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A posteriori error estimates for a multi-scale finite-element method

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Abstract
We are interested in the discretization of a diffusion problem with highly oscillating coefficient, by a multi-scale finite-element method (MsFEM). The objective of this method is to capture the multi-scale structure of the solution via local basis functions which contain the essential information on small scales. In this paper, we perform an *a posteriori* analysis of this discretization. The main result consists of building error indicators with respect to both small and large meshes used in this method. We present a numerical test in which the experiments are in good coherency with the results of analysis.

Keywords Finite element · Multi-scale finite-element method · *A posteriori* error estimates · Error indicators

Mathematics Subject Classification 65N30 · 65N50

1 Introduction

The *a posteriori* error analysis is a powerful tool to improve the quality of approximated solutions of a model of partial differential equations. The estimates obtained in this context allow us to perform a self-adaptation of the mesh and to reach a desired accuracy, via a fixed tolerance. The norm of the error is bounded by estimators which depend only on the mesh size, the data, the approximated solution, and which are explicitly computable. The

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pioneering work of the *a posteriori* error estimates, in the context of finite-element method, was done by Babu$\check{s}$ka and Rheinbolt in 1978 (see Babuška et and Rheinboldt 1978a, b).

In this work, we are interested in the methods of multi-scale finite element (MsFEM) treated in Carballal Perdiz (2010), Efendiev and Hou (2009), Hou and Wu (1997), Lozinski et al. (2013), and Ahmed Blal (2014). Generally, this method is based on two main ingredients: building multi-scale local basis functions on a coarse mesh $\mathcal{T}_H$, and coupling them with a global variational formulation on a fine mesh $\mathcal{T}_h$, providing an accurate approximation of the solution. Multi-scale basis functions are designed to capture the characteristics of the multi-scale solution and contain information on small scales. They are built from those of standard finite element in the coarse mesh, such that they have the same support and satisfy the equation $\mathbf{L}\,\phi = 0$ on each element of $\mathcal{T}_H$, where $\mathbf{L}$ is the principal operator in the model. There are also variants of MsFEM: MsFVM ("multi-scale finite volume method", see Lee et al. 2002), MsMFEM ("multi-scale mixed finite-element method", see Chen and Hou 2003), MsFVEM ("multi-scale finite volume element method", see Hou 2009), and DG-MsFEM ("discontinuous Galerkin multi-scale finite-element method", see Efendiev and Hou 2009).

In this work, we develop *a posteriori* error estimates for finite-element multi-scale method for a diffusion problem with a mesh adaptation. For this, as in the construction of the method, we will develop estimates for the overall solution related to the coarse mesh and couple these estimates with those related to the fine mesh. In Aarnes and Efendiev (2006), we find adaptive techniques for a Finite Volume Multi-scale method based on indicators on some physical criteria. In Abdulle and Nonnenmacher (2011), the authors give residual type indicators for homogenization problems. In Henning et al. (2014), we find a residual error estimate for another type of MsFEM, that is a Petrov–Galerkin MsFEM with over-sampling. In this method, the finite-dimensional "coarse-scale" subspace and the continuous fine-scale space are defined through a projection operator from $H_0^1(\Omega)$, and the authors use a reconstruction operator and a corrector operator to define the discrete problem. The *a posteriori* estimates are on the error between the exact solution and the reconstructed discrete one, and the indicators are expressed with these operators. There are three types of indicators. The first one is related to the coarse mesh and is similar to one of ours. The second is based on the jump of the fluxes through the inter-element of the fine mesh and the last one is retard the over-sampling.

2 Formulations

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain with a Lipschitz-continuous boundary $\Gamma = \partial\Omega$. We use the standard space $\mathbb{L}^2(\Omega)$ equipped with the usual norm $\|\cdot\|_0$, together with the Sobolev space $\mathbb{H}^1(\Omega)$ of functions in $\mathbb{L}^2(\Omega)$, such that their first derivatives (in distribution sense) belong to $\mathbb{L}^2(\Omega)$, equipped with the norm $\|v\|_1 := \left(\|v\|_0^2 + \|\nabla v\|_0\right)^{1/2}$. As usual, $\mathbb{H}_0^1(\Omega) = \{v \in H^1(\Omega) \ / \ v = 0 \text{ on } \Gamma\}$ is equipped with the norm $|v|_1 := \|\nabla v\|_0$.
We consider the diffusion problem associated with the operator $\mathbf{L}v := -\text{div}\,(\nu\nabla v)$ defined by:

$$\begin{cases} \text{find } u \in \mathbb{V}, \\ a(u,v) = F(v), \ \forall\, v \in \mathbb{V}, \end{cases} \tag{1}$$

where $\mathbb{V} = \mathbb{H}_0^1(\Omega)$, $a(u,v) = \displaystyle\int_\Omega \nu\nabla u \cdot \nabla v dx$ and $F(v) = \displaystyle\int_\Omega f v dx$, for all u and v in $\mathbb{V}$.
We assume that $f \in \mathbb{L}^2(\Omega)$, $\nu \in \mathbb{C}^{0,1}(\Omega)$, the space of Lipschitz-continuous functions, and

there exist two positive constants ν_m and ν_M satisfying:

$$\forall x \in \Omega, \quad 0 < \nu_m \leq \nu(x) \leq \nu_M.$$

Let us note that $\nu \in \mathbb{L}^\infty(\Omega)$ is sufficient to have the existence and uniqueness of the solution in $H_0^1(\Omega)$, but for the need of the *a posteriori* error analysis, we assume more regularity for ν.

We denote by $|\cdot|_\nu$ the energy norm defined by:

$$|v|_\nu^2 := \int_\Omega \nu |\nabla u|^2 \mathrm{d}x.$$

If D is a subset of Ω, we use the notation:

$$|v|_{\nu,D}^2 := \int_D \nu |\nabla u|^2 \mathrm{d}x,$$

and also the following notations:

$$\underline{\nu}_D = \inf_{x \in D} \nu(x), \quad \overline{\nu}_D = \sup_{x \in D} \nu(x). \tag{2}$$

We remark that:

$$|v|_{1,D} \leq \frac{1}{\underline{\nu}_D^{1/2}} |v|_{\nu,D}. \tag{3}$$

The existence and uniqueness of the solution of (1) are obtained using the Lax–Milgram theorem.
In many applications, the coefficient ν may present a highly oscillatory character. To obtain the large-scale solutions accurately and efficiently without resolving the small-scale details, we will use a Multi-scale Finite-Element Method (MsFEM) (see Babuška and Osborn 1983; Carballal Perdiz 2010; Efendiev and Hou 2009; Hou and Wu 1997; Lozinski et al. 2013; Ahmed Blal 2014), an approach that captures the multi-scale structure of the solution. Let $\mathcal{T}_H$ a regular mesh of Ω:

$$\overline{\Omega} = \bigcup_{K \in \mathcal{T}_H} K,$$

where $H = \max_K H_K$ and H_K is the diameter of the element K. We denote by $\mathcal{N}_H$ and $\mathcal{E}_H$, respectively, the set of all internal nodes x_i for $i = 1, 2, \ldots, N_H$, and the set of all edges E_i for $i = 1, 2, \ldots, N_E$ of $\mathcal{T}_H$, excluding the edges on the boundary $\partial\Omega$. The first-order standard finite-element space based on $\mathcal{T}_H$ and approximating $\mathbb{V}$ is denoted by $\mathbb{P}_1(\mathcal{T}_H)$ and defined by:

$$\mathbb{P}_1(\mathcal{T}_H) := \{v_H \in C^0(\bar{\Omega}) \,/\, v_{H|_K} \in \mathbb{P}_1(K), \forall K \in \mathcal{T}_H, (v_H)_{|\Gamma} = 0\} \subset \mathbb{V},$$

where $\mathbb{P}_1(K)$ and $\mathbb{P}_1(E)$) are the spaces of polynomial functions with degree ≤ 1 in K and E, respectively. As usual, the basis functions $\{\psi_H^i\}_{i=1}^N$ associated with $\mathbb{P}_1(\mathcal{T}_H)$ satisfy $\{\psi_H^i\}(x_j) = \delta_{ij}$, and the support of $\{\psi_H^i\}$, denoted by ω_i, is the union of the β_i elements $K_{i,d}$, such that x_i is one of their vertices:

$$\omega_i = \bigcup_{d=1}^{\beta_i} K_{i,d}.$$

The multi-scale basis functions are defined (see Efendiev and Hou 2009; Hou and Wu 1997), for $i = 1, \ldots, N_H$, as the function $\Phi_H^i \in \mathbb{H}^1(\omega_i)$ satisfying the following problems, for $d = 1, \ldots, \beta_i$:

$$\begin{cases} \mathbf{L}_{K_{i,d}} \Phi_H^i = 0 & \text{in } K_{i,d}, \\ \qquad\quad \Phi_H^i = \psi_H^i & \text{on } \partial K_{i,d}, \end{cases} \tag{4}$$

with the notation $\mathbf{L}_K := \mathbf{L}|_K$. The Multi-scale Finite-Element (MsFEM) space adapted to the operator $\mathbf{L}$ is then defined by:

$$\begin{aligned} \mathbb{V}_H &= \mathbb{S}pan\left\{\Phi_H^i,\ i = 1, ..., \mathcal{N}_H\right\} \\ &= \left\{v_H \in \mathbb{C}^0(\Omega) \cap \mathbb{H}_0^1(\Omega),\ \text{ such that } \mathbf{L}_K v_H = 0\right. \\ &\qquad \left.\text{in } K,\ \forall\, K \in \mathcal{T}_H, \ \text{ and } v_H|_E \in \mathbb{P}_1(E),\ \forall E \in \mathcal{E}_H\right\}. \end{aligned}$$

The MsFEM approximation of the problem (1) is now given by:

$$\begin{cases} \text{find } u_H \in \mathbb{V}_H, \\ a(u_H, v_H) = F(v_H),\ \forall\, v_H \in \mathbb{V}_H\ . \end{cases} \tag{5}$$

To determine the multi-scale basis functions, we have now to solve the problems (4). To do so, we consider a new finer mesh. For each $K \in \mathcal{T}_H$, we consider a regular conforming mesh $\mathcal{T}_h(K)$ all independently, and define $\mathcal{T}_h$ as the overall fine mesh:

$$K = \bigcup_{T \in \mathcal{T}_h(K)} T \qquad \text{and} \qquad \mathcal{T}_h = \bigcup_{K \in \mathcal{T}_H} \mathcal{T}_h(K). \tag{6}$$

Let us note that each $\mathcal{T}_h(K)$ is a conforming mesh, but the global mesh can be non-conforming one.

