Modified Lomax Model: A heavy-tailed distribution for fitting large-scale real-world complex networks

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Abstract

Real-world networks are generally claimed to be scale-free, meaning that the degree distributions follow the classical power-law, at least asymptotically. Yet, closer observation shows that the classical power-law distribution is often inadequate to meet the data characteristics due to the existence of a clearly identifiable non-linearity in the entire degree distribution in the log-log scale. The present paper proposes a new variant of the popular heavy-tailed Lomax distribution which we named as the Modified Lomax (MLM) distribution that can efficiently capture the crucial aspect of heavy-tailed behavior of the entire degree distribution of real-world complex networks. The proposed MLM model, derived from a hierarchical family of Lomax distributions, can efficiently fit the entire degree distribution of real-world networks without removing lower degree nodes as opposed to the classical power-law based fitting. The MLM distribution belongs to the maximum domain of attraction of the Frechet distribution and is right tail equivalent to Pareto distribution. Various statistical properties including characteristics of the maximum likelihood estimates and asymptotic distributions have also been derived for the proposed MLM model. Finally, the effectiveness of the proposed MLM model is demonstrated through rigorous experiments over fifty real-world complex networks from diverse applied domains.

Keywords: Complex networks; Degree distribution; Lomax distribution; Heavy-tailed distribution; Power-law; Statistical properties.

1. Introduction

The modeling and structural aspects of large scale real-world complex networks, including social, information, collaboration, communication, etc. have been well studied during the past decade [1, 2, 3, 4, 5] by many researchers. The World Wide Web, Twitter, Orkut, Youtube, DBLP, Wiki talk, Facebook, LinkedIn are examples of such large scale real-world complex networks. These networks are characterized by several important structural, emergent properties like degree distribution, correlation coefficient, average nearest neighbor, average path length, clustering coefficient, community structure, etc. Recently, the modeling and statistical aspects of such emergent structural properties, therefore, remain an important research area in the study of large scale real-world complex networks [6, 4, 7, 8, 9]. In this regard, the node degree distribution has been well studied and viewed as an important structural characteristic of real-world networks [10]. In 1999, Barabasi and Albert [11, 12] modeled the node degree distribution of the World Wide Web (WWW) using a power-law. Since then, many researchers have also favored the use of heavy tailed power-law in

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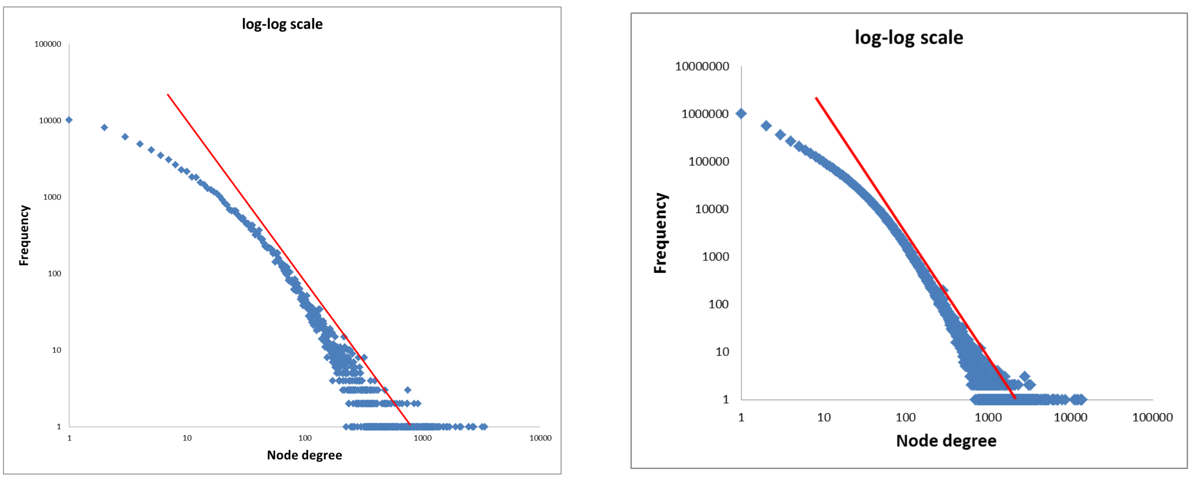
modeling the node degree distribution of real-world networks such as collaboration networks, communication networks, social networks, biological networks, etc [13, 14]. Mathematically, a quantity x follows a power-law if it is drawn from a probability distribution $P(x) \propto x^{-\alpha}$ where, the parameter α is a positive constant and is known as exponent or scaling parameter of the distribution. Thus it is common to encounter the claim that most of the real-world networks are scale-free, meaning that the degree distributions follow single power-law. Despite this, a closer observation, while fitting, shows that the classical power-law distribution is often inadequate to meet the data characteristics adequately because of the existence of an identifiable non-linearity (bend) when the entire degree distribution is considered in log-log scale as shown in Figures 1a and 1b (elaborated later).

This feature (non-linearity) of the entire degree distribution, depending on when and where it is considered or ignored, possibly constitute the reason why the universality vis-a-vis scarcity of scale-free networks has remained controversial ever since its inception [15, 16]. The debate has continued to crop up time and again throughout the last twenty-one years [17, 18, 13, 14] and in very recent times too whence it has been claimed through an empirical and extensive study that the power-law distribution does not fit well in most cases and thereby produces a significant fitting error, followed by counter-claims [19].

This apart, researchers have also argued differently in favor of scale-free structure while suggesting some softer statistical criteria for scale-freeness [20, 21, 22]. Especially significant in this context is the following quote [22]: "The fact that heavy-tailed distributions occur in complex systems is certainly important (because it implies that extreme events occur more frequently than would otherwise be the case)... However, a statistically sound power-law is no evidence of universality without a concrete underlying theory to support it. Moreover, knowledge of whether or not a distribution is heavy-tailed is far more important than whether it can be fit using a power-law".

Several other heavy-tailed distributions such as lognormal, Pareto lognormal (PLN), double Pareto lognormal (DPLN), etc. also have been proposed in modeling the degree distribution of real-world networks instead of power-law [18, 23]. Recent research also recognized the deviations from a pure power-law distribution over various network data sets and recommended some other distributions for better modeling the heavy-tailed node degree distribution [21, 24, 25]. Thus, identifying the reasons for deviation of single power-law while fitting and looking for the alternative models which can efficiently capture the crucial aspect of heavy-tailed and long-tailed behaviour of the entire degree distribution of real-worlld complex networks continue to remain a challenging task of current research in the field of complexity science even as it steadily gravitates toward data science [20, 21, 22].



(a) Twitter network

(b) LiveJournal network

Figure 1: Plot of degree distribution

Motivation: Networks are a powerful way to represent and study the structure of real-world complex systems. Across various applied domains of networks, it is common to encounter the claim that most of the real-world networks are scale-free, meaning that the degree distributions follow single power-law, though the universality of scale-free networks remains controversial as already discussed above.

Now consider an example where Figures 1a and 1b that depict the plot of entire degree distribution in the log-log scale of the Twitter and LiveJournal social networks. The horizontal axis represents the unique degree value (x), and the vertical axis represents the corresponding frequency. In these networks, a node represents a single user, and an edge represents a follower of that user. From these figures, it is clear that the pattern of the degree distribution of these networks does not match with the straight-line representation in the log-log scale through a single power-law. Usually, while fitting the node degree distribution, the single power-law is applied only for values of degree higher than some minimum (say, x_{min}) and the exponent α is estimated from the data using MLE accordingly. Thus power-law distribution provides better fitting or in other words better inclined to the right tail of the data unless otherwise, some "unimportant" (i.e., lower degree) nodes are left out. Analytically, we can say that this inadequacy of fitting a single power-law occurs because of nonlinear behavior of the degree distribution curve in the log-log scale. This motivates the researchers to use other heavy-tailed probability models with non-negative exponent for better modeling the entire degree distribution of real-world networks. To capture these nonlinearities in the degree distribution of the real-world complex networks in a log-log scale, previous studies used various heavy-tailed probability distributions [21, 18, 24]. In this current research, we study the behavior of the entire degree distributions with a new variant of Lomax distribution that has wide applications in the field of actuarial science, reliability modeling, economics and computer science [26, 27, 28, 29]. The Lomax distribution is essentially a Pareto Type-II distribution that has been shifted so that its support begins at zero [27, 26]. Some extension and generalization of the Lomax distribution has been carried out for analyzing reliability and survival data sets in the past [27, 30, 31]. Recent research also focused on a new generalization of Pareto distribution with application to the breaking stress data [32]. This paper proposes a modified Lomax (MLM) distribution to be derived from a hierarchical family of Lomax distributions where the non-negative shape parameter is assumed to be expressible as a nonlinear function of the data.

Our contribution: The major contribution here is to develop a modified Lomax (MLM) distribution from a hierarchical family of Lomax distributions for efficient modeling of the entire degree distribution of real-world complex networks [27, 26]. The reasons for introducing MLM distribution is to provide greater flexibility and better fitting to the entire node degree distribution of complex networks compared to other popularly used heavy-tailed distributions. In other words, the proposed MLM model can be used for effective modeling the degree distribution of complex networks, coming from different disciplines, in the whole range of the data without discarding some of the lower degree nodes. Moreover, some statistical properties including extreme value and asymptotic behavior of the proposed MLM distribution have been studied in this context. We also provide mathematical arguments to explain the behavior of the likelihood surface for this nonlinear variant of the Lomax distribution, i.e., MLM distribution. A sufficient condition for the existence of the global maximum for the likelihood estimates is given using the notion of the coefficient of variations (CV) and discuss the parameter estimation procedures of the proposed MLM distribution. In order to justify the effectiveness of the proposed MLM distribution, we have compared it with the other common power-law-type distributions, viz. power-law, Pareto, lognormal, exponential, power-law with exponential cutoff and Poisson [23, 13, 17]. The goodness-of-fit of the observed degree distribution is evaluated and compared using a few statistical measures, viz. bootstrap Chi-square, KL-divergence (KLD), mean absolute error (MAE) and root mean square error (RMSE). Several real-world complex networks from diverse fields have been used for experimental evaluation. Empirical results confirm the effectiveness of the proposed MLM distribution compared to other common distributions.

The remainder of the paper is organized as follows. Section 2 provides the details of the hierarchical family of Lomax distributions. We propose and interpret proposed modified Lomax (MLM) distribution in Section 3. Section 4 discusses the statistical properties, including extreme value and asymptotic behaviors of the proposed MLM distribution. Section 5 is devoted to the experimental results with a detailed analysis of the results over several real-world complex networks. Finally, Section 6 concludes the paper with a brief discussion.

2. Model

In this section, we first introduce a new family of heavy-tailed Lomax (HLM) distributions. Further, we propose a relevant model from this newly introduced family to model the real-world heavy-tailed network data sets in the whole range.

2.1. Genesis

Lomax distribution has been used as an alternative to exponential, power-law, gamma and weibull distribution for modeling heavy tailed data sets [33, 34, 35, 36]. The cumulative distribution function (CDF) and the probability density function (PDF) of the Lomax model are defined as follows:

Definition 1. A random variable Z follows Lomax distribution with parameters α and σ if the CDF is of the form:

$$F(z) = 1 - \left(1 + \frac{z}{\sigma}\right)^{-\alpha}; \; z \geq 0,$$

where α (> 0) is the shape parameter (real) and σ (> 0) is the scale parameter (real). The corresponding PDF is defined as follows:

$$f(z) = \frac{\alpha}{\sigma}\left(1 + \frac{z}{\sigma}\right)^{-\alpha-1}; \; z \geq 0 \tag{1}$$

Below we introduce a new family of heavy tailed Lomax distributions which is right tail-equivalent to a power-law distribution.

