where K_0 is as in (F4) and L_m is defined in (3.1).

Remark 4.1. We observe that assumption (F5) holds true for $L_m(x, v) = L(v) + f(x)g(m)$ as in Example 3.1.

Definition 4.2. A pair $(u^T, m^T) \in W^{1,\infty}_{loc}([0,T] \times \mathbb{R}^n) \times L^1([0,T] \times \mathbb{R}^n)$ is said to be a solution of system (1.1) if:

- (i) the first equation in (1.1) is satisfied in the viscosity sense;
- (ii) the second equation in (1.1) is satisfied in the sense of distributions.

Under assumptions (F1), (F2), (F3), (U), and (M), for any given T > 0, there exists a unique solution of (1.1) (see, for instance, [8, Theorem 4.1 and Remark 4.2]). From now on, for any given T > 0 we denote by (u^T, m^T) the unique solution of (1.1).

Let $R_0 > 0$ be such that $K_0 \subset \overline{B}_{R_0}$, where K_0 is as in (F4).

Let ℓ be a time-dependent Tonelli Lagrangian and let u be a continuous, bounded below function on \mathbb{R}^n . For any given $x \in \mathbb{R}^n$, and $t, T \in \mathbb{R}$ with 0 < t < T, classical results (see, for instance, [6, Theorem 6.1.2]) ensure the existence of solutions of the following minimization problem

$$\inf_{\xi\in\Gamma_{t,T}(x)}\left\{\int_t^T\ell(s,\xi(s),\dot{\xi}(s))ds+u(\xi(T))\right\},\,$$

where

$$\Gamma_{t,T}(x) := \{ \gamma \in AC([t,T]; \mathbb{R}^n) : \gamma(t) = x \}.$$

For any given T > 0, let (u^T, m^T) be the unique solution of (1.1). For each $x \in \mathbb{R}^n$, each $t \in [0, T]$, consider the minimization problem

$$\inf_{\xi\in\Gamma_{t,T}(x)}\left\{\int_{t}^{T}L_{m^{T}(s)}\left(\xi(s),\dot{\xi}(s)\right) ds + u^{f}(\xi(T))\right\}.$$
(4.1)

Define

 $\Gamma_{t,T}^*(x) := \left\{ \xi^* \in \Gamma_{t,T}(x) : \ \xi^* \text{ is a solution of problem (4.1)} \right\}.$

Consider the set-valued map

$$\Gamma^*_{t,T}: K_0 \rightrightarrows C^1([t,T];\mathbb{R}^n), \quad x \mapsto \Gamma^*_{t,T}(x).$$

It is easy to check that $\Gamma_{t,T}^*$ has closed graph with respect to the C^1 -topology, which implies that $\Gamma_{t,T}^*$ is Borel measurable with closed values (see, for instance, [7, Proposition 9.5]). Therefore, by the measurable selection theorem (see, for instance, [2]), there exists a measurable selection of $\Gamma_{t,T}^*$, that is, $\gamma^* : K_0 \to C^1([t,T];\mathbb{R}^n)$ such that $\gamma^*(x) \in \Gamma_{t,T}^*(x)$ for all $x \in K_0$. For any $s \in [t,T]$, let us consider the *evaluation map* $e_s : C^1([t,T];\mathbb{R}^n) \to \mathbb{R}^n$, that is,

$$e_s(\gamma) = \gamma(s).$$

Then, we define the *optimal flow* as follows:

$$\phi: [t,T] \times K_0 \to \mathbb{R}^n, \quad \phi(s,x) = e_s(\gamma^*(x)) \quad (s \in [t,T], x \in K_0).$$

Moreover, from [9, Lemma 4.15], we have that, for any T > 0,

$$m^{T}(s) = \phi(s, \cdot) \sharp m_{0}, \quad \forall s \in [t, T].$$

$$(4.2)$$

4.1 Excursion time of minimizers

Before proving Theorem 4.13 below, we derive preliminary results of interest in their own right.

Theorem 4.3 (Excursion time from a compact set). For any $R \ge R_0$ there is $M_R > 0$ such that for any T > 1, any $\tilde{x} \in \overline{B}_R$, and any $\xi^* \in \Gamma^*_{0,T}(\tilde{x})$, we have that

$$\mathcal{L}^1\Big(\{s\in[0,T]:\ \xi^*(s)\in\overline{B}_R\}\Big)\geq T-M_R.$$

Proof. Define

$$b = \min \left\{ 0, \inf_{\substack{(x,v) \in \mathbb{R}^n \times \mathbb{R}^n \\ m \in \mathcal{P}_1(\mathbb{R}^n)}} L_m(x,v) \right\}.$$

By assumption (F5), there exists $x_T \in K_0$ such that

$$\min_{x \in K_0} L_{m^T(s)}(x,0) = L_{m^T(s)}(x_T,0)$$

for $s \in [T_0, T]$. Consider the curve

$$\xi_0(s) := \begin{cases} \tilde{x} + \frac{x_T - \tilde{x}}{T_0} \cdot s, & s \in [0, T_0), \\ x_T, & s \in [T_0, T]. \end{cases}$$

From the minimality of ξ^* , we have that

$$\int_{0}^{T} L_{m^{T}(s)}(\xi^{*}(s), \dot{\xi}^{*}(s)) \, ds + u^{f}(\xi^{*}(T))$$

$$\leq \int_{0}^{T_{0}} L_{m^{T}(s)}(\xi_{0}(s), \dot{\xi}_{0}(s)) \, ds + \int_{T_{0}}^{T} L_{m^{T}(s)}(x_{T}, 0) \, ds + u^{f}(\bar{x})$$

$$= c(T_{0}, \tilde{x}, m^{T}) + \int_{T_{0}}^{T} L_{m^{T}(s)}(x_{T}, 0) \, ds,$$

where

$$c(T_0, \tilde{x}, m^T) = \int_0^{T_0} L_{m^T(s)}(\xi_0(s), \dot{\xi}_0(s)) \, ds + u^f(x_T)$$

By our assumptions, we deduce that $|c(T_0, \tilde{x}, m^T)| \leq C(T_0, R)$, where $C(T_0, R) > 0$ depends only on T_0 and R. On the other hand, by the convexity and reversibility of L with respect to the v, we deduce that $L(x, v) \geq L(x, 0)$. Thus, we have that

$$\int_{0}^{T} L_{m^{T}(s)}(\xi^{*}(s), \dot{\xi}^{*}(s)) ds$$

$$\geq bT_{0} + \int_{T_{0}}^{T} L_{m^{T}(s)}(\xi^{*}(s), 0) \mathbf{1}_{\overline{B}_{R}}(\xi^{*}(s)) ds + \int_{T_{0}}^{T} L_{m^{T}(s)}(\xi^{*}(s), 0) \mathbf{1}_{\mathbb{R}^{n} \setminus \overline{B}_{R}}(\xi^{*}(s)) ds$$

$$\geq bT_{0} + \int_{T_{0}}^{T} L_{m^{T}(s)}(x_{T}, 0) \mathbf{1}_{\overline{B}_{R}}(\xi^{*}(s)) ds + \int_{T_{0}}^{T} \inf_{x \in \mathbb{R}^{n}} L_{m^{T}(s)}(x, 0) \mathbf{1}_{\mathbb{R}^{n} \setminus \overline{B}_{R}}(\xi^{*}(s)) ds.$$

By combining the above inequalities, we deduce that

$$\int_{T_0}^T \left(L_{m^T(s)}(x_T, 0) - \inf_{x \in \mathbb{R}^n \setminus \overline{B}_R} L_{m^T(s)}(x, 0) \right) \left(1 - \mathbf{1}_{\overline{B}_R}(\xi^*(s)) \right) ds \ge bT_0 - C(T_0, R).$$

By assumption (F4) and the above inequality, we get

$$\int_{T_0}^T \left(1 - \mathbf{1}_{\overline{B}_R}(\xi^*(s))\right) ds \le \frac{C(T_0, R) - bT_0}{\delta_0} =: M_R$$

which yields the conclusion.