We denote by $\mathcal{E}_h(K)$ the set of all edges of $\mathcal{T}_h(K)$ and by $\mathcal{E}_h$ the set of all edges of $\mathcal{T}_h$ excluding the edges on the boundary $\partial\Omega$. The edges from $\mathcal{E}_h$ can be divided into two groups: those forming the boundary of coarse triangles and thus constituting the coarse mesh edges (denoted by $\mathcal{E}_H$) and those internal for some triangles $K \in \mathcal{T}_H$ forming the set $\mathcal{E}_h^0(K)$. In the sequel, we need also the following notations. For any $E \in \mathcal{E}_H$, we denote by $K_1(E)$ and $K_2(E)$, the two coarse adjacent triangles, such that $E = K_1(E) \cap K_2(E)$. Let, moreover, $\mathcal{T}_h^b(E)$ be the set of the triangles of both fine meshes $\mathcal{T}_h(K_1)$ and $\mathcal{T}_h(K_2)$ that touch E. We define also $\mathcal{E}_h^b(E)$ as the set of internal edges of both fine meshes $\mathcal{T}_h(K_1)$ and $\mathcal{T}_h(K_2)$ that touch E and are not in E. For $i = 1, \ldots, N_H$, the sets ω_i are also the supports of the multi-scale basis functions Φ_H^i and we put:

$$\begin{aligned} \mathbb{P}_H^1(\omega_i) := \{&v^h \in C^0(\bar{\omega}_i)\,/\,v^h|_E \in P_1(E),\ \forall E \in \mathcal{E}_H \cap \overline{\omega_i}, \\ &v^h|_T \in \mathbb{P}_1(T),\ \text{and } v^h_{|\partial T \cap \Gamma} = 0,\ \forall T \in \mathcal{T}_h \cap \omega_i\}. \end{aligned}$$

The approximated multi-scale function is $\Phi_{h,H}^i \in \mathbb{P}_H^1(\omega_i)$, such that the restriction on K_{id}, $\Phi_{h,H}^{i,d} := \Phi_{h,H}^i|_{K_{id}}$, for $d = 1, \ldots, \beta_i$, is the solution in $\mathbb{P}_1(\mathcal{T}_h(K_{i,d}))$, of the problem:

$$\begin{cases} a_K(\Phi_{h,H}^{i,d}, v_H^h) = 0, \forall\, v_H^h \in \mathbb{V}_0^h(K_{i,d}), \\ \Phi_{h,H}^{i,d}\Big|_{\partial K_{i,d}} = \psi_H^i\big|_{\partial K_{i,d}}, \end{cases} \tag{7}$$

with the notations $a_K(u, v) = \displaystyle\int_K \nu \nabla u \cdot \nabla v$, and for $K \in \mathcal{T}_H$:

$$\mathbb{V}_0^h(K) := \mathbb{P}_1(\mathcal{T}_h(K)) \cap \mathbb{H}_0^1(K). \tag{8}$$

We define the MsFEM discrete space by:

$$\begin{aligned}\mathbb{V}_H^h &= \mathbb{S}pan\left\{\Phi_{h,H}^i,\ i=1,\dots,\mathcal{N}_H\right\} \\ &= \Big\{v^h \in C^0(\overline{\Omega}),\ \text{such that}\ a_K(v^h,w^h)=0,\ \forall\, w^h \in \mathbb{V}_0^h(K), \\ &\qquad \forall\, K \in \mathcal{T}_H,\ v^h\Big|_E \in \mathbb{P}_1(E),\ \forall E \in \mathcal{E}_H\Big\} \cap \mathbb{H}_0^1(\Omega),\end{aligned} \tag{9}$$

and the discrete MsFEM approximation of the problem (1) is defined by:

$$\begin{cases}\text{find } u_H^h \in \mathbb{V}_H^h, \\ a(u_H^h, v_H^h) = F(v_H^h),\ \forall\, v_H^h \in \mathbb{V}_H^h\ .\end{cases} \tag{10}$$

The multi-scale solution u_H^h of the global discrete problem (10) is therefore written as a linear combination of multi-scale basis functions:

$$u_H^h = \sum_{i=1}^{\mathcal{N}_H} \alpha_i \Phi_{h,H}^i.$$

As in Carballal Perdiz (2010) and Lozinski (2010), we can prove the following *a priori* error estimate.

Proposition 1 *Let u be the solution of (1) and u_H^h be the solution of (10).*
We have the following error estimate:

$$|u-u_H^h|_\nu \le \inf_{\varphi_H\in\mathbb{V}_L} |u-\varphi_H|_\nu + \inf_{w_H^h\in\mathbb{V}_H^h} |u_H - w_H^h|_\nu + \frac{C_0}{\sqrt{\nu_m}} H\|f\|_0. \tag{11}$$

where $\mathbb{V}_L := \left\{v \in H_0^1(\Omega) \text{ such that } \forall E \in \mathcal{E}_H\ v\,|_E \ \text{ is linear}\right\}$, and C_0 be a constant which depends only on the geometry of K.

Proof Since $\mathbb{V}_H^h \subset H_0^1(\Omega)$, $u-u_H^h$ satisfies the orthogonality condition $a(u-u_H^h, w_H^h)=0$ for all $w_H^h \in \mathbb{V}_H^h$. Hence, we have:

$$\begin{aligned}|u-u_H^h|_\nu &\le |u-w_H^h|_\nu \\ &\le |u-u_H|_\nu + |u_H - w_H^h|_\nu, \quad \forall w_H^h \in \mathbb{V}_H^h.\end{aligned}$$

In the other hand, since $\mathbb{V}_H$ is also a subspace of $H_0^1(\Omega)$, by Cea's Lemma, we have:

$$|u-u_H|_\nu \le |u-v_H|_\nu, \qquad \forall v_H \in \mathbb{V}_H.$$

Let $\varphi_H^u \in \mathbb{V}_L$ be the energy norm-projection of u in $\mathbb{V}_L$, and $v_H \in \mathbb{V}_H$ such as $v_H\,\big|_{\mathcal{E}_H} = \varphi_H^u\big|_{\mathcal{E}_H}$, and then, for all $K \in \mathcal{T}_H$, the local error $e_K = \varphi_H^u|_K - v_H|_K$, which is in $\mathbb{H}_0^1(K)$, satisfies the problem:

$$\begin{cases}\mathbf{L}_K e_K = f|_K, & \text{on } K, \\ e_K \quad\ = 0 & \text{on } \partial K,\end{cases} \tag{12}$$

and by applying the Poincaré's inequality and (3):

$$\|e_K\|_{0,K} \le C(K)\frac{1}{\sqrt{\underline{\nu}_K}}|e_K|_{\nu,K}, \tag{13}$$

where $C(K)$ is the Poincaré's constant in K which can be written as $C(K) = C_0 H_K$, where C_0 depends only on the geometry of K and is independent of H_K, its diameter.

By the problem (12), we deduce:

$$|e_K|^2_{\nu,K} \leq \|f|_K\|_{0,K}\|e_K\|_{0,K} \leq \|f|_K\|_{0,K}\frac{C_0}{\sqrt{\underline{\nu}_K}}H_K|e_K|_{\nu,K},$$

and so:

$$|e|_\nu \leq \frac{C_0}{\sqrt{\nu_m}}H\|f\|_0,$$

where $H = \max_K H_K$.

We obtain finally:

$$\begin{aligned}
|u-u^h_H|_\nu &\leq |u-u_H|_\nu + \inf_{w^h_H\in\mathbb{V}^h_H}|u_H-w^h_H|_\nu,\\
&\leq |u-\varphi^u_H|_\nu + |\varphi^u_H - v_H|_\nu + \inf_{w^h_H\in\mathbb{V}^h_H}|u_H-w^h_H|_\nu,\ \forall v_H\in V_H,\\
&\leq \inf_{v_H\in\mathbb{V}_L}|u-v_H|_\nu + \frac{C_0}{\sqrt{\nu_m}}H\|f\|_0 + \inf_{w^h_H\in\mathbb{V}^h_H}|u_H-w^h_H|_\nu,
\end{aligned}$$

which gives (11) □

3 Interpolation operators and basic inequalities

In this section, we recall some basic inequalities and give interpolation operators which are essential tools to derive *a posteriori* error estimates .

BASIC INEQUALITIES

Lemma 1 (Local trace inequality, see Verfürth (2013), Adams (1995)) *For every $E \in \partial K$ and all $\mathbf{w} \in \mathbb{H}(div;\Omega)$, such that $\mathbf{w}\cdot\mathbf{n} \in \mathbb{L}^2(E)$, we have:*

$$\|\mathbf{w}\cdot\mathbf{n}\|_{\mathbb{L}^2(E)} \preceq \left(h_E^{-1/2}\|\mathbf{w}\|_{\mathbf{L}^2(K)} + h_E^{1/2}\|div\mathbf{w}\|_{\mathbb{L}^2(K)}\right),$$

where $\mathbb{H}(\mathrm{div};\Omega) := \{\mathbf{w} \in \left(\mathbb{L}^2(\Omega)\right)^2 / \mathrm{div}\mathbf{w} \in \mathbb{L}^2(\Omega)\}$, $\mathbf{n}$ the vector normal to ∂K and where we use the shorthand notation:

$$x \preceq y, \tag{14}$$

for $x \leq Cy$ with positive constant independent of x, y, and meshes.