Definition 2. A continuous random variable X follows a family of heavy-tailed Lomax (HLM) distributions if and only if it has the following CDF:

$$F(x) = 1 - \left(1 + x\right)^{-m(x)}; \; x \geq 0 \tag{2}$$

and $F(x) = 0$ if $x \leq 0$, where $m : (0, \infty) \to \mathbb{R}^+$ is a real, continuous, positive function which is differentiable on $(0, \infty)$ and satisfies the following conditions:

1. The function m is strictly positive and have finite limit at infinity, i.e., $\lim_{x\to\infty} m(x) = \alpha \; (> 0)$.
2. $\lim_{x\to 0^+} (1 + x)^{m(x)} = 1$ and $\lim_{x\to\infty} (1 + x)^{m(x)} = \infty$.
3. $\dfrac{m'(x)}{m(x)} \geq -\dfrac{1}{(1 + x)\log(1 + x)}, \; x > 0.$

It can be easily verified that the CDF in (2) satisfying conditions (1), (2) and (3) is a genuine CDF which can also be expressed as follows:

$$F(x) = 1 - \exp\left[-m(x)\log(1 + x)\right], \; x > 0$$

The PDF of this new family of heavy-tailed Lomax distribution is of the form:

$$f(x) = (1 + x)^{-m(x)}\left[\frac{m(x)}{(1 + x)} + m'(x)\log(1 + x)\right], \quad x > 0 \quad \text{and} \quad f(x) = 0, \quad x \leq 0.$$

There can be a wide variety of choices of $m(x)$ satisfying $\lim_{x\to\infty} m(x) = \alpha \; (> 0)$. It is noted that the simplest choice of $m(x) = \alpha$ and $x = \frac{z}{\sigma}$ corresponds to the Lomax distribution. We further represent this newly introduced family of Lomax distributions as a hierarchical family in accordance with Pareto distribution [37].

Definition 3. (HLM Type-I family of distributions) Supposed that a random variable X folows HLM family of distributions as defined in (2). Then with a scale parameter $\sigma \in (0, \infty)$, the CDF of HLM Type-I family of distributions takes the following form:

$$F(x) = 1 - \left[1 + \left(\frac{x}{\sigma} - 1\right)\right]^{-m\left(\frac{x}{\sigma}-1\right)}, \quad x > \sigma$$

By taking $m\left(\frac{x}{\sigma} - 1\right) = \alpha\ (> 0)$, we obtain the classical Pareto Type-I distribution.

Definition 4. (HLM Type-II family of distribution) Supposed that a random variable X folows HLM family of distributions as defined in (2). Then with a location parameter $\mu \in \mathbb{R}$ and a scale parameter $\sigma \in (0, \infty)$, the CDF of HLM Type-II family of distributions takes the following form:

$$F(x) = 1 - \left(1 + \frac{x - \mu}{\sigma}\right)^{-m\left(\frac{x-\mu}{\sigma}\right)}, \quad x > \mu$$

By taking $m\left(\frac{x-\mu}{\sigma}\right) = \alpha\ (> 0)$, we obtain the Pareto Type-II distribution. Also, in addition $\mu = 0$ corresponds to the Lomax distribution.

Definition 5. (HLM Type-III family of distribution) Supposed that a random variable X folows HLM family of distributions as defined in (2). Then with a location parameter $\mu \in \mathbb{R}$, scale parameters $\sigma \in (0, \infty)$ and a shape parameter $\gamma\ (> 0)$, the CDF of HLM Type-III family of distributions takes the following form:

$$F(x) = 1 - \left[1 + \left(\frac{x - \mu}{\sigma}\right)^{\frac{1}{\gamma}}\right]^{-m\left[\left(\frac{x-\mu}{\sigma}\right)^{\frac{1}{\gamma}}\right]}, \quad x > \mu$$

By taking $\left[m\left(\frac{x-\mu}{\sigma}\right)^{\frac{1}{\gamma}}\right] = 1$, we obtain the Pareto Type-III distribution.

Obviously, the choice of $m(\cdot)$ function is subjective and any function m satisfying conditions (1), (2) and (3) will give some known (unknown) heavy-tail Lomax distributions.

3. Modified Lomax (MLM) Model

The Lomax distribution does not provide great flexibility in modeling heavy-tailed data sets in the whole range similar to the power-law distribution. Due to this, the trend of parameter(s) induction to the baseline Lomax distribution has received increased attention in the recent years. Several generalized classes of distributions by adding additional parameters such as shape and or scale and or location in the distribution are available such as exponentiated Lomax (EL) [38], Beta-Lomax (BL) [39], exponential Lomax (ELomax) [40], Gamma-Lomax (GL) [41] and Gumbel-Lomax (GuLx) model [42].

This paper provides a new modified version of the Lomax distribution called modified Lomax (MLM) distribution. MLM distribution is shown to be an asymmetric distribution, which provides great fit in modeling large-scale heavy-tailed data sets. The proposed MLM model is derived from the HLM family of distributions (in particular, HLM Type-II model) that can efficiently model the entire degree distribution of real-world networks. In other words, the proposed MLM model can be used for effective modeling the degree distribution of real-worlld complex networks in the whole range without discarding lower degree nodes. We define a relevant model from the newly introduced HLM Type-II family with the location parameter $\mu = 0$ and we choose a flexible $m(\cdot)$ function that depends on two shape parameters α and β satisfying $\lim_{x\to\infty} m(x) = \alpha$. The rational behind adding an additional shape parameter in the HLM Type-II family of distribution will make the statistical model more flexible, simple and have physical interpretation. This idea of generalization should suffice the practical needs of working with the non linear exponent to address the structural issue (degree distribution) of real-world complex networks.

Now we choose a nonlinear function m that adds a nonlinear exponents while fitting heavy-tailed HLM Type-II model in the degree distributions is as follows:

$$m(x) = \alpha\left(\frac{\log\left(1+x\right)}{1+\log\left(1+x\right)}\right)^{\beta}.$$

The chosen $m(x)$ approaches to α from below if $-1<\beta<0$ as $x\to\infty$ and approaches to α from above for $\beta>0$ as $x\to\infty$. Note that, the function $m(x)$ as defined above includes the constant function (in this case α) as special cases by setting $\beta=0$. The derivative of $m(x)$ is given by

$$m'(x) = \frac{\alpha\beta}{x+1}\left(\frac{\log\left(1+x\right)}{1+\log\left(1+x\right)}\right)^{\beta-1}.\left(1+\log\left(1+x\right)\right)^{-2}.$$

Now, we define a relevant model with the above choice of $m(\cdot)$ in the HLM Type-II model with $\mu=0$ and name it as Modified Lomax Model to be denoted by $MLM(\alpha,\beta,\sigma)$. This modification to the Lomax distribution provides more flexibility in the data modeling since the non-negative shape parameter are assumed to be expressed as a nonlinear function of the empirical data. Thus the proposed MLM model with parameters α,β,σ could be useful for modeling the heavy-tailed degree distribution of real-world complex network data sets in the whole range.

Definition 6. (Modified Lomax Distribution) A continuous random variable X follows $MLM(\alpha,\beta,\sigma)$ distribution with α (>0) and β (>-1) as the shape parameters and σ (>0) as the scale parameter if the CDF takes the following form:

$$F(x) = 1-\exp\left[-\alpha\frac{\log^{\beta+1}(1+x/\sigma)}{[1+\log(1+x/\sigma)]^{\beta}}\right],\ x>0, \tag{3}$$

and $F(x)=0$ if $x\leq 0$. The corresponding PDF is given by,

$$f(x) = \frac{\alpha\left[\beta+1+\log\left(1+\frac{x}{\sigma}\right)\right]\left[\log\left(1+\frac{x}{\sigma}\right)\right]^{\beta}}{\sigma\left(1+\frac{x}{\sigma}\right)\left[1+\log\left(1+\frac{x}{\sigma}\right)\right]^{\beta+1}}\exp\left[-\alpha\frac{\left[\log\left(1+\frac{x}{\sigma}\right)\right]^{\beta+1}}{\left[1+\log\left(1+\frac{x}{\sigma}\right)\right]^{\beta}}\right],\ x>0 \tag{4}$$

and $f(x)=0$ if $x\leq 0$.

This MLM model includes Lomax distribution $\beta=0$ as particular case. In addition, it belongs to the new family of HLM Type-II distribution satisfying the condition: $\lim_{x\to\infty} m(x)=\alpha$ (>0). Due to the addition of an additional parameter β in the exponents of the Lomax distribution generates various shapes (unimodal and bimodal) and provides greater flexibility (nonlinearity and heavy-tail) as shown in Figure 2. We study the monotonicity for the PDF of the proposed MLM model in Theorem 1 below.

Theorem 1. Let X be the random variable follows $MLM(\alpha,\beta,\sigma)$ distribution, then the PDF as in (4) is a decreasing function for $-1<\beta<0$.

Proof. Differentiating (4) w.r.t. x, we have

$$\begin{aligned} f'(x) = &-\frac{\alpha^2\left[1-F(x)\right]\left[\beta+1+\log\left(1+\frac{x}{\sigma}\right)\right]^2\left[\log\left(1+\frac{x}{\sigma}\right)\right]^{2\beta}}{\sigma^2\left(1+\frac{x}{\sigma}\right)^2\left[1+\log\left(1+\frac{x}{\sigma}\right)\right]^{2\beta+2}} \\ &-\frac{\alpha\left[1-F(x)\right]\left\{\left[\beta+1+\log\left(1+\frac{x}{\sigma}\right)\right]\left[\log\left(1+\frac{x}{\sigma}\right)\right]^{\beta}+(1+\beta)\left[\log\left(1+\frac{x}{\sigma}\right)\right]^{\beta-1}\right\}}{\sigma^2\left(1+\frac{x}{\sigma}\right)^2\left[1+\log\left(1+\frac{x}{\sigma}\right)\right]^{\beta+1}} \\ &+\frac{\alpha\left[1-F(x)\right](1+\beta)\left[\beta+1+\log\left(1+\frac{x}{\sigma}\right)\right]\left[\log\left(1+\frac{x}{\sigma}\right)\right]^{\beta-1}}{\sigma^2\left(1+\frac{x}{\sigma}\right)^2\left[1+\log\left(1+\frac{x}{\sigma}\right)\right]^{\beta+2}} \end{aligned} \tag{5}$$

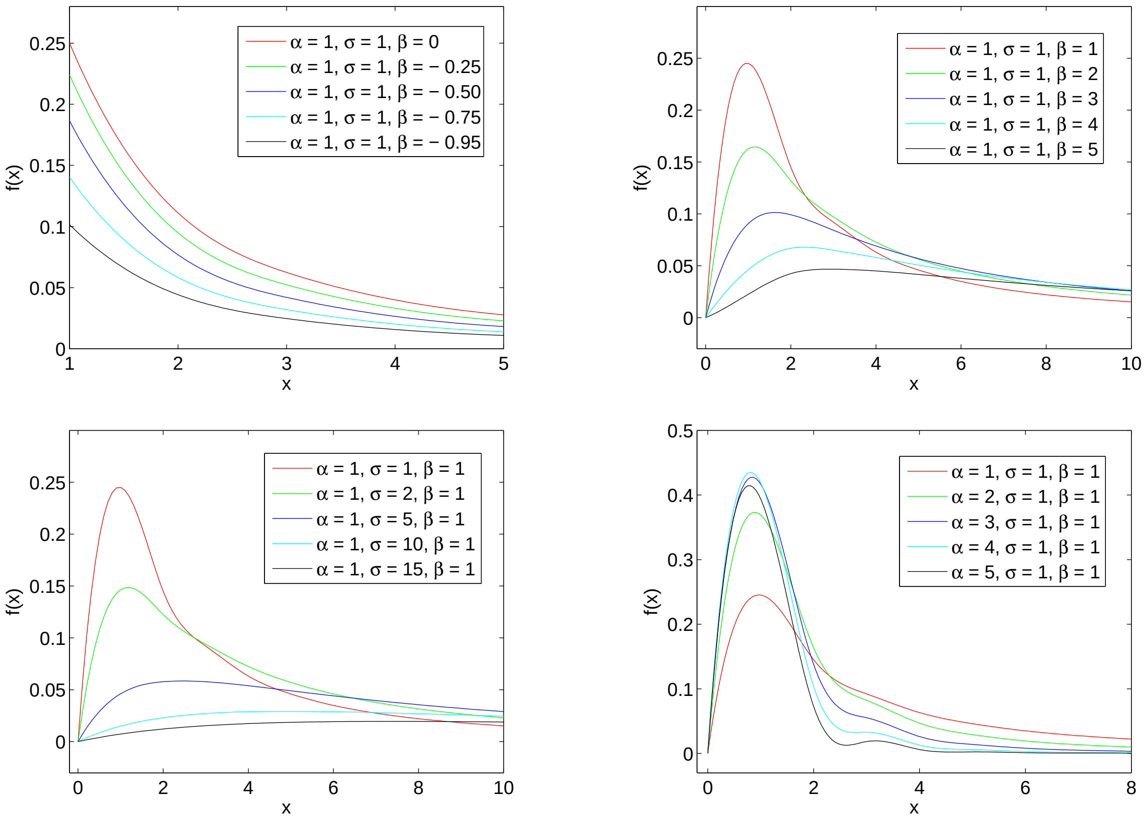


Figure 2: Plot of the PDFs of MLM distribution

Trivially, if $-1 < \beta < 0$, then $f'(x) < 0$. Thus, $f(x)$ is decreasing function if $\beta \in (-1, 0)$. □