Remark 4.4. In view of the proof of Theorem 4.3, it is clear that the result still holds true if assumption (**F5**) is replaced by the following

(F5') Let (u^T, m^T) be a solution of system (1.1). There exists $T_0 \in (0, T)$ such that

$$\bigcap_{T_0 < s < T} \operatorname*{argmin}_{x \in K_0} L_{m^T(s)}(x, 0) \neq \emptyset.$$

It is notable that (F5) is equivalent to the assumption: for every compact subset \mathcal{J} of $\mathcal{P}_1(\mathbb{R}^n)$,

$$\bigcap_{m \in \mathcal{J}} \operatorname*{argmin}_{x \in K_0} L_m(x, 0) \neq \emptyset,$$
(4.3)

where K_0 is as in (F4) and L_m is defined in (3.1). Since $\{m^T(s)\}_{s \in [T_0,T]}$ is a compact subset of $\mathcal{P}_1(\mathbb{R}^n)$, then assumption (F5') is just an application of condition (4.3) to $\{m^T(s)\}_{s \in [T_0,T]}$.

Lemma 4.5. Let $R \ge R_0$. Then there exists a constant $\kappa(R) > 0$ such that for any T > 1, any $0 \le t \le T$, any $x \in \overline{B}_R$, and any $\xi^* \in \Gamma^*_{t,T}(x)$ we have that

$$\int_t^{(t+1)\wedge T} |\dot{\xi}^*(s)|^2 \, ds \le \kappa(R).$$

Proof. We consider two cases.

CASE 1: $t \in [0, T - 1)$.

By assumptions (F4) and (F5), for any $T \ge 1$ there exists $x_T \in K_0$ such that for any $s \in [0, T]$ we have that

$$L_{m^{T}(s)}(x_{T},0) = \min_{x \in K_{0}} L_{m^{T}(s)}(x,0) \le \inf_{x \in \mathbb{R}^{n} \setminus K_{0}} L_{m^{T}(s)}(x,0) - \delta_{0}.$$

So, in view of (f) in Remark 2.4 and the reversibility of L, we get that

$$u^{T}(t,x) = \int_{t}^{T} L_{m^{T}(s)}(\xi^{*}(s), \dot{\xi}^{*}(s)) \, ds + u^{f}(\xi^{*}(T))$$

$$= \int_{t}^{t+1} L_{m^{T}(s)}(\xi^{*}(s), \dot{\xi}^{*}(s)) \, ds + \int_{t+1}^{T} L_{m^{T}(s)}(\xi^{*}(s), \dot{\xi}^{*}(s)) \, ds + u^{f}(\xi^{*}(T)) \quad (4.4)$$

$$\geq \frac{1}{4\beta} \int_{t}^{t+1} |\dot{\xi}^{*}(s)|^{2} \, ds - \alpha + \int_{t+1}^{T} L_{m^{T}(s)}(x_{T}, 0) \, ds - c_{0}.$$

For $x \in \overline{B}_R$, define

$$\sigma_T^x(s) = ((s-t) \land 1) x_T + ((1-(s-t)) \land 1)) x, \quad t \le s \le T.$$

Observe that $\sigma_T^x(t) = x$ and $\sigma_T^x(s) = x_T$ for any $s \ge t + 1$. We deduce that

$$u^{T}(t,x) = \int_{t}^{T} L_{m^{T}(s)}(\xi^{*}(s), \dot{\xi}^{*}(s)) \, ds + u^{f}(\xi^{*}(T))$$

$$\leq \int_{t}^{t+1} L_{m^{T}(s)}(\sigma_{T}^{x}(s), \dot{\sigma}_{T}^{x}(s)) \, ds + \int_{t+1}^{T} L_{m^{T}(s)}(x_{T}, 0) \, ds + u^{f}(x_{T}).$$

Since $|\sigma_T^x(s)| \leq 2R$ and $|\dot{\sigma}_T^x(s)| \leq 2R$ for all $s \in [t, T]$, we have that

$$u^{T}(t,x) \leq C(R) + u^{f}(x_{T}) + \int_{t+1}^{T} L_{m^{T}(s)}(x_{T},0) \, ds, \qquad (4.5)$$

where

$$C(R) = \max_{|x|, |v| \le 2R} |L(x, v)| + ||F(\cdot, m^T)||_{\infty}.$$

Note that $x_T \in K_0$. Since K_0 is compact, by (U) we deduce that $u^f(x_T)$ is bounded. So, combining (4.4) and (4.5), we conclude that

$$\int_{t}^{t+1} |\dot{\xi}^{*}(s)|^{2} ds \leq 4\beta \left(\alpha + C(R) + 2c_{0} \right) =: \kappa_{1}(R).$$

CASE 2: $t \in [T - 1, T]$.

The proof is similar to the one above. On the one hand, one has that

$$u^{T}(t,x) \ge \frac{1}{4\beta} \int_{t}^{T} |\dot{\xi}^{*}(s)|^{2} ds - \alpha - c_{0}.$$

On the other hand, by using the curve $\rho(s) \equiv x$ we get an upper bound of the form

$$u^{T}(t,x) \leq \int_{t}^{T} L_{m^{T}(s)}(x,0) \, ds + u^{f}(x).$$

Therefore, combining the above inequalities we obtain the desired result.

Remark 4.6.

(a) By minor adaptations of the above proof one can extend the conclusion of Lemma 4.5 as follows: for every $R \ge R_0$ and M > 0 there exists a constant $\kappa(R, M) > 0$ such that for any T > 1, any $0 \le t \le T$, any $x \in \overline{B}_R$, and any $\xi^* \in \Gamma^*_{t,T}(x)$ we have that

$$\int_t^{(t+M)\wedge T} |\dot{\xi}^*(s)|^2 \, ds \le \kappa(R,M).$$

(b) The proof of Lemma 4.5 shows that $\kappa(R) = CR^2$ for some constant C > 0.

Corollary 4.7. Let $R \ge R_0$. Then, there exists a constant $\chi(R) > 0$ such that for any T > 1, any $0 \le t \le T$, any $x \in \overline{B}_R$, and any $\xi^* \in \Gamma^*_{t,T}(x)$ we have that

$$\sup_{s \in [t,T]} |\xi^*(s)| \le \chi(R).$$

Proof. Fix $x \in \overline{B}_R$. Let $\overline{t} \in [0,T]$ be such that $\xi^*(\overline{t},x) \in \mathbb{R}^n \setminus \overline{B}_R$. Define

$$s_0 = \sup\{s \in [0, \overline{t}] : \xi^*(s) \in \overline{B}_R\}.$$

Then, we have that

$$|\xi^*(\bar{t}) - \xi^*(s_0)| = \int_{s_0}^t \left| \dot{\xi}^*(s) \right| \, ds \le M_R^{\frac{1}{2}} \left(\int_{s_0}^t \left| \dot{\xi}^*(s) \right|^2 \, ds \right)^{\frac{1}{2}} \le M_R^{\frac{1}{2}} \kappa(R, M_R)^{\frac{1}{2}},$$

where we have used Hölder's inequality and $\kappa(R, M_R)$ is as in Remark 4.6. Therefore, we get $\chi(R) = R + (M_R \kappa(R, M_R))^{\frac{1}{2}}$.