To derive the lower bound of the error, we need special functions introduced by Verfürth (1996) and called bubble functions. Using the barycentric coordinates, λ_i^K, for $i = 1, 2, 3$, associated with an element $K \in \mathcal{T}_H$, these bubble functions are defined by:

$$b_K := \begin{cases} 27\lambda_1^K\lambda_2^K\lambda_3^K & \text{on } K,\\ 0 & \text{everywhere else .}\end{cases} \tag{15}$$

$$b_E := \begin{cases} 4\lambda_1^{K_i}\lambda_2^{K_i} & \text{on } K_i, \quad \text{for } i = 1, 2\\ 0 & \text{everywhere else ,}\end{cases} \tag{16}$$

where $E = K_1 \cap K_2$. We have the following lemma (see Verfürth 1996, lemma 3.3, page 66):

Lemma 2 *Let $K \in \mathcal{T}_H$, $E \in \mathcal{E}_h$, v be a polynomial function in K, and σ be a polynomial function on E. We have:*

$$\|v\|_{0,K} \preceq \left\| b_K^{1/2} v \right\|_{0,K} \quad \textit{and} \quad \|\sigma\|_{0,E} \preceq \left\| b_E^{1/2} \sigma \right\|_{0,E}, \tag{17}$$

$$|b_K v|_{1,K} \preceq h_K^{-1} \|v\|_{0,K} \quad \textit{and} \quad |\mathcal{P}(b_E\sigma)|_{1,K} \preceq h_E^{-1/2} \|\sigma\|_{0,E}, \tag{18}$$

$$\|\mathcal{P}(b_E\sigma)\|_{0,K} \preceq h_E^{1/2} \|\sigma\|_{0,E}, \tag{19}$$

where $\mathcal{P}(\cdot)$ is a continuation operator from $\mathbb{L}^\infty(E)$ to $\mathbb{L}^\infty(K)$.

INTERPOLATION OPERATORS

We consider here operators of Clément type, related, respectively, to the spaces $\mathbb{V}_H$ and $\mathbb{V}_H^h$. Let $\mathcal{A}_H^h : \mathbb{V} \to \mathbb{V}_H^h$ and $\mathcal{A}_H : \mathbb{V} \to \mathbb{V}_H$ defined, for all $v \in \mathbb{V}$ by:

$$\mathcal{A}_H^h(v) = \sum_{i=1}^{\mathcal{N}_H} \left(\frac{1}{|\omega_i|} \int_{\omega_i} v dx \right) \Phi_{h,H}^i, \tag{20}$$

$$\mathcal{A}_H(v) = \sum_{i=1}^{\mathcal{N}_H} \left(\frac{1}{|\omega_i|} \int_{\omega_i} v dx \right) \Phi_H^i, \tag{21}$$

where Φ_H^i and $\Phi_{h,H}^i$, $i = 1, \ldots \mathcal{N}_H$ are the multi-scale basis functions defined, respectively, by (4) and (7). These operators satisfy the following properties.

Proposition 2 *Let $K \in \mathcal{T}_H$, $E \in \mathcal{E}_H$ and $v \in \mathbb{V}$. Then, for $B = \mathcal{A}_H$ or $B = \mathcal{A}_H^h$, we have:*

$$\|Bv\|_{0,K} \preceq \|v\|_{0,\omega_K}, \tag{22}$$

$$\|v - Bv\|_{0,K} \preceq H_K |v|_{1,\omega_K}, \tag{23}$$

$$\|v - Bv\|_{0,E} \preceq {H_E}^{1/2} |v|_{1,\omega_E}, \tag{24}$$

where ω_K (resp. ω_E) is the union of all the triangles of $\mathcal{T}_H$ that share at least one node with K (resp. E).

The proof is given in the Annex.

We shall also need a Clément interpolation operator $\Pi_h : H^1(K) \to \mathbb{P}_1(\mathcal{T}_h(K))$, defined relatively to the P_1 finite elements on the fine mesh $\mathcal{T}_h(K)$ for any coarse triangle K. We take a version of such an operator constructed by (see Scott and Zhang (1990)), such that $\Pi_h v_h = v_h$ for all $v_h \in \mathbb{P}_1(\mathcal{T}_h(K))$, with the following properties for all $v \in H^1(K)$:

$$\left.\begin{array}{l} \|v - \Pi_h v\|_{0,T} \preceq h_T |v|_{1,\omega_T}, \\ \|v - \Pi_h v\|_{0,e} \preceq h_e^{1/2} |v|_{1,\omega_e}, \\ \|\Pi_h v\|_{0,\tilde{e}} \preceq ||v||_{0,\gamma_{\tilde{e}}}, \end{array}\right\} \tag{25}$$

where T is any triangle from $\mathcal{T}_h(K)$, e any internal edge from $\mathcal{E}_h^0(K)$, $\tilde{e}$ any boundary edge from the mesh $\mathcal{T}_h(K)$, ω_T (resp. ω_e) is the union of all the triangles of $\mathcal{T}_h(K)$ that share at least one node with T (resp. e) and $\gamma_{\tilde{e}}$ is the union of all the boundary edges that share at least one node with $\tilde{e}$.

We define now another operator. For $v \in H^1(K)$, let $\widetilde{\Pi}_h v$, such that:

$$\left.\begin{array}{r} \widetilde{\Pi}_h v \in \mathbb{P}_1(\mathcal{T}_h(K)), \\ \widetilde{\Pi}_h v(a) = \Pi_h v(a), \text{ for all internal node } (a \notin \partial K), \\ \widetilde{\Pi}_h v(a) \text{ arbitrary, for all boundary node } (a \in \partial K). \end{array}\right\} \tag{26}$$

Proposition 3 *Let $v \in H^1(K)$, $T \in \mathcal{T}_h(K)$ and $e \in \mathcal{E}_h(K)$. Then, $\widetilde{\Pi}_h v$ defined by (26) satisfies the following estimates:*

$$\|v - v_h\|_{0,T} \preceq h_T |v|_{1,\omega_T} + \alpha_T h_T^{1/2} \|v - v_h\|_{0,\partial\omega_T \cap \partial K}, \tag{27}$$

$$\|v - v_h\|_{0,e} \preceq h_e^{1/2} |v|_{1,\omega_e} + \alpha_e \|v - v_h\|_{0,\partial\omega_e \cap \partial K}, \tag{28}$$

where α_T and α_e are boundary switches, i.e., $\alpha_T = 1$ (resp. $\alpha_e = 1$) if T (resp. e) touches ∂K; otherwise, $\alpha_T = \alpha_e = 0$, and where ω_T (resp. ω_e) is the union of all the triangles of $\mathcal{T}_h(K)$ that share at least one node with T (resp. e).

The proof is in the Annex.

4 *A posteriori* error estimates

In this section, we will give different *a posteriori* error estimates. The first ones are related to the approximation of the problem (1) in the space $\mathbb{V}_H$ spanned by the multi-scale functions, and given by the problem (5). Obviously, the function u_H will not be computed directly, but over the approximation of the multi-scale functions. Here and below, we use the notation (14) and the notation (2) for D equal K, E, T, e, etc.... For $E \in \mathcal{E}_h$, we denote by $\mathbf{n}_E$ the unit, normal, outward-pointing vector field and by $\left[\nu \nabla_{\mathbf{n}_E} u_H\right]_E$ the jump of $\nu \nabla u_H \cdot \mathbf{n}_E$ across the element E.

In the following, the notation $x \preceq y$ means that $x \leq Cy$ with positive constant independent of x, y, ν, and meshes.

4.1 Estimates related to multi-scale basis functions

Theorem 1 *Let $f \in \mathbb{L}^2(\Omega)$, u be the solution of (1), and u_H be the solution of (5), and let η_K, for $K \in \mathcal{T}_H$ defined by:*

$$\eta_K^2 := \frac{H_K^2}{\underline{\nu}_{\omega_K}} \|f_K\|_{0,K}^2 + \frac{1}{2} \sum_{E \in \mathcal{E}_K} \frac{H_E}{\underline{\nu}_{\omega_E}} \left\|\left[\nu \nabla_{\mathbf{n}_E} u_H\right]_E\right\|_{0,E}^2, \tag{29}$$

where $f_K := \dfrac{1}{|K|} \displaystyle\int_K f dx$.

Then, we have:

$$|u - u_H|_\nu \preceq \left\{ \sum_{K \in \mathcal{T}_H} \eta_K^2 + \frac{H_K^2}{\underline{\nu}_{\omega_K}} \|f - f_K\|_{0,K}^2 \right\}^{1/2}, \tag{30}$$

$$\eta_K \preceq \left\{ \frac{H_K^2}{\underline{\nu}_{\omega_K}} ||f - f_K||_{0,\omega_K}^2 + \frac{\overline{\nu}_K}{\underline{\nu}_{\omega_K}} |u - u_H|_{\nu,\omega_K}^2 \right\}^{1/2}, \tag{31}$$

where f_K is an approximation of f.

Proof 1. We begin by proving the upper bound of the error (30). By taking $w = u - u_H$ and for any $w_H \in \mathbb{V}_H$, the Galerkin orthogonality and an integration by parts in each $K \in \mathcal{T}_H$ give:

$$|u - u_H|_\nu^2 = \sum_{K \in \mathcal{T}_H} \left\{ \int_K (f + \operatorname{div}(\nu \nabla u_H))(w - w_H) dx \right.$$

$$+\frac{1}{2}\sum_{E\in\mathcal{E}_K}\int_E\left[\nu\nabla_{\mathbf{n}_E}u_H\right]_E(w-w_H)ds\Big\}.$$

By Cauchy–Schwarz inequality and since $\mathrm{div}(\nu\nabla u_H)=0$ in K, we obtain:

$$|u-u_H|_\nu^2\leq\sum_{K\in\mathcal{T}_H}\Big\{\left[\|f_K\|_{0,K}+\|f-f_K\|_{0,K}\right]\|w-w_H\|_{0,K}$$
$$+\frac{1}{2}\sum_{E\in\mathcal{E}_K}\|\left[\nu\nabla_{\mathbf{n}_E}u_H\right]_E\|_{0,E}\|w-w_H\|_{0,E}\Big\}.$$

Now, we take $w_H=\mathcal{A}_H w$, where $\mathcal{A}_H$ is defined in (21) and use the estimates of Proposition 2 and (3) to obtain:

$$|u-u_H|_\nu^2\preceq\sum_{K\in\mathcal{T}_H}\frac{H_K}{\underline{\nu}_{\omega_K}^{1/2}}\Big\{\|f_K\|_{0,K}+\|f-f_K\|_{0,K}\Big\}|w|_{\nu,\omega_K}$$
$$+\sum_{E\in\mathcal{E}_K}\frac{H_E^{1/2}}{\underline{\nu}_{\omega_E}^{1/2}}\|\left[\nu\nabla_{\mathbf{n}_E}u_H\right]_E\|_{0,E}|w|_{\nu,\omega_E},$$

which gives (30) since $w=u-u_H$.