4. Statistical Properties of the MLM distribution

4.1. Characterization and existence of the likelihood

Initially we characterize the maximum likelihood estimates (MLEs) of the parameters α and σ of a Lomax distribution. Subsequently, we derived a sufficient condition for the existence of MLEs of the MLM distribution using coefficient of variation (CV). Given a set of samples $\{x_i\}$ of size n, the log-likelihood function for the Lomax distribution, after dividing it by the sample size n, is given by

$$\ell(\alpha, \sigma) = log\alpha - log\sigma - \frac{(\alpha+1)}{n} \sum_{i=1}^{n} \log\left(1 + \frac{x_i}{\sigma}\right) \tag{6}$$

Differentiating (6) w.r.t. α and σ, respectively, we have:

$$\frac{\partial \ell(\alpha, \sigma)}{\partial \alpha} = \frac{1}{\alpha} - \frac{1}{n} \sum_{i=1}^{n} \log\left(1 + \frac{x_i}{\sigma}\right) \tag{7}$$

$$\frac{\partial \ell(\alpha, \sigma)}{\partial \sigma} = -\frac{1}{\sigma} + \frac{(1+\alpha)}{n\sigma} \sum_{i=1}^{n} \left(\frac{x_i}{\sigma + x_i}\right) \tag{8}$$

Equating to zero the derivative of $\ell(\alpha,\sigma)$ w.r.t. α in (7), we obtain $\hat{\alpha}=\alpha(\sigma)$ as follows:

$$\hat{\alpha}=\alpha(\sigma)=\frac{n}{\sum_{i=1}^{n}\log\left(1+\frac{x_i}{\sigma}\right)} \tag{9}$$

Differentiating (9) w.r.t. σ we have,

$$\alpha'(\sigma)=\frac{\hat{\alpha}^2}{n\sigma}\sum_{i=1}^{n}\frac{x_i}{\sigma+x_i} \tag{10}$$

It is important to note that there is no closed form solution to the likelihood based on (7) and (8), and a suitable numerical algorithm (for example, Newton-Raphson method) can be employed to obtain the maximum likelihood estimates (MLEs) of the α and σ. Different estimation procedures of the MLEs have been discussed in previous literature, for example see [43]. But for small or medium-sized samples, anomalous behavior of the likelihood surface can be encountered when sampling from the Lomax distribution. In this paper, we characterize the profile log-likelihood function in terms of the coefficient of variation (CV), defined as follows:

Definition 7. The CV is the ratio of the standard deviation (s) to the mean (μ),

$$CV=\frac{s}{\mu};$$

where $\mu=\frac{1}{n}\sum_{i=1}^{n}x_i$ and $s=\sqrt{\frac{1}{n}\sum_{i=1}^{n}x_i^2-\mu^2}$.

By using standard notation, the profile log-likelihood function based on equation 6, is given by

$$\ell_p(\sigma)=\sup\ell(\hat{\alpha},\sigma)=log(\alpha(\sigma))-log\sigma-1-\frac{1}{\alpha(\sigma)} \tag{11}$$

Differentiating (11) w.r.t. σ, we have the following:

$$\ell_p'(\sigma)=\frac{\alpha'(\sigma)}{\alpha(\sigma)}-\frac{1}{\sigma}+\frac{\alpha'(\sigma)}{\left[\alpha(\sigma)\right]^2} \tag{12}$$

Below we present the following lemmas which will be useful to find the sufficient condition for the existence for the global maximum of the profile log-likelihood function (11).

Lemma 1. The following limit holds:

1. $\lim_{\sigma\to\infty}\sigma\log\left(1+\frac{x}{\sigma}\right)=x$;
2. $\lim_{\sigma\to\infty}\frac{\sigma x}{\sigma+x}=x$;
3. $\lim_{\sigma\to\infty}\sigma^2\left(\log\left(1+\frac{x}{\sigma}\right)-\frac{x}{\sigma+x}\right)=\frac{x^2}{2}$.

Proof. The proof is elementary and can easily be done using series expansions. □

Lemma 2. The following limit holds:

1. $\lim_{\sigma\to\infty}\frac{1}{\alpha(\sigma)}=0$;
2. $\lim_{\sigma\to\infty}\frac{\alpha(\sigma)}{\sigma}=\frac{1}{\bar{x}}$, where $\bar{x}$ is the sample mean;
3. $\ell_0\equiv\lim_{\sigma\to\infty}\ell_p(\sigma)=\log\left(\frac{1}{\bar{x}}\right)-1$.

Proof. The proofs are straightforward and can be done using Lemma (1).

1. $\lim_{\sigma\to\infty} \dfrac{1}{\alpha(\sigma)} = \lim_{\sigma\to\infty} \dfrac{1}{n}\sum_{i=1}^{n} \log\left(1+\dfrac{x_i}{\sigma}\right) = \lim_{\sigma\to\infty} O\left(\dfrac{1}{\sigma}\right) = 0.$
2. $\lim_{\sigma\to\infty} \dfrac{\alpha(\sigma)}{\sigma} = \lim_{\sigma\to\infty} \dfrac{n}{\sigma\sum_{i=1}^{n}\log\left(1+\frac{x_i}{\sigma}\right)} = \dfrac{n}{\sum_{i=1}^{n} x_i} = \dfrac{1}{\bar{x}}.$
3. $\lim_{\sigma\to\infty} \ell_p(\sigma) = \lim_{\sigma\to\infty}\left[\log\left(\dfrac{\alpha(\sigma)}{\sigma}\right) - 1 - \dfrac{1}{\alpha(\sigma)}\right] = \log\left(\dfrac{1}{\bar{x}}\right) - 1.$

□

A sufficient condition for monotonic increasing (decreasing) for the profile log-likelihood function is presented in Theorem (2) below, for sufficiently large σ. Also, we present a sufficient condition for the existence of global maximum corresponding to the likelihood function for the Lomax distribution to be at a finite point in Corollary (1).

Theorem 2. Let X follows $LM(\alpha,\sigma)$ distribution with $\alpha,\sigma > 0$. A sufficient condition for $\ell_p(\sigma)$ to be monotonically decreasing function is CV>1 for $\sigma\to\infty$, and if CV<1, it is monotonically increasing.

Proof. Using (9) and (10) in Eqn. (12), we can write $\ell_p^{'}(\sigma)$ as:

$$\ell_p^{'}(\sigma) = -\frac{1}{\sigma}\left[\frac{\sum_{i=1}^{n}\log\left(1+\frac{x_i}{\sigma}\right) - \sum_{i=1}^{n}\frac{x_i}{\sigma+x_i}}{\sum_{i=1}^{n}\log\left(1+\frac{x_i}{\sigma}\right)}\right] + \frac{1}{n\sigma}\sum_{i=1}^{n}\frac{x_i}{\sigma+x_i} \tag{13}$$

Using the limits of Lemma (1) in Eqn. (13), we have

$$-\lim_{\sigma\to\infty}\sigma^2\ell_p^{'}(\sigma) = \frac{1}{2}\times\frac{\sum_{i=1}^{n}x_i^2}{\sum_{i=1}^{n}x_i} - \bar{x}. \tag{14}$$

Finally, we note that $-\lim_{\sigma\to\infty}\sigma^2\ell_p^{'}(\sigma) > 0$ when the R.H.S of Eqn.(14) is strictly greater than 0. Alternatively, the likelihood function is monotonic decreasing when $\frac{1}{2n}\sum_{i=1}^{n}x_i^2 - \bar{x}^2 > 0$, or, equivalently, CV$>1$. In a similar way, we can show that if CV<1, then the $\ell_p(\sigma)$ is monotonic increasing function for sufficiently large σ. □

Remark 1. As a consequence of Theorem (2), it can be immediately concluded that $\ell_p(\sigma)$ tends to ℓ_0 based on Lemma (2) and $\ell_p(\sigma)$ is a monotonic function for sufficiently large σ. The value of CV as a measure that can be useful to determine when $\ell_p(\sigma)$ will be monotonic increasing or decreasing function for sufficiently large σ.

Corollary 1. Given a set of samples $\{x_i\}$ of $(+)ve$ numbers with CV>1, the profile likelihood function for the $LM(\alpha,\sigma)$ distribution has a global maximum at a finite point.

Proof. For small or moderate values of σ, using (9), we have

$$\lim_{\sigma\to 0}\alpha(\sigma) = \lim_{\sigma\to 0}\frac{n}{\sum_{i=1}^{n}\log\left(1+\frac{x_i}{\sigma}\right)} = 0. \tag{15}$$

Now, using (15) in (11) we have the following:

$$\lim_{\sigma\to 0}\ell_p(\sigma) = -\infty. \tag{16}$$

Since $\ell_p(\sigma)$ is a continuous and monotonic decreasing function for sufficiently large σ (as in Theorem 2) and using (16), we can conclude that a global maximum exists at a finite point when CV>1. □

Remark 2. Corollary 1 shows that the likelihood function for the Lomax distribution has a global maximum for the samples $\{x_i\}$ with CV>1 at a finite point. The calculation of CV is completely based on available empirical data and easy to compute. The existence of MLE based on CV for the MLM distribution will also holds as because MLM model reduce to Lomax distribution when $\lim_{x\to\infty} m(x) = \alpha$. This can be empirically validated in section 55.3 and will be useful useful from practitioner's point of view.

4.2. MLE of parameters

In this section, the maximum likelihood estimates are derived for parameters $\alpha, \beta,$ and σ of MLM distribution. Let $x_1, x_2, ..., x_n$ be a sample of size n from MLM(α, β, σ) distribution. Then the log-likelihood function for the vector of parameters $\Theta = (\alpha, \beta, \sigma)^T$ is given by

$$\begin{aligned}\ell \equiv \ell(x; \alpha, \beta, \sigma) &= n\log(\alpha) - \sum_{i=1}^{n} \log(\sigma + x_i) + \sum_{i=1}^{n} \log\left[\beta + 1 + \log\left(1 + \frac{x_i}{\sigma}\right)\right] \\ &+ \beta\sum_{i=1}^{n} \log\left[\log\left(1 + \frac{x_i}{\sigma}\right)\right] - (\beta+1)\sum_{i=1}^{n} \log\left[1 + \log\left(1 + \frac{x_i}{\sigma}\right)\right] \\ &- \alpha\sum_{i=1}^{n} \frac{[\log\left(1 + \frac{x_i}{\sigma}\right)]^{\beta+1}}{[1 + \log\left(1 + \frac{x_i}{\sigma}\right)]^{\beta}},\end{aligned} \tag{17}$$

The maximum likelihood estimate for the parameters α,β, and σ are given by $\hat{\alpha}$,$\hat{\beta}$, and $\hat{\sigma}$, are obtained by maximizing the likelihood function in Equation (17). The first-order partial derivatives of (1) with respect to α, β, and σ are

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} \frac{\left[\log\left(1 + \frac{x_i}{\sigma}\right)\right]^{\beta+1}}{\left[1 + \log\left(1 + \frac{x_i}{\sigma}\right)\right]^{\beta}} \tag{18}$$

$$\frac{\partial \ell}{\partial \beta} = \sum_{i=1}^{n} \frac{1}{(1 + \beta + w_i)} + \sum_{i=1}^{n} \log\left(\frac{w_i}{1 + w_i}\right) \times \left[1 - \frac{\alpha w_i^{\beta+1}}{(1 + w_i)^{\beta}}\right] \tag{19}$$

$$\frac{\partial \ell}{\partial \sigma} = -\sum_{i=1}^{n} \frac{1}{(\sigma + x_i)} + \sum_{i=1}^{n} \frac{x_i}{\sigma(\sigma + x_i)}\left[\frac{\beta+1}{(1 + w_i)} - \frac{\beta}{w_i} - \frac{1}{(1 + \beta + w_i)}\right] + \alpha\sum_{i=1}^{n} \frac{x_i}{\sigma(\sigma + x_i)}\left[\frac{(1 + \beta + w_i) w_i^{\beta}}{(1 + w_i)^{\beta+1}}\right], \tag{20}$$

where $w_i = \log\left(1 + \frac{x_i}{\sigma}\right)$.