Proposition 4.8 (Attainable set from K_0). There exists a constant $R_1 > 0$ such that for each $T \ge 1$ the solution (u^T, m^T) of system (1.1) satisfies

$$\operatorname{supp}\left(m^{T}(t)\right) \subset \overline{B}_{R_{1}}, \quad \forall t \in [0, T]$$

Proof. Recall that R_0 has been fixed so that $K_0 \subset \overline{B}_{R_0}$ and $m^T(t) = \phi(t, \cdot) \sharp m_0$. So, by assumption (**M**), Corollary 4.7, and the definition of ϕ we conclude that

$$\operatorname{supp}\left(m^{T}(t)\right) \subset \overline{B}_{R_{1}}, \quad \forall T > 0, \quad \forall t \in [0, T],$$

where $R_1 := \chi(R_0)$.

4.2 Uniform Lipschitz continuity

Proposition 4.9. Let $R \ge R_0$. Then, there exists a constant $L_R > 0$ such that for all T > 1

$$|u^{T}(t, x+h) - u^{T}(t, x)| \le L_{R}|h|, \quad \forall x, x+h \in \overline{B}_{R}, \, \forall t \in [0, T]$$

Proof. Since (u^T, m^T) is the solution of (1.1), we have that

$$\begin{cases} -\partial_t u^T + H(x, Du^T) = F(x, m^T) & \text{in } [0, T] \times \mathbb{R}^n, \\ u^T(T, x) = u^f(x) & \text{in } \mathbb{R}^n, \end{cases}$$

and

$$u^{T}(t,x) = \inf_{\xi \in \Gamma_{t,T}(x)} \left\{ \int_{t}^{T} L_{m^{T}(s)} \left(\xi(s), \dot{\xi}(s) \right) ds + u^{f}(\xi(T)) \right\}.$$
 (4.6)

Fix $R \ge R_0$ and $x \in \overline{B}_R$. Let $h \in \mathbb{R}^n$ be such that $x + h \in \overline{B}_R$. We consider two cases. CASE 1: $t \in [0, T - 1)$.

Let $\xi^* \in \Gamma^*_{t,T}(x)$. By testing with the curve

$$\xi_h(s) := \xi^*(s) + (t+1-s)^+ h \quad (s \in [t,T]),$$

we obtain the upper bound

$$u^{T}(t, x+h) - u^{T}(t, x) \leq \int_{t}^{t+1} \left(L_{m^{T}(s)}(\xi_{h}(s), \dot{\xi}_{h}(s)) - L_{m^{T}(s)}(\xi^{*}(s), \dot{\xi}^{*}(s)) \right) ds$$

$$\leq \underbrace{\int_{t}^{t+1} \left(L_{m^{T}(s)}(\xi_{h}(s), \dot{\xi}_{h}(s)) - L_{m^{T}(s)}(\xi^{*}(s), \dot{\xi}_{h}(s)) \right) ds}_{A}$$

$$+ \underbrace{\int_{t}^{t+1} \left(L_{m^{T}(s)}(\xi^{*}(s), \dot{\xi}_{h}(s)) - L_{m^{T}(s)}(\xi^{*}(s), \dot{\xi}^{*}(s)) \right) ds}_{B}$$

We estimate term A first. Recall that, for any $m \in \mathcal{P}_1(\mathbb{R}^n)$, L_m is a strict Tonelli Lagrangian. By (e) in Remark 2.4 and (F1), there exists a constant $\alpha_1 > 0$ such that

$$|D_x L_m(x,v)| \le \alpha_1 (1+|v|^2), \quad \forall (x,v) \in \mathbb{R}^n \times \mathbb{R}^n, \ \forall m \in \mathcal{P}_1(\mathbb{R}^n).$$
(4.7)

Then, by Lemma 4.5, we get

$$\begin{split} A &\leq \int_{t}^{t+1} \int_{0}^{1} |h| \left| D_{x} L_{m^{T}(s)} \left(\lambda \xi_{h}(s) + (1-\lambda) \xi^{*}(s), \dot{\xi}_{h}(s) \right) \right| \, d\lambda ds \\ &\leq \alpha_{1} |h| \int_{t}^{t+1} \left(1 + |\dot{\xi}_{h}(s)|^{2} \right) ds = \left| \alpha_{1} |h| \int_{t}^{t+1} (1 + |\dot{\xi}^{*} - h|^{2}) \, ds \\ &\leq (3 + 2R) \alpha_{1} |h| \left(1 + \int_{t}^{t+1} |\dot{\xi}^{*}(s)|^{2} \, ds \right) \leq \kappa'(R) |h|, \end{split}$$

where $\kappa'(R) = (3 + 2R)\alpha_1(1 + \kappa(R))$ and $\kappa(R)$ is as in Lemma 4.5. Similarly, by Lemma 4.5 and (d) in Remark 2.4 we get $B \leq \kappa''(R)|h|$, where $\kappa''(R)$ is a positive constant depending only on R.

CASE 2: $t \in [T - 1, T]$.

Let $\xi^* \in \Gamma^*_{t,T}(x)$. Define the curve $\xi(s) = \xi^*(s) + h$, for $s \in [t, T]$. Then, we have that

$$u^{T}(t, x+h) - u^{T}(t, x)$$

$$\leq \int_{t}^{T} \left(L_{m^{T}(s)}(\xi(s), \dot{\xi}(s)) - L_{m^{T}(s)}(\xi^{*}(s), \dot{\xi}^{*}(s)) \right) ds + \operatorname{Lip}(u^{f})|h|.$$

To conclude the proof, we only need to estimate

$$\int_{t}^{T} \left(L_{m^{T}(s)}(\xi(s), \dot{\xi}(s)) - L_{m^{T}(s)}(\xi^{*}(s), \dot{\xi}^{*}(s)) \right) ds$$

Again by Lemma 4.5 and (4.7) we get

$$\int_{t}^{T} \left(L_{m^{T}(s)}(\xi(s), \dot{\xi}(s)) - L_{m^{T}(s)}(\xi^{*}(s), \dot{\xi}^{*}(s)) \right) ds$$

$$\leq \int_{t}^{T} \int_{0}^{1} |h| \left| D_{x} L_{m^{T}(s)}(\lambda h + \xi^{*}(s), \dot{\xi}^{*}(s)) \right| d\lambda ds$$

$$\leq \alpha_{1} |h| \left(1 + \int_{t}^{T} |\dot{\xi}^{*}(s)|^{2} ds \right) \leq \kappa'''(R) |h|,$$

where $\kappa'''(R) = \alpha_1(1+\kappa(R))$. So, $u^T(t, x+h) - u^T(t, x) \le L_R|h|$ with $L_R = \kappa'''(R) + \text{Lip}(u^f)$. This suffices to get the conclusion.