2. To prove (31), the upper bound of the indicator, we use the technique of bubble functions which is now standard. Let $w_K=b_K f_K$, where b_K is the bubble function in K defined by (15). By (17), we have:

$$\|f_K\|_{\mathbb{L}^2(K)}^2\preceq\int_K b_K f_K^2\mathrm{d}x.$$

Since $\mathbf{L}_K u_H=0$ in K, and $\int_K w_K f dx-\int_K \nu\nabla u\cdot\nabla w_K dx=0$, and then, using Green's formula, we have:

$$\begin{aligned}\|f_K\|_{\mathbb{L}^2(K)}^2&\preceq\int_K w_K(f_K+\mathrm{div}(\nu\nabla u_H))\mathrm{d}x,\\&\preceq\left[\int_K w_K(f_K-f)+\int_K w_K\mathrm{div}(\nu\nabla u_H)dx\right]+\int_K\nu\nabla u\cdot\nabla w_K\mathrm{d}x,\\&\preceq\left[\int_K w_K(f_K-f)\mathrm{d}x+\int_K\nu\nabla(u-u_H)\cdot\nabla w_K\mathrm{d}x\right].\end{aligned}$$

The Cauchy–Schwarz and inverse inequalities, the definition of w_K, and (3) give:

$$\|f_K\|_{\mathbb{L}^2(K)}\preceq\left(\|f_K-f\|_{\mathbb{L}^2(K)}^2+H_K^{-2}\overline{\nu}_K|u-u_H|_{\nu,K}^2\right)^{1/2},$$

and then:

$$\frac{H_K^2}{\underline{\nu}_{\omega_K}}\|f_K\|_{\mathbb{L}^2(K)}^2\preceq\frac{H_K^2}{\underline{\nu}_{\omega_K}}\|f_K-f\|_{\mathbb{L}^2(K)}^2+\frac{\overline{\nu}_K}{\underline{\nu}_{\omega_K}}|u-u_H|_{\nu,K}^2.\tag{32}$$

Let now $\mathbf{w}=\nu\nabla(u_H-u)$. First of all, we remark that for all element $K\in\mathcal{T}_H$ and all $E\in\partial K$, we have $\mathrm{div}\mathbf{w}=f$ in K and $\left[\nu\nabla_{\mathbf{n}_E}u\right]_E=0$. In the other hand, since $\mathbf{L}_K u_H=0$ in K, $u_H|_{\partial K}$ is regular and K is convex, we deduce by regularity theorem

(Cf. Grisvard 1985) that $u_H|_K \in H^2(K)$, and so, $\mathbf{w}\cdot\mathbf{n}$ is in $L^2(E)$. By Lemma 1, we therefore have:

$$\begin{aligned}\left\|\left[\nu\nabla_{\mathbf{n}_E}u_H\right]_E\right\|_{\mathbb{L}^2(E)} &\preceq H_E^{1/2}||f||_{0,K} + H_E^{-1/2}||\nu\nabla(u-u_H)||_{0,K},\\ &\preceq H_E^{1/2}||f_K||_{0,K} + H_E^{1/2}||f-f_K||_{0,K}\\ &\quad+\frac{\overline{\nu}_K^{1/2}}{H_E^{1/2}}|u-u_H|_{\nu,K}.\end{aligned}$$

Multiplying the previous inequality by $H_E^{1/2}/\underline{\nu}_E^{1/2}$ and using the fact that $H_E \le H_K$ and $\underline{\nu}_E \ge \underline{\nu}_K$, and the inequality (32), we find:

$$\frac{H_E^{1/2}}{\underline{\nu}_{\omega_E}^{1/2}}\left\|\left[\nu\nabla_{\mathbf{n}_E}u_H\right]_E\right\|_{\mathbb{L}(E)} \preceq \frac{H_K}{\underline{\nu}_{\omega_K}^{1/2}}||f-f_K||_{0,K} + \frac{\overline{\nu}_K^{1/2}}{\underline{\nu}_{\omega_K}^{1/2}}|u-u_H|_{\nu,K},$$

or

$$\frac{H_E}{\underline{\nu}_{\omega_E}}\left\|\left[\nu\nabla_{\mathbf{n}_E}u_H\right]_E\right\|^2_{\mathbb{L}^2(E)} \preceq \frac{H_K^2}{\underline{\nu}_{\omega_K}}||f-f_K||^2_{0,\omega_K} + \frac{\overline{\nu}_K}{\underline{\nu}_{\omega_K}}|u-u_H|^2_{\nu,\omega_K},$$

and therefore:

$$\sum_{E\in\mathcal{E}_K}\frac{H_E}{\underline{\nu}_{\omega_E}}\left\|\left[\nu\nabla_{\mathbf{n}_E}u_H\right]_E\right\|^2_{\mathbb{L}^2(E)} \preceq \frac{H_K^2}{\underline{\nu}_{\omega_K}}||f-f_K||^2_{0,\omega_K} + \frac{\overline{\nu}_K}{\underline{\nu}_{\omega_K}}|u-u_H|^2_{\nu,\omega_K}.$$

□

4.2 Estimates for the fully approximated problem

In this section, we give *a posteriori* error estimates for the discrete MsFEM approximation problem given by (10).

The *a posteriori* indicators are defined, for all $K \in \mathcal{T}_H$, all $T \in \mathcal{T}_h(K)$, and all $E \in \mathcal{E}_H$, by:

$$\eta^2_{K,h} := \frac{H_K^2}{\underline{\nu}_{\omega_K}}\|f_K\|^2_{0,K} + \sum_{E\in\mathcal{E}_K}\frac{H_E}{\underline{\nu}_{\omega_E}}\left\|\left[\nu\nabla_{\mathbf{n}_E}u_H^h\right]_E\right\|^2_{0,E}. \tag{33}$$

$$\delta^2_{K,h} := \frac{1}{\underline{\nu}_K}\sum_{T\in\mathcal{T}_h(K)} h_T^2\|\mathrm{div}(\nu\nabla u_H^h)\|^2_{0,T} + \sum_{e\in\mathcal{E}_h^0(K)} h_e\left\|\left[\nu\nabla_{\mathbf{n}_e}u_H^h\right]_E\right\|^2_{0,e}. \tag{34}$$

$$\xi^2_{E,h} := \frac{H_E}{\underline{\nu}_{\omega_E}}\left(\sum_{T\in\mathcal{T}_h^b(E)} h_T||div(\nu\nabla u_H^h)||^2_{0,T} + \sum_{e\in\mathcal{E}_h^b(E)}\left\|\left[\nu\nabla_{\mathbf{n}_e}u_H^h\right]_e\right\|^2_{0,e}\right). \tag{35}$$

We will prove the upper and lower bounds of the error by these indicators.

4.2.1 An upper bound for the error

Theorem 2 *Let $f \in \mathbb{L}^2(\Omega)$ and u_H^h be the solution of (10). We have:*

$$|u-u_H^h|_\nu \preceq \left\{\sum_{K\in\mathcal{T}_H}\left(\eta^2_{K,h} + \frac{H_K^2}{\underline{\nu}_{\omega_K}}\|f-f_K\|^2_{0,K}\right) + \sum_{K\in\mathcal{T}_H}\delta_K^2 + \sum_{E\in\mathcal{E}_H}\xi_E^2\right\}^{1/2}; \tag{36}$$

Remark 1 the first sum in (36) is the indicator related to the error due to the coarse mesh $\mathcal{T}_H$. The others sums should be interpreted as the indicators of the error due to the fine meshes $\mathcal{T}_h(K)$.

Proof of Theorem 2 Let u_H^h be the solution of (10) and $w = u - u_H^h$. Then, by the Galerkin orthogonality, we have for any $w_H^h \in \mathbb{V}_H^h$:

$$|u - u_H^h|_\nu^2 = \int_\Omega \nu\nabla(u - u_H^h) \cdot \nabla(w - w_H^h)\mathrm{d}x,$$
$$= \sum_{K\in\mathcal{T}_H} \left[\sum_{T\in\mathcal{T}_h(K)} \int_T \nu\nabla(u - u_H^h) \cdot \nabla(w - w_H^h)\mathrm{d}x \right].$$

Using (1) and an integration by parts over all the triangles $T \in \mathcal{T}_h(K)$, for all $K \in \mathcal{T}_H$, yields:

$$|u - u_H^h|_\nu^2 = \sum_{K\in\mathcal{T}_H} \left\{ \sum_{T\in\mathcal{T}_h(K)} \int_T (f + \mathrm{div}(\nu\nabla u_H^h))(w - w_H^h)\mathrm{d}x \right.$$
$$\left. + \sum_{e\in\mathcal{E}_h(K)} \int_e \left[\nu\nabla_{\mathbf{n}_e} u_H^h\right]_e (w - w_H^h)\mathrm{d}s \right\}.$$

Since the edges from $\mathcal{E}_h$ is divided into two groups: $\mathcal{E}_H$, the set of the boundary of coarse triangles, and $\mathcal{E}_h^0(K)$, the set of internal edges from $\mathcal{E}_h(K)$ for some triangle $K \in \mathcal{T}_H$, the last sum can be rewritten as:

$$|u - u_H^h|_\nu^2 = \left\{ \sum_{K\in\mathcal{T}_H} \int_K f(w - w_H^h)\mathrm{d}x \right.$$
$$\left. + \sum_{E\in\mathcal{E}_H} \int_E \left[\nu\nabla_{\mathbf{n}_E} u_H^h\right]_E (w - w_H^h)\mathrm{d}s \right\}$$
$$+ \left\{ \sum_{K\in\mathcal{T}_H} \left[\sum_{T\in\mathcal{T}_h(K)} \int_T \mathrm{div}(\nu\nabla u_H^h)(w - w_H^h)\mathrm{d}x \right.\right.$$
$$\left.\left. + \sum_{e\in\mathcal{E}_h^0(K)} \int_e \left[\nu\nabla_{\mathbf{n}_e} u_H^h\right]_e (w - w_H^h)\mathrm{d}s \right] \right\},$$
$$:= T_1 + T_2.$$