The MLEs of the three parameters of the MLM(α, β, σ) distributions are obtained by setting these above equations to zero and solving them simultaneously. Closed forms of the solutions are not available for the equations (18), (19) and (20). So, iterative methods will be applied to solve these equations numerically.

4.3. Asymptotic distribution

Fisher information matrix, a measure of the information content of the data relative to the parameters to be estimated, plays an important role in parameter estimation. The Fisher information matrix (F) can be obtained by taking the expected values of the second-order and mixed partial derivatives of $\ell(\alpha, \beta, \sigma)$ w.r.t. α, β, and σ. Since, the analytical expression is hard to compute. Thus, it can be approximated by numerically investing the the $F = (F_{ij})$ matrix. The asymptotic F matrix can be given as follows:

$$F = \begin{bmatrix} -\frac{\partial^2 \ell}{\partial \alpha^2} & -\frac{\partial^2 \ell}{\partial \alpha \partial \beta} & -\frac{\partial^2 \ell}{\partial \alpha \partial \sigma} \\ -\frac{\partial^2 \ell}{\partial \alpha \partial \beta} & -\frac{\partial^2 \ell}{\partial \beta^2} & -\frac{\partial^2 \ell}{\partial \beta \partial \sigma} \\ -\frac{\partial^2 \ell}{\partial \alpha \partial \sigma} & -\frac{\partial^2 \ell}{\partial \beta \partial \sigma} & -\frac{\partial^2 \ell}{\partial \sigma^2} \end{bmatrix}$$

The second and mixed partial derivatives of the log likelihood function are obtained as follows:

$$\frac{\partial^2 \ell}{\partial \alpha^2} = -\frac{n}{\alpha^2} \tag{21}$$

$$\frac{\partial^2 \ell}{\partial \alpha \partial \beta} = \sum_{i=1}^{n} \log\left(\frac{1+w_i}{w_i}\right) \times \left[\frac{w_i^{\beta+1}}{(1+w_i)^{\beta}}\right] \tag{22}$$

$$\frac{\partial^2 \ell}{\partial \alpha \partial \sigma} = \sum_{i=1}^{n} \frac{x_i}{\sigma(\sigma+x_i)} \times \left[\frac{w_i^{\beta}(1+\beta+w_i)}{(1+w_i)^{\beta+1}}\right] \tag{23}$$

$$\frac{\partial^2 \ell}{\partial \beta^2} = -\sum_{i=1}^{n} \frac{1}{(1+\beta+w_i)^2} - \alpha \sum_{i=1}^{n} \log^2\left(\frac{w_i}{1+w_i}\right) \left[\frac{w_i^{\beta+1}}{(1+w_i)^{\beta}}\right] \tag{24}$$

$$\begin{aligned} \frac{\partial^2 \ell}{\partial \beta \partial \sigma} = & \sum_{i=1}^{n} \frac{x_i}{\sigma(\sigma+x_i)(1+\beta+w_i)^2} - \sum_{i=1}^{n} \frac{x_i}{\sigma(\sigma+x_i)} \times \left[\frac{(1+w_i)^{\beta} - \alpha w_i^{\beta+1}}{w_i(1+w_i)^{\beta+1}}\right] \\ & + \alpha \sum_{i=1}^{n} \frac{x_i}{\sigma(\sigma+x_i)} \left[\frac{(1+\beta+w_i)w_i^{\beta}}{(1+w_i)^{\beta+1}}\right] \left[\log\left(\frac{w_i}{1+w_i}\right)\right] \end{aligned} \tag{25}$$

$$\begin{aligned} \frac{\partial^2 \ell}{\partial \sigma^2} = & \sum_{i=1}^{n} \frac{1}{(\sigma+x_i)^2} + \sum_{i=1}^{n} \frac{x_i^2}{\sigma^2(\sigma+x_i)^2} \left[\frac{\beta+1}{(1+w_i)^2} - \frac{\beta}{w_i^2} - \frac{1}{(1+\beta+w_i)^2}\right] \\ & + \sum_{i=1}^{n} \frac{x_i(2\sigma+x_i)}{\sigma^2(\sigma+x_i)^2} \left[\frac{\beta}{w_i} + \frac{1}{(1+\beta+w_i)} - \frac{\beta+1}{(1+w_i)}\right] - \alpha \sum_{i=1}^{n} \frac{x_i^2(1+\beta)}{\sigma^2(x_i+\sigma)^2} \left[\frac{w_i^{\beta-1}(\beta+w_i)}{(1+w_i)^{\beta+1}}\right] \\ & - \alpha \sum_{i=1}^{n} \frac{x_i}{\sigma^2(\sigma+x_i)^2} \left[\frac{w_i^{\beta}(1+\beta+w_i)\left[(2\sigma+x_i)(1+w_i) - x_i(\beta+1)\right]}{(1+w_i)^{\beta+2}}\right] \end{aligned} \tag{26}$$

The variance-covariance matrix is approximated by $M = (M_{ij})$ where $M_{ij} = F_{ij}^{-1}$. The asymptotic distribution of MLEs for α, β, and σ can be written as

$$\left[(\hat{\alpha}-\alpha), (\hat{\beta}-\beta), (\hat{\sigma}-\sigma)\right] \sim N_3(0, F^{-1}(\hat{\theta}))$$

Then the approximate $100(1-k)\%$ confidence intervals for α, β, and σ are given by $\hat{\alpha} \pm \mathcal{Z}_{\frac{k}{2}}\sqrt{Var(\hat{\alpha})}$, $\hat{\beta} \pm \mathcal{Z}_{\frac{k}{2}}\sqrt{Var(\hat{\beta})}$, and $\hat{\sigma} \pm \mathcal{Z}_{\frac{k}{2}}\sqrt{Var(\hat{\sigma})}$, where $\hat{\Theta} = (\hat{\alpha}, \hat{\beta}, \hat{\sigma})$ and $\mathcal{Z}_k$ is the upper 100 k-th percentile of the standard normal distribution.

4.4. Extreme value properties

Here we study some of the interesting extreme value theoretic properties. The concept of regular variation is an important notion of extreme value theory. Below we show the extreme value results for the MLM distribution that can characterize the asymptotic behavior of extremes along with well grounded statistical theory.

Definition 8. (Maximum domain of attraction) A function F is said to be regularly varying at infinity, if for every $t > 0$,

$$\lim_{x\to\infty} \frac{1-F(tx)}{1-F(x)} = t^{-\alpha};\ \alpha > 0.$$

Then we say that F is a function with regularly varying tails with $\alpha > 0$ as the tail index and F belongs to the maximum domain of attraction (MDA) of the Frechet distribution with index α.

Theorem 3. The CDF (Eqn. 3) of the MLM distribution is a function with regularly varying tails and it belongs to MDA of the Frechet distribution with index α.

Proof.

$$1 - F(tx) = \exp\left[-\alpha.\frac{\log^{\beta+1}\left(1+\frac{tx}{\sigma}\right)}{\left(1+\log\left(1+\frac{tx}{\sigma}\right)\right)^{\beta}}\right];\ t > 0 \tag{27}$$

Now, we have (using expansions of $\log(1-x)$ and $\exp(x)$):

$$\begin{aligned}
\left(\frac{\log\left(1+\frac{tx}{\sigma}\right)}{1+\log\left(1+\frac{x}{\sigma}\right)}\right)^{\beta} &= \left(1 - \frac{1}{1+\log\left(1+\frac{tx}{\sigma}\right)}\right)^{\beta} \\
&= \exp\left[\beta\log\left(1-\frac{1}{1+\log\left(1+\frac{tx}{\sigma}\right)}\right)\right] \\
&= \exp\left[\beta\left(-\frac{1}{\log\left(1+\frac{tx}{\sigma}\right)} + O\left(\frac{1}{\log^2\left(1+\frac{tx}{\sigma}\right)}\right)\right)\right] \\
&= 1 - \frac{\beta}{\log\left(1+\frac{tx}{\sigma}\right)} + O\left(\frac{\beta}{\log^2\left(1+\frac{tx}{\sigma}\right)}\right)
\end{aligned} \tag{28}$$

Using Eqn. (27) and (28) together, we get

$$1 - F(tx) = \exp\left[-\alpha\log\left(1+\frac{tx}{\sigma}\right)\left\{1 - \frac{\beta}{\log\left(1+\frac{tx}{\sigma}\right)} + O\left(\frac{\beta}{\log^2\left(1+\frac{tx}{\sigma}\right)}\right)\right\}\right] \tag{29}$$

Similarly for $t = 1$, Eqn. (29) becomes

$$1 - F(x) = \exp\left[-\alpha\log\left(1+\frac{x}{\sigma}\right)\left\{1 - \frac{\beta}{\log\left(1+\frac{x}{\sigma}\right)} + O\left(\frac{\beta}{\log^2\left(1+\frac{x}{\sigma}\right)}\right)\right\}\right] \tag{30}$$

Now,

$$\begin{aligned}
\lim_{x\to\infty}\frac{1-F(tx)}{1-F(x)} &= \lim_{x\to\infty}\exp\left[-\alpha\log\left(\frac{1+\frac{tx}{\sigma}}{1+\frac{x}{\sigma}}\right) + O\left(\frac{1}{\log^2\left(1+\frac{tx}{\sigma}\right)} + \frac{1}{\log^2\left(1+\frac{x}{\sigma}\right)}\right)\right] \\
&= \exp\left(-\alpha\log t\right) \\
&= t^{-\alpha}.
\end{aligned}$$

Thus, $F \in MDA(\Phi_\alpha)$. $\square$

Now we study the tail-equivalent and heavy-tailed behaviour of the proposed MLM distribution as follows:

Definition 9. (Tail-equivalent) Two distributions F and G are said to be tail-equivalent if

$$\lim_{x\to\infty}\frac{1-F(x)}{1-G(x)} = c;\ 0 < c < \infty.$$

Theorem 4. The $MLM(\alpha, \beta, \sigma)$ distribution, defined in Eqn. (3), is right tail-equivalent to the power-law distribution.

Proof. Let $G(x)$ be the CDF of the power-law distribution, i.e.,

$$1 - G(x) = \left(1 + \frac{x}{\sigma}\right)^{-\alpha}$$

and $F(x)$ is the CDF of MLM distribution as given in Eqn.(3). Then,

$$\begin{aligned}
\lim_{x\to\infty} \frac{1-F(x)}{1-G(x)} &= \lim_{x\to\infty} \frac{\exp\left[-\alpha\log\left(1+\frac{x}{\sigma}\right) + \alpha\beta + O\left(\frac{1}{\log\left(1+\frac{x}{\sigma}\right)}\right)\right]}{\exp\left[-\alpha\log\left(1+\frac{x}{\sigma}\right)\right]} \quad \text{(Using Eqn. (30))} \\
&= \lim_{x\to\infty} \exp\left[\alpha\beta + O\left(\frac{1}{\log\left(1+\frac{x}{\sigma}\right)}\right)\right] \\
&= \exp\left(\alpha\beta\right) \\
&= c < \infty.
\end{aligned}$$

□

Definition 10. (Heavy-tailed distribution) A distribution function F is heavy-tailed if

$$\lim_{x\to\infty} \exp\{\lambda x\}\big(1 - F(x)\big) = \infty, \text{ for any } \lambda > 0.$$

Theorem 5. The $MLM(\alpha, \beta, \sigma)$ distributions, defined in Eqn. (3), are heavy-tailed distributions.