Corollary 4.10. Let $R \ge R_0$. Then there exists a constant $\chi'(R) > 0$ such that for any T > 1, any $0 \le t \le T$, any $x \in \overline{B}_R$, and any $\xi^* \in \Gamma^*_{t,T}(x)$, we have that

$$\sup_{s \in [t,T]} |\dot{\xi}^*(s)| \le \chi'(R).$$

Proof. Let p^* be the dual arc of ξ^* , that is, $p^*(s) = D_v L_{m^T(s)}(\xi^*(s), \dot{\xi}^*(s))$ for any $s \in [t, T]$. Then the pair (ξ^*, p^*) satisfies the maximum principle in Hamiltonian form

$$\begin{cases} \dot{\xi}^*(s) = D_p H_{m^T(s)}(\xi^*(s), p^*(s)) \\ \dot{p}^*(s) = -D_x H_{m^T(s)}(\xi^*(s), p^*(s)) \end{cases} \quad (s \in [t, T]) \end{cases}$$

Moreover, by [6, Theorem 6.4.8] we know that

$$p^*(s) = D_x u^T(s, \xi^*(s)), \quad t < s \le T.$$

Now, observe that, in view of Corollary 4.7, $\xi^*(s) \in \overline{B}_{\chi(R)}$ for all $s \in [t, T]$ and, on account of Proposition 4.9, $\{u^T(s, \cdot)\}_{s \in [t,T]}$ is equi-Lipschitz continuous on $\overline{B}_{\chi(R)}$. Therefore, there exists a positive constant c_R , independent of T, such that $|p^*(s)| \leq c_R$ for every $s \in [t, T]$. Consequently,

$$\sup_{s \in [t,T]} |\dot{\xi}^*(s)| = \sup_{s \in [t,T]} |D_p H_{m^T(s)}(\xi^*(s), p^*(s))| \le \alpha_2 (1 + c_R) := \chi'(R),$$

where the inequality follows from the fact that $D_p H_{m^T(s)}(x,p) \le \alpha_2(1+|p|)$ for some $\alpha_2 > 0$ and all $(x,p) \in \mathbb{R}^n \times \mathbb{R}^n$ and $s \in [t,T]$.

Remark 4.11. Owing to Corollary 4.10, for any T > 1 we have that $m^T : [0,T] \to \mathcal{P}_1(\mathbb{R}^n)$ is Lipschitz continuous. Indeed, by (4.2) we deduce that, for all $s', s \in [0,T]$,

$$d_1(m^T(s'), m^T(s)) \le \int_{K_0} |\phi(s', x) - \phi(s, x)| \, dm_0 \le \chi'(R_0) |s' - s|,$$

where R_0 is such that $K_0 \subset \overline{B}_{R_0}$.

4.3 Main result 2

Before proving our main result, we show the following lemma. We recall that R_1 is the constant given by Proposition 4.8.

Lemma 4.12. For any $R \ge R_1$ there exists a constant C(R) > 0, such that for any T > 0, and any $(\bar{\lambda}, \bar{u}, \bar{m}) \in S$, the solution (u^T, m^T) of (1.1) satisfies

$$\int_0^T \int_{\overline{B}_R} \left(F(x, m^T(t)) - F(x, \overline{m}) \right) d(m^T(t) - \overline{m}) dt \le \tilde{C}(R),$$

where R_1 is as in Proposition 4.8.

Proof. Fix $R \ge R_1$. Then, $K_0 \subset \overline{B}_R$ and $\partial \overline{B}_R \cap K_0 = \emptyset$. Let $\epsilon > 0$ and let $\xi : \mathbb{R}^n \to \mathbb{R}$ be a smooth, nonnegative, symmetric kernel of integral one, with support contained in the unit ball. Fix $(\bar{\lambda}, \bar{u}, \bar{m}) \in S$ and define $\bar{m}^{\epsilon} := \xi^{\epsilon} \star \bar{m}$ where $\xi^{\epsilon}(x) = \frac{1}{\epsilon^n} \xi(\frac{x}{\epsilon})$. Then

$$\int_0^T \int_{\overline{B}_R} \left(F(x, m^T(t)) - F(x, \bar{m}) \right) d(m^T(t) - \bar{m}) dt$$
$$= \int_0^T \int_{\overline{B}_R} \left(F(x, m^T(t)) - F(x, \bar{m}) \right) d(m^T(t) - \bar{m}^\epsilon) dx dt$$
$$+ \int_0^T \int_{\overline{B}_R} \left(F(x, m^T(t)) - F(x, \bar{m}) \right) d(\bar{m}^\epsilon - \bar{m}) dt.$$

Since $\bar{m}^{\epsilon} \to \bar{m}$ as $\epsilon \to 0$, we have that there exists $\bar{\epsilon} > 0$ such that, for every $\epsilon \leq \bar{\epsilon}$,

$$\int_0^T \int_{\overline{B}_R} \left(F(x, m^T(t)) - F(x, \overline{m}) \right) d(\overline{m}^{\epsilon} - \overline{m}) \, dt \le 1.$$

On the other hand, by the convexity of H, we obtain

$$\int_0^T \int_{\overline{B}_R} \left(F(x, m^T(t)) - F(x, \bar{m}) \right) d(m^T(t) - \bar{m}^{\epsilon}) \, dx dt$$

$$\leq \int_0^T \int_{\overline{B}_R} \left(H(x, D\bar{u}) - H(x, Du^T) - \left\langle D_p H(x, Du^T), D(\bar{u} - u^T) \right\rangle \right) \, m^T(t) \, dx dt$$

$$+ \int_0^T \int_{\overline{B}_R} \left(H(x, Du^T) - H(x, D\bar{u}) - \left\langle D_p H(x, D\bar{u}), D(u^T - \bar{u}) \right\rangle \right) \, \bar{m}^{\epsilon} \, dx dt$$

$$+ \int_0^T \int_{\overline{B}_R} \left(F(x, m^T(t)) - F(x, \bar{m}) \right) \, (m^T(t) - \bar{m}^{\epsilon}) \, dx dt.$$

Recombining the terms on right hand-side of the above expression, we get

$$\int_{0}^{T} \int_{\overline{B}_{R}} \left(F(x, m^{T}(t)) - F(x, \bar{m}) \right) d(m^{T}(t) - \bar{m}^{\epsilon}) \, dx dt$$

$$\leq \underbrace{\int_{0}^{T} \int_{\overline{B}_{R}} (H(x, D\bar{u}(x)) - F(x, \bar{m})) \, (m^{T}(t) - \bar{m}^{\epsilon}) \, dx dt}_{A} + \underbrace{\int_{0}^{T} \int_{\overline{B}_{R}} \left\langle D \left(u^{T}(t, x) - \bar{u}(x) \right) , D_{p} H(x, D\bar{u}(x)) \right\rangle \bar{m}^{\epsilon} \, dx dt}_{B} \qquad (4.8)$$

$$+ \underbrace{\int_{0}^{T} \int_{\overline{B}_{R}} \left\langle D \left(u^{T}(t, x) - \bar{u}(x) \right) , D_{p} H(x, D\bar{u}^{T}(t, x)) \right\rangle \bar{m}^{T}(t) \, dx dt}_{C} + \underbrace{\int_{0}^{T} \int_{\overline{B}_{R}} - \left(H(x, Du^{T}(t, x)) - F(x, m^{T}(t)) \right) \, (m^{T}(t) - \bar{m}^{\epsilon}) \, dx dt .$$