We now take $w_H^h = \mathcal{A}_H^h w$ where $\mathcal{A}_H^h$ is defined in (20), and use Proposition 2 and the inequality (3) to estimate the terms in T_1:

$$T_1 \preceq \sum_{K\in\mathcal{T}_H} \frac{H_K}{\underline{\nu}_{\omega_K}^{1/2}} \|f_K\|_{0,T} |w|_{\nu,\omega_K} + \sum_{E\in\mathcal{E}_H} \frac{H_E^{1/2}}{\underline{\nu}_{\omega_E}^{1/2}} \left|\left| \left[\nu\nabla_{\mathbf{n}_E} u_H^h\right] \right|\right|_{0,E} |w|_{\nu,\omega_E}$$
$$+ \sum_{K\in\mathcal{T}_H} \frac{H_K}{\underline{\nu}_{\omega_K}^{1/2}} \|f - f_K\|_{0,K} |w|_{\nu,\omega_K},$$

$$\preceq \left\{ \sum_{K\in\mathcal{T}_H} \left(\eta^2_{K,h} + \frac{H_K}{\underline{\nu}^{1/2}_{\omega_K}} \|f - f_K\|^2_{0,K} \right) \right\}^{1/2} |w|_\nu .$$

To estimate the terms in T_2, we shall use the definition of the space $\mathbb{V}^h_H$. Since u^h_H is an element of $\mathbb{V}^h_H$, it satisfies on any $K \in \mathcal{T}_H$ [see (7), (8), and (9)]:

$$\int_K \nu \nabla u^h_H \cdot \nabla v^h \mathrm{d}x = 0, \quad \forall v^h \in \mathbb{V}^0_h(K). \tag{37}$$

An integration by parts over all the triangles $T \in \mathcal{T}_h(K)$ yields for all $v^h \in \mathbb{V}^0_h(K)$:

$$\sum_{e\in\mathcal{E}^0_h(K)} \int_e \left[\nu \nabla_{\mathbf{n}_e} u^h_H\right]_e v_h \mathrm{d}s + \sum_{T\in\mathcal{T}_h(K)} \int_T \mathrm{div}(\nu\nabla u^h_H) v_h \mathrm{d}x = 0.$$

Thus, one can write for any $v^h \in C(\Omega)$, such that $v^h|_K \in \mathbb{V}^0_h(K)$, for all $K \in \mathcal{T}_H$:

$$T_2 = \sum_{K\in\mathcal{T}_H} \left[\sum_{T\in\mathcal{T}_h(K)} \int_T \mathrm{div}(\nu\nabla u^h_H) \left((w - w^h_H) - v^h \right) \mathrm{d}x \right.$$
$$\left. + \sum_{e\in\mathcal{E}^0_h(K)} \int_e \left[\nu \nabla_{n_e} u^h_H\right]_e \left((w - w^h_H) - v^h \right) \mathrm{d}s \right].$$

Let us now choose $\widetilde{\Pi}_h(w) = v^h + w^h_H$, such that $v^h|_{\partial K} = 0$, where $\widetilde{\Pi}_h$ is the interpolation operator defined in (26). With the help of Proposition 3, one gets:

$$T_2 \preceq \sum_{K\in\mathcal{T}_H} \left[\sum_{T\in\mathcal{T}_h(K)} ||div(\nu\nabla u^h_H)||_{0,T} \left\| w - \widetilde{\Pi}_h(w) \right\|_{0,T} \right.$$
$$\left. + \sum_{e\in\mathcal{E}_h(K)} || \left[\nu n_e \cdot \nabla u^h_H\right]_e ||_{0,e} \left\| w - \widetilde{\Pi}_h(w) \right\|_{0,e} \right].$$

We have $v^h \in \mathbb{V}^0_h(K)$ and $w^H_h \ |_K \in \mathbb{V}_h(K)$, and then, $v^h + w^h_H \ |_K \in \mathbb{V}_h(K)$.

Let $\widetilde{\Pi}_h : \mathbb{V} \to \mathbb{V}_h(K)$, such as $\widetilde{\Pi}_h(w) = v^h + w^h_H \ |_K$. We make the estimates taking into account that $v^h|_{\partial K} = 0$ and $\widetilde{\Pi}_h(w) = v^h + w^h_H$, and using Proposition 3, we obtain:

$$\left\| w - \widetilde{\Pi}_h(w) \right\|_{0,T} \preceq h_T |w|_{1,\omega_T} + \alpha_T h_T^{1/2} \left\| w - \widetilde{\Pi}_h(w) \right\|_{0,\partial\omega_T\cap\partial K},$$
$$\left\| w - \widetilde{\Pi}_h(w) \right\|_{0,e} \preceq h_e^{1/2} |w|_{1,\omega_e} + \alpha_e \left\| w - \widetilde{\Pi}_h(w) \right\|_{0,\partial\omega_e\cap\partial K},$$

which give:

$$\|w - \Pi_h w\|_{0,T} \preceq h_T |w|_{1,\omega_T} + \alpha_T h_T^{1/2} \left\| w - (v^h + w^h_H) \right\|_{0,\partial\omega_T\cap\partial K},$$
$$\left\| w - \widetilde{\Pi}_h(w) \right\|_{0,e} \preceq h_e^{1/2} |w|_{1,\omega_e} + \alpha_e \left\| w - (v^h + w^h_H) \right\|_{0,\partial\omega_e\cap\partial K},$$

and since $v^h|_{\partial K} = 0$, we obtain:

$$\left\| w - \widetilde{\Pi}_h(w) \right\|_{0,T} \preceq h_T |w|_{1,\omega_T} + \alpha_T h_T^{1/2} \left\| w - w^h_H \right\|_{0,\partial\omega_T\cap\partial K},$$
$$\left\| w - \widetilde{\Pi}_h(w) \right\|_{0,e} \preceq h_e^{1/2} |w|_{1,\omega_e} + \alpha_e \left\| w - w^h_H \right\|_{0,\partial\omega_e\cap\partial K}.$$

$$T_2 \preceq \sum_{K\in\mathcal{T}_H}\left[\sum_{T\in\mathcal{T}_h(K)} ||\mathrm{div}(\nu\nabla u_H^h)||_{0,T}\left(h_T|w|_{1,\omega_T} + \alpha_T h_T^{1/2}\left\|w-w_H^h\right\|_{0,\gamma_{TK}}\right)\right.$$
$$\left. + \sum_{e\in\mathcal{E}_h(K)} ||\left[\nu\nabla_{\mathbf{n}_e} u_H^h\right]_e||_{0,e}\left(h_e^{1/2}|w|_{1,\omega_e} + \alpha_e\left\|w-w_H^h\right\|_{0,\gamma_{eK}}\right)\right],$$

where $\gamma_{TK} = \partial\omega_T \cap \partial K$, $\gamma_{eK} = \partial\omega_e \cap \partial K$, α_T, and α_e are defined in Proposition 3. We now gather the terms in the sums above and use Cauchy–Schwarz, and then, (24) to get:

$$T_2 \preceq \sum_{K\in\mathcal{T}_H}\left(\sum_{T\in\mathcal{T}_h(K)} h_T^2||div(\nu\nabla u_H^h)||_{0,T}^2 + \sum_{e\in\mathcal{E}_h^0(K)} h_e||\left[\nu n_e\cdot\nabla u_H^h\right]_e||_{0,e}^2\right)^{1/2}|w|_{1,K}$$
$$+\sum_{E\in\mathcal{E}_H}\left(\sum_{T\in\mathcal{T}_h^b(E)} h_T||div(\nu\nabla u_H^h)||_{0,T}^2 + \sum_{e\in\mathcal{E}_h^b(E)} ||\left[\nu n_e\cdot\nabla u_H^h\right]_e||_{0,e}^2\right)^{1/2} H_E^{1/2}|w|_{1,\omega_E}.$$

Using again the inequality (3), we obtain:

$$T_2 \preceq \left(\sum_{K\in\mathcal{T}_H}\delta_K^2 + \sum_{E\in\mathcal{E}_H}\xi_E^2\right)^{1/2} |w|_\nu.$$

Combining together the bounds for T_1 and T_2 and recalling that $w = u - u_H^h$, this leads to (36) □

4.2.2 A lower bound for the error

To have an optimal a posteriori error estimate of the error, we have to prove that the indicators $\eta_{K,h}$, δ_K, and ξ_E defined in (33), (34), and (34), respectively, are locally lower bounds of the error.

Theorem 3 *Let $f \in \mathbb{L}^2(\Omega)$, u_H^h the solution of (10) and $\eta_{K,h}$, δ_K, and ξ_E defined by (33), (34), and (34), respectively. The following estimates hold for all $K\in\mathcal{T}_H$ and all $E\in\mathcal{E}_H$:*

$$\eta_{K,h} \preceq \left\{\left(\frac{\overline{\nu}_K}{\underline{\nu}_{\omega_K}}\right)|u-u_H^h|_{\nu,K}^2 + \frac{H_K^2}{\underline{\nu}_{\omega_K}}\|f-f_K\|_{0,K}^2\right.$$
$$+\sum_{E\in\mathcal{E}_H(K)}\sum_{T\in\mathcal{T}_h^b(E)}\left\{\left(\frac{\overline{\nu}_T}{\underline{\nu}_{\omega_K}}\right)\left(\frac{H_K}{h_T}\right)|u-u_H^h|_{\nu,T}^2\right\}$$
$$\left.+\frac{1}{\underline{\nu}_{\omega_K}}\left(\sum_{T\in\mathcal{T}_h(K)} h_T^2\|f\|_{0,T}^2\right)\right\}^{1/2},$$

$$\delta_K \preceq \left\{\left(\frac{\overline{\nu}_K}{\underline{\nu}_K}\right)|u-u_H^h|_{\nu,K}^2 + \frac{H_K^2}{\underline{\nu}_K}\|f-f_K\|_{0,K}^2 + \frac{1}{\underline{\nu}_K}\left(\sum_{T\in\mathcal{T}_h(K)} h_T^2\|f\|_{0,T}^2\right)\right\}^{1/2},$$

$$\xi_E \preceq \left\{\sum_{T\in\mathcal{T}_h^b(E)}\left(\frac{\overline{\nu}_T}{\underline{\nu}_{\omega_E}}\right)\left(\frac{H_E}{h_T}\right)|u-u_H^h|_{\nu,T}^2 + \sum_{K\subset\omega_K}\frac{H_K^2}{\underline{\nu}_{\omega_E}}\|f-f_K\|_{0,K}^2\right.$$

$$\sum_{K\in\omega_K}\left(\left(\frac{\overline{\nu}_K}{\underline{\nu}_{\omega_E}}\right)|u-u_H^h|^2_{\nu,K}+\frac{1}{\underline{\nu}_{\omega_E}}\sum_{T\in\mathcal{T}_h(K)}h_T^2\|f\|^2_{0,T}\right)\Bigg\}^{1/2}.$$

Proof We will successively give upper bounds of $\|\mathrm{div}(\nu\nabla u_H^h)\|_{0,T}$, $\|\left[\nu\nabla u_H^h\cdot n_e\right]_e\|_{0,e}$, $\|\left[\nu\nabla u_H^h\cdot n_E\right]_E\|_{0,E}$, and $\|f_K\|_{0,K}$, for arbitrary $T\in\mathcal{T}_h(K)$, $e\in\mathcal{E}_h(K)$, $E\in\mathcal{E}_H$, and $K\in\mathcal{T}_H$.