Proof.

$$\begin{aligned}
\lim_{x\to\infty} \exp\{\lambda x\}\big(1 - F(x)\big) &= \lim_{x\to\infty} \exp\left[\lambda x - \alpha\log\left(1 + \frac{x}{\sigma}\right) + \alpha\beta + O\left(\frac{1}{\log\left(1+\frac{x}{\sigma}\right)}\right)\right] \\
&= \infty,
\end{aligned}$$

since $\log\left(1+\frac{x}{\sigma}\right) \asymp x^{\epsilon}$ for any $\epsilon > 0$ and for sufficiently large x. □

There are two other important class of distributions [44] viz. the class $\mathbb{D}$ of dominated-variation distributions and and the class $\mathbb{L}$ of long-tailed distributions that are used in the risk theory and queueing theory. The proposed MLM distributions also follows these two properties.

Definition 11. A distribution F belong to the class $\mathbb{D}$ of dominated-variation distributions if

$$\limsup_{x\to\infty} \frac{1-F(x)}{1-F(2x)} < \infty.$$

Theorem 6. If $\alpha > 0$, then $MLM(\alpha, \beta, \sigma)$ distribution, defined in Eqn. (3), belongs to the class $\mathbb{D}$ of dominated-variation distributions.

Proof.

$$\lim_{x\to\infty}\frac{1-F(x)}{1-F(2x)} = \lim_{x\to\infty}\frac{\exp\left[-\alpha\log\left(1+\frac{x}{\sigma}\right)+\alpha\beta+O\left(\frac{1}{\log\left(1+\frac{x}{\sigma}\right)}\right)\right]}{\exp\left[-\alpha\log\left(1+\frac{2x}{\sigma}\right)+\alpha\beta+O\left(\frac{1}{\log\left(1+\frac{2x}{\sigma}\right)}\right)\right]}$$

$$= \lim_{x\to\infty}\exp\left[\alpha\log\left(\frac{1+\frac{2x}{\sigma}}{1+\frac{x}{\sigma}}\right)+O\left(\frac{1}{\log\left(1+\frac{x}{\sigma}\right)}+\frac{1}{\log\left(1+\frac{2x}{\sigma}\right)}\right)\right]$$

$$= \exp\left(\alpha\log 2\right)$$

$$= 2^{\alpha} < \infty.$$

where $\alpha > 0$. □

Definition 12. A distribution F is said to belong to the class $\mathbb{L}$ of long-tailed distributions if F has right unbounded support and, for any fixed $y > 0$,

$$\lim_{x\to\infty}\frac{1-F(x+y)}{1-F(x)} = 1.$$

Theorem 7. The $MLM(\alpha, \beta, \sigma)$ distribution, defined in Eqn. (3), belongs to the class $\mathbb{L}$ of long-tailed distributions.

Proof.

$$\lim_{x\to\infty}\frac{1-F(x+y)}{1-F(x)} = \lim_{x\to\infty}\exp\left[-\alpha\log\left(1+\frac{(y/\sigma)}{(1+\frac{x}{\sigma})}\right)+O\left(\frac{1}{\log\left(1+\frac{x}{\sigma}\right)}+\frac{1}{\log\left(1+\frac{x+y}{\sigma}\right)}\right)\right]$$

$$= 1, \quad \text{since} \quad \sigma > 0.$$

□

We have shown that the proposed MLM distributions are heavy-tailed and also possess the additional regularity property of subexponentiality [45] as given below. Essentially this corresponds to good tail behaviour under the operation of convolution.

Definition 13. (Subexponential distribution) We say that a distribution F is subexponential if

$$\lim_{x\to\infty}\frac{1-F*F(x)}{1-F(x)} = 2,$$

where $*$ denotes the convolution operation.

Theorem 8. The $MLM(\alpha, \beta, \sigma)$ distribution, defined in Eqn. (3), is subexponential.

Proof. Form Theorem 6 and Theorem 7, the $MLM(\alpha, \beta, \sigma)$ distribution belongs to $\mathbb{D}\cap\mathbb{L}$. Using [46], $\mathbb{D}\cap\mathbb{L}\subset\mathbb{S}$, where S is the class of subsexponetial distribution. Hence the theorem. □

Definition 14. (Von-Mises type function) A distribution function F is called a Von-Mises type function if

$$\lim_{x\uparrow r(F)} x\frac{d}{dx}\left[\frac{1-F(x)}{xf(x)}\right] = 0,$$

where $r(F) = \sup\{x : F(x) < 1\}$ denotes the right extremity of the distribution function F [44].

Theorem 9. The $MLM(\alpha, \beta, \sigma)$ distribution, defined in Eqn. (3), satisfies the Von-Mises condition.

Proof.

$$
\begin{aligned}
\lim_{x\to\infty} x\frac{d}{dx}\left[\frac{1-F(x)}{xf(x)}\right] &= \frac{\alpha}{\sigma}\lim_{x\to\infty} x\frac{d}{dx}\left[\frac{\left(1+\frac{x}{\sigma}\right)\left[1+\log\left(1+\frac{x}{\sigma}\right)\right]^{\beta+1}}{x\left[\beta+1+\log\left(1+\frac{x}{\sigma}\right)\right]\left[\log\left(1+\frac{x}{\sigma}\right)\right]^{\beta}}\right] \\
&= \frac{\alpha}{\sigma}\lim_{x\to\infty}\frac{\left[1+\log\left(1+\frac{x}{\sigma}\right)\right]^{\beta+1}}{\left[\beta+1+\log\left(1+\frac{x}{\sigma}\right)\right]\left[\log\left(1+\frac{x}{\sigma}\right)\right]^{\beta}} \\
&\times\left[-\frac{1}{x}+\frac{1+\beta}{\sigma\left(1+\log\left(1+\frac{x}{\sigma}\right)\right)}-\frac{1}{\sigma\left(1+\beta+\log\left(1+\frac{x}{\sigma}\right)\right)}-\frac{\beta}{\sigma\log\left(1+\frac{x}{\sigma}\right)}\right] \\
&= 0.
\end{aligned}
$$

□

5. Experimental Analysis

5.1. Description of data sets

We present here the results of fitting modified Lomax (MLM) distribution over 50 real-worlld complex networks [47, 48] coming from broad variety of different disciplines such as Social Networks, Collaboration Networks, Communication Networks, Citation Networks, Temporal Networks, Web Graphs, Product co-purchasing Networks, Biological Networks, Brain Networks, etc. Please go through the supplementary materials for more details about the data sets under consideration. Some statistical measures of the data sets and the detailed experimentation of the performances of the proposed MLM distribution compared to the other common power-law related distribution such as Lomax, Pareto, Log-normal, power-law cutoff, Exponential and Poisson are discussed in the following sub sections.

5.2. Performance measures

Here we use some evaluation measures which justify that the degree distribution of a real-world complex network can plausibly been drawn from the proposed MLM distribution. As here the actual distribution is discrete, we can quantify the goodness-of-fit test (i.e., how closely a hypothesized distribution resembles the actual distribution) by calculating the Chi-square statistic value based on bootstrap resampling by generating 50000 synthetic data sets. The Chi-square test will return a p value which quantifies the probability that our data were drawn from the hypothesized distribution. If the p value is small (less than the significance level), we can reject the null hypothesis that the data come from the MLM distribution. We have also computed few other statistical measures such as KL-divergence, RMSE and MAE for quantifying the goodness-of-fit of the proposed MLM distribution model in comparison to the other standard distribution functions related to other heavy-tailed distributions.

5.3. Analysis of results

Table 1 represents some of the statistical measures corresponding to the network data and also provides the statistical evidences of the proposed fitting over the node degree distribution in the whole range using MLM distribution. CV is also calculated corresponding to each of the degree distribution data and it gives us the sufficient condition for the existence of the global maximum at finite point of the $MLM(\alpha, \beta, \sigma)$ distribution. From Table 1 it is clear that that the value of CV is greater than 1 in all the network data sets under consideration. Thus it confirms that the maximum likelihood estimates for the parameters (α, β, σ) of the proposed MLM distribution attain at the finite points which has been theoretically described in Section 44.1. To estimate the parameters (α, β, σ) of the MLM distribution numerically, we have used "optim" function along with the quasi-Newton L-BFGS-B algorithm in R statistical software by taking the initial parameters value $(\alpha, \beta, \sigma) = (1, 0, 1)$. The estimated values of the parameters for all the data sets satisfied

Table 1: Performance of the proposed MLM model over different real-worlld networks

Data sets		No. of nodes	No. of edges	Stat. Prop.			Estimated parameters			Bootstrap chi-square value
				s	μ	$\frac{s}{\mu}$	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\sigma}$	(p)
Social Networks	ego-Twitter(In)	81,306	1,768,149	57.965	21.747	2.6654	1.9922	-0.3591	30.543	0.9920
	ego-Gplus(In)	107,614	13,673,453	1404.8	283.42	4.9568	0.7108	-0.4983	23.077	0.9963
	soc-Slashdot	70,068	358,647	35.069	10.237	3.426	0.8663	-0.6228	1.0461	0.9955
	soc-Delicious(In)	536,108	1,365,961	39.826	10.673	3.7312	1.3630	-0.6819	5.3709	0.9960
	soc-Digg(In)	770,799	5,907,132	166.61	46.584	3.5765	0.7931	-0.6928	5.5163	0.9890
	soc-Academia	200,169	1,398,063	48.297	14.259	3.3871	2.7429	-0.3737	36.644	0.6087
	LiveJournal(In)	4,847,571	68,993,773	44.969	15.368	2.926	2.6892	-0.7272	51.933	0.8983
	Dogster-Friendship	426,821	8,546,581	284.06	40.033	7.095	1.5634	0.3108	14.057	0.9500
	Higgs-Twitter(In)	456,626	14,855,842	350.91	54.786	6.4051	1.6797	-0.0347	36.204	0.9870
	Artist-Facebook	50,615	819,307	63.427	32.366	1.9596	2.0117	-0.1445	39.337	0.9812
	Athletes-Facebook	13,866	86,859	17.978	12.438	1.4453	3.1229	0.1406	21.180	0.9640
Citation Networks	cit-HepTh(In)	27,770	352,807	43.139	15.220	2.8342	1.8410	-0.3093	16.416	0.8730
	cit-HepPh(In)	34,546	421,578	27.286	14.933	1.8271	2.5553	-0.3622	34.349	0.9900
	cit-Patents(In)	3,774,768	16,518,948	6.9125	5.0687	1.3637	4.4822	-0.2534	21.689	0.8080
	cit-Citeseer(In)	227,320	814,134	9.8260	5.4322	1.8088	2.2630	-0.2788	7.4150	0.6350
Collaboration Networks	ca-CondMat	23,133	93,497	10.671	8.0189	1.3308	3.1068	0.3615	10.5353	0.9896
	ca-AstroPh	18,772	198,110	30.568	21.103	1.4484	16.434	37.276	0.0101	0.9990
	ca-GrQc	5,242	14,496	7.9186	5.5284	1.4322	2.2624	3.5861	0.6765	0.7849
	ca-HepPh	12,008	118,521	46.654	19.696	2.3687	0.9798	2.8780	0.6791	0.8163
	ca-HepTh	9,877	25,998	6.1867	5.2618	1.1757	2.9417	5.2791	0.4825	0.9332
Web Graphs	Google(In)	875,713	5,105,039	43.320	7.1444	6.0634	1.1999	-0.6399	2.0429	0.9780
	BerkStan(In)	685,230	7,600,595	300.08	12.316	24.364	1.4129	1.8449	0.7592	0.6250
	Wikipedia2009(In)	1,864,433	4,507,315	12.846	4.8903	2.6268	1.3988	-0.6291	1.9658	0.9891
	WikipediaLinkFr(In)	4,906,478	113,122,279	1864.4	48.608	38.356	1.0988	-0.7123	9.8888	0.9152
	Hudong(In)	1,984,484	14,869,483	199.28	16.467	12.101	1.1567	10.921	0.0013	0.9883
Biological Networks	Yeast-PPIN	2,361	7,182	8.0800	6.0838	1.3281	10.535	-0.4527	175.29	0.9930
	Diseasome	3,926	7,823	9.1009	5.5334	1.6447	10.9688	-0.9493	134.52	0.8090
	Bio-Mouse-Gene	45,101	14,506,199	856.67	643.27	1.3317	6.3e-08	-1.2e-02	2.1e+00	0.9898
	Bio-Dmela	7,393	25569	10.782	6.9170	1.5587	14.979	-0.5053	498.27	0.9806
	Bio-WormNet-v3	16,347	762,822	138.17	93.328	1.4805	5.6496	-0.9801	704.71	0.9938
Product co-purchasing networks	amazon0601(In)	403,394	3,387,388	15.279	8.3989	1.8191	3.8261	-.7137	19.522	0.6010
	amazon0505(In)	410,236	3,356,828	15.313	8.1826	1.8714	3.8367	-0.8006	19.984	0.6880
	amazon0312(In)	400,727	3,200,444	15.073	7.9865	1.8873	3.7631	-0.8179	18.747	0.5890
Temporal Networks	sx-mathoverflow(In)	24,818	506,550	31.476	10.424	3.0195	1.4452	2.4236	0.8241	0.9846
	sx-stackoverflow(In)	2,601,977	63,497,050	186.00	27.647	6.7278	1.0218	-0.8224	4.4865	0.9490
	sx-superuser(In)	194,085	1,443,339	23.782	5.8239	4.0836	1.7401	2.1405	0.7284	0.9780
	sx-askubuntu (In)	159,316	964,437	18.404	4.3856	4.1966	2.1923	2.2069	0.7665	0.9300
Communication Networks	Email-Enron	36,692	183,831	36.100	10.021	3.6027	1.2417	-0.1275	2.9045	0.9641
	Wiki-Talk(In)	2,394,385	5,021,410	12.259	2.1195	5.7844	1.5167	-0.2846	0.0016	0.9900
	Rec-Libimseti(In)	220,970	17,359,346	413.71	102.85	4.0227	2.5008	-0.8496	331.18	0.9670
Ground-truth Networks	Wiki-Topcats	1,791,489	28,511,807	283.78	15.915	17.831	1.1811	0.1998	2.6412	0.8310
	com-Friendster	65,608,366	1,806,067,135	137.81	55.056	2.5031	4.5863	-0.9188	590.01	0.9000
	com-LiveJournal	3,997,962	34,681,189	42.957	17.349	2.4759	2.8206	-0.6020	65.638	0.7980
	com-Orkut	3,072,441	117,185,083	154.78	76.281	2.0291	3.7049	0.1292	167.93	0.9890
	com-Youtube	1,134,890	2,987,624	50.754	5.2650	9.6398	1.6113	8.3355	0.0094	0.8410
Brain Networks	Human25890-session1	177,584	15,669,036	319.01	176.47	1.8078	1.6098	-0.2076	168.75	0.8710
	Human25890-session2	723,881	158,147,409	667.91	436.94	1.5286	14.423	-0.3466	18886.3	0.9980
	Human25864-session2	692,957	133,727,516	554.48	385.96	1.4366	16.250	-0.3379	19217.8	0.9660
	Human25913-session2	726,197	183,978,766	446.92	258.99	1.7256	7.1013	-0.4681	5779.8	0.9290
	Human25886-session1	780,185	158,184,747	558.41	405.50	1.3771	21.591	-0.3119	26975.9	0.9768