In the following, we analyze each term on the right hand-side of (4.8). Since $m^T(t)$, \bar{m}^{ϵ} are probability measures and B_{R_1} , $K_0 \subset \overline{B}_R$ for every $t \in [0, T]$, we have that

$$A = \int_0^T \int_{\overline{B}_R} \left(H(x, D\bar{u}(x)) - F(x, \bar{m}) \right) \left(m^T(t) - \bar{m}^\epsilon \right) dx dt = 0.$$
(4.9)

In order to study term B, define

$$V^{\epsilon} := \frac{\xi^{\epsilon} \star (\bar{m}D_pH(\cdot, D\bar{u}))}{\bar{m}^{\epsilon}}.$$

Then, we have that $-\operatorname{div}(\bar{m}^{\epsilon}V^{\epsilon}) = 0$ in \mathbb{R}^n . We multiply this equality by $u^T(t, x) - \bar{u}(x)$ and

integrate on $(0,T) \times \overline{B}_R$, to obtain

$$0 = \int_0^T \int_{\overline{B}_R} \left\langle D(u^T(t,x) - \bar{u}(x)), V^{\epsilon}(x) \right\rangle \bar{m}^{\epsilon} \, dx dt - \int_0^T \int_{\partial \overline{B}_R} \left(u^T(t,x) - \bar{u}(x) \right) \left\langle \hat{\nu}, V^{\epsilon}(x) \right\rangle \, \bar{m}^{\epsilon} \, dS dt,$$
(4.10)

where $\hat{\nu}(x) = \frac{x}{R}$ is the outward unit normal to $\partial \overline{B}_R$. Since $\partial \overline{B}_R \cap \operatorname{supp}(\overline{m}) = \emptyset$, we get

$$\int_0^T \int_{\partial \overline{B}_R} \left(u^T(t,x) - \overline{u}(x) \right) \left\langle \nu, V^{\epsilon}(x) \right\rangle \ \overline{m}^{\epsilon} \ dxdt = 0.$$

Thus, (4.10) can be rewritten as

$$0 = \int_0^T \int_{\overline{B}_R} \left\langle D(u^T(t,x) - \bar{u}(x)), D_p H(x, D\bar{u}(x)) \right\rangle \bar{m}^\epsilon \, dx dt + R_\epsilon, \tag{4.11}$$

where

$$R_{\epsilon} = \int_{0}^{T} \int_{\overline{B}_{R}} \int_{\mathbb{R}^{n}} \xi^{\epsilon}(x-y) \left\langle D\left(u^{T}(t,x) - \bar{u}(x)\right), D_{p}H(y,D\bar{u}(y)) - D_{p}H(x,D\bar{u}(x))\right\rangle \bar{m}(dy) dx dt.$$

By the definition of R_{ϵ} , we have that

$$\begin{aligned} |R_{\epsilon}| \leq & T \sup_{t \in [0,T]} \|D\left(u^{T}(t,\cdot) - \bar{u}(\cdot)\right)\|_{\infty,\overline{B}_{R}} \cdot \\ & \int_{\overline{B}_{R}} \int_{\mathbb{R}^{n}} \xi^{\epsilon}(x-y) |D_{p}H(y, D\bar{u}(y)) - D_{p}H(x, D\bar{u}(x))| \bar{m}(dy) dx. \end{aligned}$$

We now prove that $R_{\epsilon} \to 0$ as $\epsilon \to 0$. First, observe that the integral term on the right-hand side of the above inequality can be rewritten as follows

$$\begin{split} &\int_{\overline{B}_R} \int_{\mathbb{R}^n} \xi^{\epsilon}(x-y) \left| D_p H(y, D\bar{u}(y)) - D_p H(x, D\bar{u}(x)) \right| \, \bar{m}(dy) dx \\ &= \int_{\mathbb{R}^n} \int_{\overline{B}_R} \xi^{\epsilon}(x-y) \left| D_p H(y, D\bar{u}(y)) - D_p H(x, D\bar{u}(x)) \right| \, dx \bar{m}(dy) \\ &\leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \xi^{\epsilon}(x-y) \left| D_p H(y, D\bar{u}(y)) - D_p H(x, D\bar{u}(x)) \right| \, dx \bar{m}(dy) \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \xi(z) \left| D_p H(y, D\bar{u}(y)) - D_p H(y + \epsilon z, D\bar{u}(y + \epsilon z)) \right| \, dz \bar{m}(dy). \end{split}$$

Since $D_pH(\cdot, D\bar{u}(\cdot))$ is bounded and ξ has compact support, for some C > 0 we have that

$$\int_{\mathbb{R}^n} \xi(z) \left| D_p H(y, D\bar{u}(y)) - D_p H(y + \epsilon z, D\bar{u}(y + \epsilon z)) \right| dz \le C.$$

Moreover, by the continuity of $D\bar{u}$ on $\operatorname{supp}(\bar{m})$, we deduce that for any $y \in \operatorname{supp}(\bar{m})$

$$\int_{\mathbb{R}^n} \xi(z) \left| D_p H(y, D\bar{u}(y)) - D_p H(y + \epsilon z, D\bar{u}(y + \epsilon z)) \right| \, dz \to 0 \quad \text{as } \epsilon \to 0$$

Therefore, by Lebesgue's dominated convergence theorem, we get

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \xi(z) \left| D_p H(y, D\bar{u}(y)) - D_p H(y + \epsilon z, D\bar{u}(y + \epsilon z)) \right| \, dz \bar{m}(dy) \to 0 \quad \text{as } \epsilon \to 0.$$

In conclusion, we have that $B \leq CT |R_{\epsilon}|$. In particular, for any T > 0 there exists $\epsilon_T > 0$ such that $B \leq 1$ for all $\epsilon \leq \epsilon_T$. Hereafter, for any T > 0 fix $\epsilon = \epsilon_T \wedge \overline{\epsilon}$.

Finally, we to bound C + D. By the continuity equation in system (1.1), we deduce that

$$\partial_t (m^T(t) - \bar{m}^{\epsilon}) - \operatorname{div} \left(m^T(t) D_p H(x, Du^T(t, x)) \right) = 0.$$

in the sense of distributions. Multiplying this equality by $u^T - \bar{u}$ and integrating in space-time, we conclude that

$$0 = \int_{\overline{B}_R} \left((u^f(x) - \bar{u}(x))(m^T(T) - \bar{m}^{\epsilon}) - (u^T(0, x) - \bar{u}(x))(m_0 - \bar{m}^{\epsilon}) \right) dx$$

+
$$\int_0^T \int_{\overline{B}_R} - \left(H(x, Du^T(t, x)) - F(x, m^T) \right) (m^T(t) - \bar{m}^{\epsilon}) dx dt$$

+
$$\int_0^T \int_{\overline{B}_R} \left\langle D(u^T(t, x) - \bar{u}(x)), D_p H(x, Du^T(t, x)) \right\rangle m^T(t) dx dt$$

-
$$\int_0^T \int_{\partial \overline{B}_R} \left(u^T(t, x) - \bar{u}(x) \right) \left\langle \hat{\nu}, D_p H(x, Du^T(t, x)) \right\rangle m^T(t) dS dt,$$

where $\hat{\nu}(x) = \frac{x}{R}$ is the outward unit normal to $\partial \overline{B}_R$. Again, since $\partial \overline{B}_R \cap \overline{B}_{R_1} = \emptyset$, the integral over $\partial \overline{B}_R$ is zero. In addition, the first integral is uniformly bounded with respect to T, because

$$\left| \int_{\overline{B}_R} \left(u^f(x) - \bar{u}(x) \right) (m^T(T) - \bar{m}^\epsilon) \, dx \right| \le 2(\|u^f\|_{\infty, \overline{B}_R} + \|\bar{u}\|_{\infty, \overline{B}_R}).$$