1. As in a standard way, we take $w_T=b_T div(\nu\nabla u_H^h)$. By (17), we have:

$$\|div(\nu\nabla u_H^h)\|^2_{0,T}\preceq\int_T w_T div(\nu\nabla u_H^h)dx.$$

The continuous problem (1) with $v=w_T$ and an integration by parts give the following relation:

$$\begin{aligned}\int_T w_T\mathrm{div}(\nu\nabla u_H^h)&=\int_T w_T\mathrm{div}(\nu\nabla u_H^h)+\int_T\nu\nabla u\cdot\nabla w_T-\int_T f w_T\\&=\int_T\nu\nabla(u-u_H^h)\cdot\nabla w_T+\int_T f_K w_T+\int_T(f_K-f)w_T,\end{aligned}$$

and by (18), we deduce the following estimate:

$$\|\mathrm{div}(\nu\nabla u_H^h)\|_{0,T}\preceq\overline{\nu}_T^{1/2}h_T^{-1}|u-u_H^h|_{\nu,T}+\|f-f_K\|_{0,T}+\|f_K\|_{0,T}. \tag{38}$$

2. With an analogous argument as before, we consider $w_e=b_e[\nu\nabla u_H^h\cdot n_e]_e$. We have:

$$\|[\nu\nabla u_H^h\cdot n_e]_e\|^2_{0,e}\preceq\int_e w_e[\nu\nabla u_H^h\cdot n_e]_e,$$

and

$$\begin{aligned}\int_e w_e[\nu\nabla u_H^h\cdot n_e]_e&=\sum_{T\subset\omega_e}\left(\int_T\mathrm{div}(\nu\nabla u_H^h)w_e-\int_T\nu\nabla u_H^h\cdot\nabla w_e\right)\\&=\sum_{T\subset\omega_e}\int_T\mathrm{div}(\nu\nabla u_H^h)w_e+\int_{\omega_e}\nu\nabla(u-u_H^h)\cdot\nabla w_e\\&\quad+\int_{\omega_e}f_K w_e-\int_{\omega_e}(f-f_K)w_e,\end{aligned}$$

and finally, by (38), (18), (19), and the fact that there exist two positive constants c_1 and c_2 with $c_1h_e\le h_T\le c_2h_e$, we obtain:

$$\|[\nu\nabla u_H^h\cdot n_e]_e\|_{0,e}\preceq h_e^{-1/2}\overline{\nu}_{\omega_e}^{1/2}|u-u_H^h|_{\nu,\omega_e}+h_e^{1/2}\left(\|f_K\|_{0,\omega_e}+\|f-f_K\|_{0,\omega_e}\right). \tag{39}$$

3. Since $E=\bigcup_{e\in\mathcal{E}_h^b(E)}e$, the bound of $|\left[\nu\nabla u_H^h\cdot n_E\right]_E\|_{0,E}$ is given using the estimate (39):

$$\begin{aligned}\frac{H_E}{\underline{\nu}_{\omega_E}}\|[\nu\nabla u_H^h\cdot n_E]_E\|^2_{0,E}&=\frac{H_E}{\underline{\nu}_{\omega_E}}\sum_{e\subset E}\|[\nu\nabla u_H^h\cdot n_e]_e\|^2_{0,e},\\&\preceq\sum_{e\subset E}\left\{\left(\frac{H_E}{h_e}\right)\left(\frac{\overline{\nu}_{\omega_e}}{\underline{\nu}_{\omega_E}}\right)|u-u_H^h|^2_{\nu,\omega_e}\right.,\end{aligned}$$

$$
\begin{aligned}
&+\frac{H_E}{\underline{\nu}_{\omega_E}} h_e\left(\|f_K\|^2_{0,\omega_e}+\|f-f_K\|^2_{0,\omega_e}\right)\Big\},\\
&\preceq \sum_{T\in\mathcal{T}_h^b(E)}\left\{\left(\frac{H_E}{h_T}\right)\left(\frac{\overline{\nu}_T}{\underline{\nu}_{\omega_E}}\right)|u-u_H^h|^2_{\nu,T}\right.\\
&\left.+\frac{H_E}{\underline{\nu}_{\omega_E}} h_T\left(\|f_K\|^2_{0,T}+\|f-f_K\|^2_{0,T}\right)\right\}. \qquad (40)
\end{aligned}
$$

4. A bound of $\frac{H_K}{\nu_K^{1/2}}\|f_K\|_{0,K}$ will be obtained by analogous arguments as the proof of Theorem 1 using u_H^h which, here, satisfies Eq. (37), a weak form of $L_K u_H = 0$ in K. We consider $w_K = b_K f_K$ and we denote by w_K^h its L^2-projection in $\mathbb{V}_0^h(K)$. By definition of the function w_K and w_K^h and using a scaling argument, we prove the following lemma:

□

Lemma 3

$$\|w_K\|_{0,K} \preceq \|f_K\|_{0,K}. \qquad (41)$$

$$\|w_K - w_K^h\|_{0,T} \preceq \frac{h_T}{H_K}\|f_K\|_{0,K}, \quad \forall T\in\mathcal{T}_h(K). \qquad (42)$$

$$|w_K^h|_{\nu,K} \preceq \frac{\overline{\nu}_K^{1/2}}{H_K}\|f_K\|_{0,K}. \qquad (43)$$

Proof of the lemma The first inequality is obvious since $|b_K|\leq 1$ in K. The second is obtained using the error interpolation estimate in each $T\in\mathcal{T}_h(K)$: $\|w_K-w_K^h\|_{0,T}\preceq h_T|w_K|_{1,\omega_T} = |f_K|h_T|b_K|_{1,\omega_T}$. Now, by scaling argument, we know that $|b_K|_{1,T}\preceq|\hat{b}_K|_{1,\hat{T}} = \hat{C}$. Furthermore, Since $|f_K| = \|f_K\|_{0,K}|K|^{-1/2}$, and $|K|^{-1/2}\preceq {H_K}^{-1}$, we obtain (42). The estimate (43) is obtained by scaling argument as previously. □

We return to the proof of the theorem. By (17), we have $\|f_K\|^2_{0,K}\preceq\int_K b_K f_K^2 dx$. Since $\mathrm{supp}(w_K^h) = K$, we have $a_K(u-u_H^h, w_K^h) = \int_K f w_H^h$. We deduce:

$$
\begin{aligned}
\int_K b_K f_K^2 dx &= \int_K w_K^h f + \int_K (w_K - w_K^h) f + \int_K w_K(f_K - f)\\
&= \int_K \nu\nabla(u-u_H^h)\cdot\nabla w_K^h + \int_K (w_K - w_K^h) f + \int_K w_K(f_K-f),\\
&=: R_1 + R_2 + R_3.
\end{aligned}
$$

$$
\begin{aligned}
R_1 &\preceq |u-u_H^h|_{\nu,K}|w_K^h|_{\nu,K} \preceq \frac{\overline{\nu}_K^{1/2}}{H_K}|u-u_H^h|_{\nu,K}\|f_K\|_{0,K} \text{ by (43)}.\\
R_3 &\preceq \|w_K\|_{0,K}\|f-f_K\|_{0,K} \preceq \|f-f_K\|_{0,K}\|f_K\|_{0,K} \text{ by (41)}.\\
R_2 &= \sum_{T\in\mathcal{T}_h(K)}\int_T (w_K-w_K^h) f \leq \sum_{T\in\mathcal{T}_h(K)}\|(w_K-w_K^h)\|_{0,T}\|f\|_{0,T},\\
&\preceq \frac{1}{H_K}\|f_K\|_{0,K}\sum_{T\in\mathcal{T}_h(K)} h_T\|f\|_{0,T} \text{ by (42)}.
\end{aligned}
$$

We obtain finally:

$$\frac{H_K}{\underline{\nu}_{\omega_K}^{1/2}}\|f_K\|_{0,K} \preceq \left(\frac{\overline{\nu}_K}{\underline{\nu}_{\omega_K}}\right)^{1/2}|u-u_H^h|_{\nu,K} + \frac{1}{\underline{\nu}_{\omega_K}^{1/2}}\left(\sum_{T\in\mathcal{T}_h(K)} h_T\|f\|_{0,T}\right) + \left(\frac{H_K}{\underline{\nu}_{\omega_K}^{1/2}}\right)\|f-f_K\|_{0,K}. \tag{44}$$

The lower bounds are now obtained by gathering all the terms.