the condition, i.e., ($\alpha > 0,\ \beta > -1$ and $\sigma > 0$) as clearly seen in Table 1, for the complete characterization of the proposed MLM distribution. Empirically it is observed that in almost all the cases the estimated value of the parameter σ attains the higher values as compared to the estimated value of α. On the other hand, the estimated value of the parameter β lies between $(0, 1)$ lies between -1 and 1 except a few which can be clearly seen from Table 1.

Furthermore, we leverage one of the popular statistical method viz. bootstrapping chi-square test to evaluate the goodness-of-fit test of the proposed MLM distribution. From Table 1, it is clear that the proposed MLM distribution produces higher p values (i.e. closure to 1) in almost all the data sets which suggest that the null hypothesis i.e. the data drawn from MLM distribution cannot be ruled out at the 0.05 level of significance. This indicates that the observed degree distribution is plausibly drawn from the MLM distribution. Thus from Table 1 it can be concluded that the proposed MLM distribution is effective in modeling the entire degree distribution of real-worlld complex networks without ignoring some of the lower degree nodes as oppose to the procedure of fitting power law distribution. In addition, we also used some other statical measures viz. KLD, RMSE and MAE in order to compare the performance of the proposed MLM distribution with the each of the other common power-law related distributions as given in the following Tables 2 and 3.

Tables 2 and 3 depict the values of different statistical measures (viz. RMSE, MAE and KLD) which has been used for the measure of performances of the MLM distribution in comparison to the competitive distributions while modeling the data. RMSE and MAE are two different variants, carrying information about the differences between actual and predicted degree frequencies corresponding to a network. Higher similarity between actual and mapped distributions is achieved by generating smaller values of RMSE and

Table 2: Table of different statistical measures of different competitive models over real-worlld networks

Data sets		MLM			Lomax			Power-law			Pareto		
		RMSE	KLD	MAE	RMSE	KLD	MAE	RMSE	KLD	MAE	RMSE	KLD	MAE
Social Networks	ego-Twitter(In)	16.800	0.00819	1.3498	29.366	0.01354	2.4701	204.35	0.1831	10.847	354.25	0.2857	15.603
	ego-Gplus(In)	1.6115	0.05601	0.1825	10.491	0.06444	0.3033	53.064	0.2299	0.9221	86.955	0.3113	1.1847
	soc-Slashdot	31.527	0.01365	2.3951	32.065	0.014102	2.4658	247.87	0.1007	10.074	247.84	0.1007	10.073
	soc-Delicious(In)	79.809	0.00839	3.7730	91.993	0.01326	4.8060	349.66	0.2021	14.867	471.02	0.1349	17.874
	soc-Digg(In)	13.634	0.02182	0.8440	24.841	0.02391	1.0269	208.01	0.1601	4.2185	212.87	0.1601	4.2312
	soc-Academia	16.323	0.00351	0.5705	48.951	0.01019	1.6178	229.54	0.2027	6.3889	440.15	0.274	10.464
	LiveJournal(In)	243.99	6.13e-04	5.4026	1764.9	0.02111	54.400	5025.2	0.1614	127.98	8100.9	0.1785	164.18
	Dogster-Friendship	32.203	0.01328	0.8502	36.449	0.01700	1.0755	358.27	0.2926	5.6815	549.57	0.4618	7.6734
	Higgs-Twitter(In)	19.821	0.00785	0.4710	20.609	0.00793	0.4621	260.32	0.2492	4.8938	524.96	0.4806	7.6938
	Artist-Facebook	11.708	0.01079	2.1381	12.923	0.01199	2.6173	100.49	0.1643	14.552	350.05	0.4010	26.467
	Athletes-Facebook	4.4304	0.00879	1.3252	9.2379	0.00966	1.8260	100.16	0.2049	13.387	204.91	0.4164	23.839
Citation Networks	cit-HepTh(In)	3.2640	0.01354	0.5071	7.9393	0.01585	0.7585	73.531	0.1741	4.0821	122.79	0.2566	5.997
	cit-HepPh(In)	9.6810	0.00821	1.9016	21.303	0.01317	3.1135	128.55	0.1825	13.234	257.41	0.2689	21.445
	cit-Patents(In)	445.80	1.61e-04	47.603	2577.5	0.00192	230.35	27.5K	0.2266	2049.5	34.8K	0.2366	2533.2
	cit-Citeseer(In)	40.728	0.00228	3.3778	28.032	0.00278	3.3902	889.88	0.3308	49.467	1156.2	0.2916	62.026
Collaboration Networks	ca-CondMat	14.830	0.00479	4.1570	36.094	0.00814	7.3904	107.86	0.1025	26.092	469.12	0.3738	63.075
	ca-AstroPh	23.890	0.02756	5.9796	32.799	0.03457	7.4448	92.255	0.1753	15.158	251.03	0.3816	27.707
	ca-GrQc	15.850	0.03055	7.2247	35.935	0.04013	12.286	124.24	0.2554	27.137	202.33	0.2741	44.221
	ca-HepPh	13.944	0.06959	4.1919	19.607	0.07266	4.7763	75.071	0.1769	8.0906	144.39	0.2569	14.668
	ca-HepTh	23.280	0.00896	10.851	61.797	0.01353	20.391	268.91	0.2346	66.108	437.19	0.2829	106.36
Web Graphs	Google(In)	360.62	0.01368	13.845	337.68	0.01546	14.201	1809.1	0.124	45.023	1809.2	0.124	45.023
	BerkStan(In)	71.819	0.03116	0.9478	105.20	0.0346	1.1962	615.03	0.1863	4.0722	615.01	0.1863	4.0721
	Wikipedia2009(In)	86.510	0.00169	7.7498	103.94	0.00197	8.5289	4371.9	0.1352	164.58	4371.9	0.1352	164.58
	WikipediaLinkFr(In)	124.98	0.01776	0.3174	146.14	0.03082	0.4465	248.09	0.1518	0.7857	397.39	0.1521	1.0815
	Hudong(In)	8.0517	0.00433	0.2508	25.163	0.00525	0.4600	587.21	0.0868	4.6828	587.22	0.0868	4.6828
Biological Networks	Yeast-PPIN	4.3766	0.01487	2.6321	12.651	0.02389	5.4529	75.325	0.1999	19.013	77.455	0.1998	19.079
	Diseasome	8.8683	0.08000	2.7445	12.451	0.10202	3.4575	26.006	0.2248	5.3567	26.005	0.2248	5.3566
	Bio-Mouse-Gene	7.6919	0.18941	2.2557	14.654	0.19473	2.3919	41.371	0.4566	3.9018	92.539	0.5373	4.7342
	Bio-Dmela	16.173	0.01305	4.0968	10.579	0.01759	3.8219	143.71	0.1907	21.415	143.67	0.1907	21.414
	Bio-WormNet-v3	14.054	0.04648	2.6066	13.018	0.09249	3.7468	46.259	0.2761	6.8867	101.89	0.3744	9.0163
Product co-purchasing networks	amazon0601(In)	94.347	0.00374	8.1863	147.602	0.00695	10.928	1495.4	0.2708	70.281	2539.8	0.4022	114.59
	amazon0505(In)	109.95	0.00412	9.1836	94.048	0.00499	8.7882	1572.9	0.2463	73.003	2494.5	0.3711	111.56
	amazon0312(In)	100.89	0.00430	8.5465	92.525	0.00495	8.5742	1564.4	0.2425	71.875	2462.9	0.3686	109.21
Temporal Networks	sx-mathoverflow(In)	19.706	0.01879	2.4647	38.764	0.02621	3.7877	213.91	0.2131	13.600	213.82	0.2132	13.612
	sx-stackoverflow(In)	39.654	0.00336	0.8694	62.254	0.00345	1.0741	1877.5	0.2016	14.007	1884.7	0.2017	14.017
	sx-superuser(In)	79.777	0.00654	4.5409	136.85	0.01045	6.8313	900.04	0.1808	33.837	900.33	0.1808	33.839
	sx-askubuntu(In)	106.04	0.01100	6.2022	176.58	0.01707	9.3509	949.66	0.2091	39.419	949.73	0.2091	39.420
Communication Networks	Email-Enron	74.667	0.03523	5.2075	76.155	0.03531	5.2347	246.51	0.1779	14.886	245.25	0.1778	14.859
	Wiki-Talk(In)	670.47	0.00356	25.871	671.76	0.00357	25.898	9669.4	0.3376	293.63	9669.4	0.3376	293.63
	Rec-Libimseti(In)	23.341	0.02163	0.4953	66.434	0.09978	1.7923	77.081	0.2198	2.1486	133.91	0.2096	2.7441
Ground-truth Networks	Wiki-Topcats	11.375	0.00190	0.1011	14.955	0.00201	0.1347	565.21	0.1377	2.6145	930.44	0.1612	3.8073
	com-Friendster	8266.8	0.00126	411.15	41.69K	0.06401	3385.2	71.5K	0.1498	4575.6	129K	0.1498	5591.7
	com-LiveJournal	165.79	0.00084	6.9832	1741.3	0.02462	50.318	4102.9	0.1823	106.85	7116.4	0.2147	150.46
	com-Orkut	197.89	0.00793	7.0113	207.43	0.01049	9.9761	2443.6	0.5498	80.712	4299.3	0.8033	101.64
	com-Youtube	53.288	0.00122	0.6984	81.409	0.00175	1.0862	1380.5	0.1342	15.690	1380.5	0.1342	15.691
Brain Networks	Human25890-session1	17.309	0.01920	3.7439	19.545	0.02264	4.0023	305.41	0.3397	22.537	588.71	0.5598	30.333
	Human25890-session2	46.276	0.01024	5.8781	97.122	0.04774	15.303	794.95	0.4462	50.379	1623.8	0.6513	63.754
	Human25864-session2	64.037	0.01321	9.8670	111.45	0.05335	20.876	1120.3	0.4967	68.172	1711.3	0.6419	78.736
	Human25913-session2	112.661	0.01347	11.574	119.54	0.04719	20.971	904.76	0.2764	58.999	1892.4	0.4566	81.206
	Human25886-session1	65.181	0.01471	12.051	116.78	0.05476	23.396	978.66	0.4664	69.517	1873.6	0.6805	86.597