Since m_0 , \bar{m}^{ϵ} are probability measures, by Poincaré's inequality we deduce that

$$\begin{aligned} \left| \int_{\overline{B}_R} \left(u^T(0,x) - \bar{u}(x) \right) (m_0 - \bar{m}^{\epsilon}) \, dx \right| \\ &\leq \left| \int_{\overline{B}_R} \left(u^T(0,x) - \frac{1}{\mathcal{L}^n(\overline{B}_R)} \int_{\overline{B}_R} u^T(0,y) \, dy \right) \, (m_0 - \bar{m}^{\epsilon}) dx \right| + \left| \int_{\overline{B}_R} \bar{u}(x) (m_0 - \bar{m}^{\epsilon}) \, dx \right| \\ &\leq N(R) (\|Du^T(0,\cdot)\|_{\infty,\overline{B}_R} + \|\bar{u}\|_{\infty,\overline{B}_R}). \end{aligned}$$

for some constant N(R) > 0. Therefore,

$$C+D \leq 2\left(\|u^f\|_{\infty,\overline{B}_R} + \|\bar{u}\|_{\infty,\overline{B}_R}\right) + N(R)\left(\|Du^T(0,\cdot)\|_{\infty,\overline{B}_R} + \|\bar{u}\|_{\infty,\overline{B}_R}\right).$$

In view of the above estimates on A, B, C, and D, we conclude that

$$\int_0^T \int_{\overline{B}_R} \left(m^T(t) - \bar{m}^\epsilon \right) \left(F(x, m^T(t)) - F(x, \bar{m}) \right) \, dx dt \le C(R),$$

for some constant C(R) > 0.

Theorem 4.13 (Convergence of solutions of (1.1)). Let $(\bar{\lambda}, \bar{u}, \bar{m}) \in S$. Let R_1 be as in Proposition 4.8. Then for any $R > R_1$, there exists a constant C(R) > 0 such that for every $T \ge 1$ the solution (u^T, m^T) of system (1.1) satisfies

$$\sup_{t\in[0,T]} \left\| \frac{u^T(t,\cdot) - \bar{u}(\cdot)}{T} + \bar{\lambda} \left(1 - \frac{t}{T} \right) \right\|_{\infty,\overline{B}_R} \le \frac{C(R)}{T^{\frac{1}{n+2}}},\tag{4.12}$$

$$\frac{1}{T} \int_0^T \left\| F(\cdot, m^T(s)) - F(\cdot, \bar{m}) \right\|_{\infty, \overline{B}_R} ds \le \frac{C(R)}{T^{\frac{1}{n+2}}}.$$
(4.13)

Proof. Fix a radius $R > R_1$. Define

$$w(t,x) := \bar{u}(x) - \bar{\lambda}(T-t), \quad \forall (x,t) \in \mathbb{R}^n \times [0,T].$$

Since $(\bar{\lambda}, \bar{u}, \bar{m})$ is a solution of (1.2), we have that w is a viscosity solution of the following Cauchy problem

$$\begin{cases} -\partial_t w + H(x, Dw) = F(x, \bar{m}) & \text{in} \quad (0, T) \times \mathbb{R}^n, \\ w(T, x) = \bar{u}(x) & \text{in} \quad \mathbb{R}^n. \end{cases}$$

So, w(t, x) can be represented as the value function of the following minimization problem

$$w(t,x) = \inf_{\gamma \in \Gamma_{t,T}(x)} \left\{ \int_t^T L_{\bar{m}}\left(\gamma(s), \dot{\gamma}(s)\right) \, ds + \bar{u}(\gamma(T)) \right\}, \quad \forall (x,t) \in \mathbb{R}^n \times [0,T].$$
(4.14)

Since (u^T, m^T) is a solution of (1.1), then we get that

$$\begin{cases} -\partial_t u^T + H(x, Du^T) = F(x, m^T) & \text{in} \quad (0, T) \times \mathbb{R}^n, \\ u^T(T, x) = u^f(x) & \text{in} \quad \mathbb{R}^n. \end{cases}$$

We prove inequality (4.13) first. For any given $(x, t) \in \overline{B}_R \times [0, T]$, let $\gamma^* : [t, T] \to \mathbb{R}^n$ be a minimizer of problem (4.14). By Lemma A.1 below and Hölder's inequality, we get

$$\begin{split} &\int_{t}^{T} \|F(\cdot, m^{T}(s)) - F(\cdot, \bar{m})\|_{\infty, \overline{B}_{R}} \mathbf{1}_{\overline{B}_{R}}(\gamma^{*}(s)) \frac{ds}{T} \\ &\leq C((\|DF\|_{\infty}) \int_{t}^{T} \|F(\cdot, m^{T}(s)) - F(\cdot, \bar{m})\|_{2, \overline{B}_{R}}^{\frac{2}{n+2}} \mathbf{1}_{\overline{B}_{R}}(\gamma^{*}(s)) \frac{ds}{T} \\ &\leq \frac{C(\|DF\|_{\infty})}{T} \left(\int_{t}^{T} \|F(\cdot, m^{T}(s)) - F(\cdot, \bar{m})\|_{2, \overline{B}_{R}}^{2} ds \right)^{\frac{1}{n+2}} \left(\int_{t}^{T} \mathbf{1}_{\overline{B}_{R}}(\gamma^{*}(s)) ds \right)^{\frac{n+1}{n+2}} \end{split}$$

Now, by assumption (F3) and Lemma 4.12 the term

$$\left(\int_t^T \|F(\cdot, m^T(s)) - F(\cdot, \bar{m})\|_{2,\overline{B}_R}^2 \, ds\right)^{\frac{1}{n+2}}$$

is bounded by a constant depending only on R, while

$$\left(\int_t^T \mathbf{1}_{\overline{B}_R}(\gamma^*(s)) \, ds\right)^{\frac{n+1}{n+2}} \le T^{\frac{n+1}{n+2}}.$$

Inequality (4.13) follows.

Next, we prove (4.12). By the definition of w, we have that

$$u^{T}(t,x) - w(t,x) \leq \int_{t}^{T} L_{m^{T}(s)}(\gamma^{*}(s),\dot{\gamma}^{*}(s)) \, ds + u^{f}(\gamma^{*}(T)) - \int_{t}^{T} L_{\bar{m}}(\gamma^{*}(s),\dot{\gamma}^{*}(s)) \, ds - \bar{u}(\gamma^{*}(T)) = u^{f}(\gamma^{*}(T)) - \bar{u}(\gamma^{*}(T)) + \int_{t}^{T} \left(F(\gamma^{*}(s),m^{T}(s)) - F(\gamma^{*}(s),\bar{m})\right) \, ds.$$

$$(4.15)$$

By (4.15), we get

$$\frac{u^{T}(t,x) - w(t,x)}{T} \leq \underbrace{\left|\frac{u^{f}(\gamma^{*}(T)) - \bar{u}(\gamma^{*}(T))\right|}{A}}_{A} + \underbrace{\frac{1}{T} \int_{t}^{T} \left|F(\gamma^{*}(s), m^{T}(s)) - F(\gamma^{*}(s), \bar{m})\right| ds}_{B}.$$