By (44) and (40), we have:

$$\eta_{K,h} \preceq \left\{\left(\frac{\overline{\nu}_K}{\underline{\nu}_{\omega_K}}\right)|u-u_H^h|^2_{\nu,K} + \frac{H_K^2}{\underline{\nu}_{\omega_K}}\|f-f_K\|^2_{0,K} + \sum_{E\in\mathcal{E}_H(K)}\sum_{T\in\mathcal{T}_h^b(E)}\left\{\left(\frac{\overline{\nu}_T}{\underline{\nu}_{\omega_K}}\right)\left(\frac{H_K}{h_T}\right)|u-u_H^h|^2_{\nu,T}\right\} + \frac{1}{\underline{\nu}_{\omega_K}}\left(\sum_{T\in\mathcal{T}_h(K)} h_T^2\|f\|^2_{0,T}\right)\right\}^{1/2}.$$

By (38), (39), (44), and using the inequalities $h_e \leq H_K$ and $h_T \leq H_K$, we obtain the estimation:

$$\delta_K \preceq \left\{\left(\frac{\overline{\nu}_K}{\underline{\nu}_K}\right)|u-u_H^h|^2_{\nu,K} + \frac{H_K^2}{\underline{\nu}_K}\|f-f_K\|^2_{0,K} + \frac{1}{\underline{\nu}_K}\left(\sum_{T\in\mathcal{T}_h(K)} h_T^2\|f\|^2_{0,T}\right)\right\}^{1/2}.$$

By (38)and (39), since $h_T \leq H_K$, and then by (44), we have:

$$\xi_E \preceq \left\{\sum_{T\in\mathcal{T}_h^b(E)}\left(\frac{\overline{\nu}_T}{\underline{\nu}_{\omega_E}}\right)\left(\frac{H_E}{h_T}\right)|u-u_H^h|^2_{\nu,T} + \sum_{K\subset\omega_K}\frac{H_K^2}{\underline{\nu}_{\omega_E}}\|f-f_K\|^2_{0,K} + \sum_{K\in\omega_K}\left(\left(\frac{\overline{\nu}_K}{\underline{\nu}_{\omega_E}}\right)|u-u_H^h|^2_{\nu,K} + \frac{1}{\underline{\nu}_{\omega_E}}\sum_{T\in\mathcal{T}_h(K)} h_T^2\|f\|^2_{0,T}\right)\right\}^{1/2}.$$

Remark 2 In Theorem 3, the error indicators are increased by a quantity related to the H/h ratio; this means that we do not need to refine the fine mesh, so that this ratio is not big enough.

5 Implementation

In this section, we will show a numerical test which confirm the reliability and effectiveness of the *a posteriori* error estimates for the MsFEM method developed in the previous section. All the calculations presented are done with the software *FreeFem++* (Cf. Hecht and Pironneau 2021).

We consider the following problem:

$$\begin{cases} -\text{div}\,(\nu\nabla u) = f & \text{in } \Omega, \\ u = g & \text{on } \Gamma, \end{cases} \tag{45}$$

with $\Omega =]0,1[\times]0,1[$, $f(x,y) = xy$, $g(x,y) = xy$, $\varepsilon = 10^{-2}$, and:

$$\nu(x,y) = \begin{cases} 1 + 10\cos^2\left(\left(\dfrac{2}{3\varepsilon} + 2)\pi x\right)\right)\sin^2\left(\dfrac{4\pi y}{3\varepsilon}\right) & \text{if } x \geq \frac{3}{4} \text{ and } y \geq \frac{3}{4}, \\ 1 & \text{if } x \leq \frac{3}{4} \text{ or } y \leq \frac{3}{4}. \end{cases}$$

which present large oscillations in the part of the domain defined by $[\frac{3}{4}, 1[\times[\frac{3}{4}, 1[$.

The reference solution (U_{Ref}) we consider is the solution obtained by the standard finite-element method of degree 1, using a fine mesh $\mathcal{T}_{h_0}$ with $h_0 = 1/200$ (see Fig. 1 (a)).

To establish our MsFEM method, we need to build two meshes of Ω, $\mathcal{T}_H$ the coarse one and $\mathcal{T}_h$ the fine one. We first build the conforming coarse mesh $\mathcal{T}_H = \bigcup K$ with $H = \max_K H_K$ and H_K is the diameter of K. The fine mesh is then constructed in the following manner. Each element K of $\mathcal{T}_H$ will be meshed with a step equal to H_K/n_K, where n_K is an integer fixed in advance, by eventually taking into account the a priori information on the problem. We thus obtain a conforming mesh $\mathcal{T}_h(K) = \bigcup\limits_{T\subset K} T$. We denote the diameter of the element T by h_T and $h = \max_T h_T$. The global fine mesh of the domain is obtained by catering these last meshes and will not be necessary conforming:

$$\mathcal{T}_h = \bigcup_{K\in\mathcal{T}_H} \mathcal{T}_h(K).$$

In the first subsection, we present the resolution algorithm for the MsFEM scheme developed in this work and then the mesh adaptation algorithm. In the second subsection, we present the numerical calculations for the problem (45).

5.1 Algorithms

The resolution algorithm has the following structure:

Algorithm 1 *($\mathcal{T}_H$, n, f) (**Resolution algorithm**)*

- *a coarse mesh $\mathcal{T}_H = \bigcup K$, representing as well as possible the geometry of the problem;*
- *$\{S_i,\ i = 1, \ldots, N\}$ the set of vertices of $\mathcal{T}_H$;*
- *n a non-negative integer;*
- *the data f;*

1. *for each $K \in \mathcal{T}_H$, build the meshes $\mathcal{T}_h(K)$ with $h_T \approx H_K/n$;*
2. *for each i, $1 \leq i \leq N$, calculate the basis function Φ_i in each K (a part of the support of this function) as the solution of (7);*
3. *solve the discrete problem (10) (which is a $N \times N$ system), to calculate the solution u_H^h.*

The indicators developed in Theorem 1 and Theorem 2 are robust tools to find the elements K and T, where the error between the exact solution and the approximated solution is too large, and then must be refined to improve the quality of the numerical solution. Let η_K, $\delta_{K,h}$, and $\xi_{E,h}$ defined respectively by (29), (34), and (34). First, we refine $\mathcal{T}_H$, using only the indicators η_K, and $\xi_{E,h}$. Indeed, $\delta_{K,h}$ is much smaller than the others and can be used to adapt the meshes related to the basis functions (fine mesh).

The mesh adaptation process has the following general structure.

Algorithm 2 *(**Algorithm of adaptation**) Given*

$\mathcal{T}_H$, the coarse mesh;
$n \in \mathbb{N}^*$;
the date f;
Tol_H desired thresholds of error;
θ, such that $0 < \theta < 1$;
m the maximum number of iterations;

Initialisation

1. *put $k = 0$;*
2. *build $\mathcal{T}_h$, and calculate u_H^h by* **Algorithm 1***($\mathcal{T}_H$, n, f).*
3. *take $\mathcal{T}_{H,k} \leftarrow \mathcal{T}_H$ and $\mathcal{T}_{h,k} \leftarrow \mathcal{T}_h$.*

The indicators and refinement *while $k \le m$ do*

1. *for each $K \in \mathcal{T}_{H,k}$, calculate the error estimators η_K and $\xi_{E,h}$ given by (33) and (34), respectively;*
2. *If $\max\limits_{K \in \mathcal{T}_H, E \in \mathcal{E}_H} \{\eta_K, \xi_{E,h}\} \le Tol_H$, then* **STOP**.
3. *else*

(a) refine the elements K, such that: $\left(\max\limits_{K \in \mathcal{T}_H, E \in \mathcal{E}_H} \{\eta_K, \xi_{E,h}\}\right) \theta \ge Tol_H$,
(b) keep conformity with neighboring triangles to obtain the new coarse mesh $\mathcal{T}_H^*$;
(c) $k \leftarrow k + 1$;
(d) $\mathcal{T}_{H,k} \leftarrow \mathcal{T}_H^*$;
(e) calculate u_H^h by **Algorithm 1***($\mathcal{T}_{H,k}$, n, f)*

end while

We take $\theta = 4/7$, $Tol_H = 0,014$, and proceed to mesh adaptivity via the *a posteriori* estimators, using **Algorithm 2**. The first iteration of this algorithm gives the estimators for the solution based on the initial meshes. The isovalues of these estimators are given in Fig. 2 (a)–(b). Based on the criteria in the algorithm, we scored by *, in Fig. 2 (c), the elements $K \in \mathcal{T}_H$ to be refined .

To compare the MsFEM adaptation with standard FEM adaptation, we use the standard estimators given, for each K in $\mathcal{T}_H$ by:

$$I_K^2 := \frac{h_K^2}{\underline{\nu}_{\omega_K}} \|\mathrm{div}(\nu \nabla u_{FEM})\|_{0,K}^2 + \frac{1}{2} \sum_{E \in \partial K} \frac{h_E}{\underline{\nu}_{\omega_E}} \| [\nu \nabla u_{FEM} \cdot n] \|_{0,E},$$

where h_K is the diameter of K and h_E is the diameter of E, and we put

$ER_{MsFEM} := U_{Ref} - u_H^h$, the error between the reference solution and the MsFEM solution, and $ER_{FEM} := U_{Ref} - u_H^{EFM}$, the error between the reference solution and the solution obtained by the standard Finite-Element Method with the coarse mesh. We will consider different norms and put:

$e_{MsFEM}^1 := \|ER_{MsFEM}\|_{\mathbb{L}^2(\Omega)}$ and $e_{FEM}^1 := \|ER_{FEM}\|_{\mathbb{L}^2(\Omega)}$,
$e_{MsFEM}^2 := |ER_{MsFEM}|_{H^1(\Omega)}$ and $e_{FEM}^2 := |ER_{FEM}|_{H^1(\Omega)}$,
$e_{MsFEM}^3 := |ER_{MsFEM}|_{\nu}$ and $e_{FEM}^3 := |ER_{FEM}|_{\nu}$,
$e_{MsFEM}^4 := \|ER_{MsFEM}\|_{\infty}$ and $e_{FEM}^4 := \|ER_{FEM}\|_{\infty}$.

5.2 Numerical test

We consider the initial coarse mesh $\mathcal{T}_H$ given in Fig. 1(c) and the corresponding fine mesh given in Fig. 1(d), where we have chosen $n = 4$. By **Algorithm 1**$(\mathcal{T}_H, n, f)$, we obtain the solution represented by its isovalues in Fig. 1(b). We note that with this initial meshes, the MsFEM solution is less regular than the reference one.

The first adaptation of the coarse mesh $\mathcal{T}_H$ is made by cutting by 4 the elements K to obtain a finer conforming mesh given in Fig. 3a and the corresponding fine mesh given in Fig. 3b. The solution related to the new meshes is given in Fig. 3f. We note that the modifications are made near the two refined elements. Figure 3 (c), (d), and (e) shows that the isovalues of indicators are smaller near the refined elements.