MAE. From Tables 2 and 3, it is clear that the proposed MLM distribution provides smaller RMSE and MAE values compared to other competitive distributions in almost all the networks except a few where the power-law cutoff distribution outperforms the others. The worst performance observed for the poisson distribution in minimizing the RMSE and MAE values compared to the other competing distributions over all the real-worlld networks as clearly seen from Table 3. The Kullback-Leibler divergence (KLD), or relative entropy, is a quantity which measures the dissimilarity between two probability distributions. Thus the smaller value of KLD represents the higher similarity between the actual and the predicted distribution. From Tables 2 and 3 it is clear that the proposed MLM distribution generates smaller KLD values compared to other competitive distributions in almost all the networks except a few where power-law cutoff distribution outperforms the others. This indicates that the observed degree distribution satisfactorily matches the proposed MLM distribution in almost all the networks. Note that, in terms of KLD, the Poisson and Exponential distributions always perform worse than the others in all the networks as in the case RMSE and MAE. The performance of the proposed MLM distribution is always superior to the competitive in terms of KLD over almost all the networks. Thus overall, by considering RMSE, MAE and KLD values, the performance of the proposed MLM distribution for all the networks is found to be better than the other competing distributions which suggest that the observed distribution plausibly comes from the proposed MLM distribution.

Table 3: Table of different statistical measures of different competitive models over real-worlld networks

Data sets		Log-normal			Poisson			Power-law Cutoff			Exponential		
		RMSE	KLD	MAE	RMSE	KLD	MAE	RMSE	KLD	MAE	RMSE	KLD	MAE
Social Networks	ego-Twitter(In)	53.863	0.0169	2.9494	410.93	10.452	36.645	68.004	0.0397	4.1974	157.98	0.2733	11.567
	ego-Gplus(In)	10.155	0.0678	0.2523	95.967	25.317	3.0371	30.925	0.1475	0.6821	50.098	1.3131	1.8328
	soc-Slashdot	237.63	0.1058	10.549	684.36	10.069	42.407	19.598	0.0075	1.3599	434.25	0.6381	22.275
	soc-Delicious(In)	281.34	0.0579	10.781	957.82	6.8432	56.634	66.896	0.0185	4.2366	535.11	0.4626	25.304
	soc-Digg(In)	69.438	0.0552	1.9087	323.59	21.541	15.134	65.713	0.0441	1.9042	204.50	0.8907	8.0015
	soc-Academia	91.003	0.0169	2.0921	542.38	7.2349	22.153	62.376	0.0255	1.9845	198.11	0.1924	6.6739
	LiveJournal(In)	3473.6	0.0355	70.64	13.61K	9.1120	481.38	808.79	0.0101	24.501	7017.9	0.3449	186.01
	Dogster-Friendship	42.539	0.0272	1.1459	494.49	14.575	15.309	182.47	0.1862	4.1579	165.19	0.4765	5.4623
	Higgs-Twitter(In)	41.955	0.0134	0.5995	448.91	16.309	14.753	118.23	0.0914	2.8051	134.68	0.3163	4.4689
	Artist-Facebook	24.071	0.0154	3.026	323.46	12.537	53.920	56.801	0.0458	6.6452	88.351	0.1799	13.623
	Athletes-Facebook	15.461	0.0127	2.5674	175.95	4.7428	35.815	25.099	0.0324	4.3180	28.388	0.0586	6.2610
Citation Networks	cit-HepTh(In)	22.59	0.0255	2.331	153.39	8.0679	13.774	25.42	0.0464	2.816	58.74	0.2778	4.5286
	cit-HepPh(In)	44.951	0.0189	4.4405	303.32	7.9234	46.775	36.887	0.0221	4.5287	107.32	0.1801	14.145
	cit-Patents(In)	9612.7	0.0192	725.71	38.2K	1.6549	3657.1	2424.5	0.0061	271.89	13.2K	0.0659	1147.5
	cit-Citeseer(In)	353.26	0.0299	21.921	1507.6	2.566	109.15	195.67	0.0131	13.877	629.02	0.1486	44.301
Collaboration Networks	ca-CondMat	42.665	0.0082	6.5746	378.65	2.7263	80.781	62.929	0.0287	13.362	64.985	0.0472	16.873
	ca-AstroPh	28.209	0.0312	6.8565	229.81	9.8703	55.51	50.604	0.0384	7.4579	68.185	0.1235	14.361
	ca-GrQc	30.184	0.0515	12.148	193.94	2.3256	63.61	58.305	0.0659	18.259	69.169	0.1418	25.759
	ca-HepPh	29.993	0.1011	6.8958	185.48	11.609	39.673	50.717	0.1128	8.2477	89.936	0.5187	17.589
	ca-HepTh	55.618	0.0178	21.032	370.15	1.5051	133.68	89.425	0.0245	27.613	109.96	0.0551	43.882
Web Graphs	Google(In)	1514.5	0.0878	40.067	4442.6	4.712	154.92	188.01	0.0157	9.6549	2589.4	0.4419	76.441
	BerkStan(In)	185.04	0.1002	2.0198	993.01	7.0379	11.9628	322.63	0.1037	2.8203	595.53	0.7438	6.6185
	Wikipedia2009(In)	2720.9	0.0798	116.43	8425.7	3.6475	398.72	781.87	0.0082	35.531	4727.1	0.3431	213.66
	WikipediaLinkFr(In)	240.18	0.0543	0.5234	762.72	25.726	4.0278	121.31	0.0622	0.5006	534.61	1.0217	2.0471
	Hudong(In)	746.47	0.1593	6.5493	1975.73	11.088	25.837	75.362	0.0063	0.8323	1494.8	1.1798	15.836
Biological Networks	Yeast-PPIN	29.928	0.0496	9.3869	109.62	2.5149	39.181	4.9595	0.0175	2.9178	45.462	0.1234	13.786
	Diseasome	23.282	0.1552	4.8906	55.985	3.0101	12.001	9.3332	0.0822	2.8587	31.709	0.2979	5.7013
	Bio-Mouse-Gene	17.199	0.1878	2.5372	101.23	15.318	10.254	9.277	0.0943	1.6036	31.649	0.4882	3.6376
	Bio-Dmela	46.271	0.0426	9.2857	206.44	3.6221	45.991	24.091	0.0162	5.0541	86.659	0.1724	18.048
	Bio-WormNet-v3	17.726	0.0851	3.9352	104.46	18.563	21.927	6.7764	0.0419	2.2424	40.795	0.3082	7.0826
Product co-purchasing networks	amazon0601(In)	286.61	0.0102	16.881	2064.6	2.7267	140.46	297.39	0.0382	22.199	308.32	0.0574	24.114
	amazon0505(In)	358.59	0.0125	19.123	2172.5	3.0551	144.34	260.85	0.0342	20.136	390.13	0.0628	26.178
	amazon0312(In)	338.03	0.0116	17.742	2131.9	2.6839	140.75	273.39	0.0352	20.381	383.82	0.0639	26.299
Temporal Networks	sx-mathoverflow(In)	41.934	0.0634	5.3161	281.78	8.0773	32.868	92.603	0.0861	7.9912	129.69	0.4636	15.172
	sx-stackoverflow(In)	341.96	0.0286	4.4267	2469.6	18.054	42.111	740.18	0.0685	7.2275	1324.4	0.6829	19.362
	sx-superuser(In)	243.42	0.0616	13.199	1246.2	3.8103	68.010	354.72	0.0570	16.613	609.40	0.3891	34.655
	sx-askubuntu(In)	212.91	0.0649	12.451	1228.7	2.6253	68.973	389.14	0.0719	20.113	555.44	0.3433	33.693
Communication Networks	Email-Enron	121.47	0.0873	8.445	426.39	6.8601	38.373	95.468	0.0689	7.664	230.41	0.5405	18.139
	Wiki-Talk(In)	7978.6	0.1902	246.26	21.9K	1.2506	646.54	672.32	0.0036	25.905	16.5K	0.4879	542.31
	Rec-Libimseti(In)	87.472	0.0755	1.4021	281.18	30.222	8.0019	28.059	0.0359	0.6971	166.18	0.8547	3.9402
Ground-truth Networks	Wiki-Topcats	272.99	0.0464	1.5159	1477.2	8.7468	12.121	389.86	0.0629	2.2289	832.23	0.6767	5.8936
	com-Friendster	101K	0.0762	4022.8	280K	24.658	22.1K	17.8K	0.0052	1025.5	193K	0.7216	10.1K
	com-LiveJournal	2629.9	0.0299	51.656	10.9K	9.5778	401.89	497.89	0.0104	18.559	5230.3	0.2889	139.74
	com-Orkut	452.92	0.0459	19.624	3118.1	11.839	135.17	261.75	0.0479	16.197	228.83	0.0599	14.496
	com-Youtube	1422.2	0.1416	17.219	3838.9	3.4522	51.118	143.79	0.0045	2.2564	2515.4	0.6241	31.101
Brain Networks	Human25890-session1	27.703	0.0222	4.1822	472.32	16.412	52.714	65.972	0.0509	7.8146	92.649	0.2272	15.977
	Human25890-session2	78.483	0.0471	14.289	1326.6	16.151	112.92	83.707	0.0162	6.709	183.06	0.1701	26.302
	Human25864-session2	83.489	0.0495	18.215	1433.3	17.298	143.91	106.20	0.0156	9.8828	212.96	0.1605	33.272
	Human25913-session2	99.615	0.0292	15.796	1614.4	18.899	171.16	223.39	0.0219	15.122	440.58	0.3331	59.629
	Human25886-session1	89.805	0.0568	20.819	1568.3	13.481	153.03	102.34	0.0154	11.296	207.52	0.1354	33.287

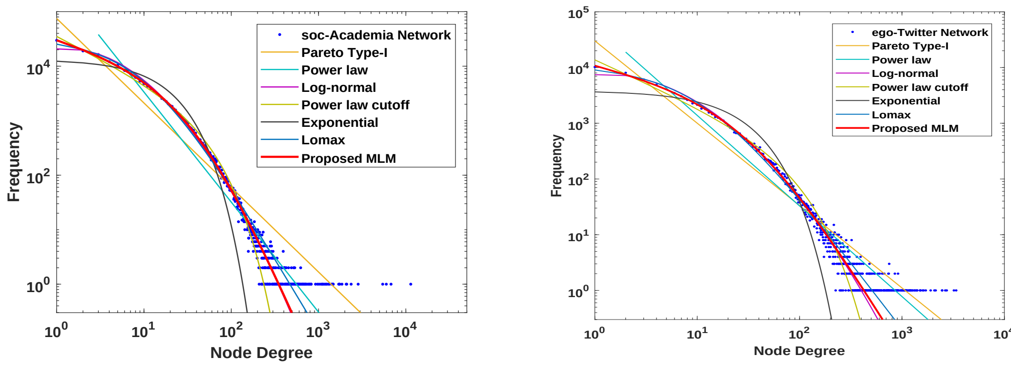


Figure 3: Degree distribution of soc-Academia and ego-Twitter networks in log-log scale