Let us first consider term B. Note that

$$\frac{1}{T} \int_{t}^{T} \left| F(\gamma^{*}(s), m^{T}(s)) - F(\gamma^{*}(s), \bar{m}) \right| ds$$

= $\int_{t}^{T} \left| F(\gamma^{*}(s), m^{T}(s)) - F(\gamma^{*}(s), \bar{m}) \right| \mathbf{1}_{\overline{B}_{R}}(\gamma^{*}(s)) \frac{ds}{T}$
+ $\int_{t}^{T} \left| F(\gamma^{*}(s), m^{T}(s)) - F(\gamma^{*}(s), \bar{m}) \right| \mathbf{1}_{\mathbb{R}^{n} \setminus \overline{B}_{R}}(\gamma^{*}(s)) \frac{ds}{T}.$

Since F is bounded, by Theorem 4.3 we know that the second integral on the right hand-side of the above equality goes to zero as $T \to \infty$. As for the first integral, observe that

$$\int_{t}^{T} \left| F(\gamma^{*}(s), m^{T}(s)) - F(\gamma^{*}(s), \bar{m}) \right| \mathbf{1}_{\overline{B}_{R}}(\gamma^{*}(s)) \frac{ds}{T}$$
$$\leq \int_{t}^{T} \| F(\cdot, m^{T}(s)) - F(\cdot, \bar{m}) \|_{\infty, \overline{B}_{R}} \mathbf{1}_{\overline{B}_{R}}(\gamma^{*}(s)) \frac{ds}{T}.$$

Thus, we estimate the term on the right-hand side of the above iniequality as we did above.

Now, we give a bound for A. Since \bar{u} and u^f are Lipschitz continuous, we deduce that both functions grow at most linearly at infinity, i.e., there exists a constant N > 0 such that

$$\bar{u}(y), u^f(y) \le N(1+|y|), \quad \forall y \in \mathbb{R}^n.$$

Also, by Corollary 4.10, $\sup_{s \in [t,T]} |\dot{\gamma}^*(s)| \le \chi'(R)$ for some constant $\chi'(R) > 0$. Therefore, by Theorem 4.3 there exists a positive constant M_R such that for any T > 0 we have that

$$|\bar{u}(\gamma^*(T))|, |u^f(\gamma^*(T))| \le N(1+|\gamma^*(T)|) \le N(1+(R+\chi'(R)M_R)).$$

Thus, we conclude that $A \leq O(\frac{1}{T})$. So, combining (4.15) and (4.13) we obtain

$$\frac{u^{T}(t,x) - w(t,x)}{T} \le \frac{C(R)}{T^{\frac{1}{n+2}}}.$$

Moreover, for any given $(x, t) \in \overline{B}_R \times [0, T]$, let $\xi^*(\cdot)$ be a minimizer of problem (4.1). Since u^T is the value function of (4.1), we have that

$$w(t,x) - u^{T}(t,x) \leq \int_{t}^{T} L_{\bar{m}}(\xi^{*}(s), \dot{\xi}^{*}(s)) \, ds + \bar{u}(\xi^{*}(T)) - \int_{t}^{T} L_{m^{T}(s)}(\xi^{*}(s), \dot{\xi}^{*}(s)) \, ds - u^{f}(\xi^{*}(T)) = \bar{u}(\xi^{*}(T)) - u^{f}(\xi^{*}(T)) + \int_{t}^{T} \left(F(\xi^{*}(s), \bar{m}) - F(\xi^{*}(s), m^{T}(s)) \right) \, ds.$$

$$(4.16)$$

So, by almost the same arguments used above, one obtains

$$\frac{w(t,x) - u^T(t,x)}{T} \le \frac{C(R)}{T^{\frac{1}{n+2}}},$$

which concludes the proof of (4.12).

Remark 4.14. In view of Remark 4.4 and the above proof, it is clear that the Theorem 4.13 still holds true if assumption (**F5**) is replaced by assumption (**F5**').

A Appendix

In this section, we first give the proof of (ii) of Theorem 3.4 and then show Lemma A.1, which was used in the proof of Theorem 4.13.

A.1 **Proof of** (*ii*) **of Theorem 3.4**

Let $(c(H_{\bar{m}_1}), \bar{u}_1, \bar{m}_1)$, $(c(H_{\bar{m}_2}), \bar{u}_2, \bar{m}_2) \in S$, where $H_{\bar{m}_i}$ is defined in (3.2) and $c(H_{\bar{m}_i})$ denotes the Mañé critical value of $H_{\bar{m}_i}$, i = 1, 2. Let $\epsilon > 0$, $\xi : \mathbb{R}^n \to \mathbb{R}$ be a smooth, nonnegative, symmetric kernel with a support contained in the unit ball and of integral one. Set $\xi^{\epsilon}(x) = \frac{1}{\epsilon^n}\xi(\frac{x}{\epsilon})$ for i = 1, 2. Define $m_i^{\epsilon} := \xi^{\epsilon} \star \bar{m}_i$, i.e., the convolution of ξ^{ϵ} and \bar{m}_i , and

$$V_i^{\epsilon}(x) := \frac{\xi^{\epsilon}(x) \star \left(\bar{m}_i D_p H(x, D\bar{u}_i(x))\right)}{m_i^{\epsilon}}$$

It is clear that $-\operatorname{div}(m_i^{\epsilon}V_i^{\epsilon}) = 0$ in \mathbb{R}^n . We multiply this equality by $\bar{u}_1 - \bar{u}_2$, integrate by parts and subtract the resulting formulas to get

$$\int_{\mathbb{R}^n} \left\langle D(\bar{u}_1 - \bar{u}_2), m_1^{\epsilon} V_1^{\epsilon} - m_2^{\epsilon} V_2^{\epsilon} \right\rangle dx = 0.$$

Hence

$$0 = \int_{\mathbb{R}^n} \left\langle D(\bar{u}_1 - \bar{u}_2), \xi^{\epsilon} \star (\bar{m}_1 D_p H(\cdot, D\bar{u}_1) - \bar{m}_2 D_p H(\cdot, D\bar{u}_2)) \right\rangle dx$$

$$= \int_{\mathbb{R}^n} \left\langle D(\bar{u}_1 - \bar{u}_2), m_1^{\epsilon} D_p H(x, D\bar{u}_1) - m_2^{\epsilon} D_p H(x, D\bar{u}_2) \right\rangle dx + R_{\epsilon},$$
 (A.1)

where we have defined

$$R_{\epsilon} := \int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{n}} \xi^{\epsilon}(x-y) \left\langle D(\bar{u}_{1}-\bar{u}_{2})(x), D_{p}H(y, D\bar{u}_{1}(y)) - D_{p}H(x, D\bar{u}_{1}(x)) \right\rangle \bar{m}_{1}(dy) dx \\ - \int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{n}} \xi^{\epsilon}(x-y) \left\langle D(\bar{u}_{1}-\bar{u}_{2})(x), D_{p}H(y, D\bar{u}_{2}(y)) - D_{p}H(x, D\bar{u}_{2}(x)) \right\rangle \bar{m}_{2}(dy) dx$$

In particular, with almost the same considerations as in Lemma 4.12, one can prove that $R_{\epsilon} \to 0$ as $\epsilon \to 0$.

Next, for i = 1, 2, we multiply $H_{\bar{m}_i}(x, D\bar{u}_i) = c(H_{\bar{m}_i})$ by $(m_1^{\epsilon} - m_2^{\epsilon})$, integrate in space and subtract the second identity from the first one to get

$$\int_{\mathbb{R}^n} \left(H(x, D\bar{u}_1) - H(x, D\bar{u}_2) - F(x, \bar{m}_1) + F(x, \bar{m}_2) \right) (m_1^{\epsilon} - m_2^{\epsilon}) dx = 0.$$

Now, combining the above equality with (A.1) we obtain

$$-R_{\epsilon} = \int_{\mathbb{R}^{n}} \left\langle D(\bar{u}_{1} - \bar{u}_{2}), m_{1}^{\epsilon} D_{p} H(x, D\bar{u}_{1}) - m_{2}^{\epsilon} D_{p} H(x, D\bar{u}_{2}) \right\rangle dx - \int_{\mathbb{R}^{n}} \left(H(x, D\bar{u}_{1}) - H(x, D\bar{u}_{2}) - F(x, \bar{m}_{1}) + F(x, \bar{m}_{2}) \right) (m_{1}^{\epsilon} - m_{2}^{\epsilon}) dx.$$
(A.2)

Hereafter, we denote $H(x, D\bar{u}_i)$ by H_i and $D_pH(x, D\bar{u}_i)$ by D_pH_i for i = 1, 2. Then, following Lasry-Lions [20], by (A.2) R_{ϵ} can be recast as

$$-R_{\epsilon} = \int_{\mathbb{R}^n} \left(H_2 - H_1 - \left\langle DH_1, D(\bar{u}_2 - \bar{u}_1) \right\rangle \right) m_1^{\epsilon} dx$$

+
$$\int_{\mathbb{R}^n} \left(H_1 - H_2 - \left\langle DH_2, D(\bar{u}_1 - \bar{u}_2) \right\rangle \right) m_2^{\epsilon} dx$$

+
$$\int_{\mathbb{R}^n} \left(F(x, \bar{m}_1) - F(x, \bar{m}_2) \right) (m_1^{\epsilon} - m_2^{\epsilon}) dx.$$

Owing to the convexity of H with respect to the second argument, the terms $H_2 - H_1 - \langle DH_1, D(\bar{u}_2 - \bar{u}_1) \rangle$ and $H_1 - H_2 - \langle DH_2, D(\bar{u}_1 - \bar{u}_2) \rangle$ are nonnegative. So we get

$$\int_{\mathbb{R}^n} \left(F(x, \bar{m}_1) - F(x, \bar{m}_2) \right) (m_1^{\epsilon} - m_2^{\epsilon}) dx \le -R_{\epsilon}.$$

Letting $\epsilon \to 0$ we have that

$$\int_{\mathbb{R}^n} \left(F(x, \bar{m}_1) - F(x, \bar{m}_2) \right) d(\bar{m}_1 - \bar{m}_2) \le 0,$$

which together with assumption (F3) implies that

$$\int_{\mathbb{R}^n} \left(F(x, \bar{m}_1) - F(x, \bar{m}_2) \right)^2 dx \le 0.$$

Hence, for every $x \in \mathbb{R}^n$, we have that $F(x, \bar{m}_1) = F(x, \bar{m}_2)$ and then, since $H_{\bar{m}_1} = H_{\bar{m}_2}$, we deduce that $c(H_{\bar{m}_1}) = c(H_{\bar{m}_2})$.

A.2 A technical lemma

Lemma A.1. For any $n \ge 1$ and $D \ge 0$ there exists a constant c = c(n, D) such that any Lipschitz continuous function $f : \mathbb{R}^n \to \mathbb{R}$, with $\operatorname{Lip}(f) \le D$, satisfies

$$||f||_{\infty} \le c(n,D) ||f||_2^{\frac{2}{n+2}}.$$
 (A.3)

Proof. Observe, first, that (A.3) is trivial if $f \notin L^2(\mathbb{R}^n)$.

For any $\delta > 0$ and any $x^0 \in \mathbb{R}^n$, we set

$$Q(x^{0}, \delta) = \{ x \in \mathbb{R}^{n} : \max_{1 \le i \le n} |x_{i} - x_{i}^{0}| \le \delta \}.$$

Then, for any such cube $Q(x^0,\delta)$, we have that

$$|f(x)| \le |f(y)| + D\delta\sqrt{n}, \quad \forall x, y \in Q(x^0, \delta).$$

Taking the square of both sides and integrating over $Q(x^0, \delta)$, we obtain

$$\int_{Q(x^0,\delta)} |f(x)|^2 dy \le 2 \int_{Q(x^0,\delta)} \left(|f(y)|^2 + D^2 \delta^2 n \right) dy,$$

which implies

$$2^{n}\delta^{n}|f(x)|^{2} \leq 2\int_{Q(x^{0},\delta)}|f(y)|^{2}dy + 2^{n+1}D^{2}n\delta^{n+2}.$$

Taking the supremum over the cube Q we have that

$$\frac{1}{2} \sup_{x \in Q(x^0,\delta)} |f(x)|^2 \le \frac{1}{2^n \delta^n} \int_{Q(x^0,\delta)} |f(y)|^2 dy + 2^n D^2 n \delta^2 \le \frac{1}{2^n \delta^n} \int_{\mathbb{R}^n} |f(y)|^2 dy + 2^n n D^2 \delta^2.$$

Since x^0 may be taken arbitrarily in \mathbb{R}^n , we deduce that

$$\frac{1}{2} \sup_{x \in \mathbb{R}^n} |f(x)|^2 \le \frac{1}{2^n \delta^n} \int_{\mathbb{R}^n} |f(y)|^2 dy + nD^2 \delta^2 =: G(\delta).$$

Taking the minimum of $G(\delta)$ for $\delta \in (0, +\infty)$, yields the conclusion.

The following example shows that the exponent in (A.3) is optimal, in the sense that such a bound would be false for any exponent $\theta > \frac{2}{n+2}$.

Example A.2. Consider a function on \mathbb{R} defined by

$$f(t) := \begin{cases} \frac{1}{k} - t, & t \in [0, \frac{1}{k}], \\ 0, & otherwise, \end{cases}$$

for some $k \in \mathbb{N}$. Then, we have

$$||f||_{\infty} = \frac{1}{k}, \quad ||f||_2 = \left(\int_0^{\frac{1}{k}} (\frac{1}{k} - t)^2 dt\right)^{\frac{1}{2}} = \left(\frac{1}{3}\right)^{\frac{1}{2}} \left(\frac{1}{k}\right)^{\frac{3}{2}}.$$

Since n = 1, then

$$\|f\|_2^{\frac{2}{n+2}} = \left(\frac{1}{3}\right)^{\frac{1}{3}} \frac{1}{k},$$

which implies that the estimate (A.3) is optimal.

Acknowledgements: Piermarco Cannarsa was partly supported by Istituto Nazionate di Alta Matematica (GNAMPA 2018 Research Projects) and by the MIUR Excellence Department Project awarded to the Department of Mathematics, University of Rome Tor Vergata, CUP E83C18000100006. Wei Cheng was partly supported by Natural Scientific Foundation of China (Grant No. 11871267, No. 11631006 and No.11790272). Kaizhi Wang was partly supported by National Natural Science Foundation of China (Grant No. 11771283) and by China Scholarship Council (Grant No. 201706235019).

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