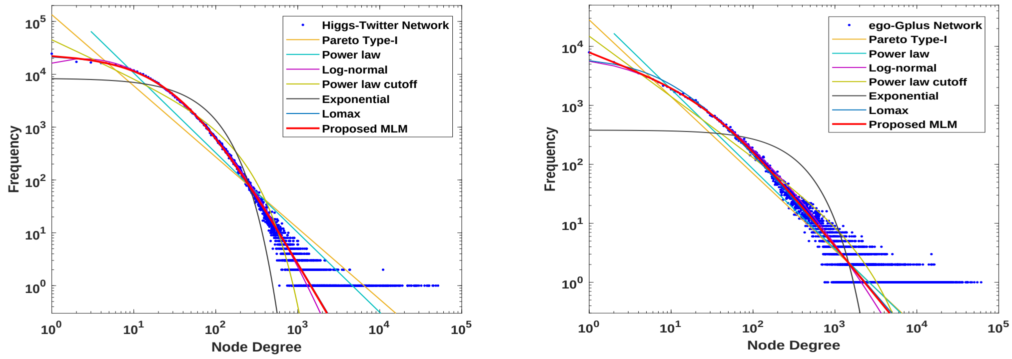


Figure 4: Degree distribution of Higgs-Twitter and ego-Gplus networks in log-log scale

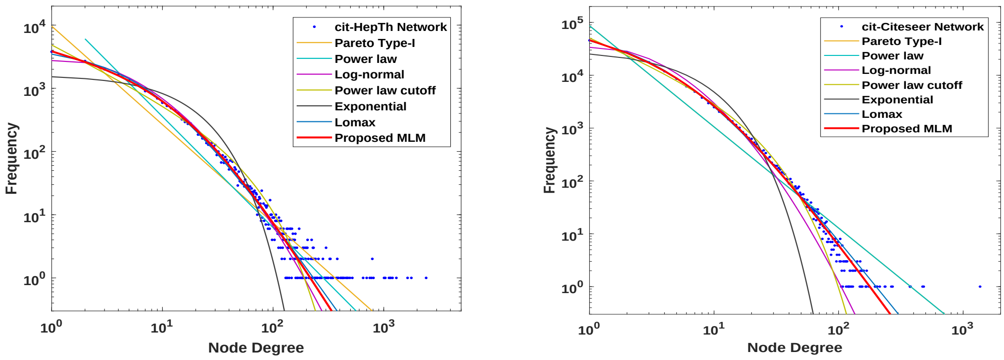


Figure 5: Degree distribution of cit-HepTh and cit-Citeseer networks in log-log scale

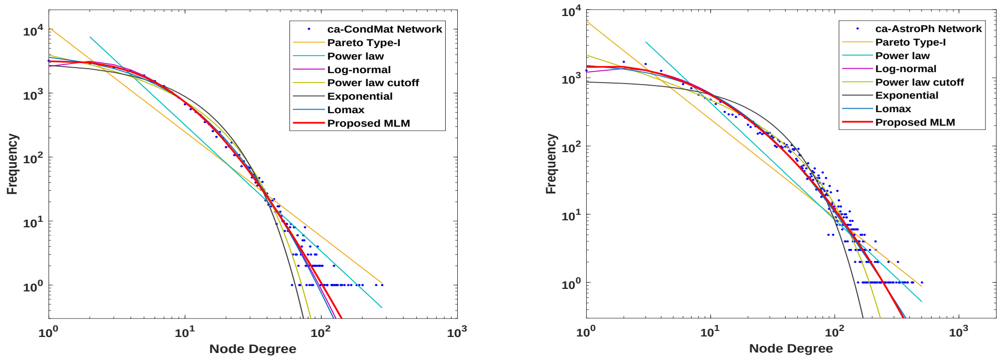


Figure 6: Degree distribution of ca-CondMat and ca-AstroPh networks in log-log scale

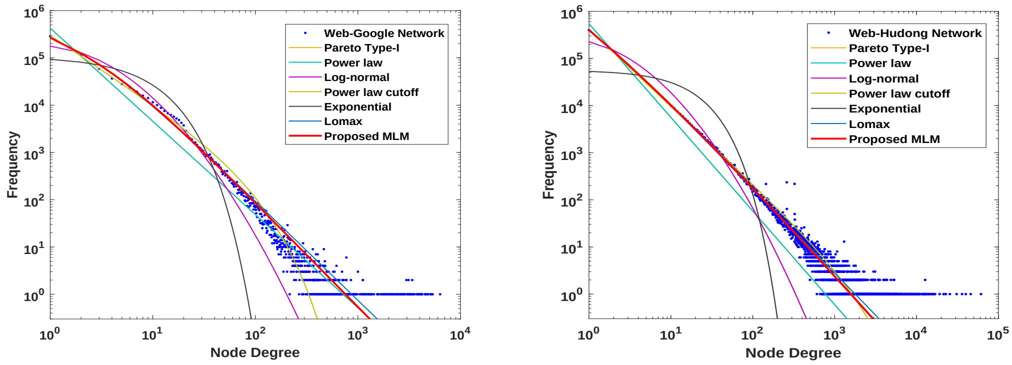


Figure 7: Degree distribution of Web-Google and Web-Hudong networks in log-log scale

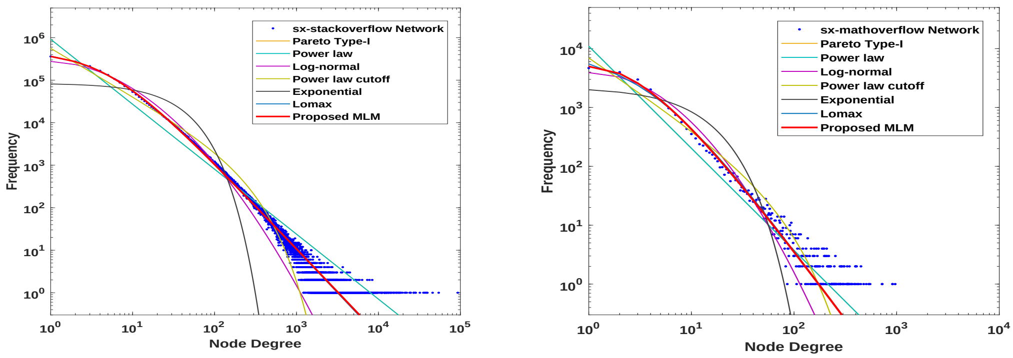


Figure 8: Degree distribution of sx-stack overflow and sx-mathoverflow networks in log-log scale

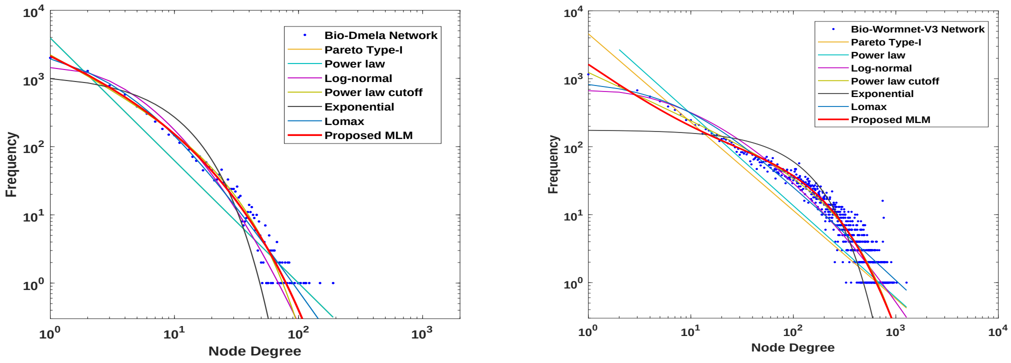


Figure 9: Degree distribution of Bio-Dmela and Bio-Wormnet-V3 networks in log-log scale

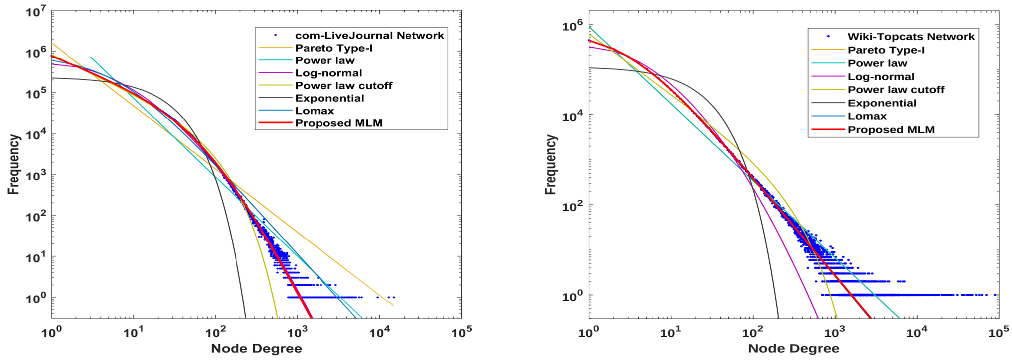


Figure 10: Degree distribution of LiveJournal and Wiki-Topcats networks in log-log scale

The effectiveness of the proposed MLM distribution can also be verified through the plotting of the fitted results of competitive distributions. For this purpose, the log-log plots of the of the original frequency distribution, the estimated frequency by MLM distribution and the frequency estimated by power-law, pareto, log-normal, power-law cutoff and exponential distributions are drawn for all the networks under consideration. Twenty four such examples have been provided in Figures 3-10. These are the soc-Academia network, ego-Twitter network, Higgs-Twitter network, ego-Gplus network, cit-HepTh network, cit-Citeseer network, ca-CondMat network, ca-AstroPh network, Web-Google network, web-Hudong network, sx-stackoverflow, sx-mathoverflow, Bio-Dmela network, Bio-Wormnet-V3 network, com-LiveJournal and- com-Wiki-Topcats network. Few more plotted results are also provided in the supplementary section. We have omitted the plot of the poisson distribution due to its poor performances over all the networks. It is visually clear From Figures 3-10 that the proposed MLM distribution provides better fit compared to the other competitive distributions in almost all of the networks since the proposed curve always passes through the middle of the scatter plot of the observed distribution. In a few cases the power-law cutoff and log-normal provide a better fit than the proposed distribution. It is visually clear from observing the social, biological, brain and citation networks that the entire node degree distribution can be better represented by the MLM distribution compared to other heavy tailed distributions. Thus the proposed MLM distribution, a modification of the Lomax distribution with non linear exponent in the shape parameter, can be used for effective and efficient modeling of the entire degree distribution of real-worlld networks without ignoring the lower degree nodes. The proposed MLM distribution provides more flexibility in the degree distribution modeling since the non-negative shape parameter are assumed to be expressed as a nonlinear function of the data. Empirical results also suggests the effectiveness of the proposed MLM distribution compared to others as depicted through Tables 1-3 and Figures 3-10.

6. Conclusion and Discussion

In this article, we have proposed a modified Lomax (MLM) distribution derived from a hierarchical family of Lomax distributions for flexible and efficient modeling of the entire node degree distribution of real-world complex networks. The proposed MLM distribution can be thought of as a generalization of the Lomax distribution with the nonlinear exponent in the shape parameter. We have theoretically established that the MLM distribution is heavy-tailed and right-tailed equivalent to the power-law distribution. Furthermore, we have shown a sufficient condition for the existence of the MLE for the parameters of MLM distribution using the notion of CV. The proposed MLM distribution can find MLE for the parameters at finite points when the value of CV>1. We also theoretically justified that the MLM distribution is a function with regularly varying tails which belongs to the Maximum domain of attraction of the Frechet distribution. We have further studied the asymptotic behaviors of the MLM distribution in this context.

The proposed MLM distribution captures the heavy-tailed and nonlinear behavior of the entire degree distributions of real-world networks in the original and the log-log scale more adroitly. It also enables us to accurately characterize the degree distribution pattern which may have a significant impact on analyzing real-world networks in terms of their social or biological aspects, as the case may be. We have applied the proposed MLM distribution in modeling the entire degree distribution over 50 different real-world empirical data sets taken from diverse fields. Empirical results suggest that as compared to the power-law distribution or any other well-known distribution, our proposed MLM distribution produces a lower fitting error in terms of three statistical tests, viz. RMSE, KL-divergence, and MAE. We also demonstrated the statistical significance of the estimated MLM distribution with the help of the bootstrap Chi-square value. This generalization of the Lomax distribution by adding an additional parameter in the base model results in flexible modeling to the entire degree distribution of a real-world network compared to other heavy-tailed distributions unlike power-law. The proposed fit distribution sometimes helps us in better characterization of the evolution process of large scale real-world networks instead of explicitly performing the empirical study at each time step. Thus, by simulating the parameters of a proposed fit MLM distribution, one can easily capture the spatial structure and dynamical pattern of a real-world network as the network evolves over time. The dynamic pattern analysis of such structural properties in real-world networks is one of the future scopes of research.

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