This procedure of **Algorithm 2** is continued, until the estimators reach Tol_H. This is done at the 6^{th} iteration for the MsFEM solution. In Fig.4, we find all the information about the solution MsFEM at this iteration. The solution FEM presented in this figure is obtained by starting with the initial coarse mesh and using adaptivity via the estimators I_K, until these estimators reach Tol_H. This is done at the 8^{th} iteration and the results are also presented in Fig. 4.

We notice that MsFEM captures the oscillations much better than the FEM, with a lower cost, since the size of the MsFEM system is smaller than the FEM system (see Table 1). We see in Fig. 4 that the error given by FEM is large in the oscillation zone, while MsFEM gives a better approximation in this zone.

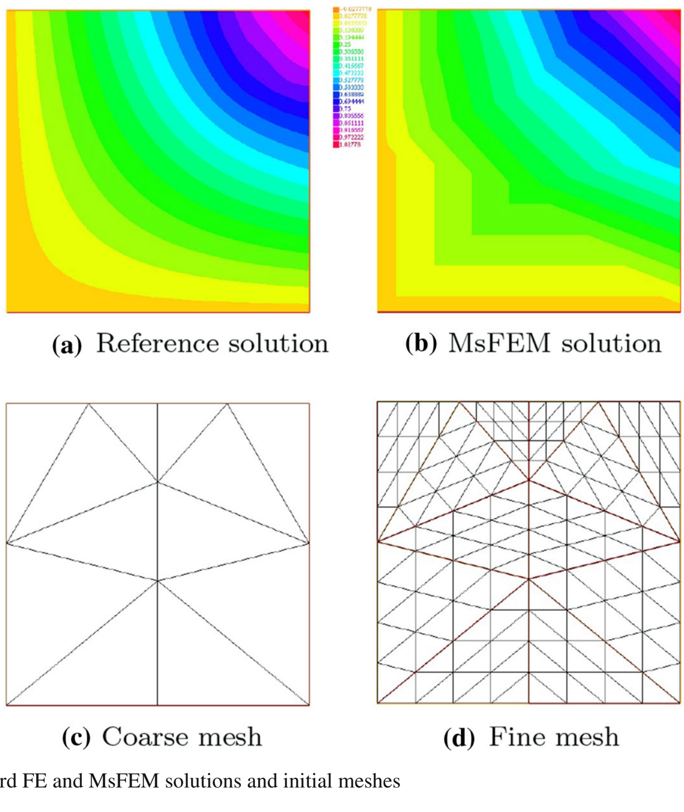
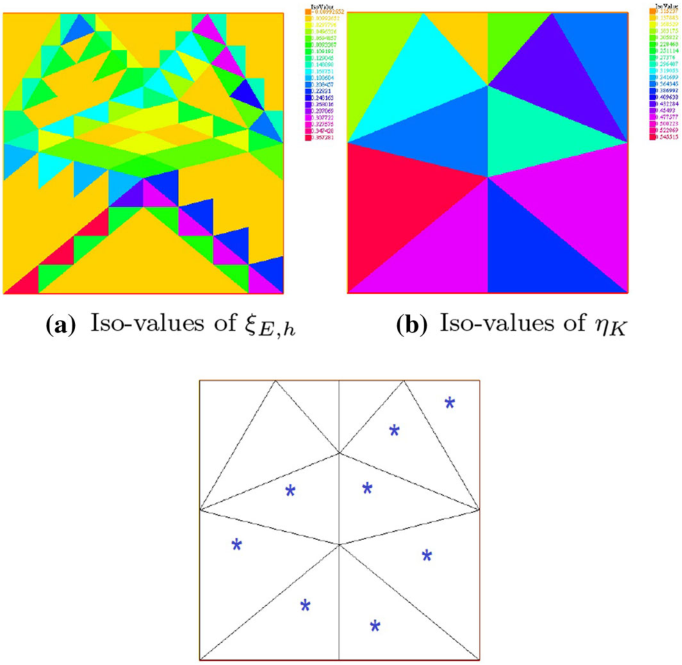


Fig. 1 Standard FE and MsFEM solutions and initial meshes



(a) Iso-values of $\xi_{E,h}$ **(b)** Iso-values of η_K

(c) Detected zones,

Fig. 2 Isovalues of the indicators related to initial mesh

In Table 1, we give a summary of the numerical results for this test. The notation ITE means "iteration". The suffix "max" or "min" means the maximum value or the minimum value, respectively.

At first sight, we note that the error decreases with the iterations of the mesh adaptation as well as the indicators and their gaps between the maximum and the minimum values. These indicators detect the location where the MsFEM solution is not accurate enough. One of the gaps can sometimes increase, but globally, it decreases. For the same tolerance of the error, MsFEM needs less iterations than FEM, and the solution is better in the different norms, as shown in Table 1. Furthermore, this strategy leads to an equidistributivity of the error, since the gap between the indicators $\eta_{K,h}$ and $\xi_{E,h}$ decreases from the first iteration to the last one as well as the others gaps (see Table 1).

In Fig. 5a, we plotted the log of the maximum of the indicator η_K as a function of the number of degree of freedom, and in Fig. 5b, we find a comparison between the error MsFEM and the error FEM as functions of the number of iterations.

We tested the robustness of our MsFEM adaptive algorithm when the parameter ε goes to 0. In Fig. 5c, we see that the error decreases in a similar way for the three values $\varepsilon = 10^{-2}$, $\varepsilon = 10^{-3}$ or $\varepsilon = 10^{-4}$. Furthermore, we do not need the have the fine mesh very fine. Indeed, in Fig. 5c, we can see that before adaptation procedure, the error is the same for the three fine meshes related to $h = H/4$, $h = H/8$, and $h = H/16$, and it is not recommended to take the mesh $\mathcal{T}_h$ very fine when we adapt $\mathcal{T}_H$.

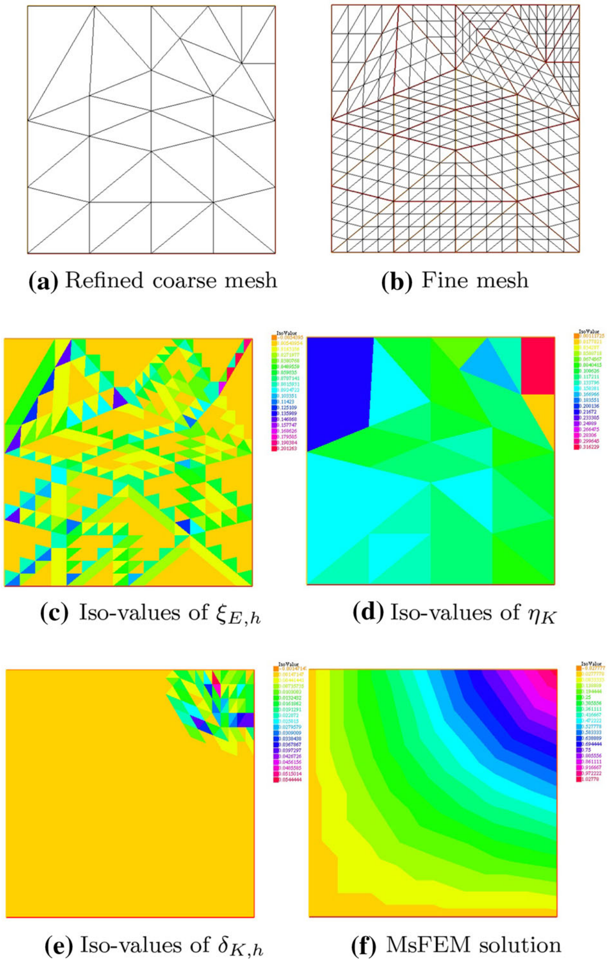
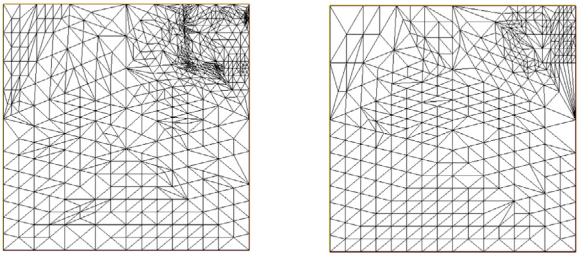


Fig. 3 Refined meshes, the indicators, and the new solution at the first iteration of adaptation

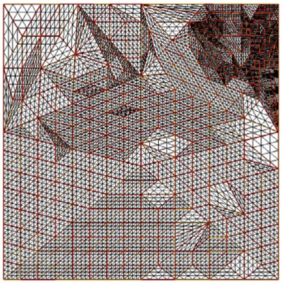
6 Conclusion

In this work, we have presented an a posteriori analysis of a multi-scale finite-element method (MsFEM), for a diffusion problem with highly oscillating coefficients. We derived upper and lower bounds for the approximation error and presented a numerical test confirming their performance in regard to their efficiency and reliability. The indicators obtained are of residual type. Those related to the fine mesh are, in some how, standard and represent the residual equation and the jump of the normal derivative of the solution through the interfaces of the fine mesh. The others are also of residual type and take into account the linearity constraint on the interfaces of the coarse mesh.

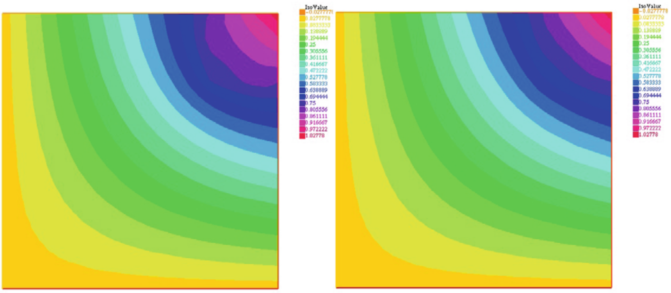
It was noticed in the work Lozinski et al. (2013) that the error decreases with the step H of the coarse mesh and a good precision is reached for a relatively small number of basis functions, whereas the difference between the errors associated with the different steps h of the fine mesh is minimal. As a consequence, in the numerical test, we opted to refine only the



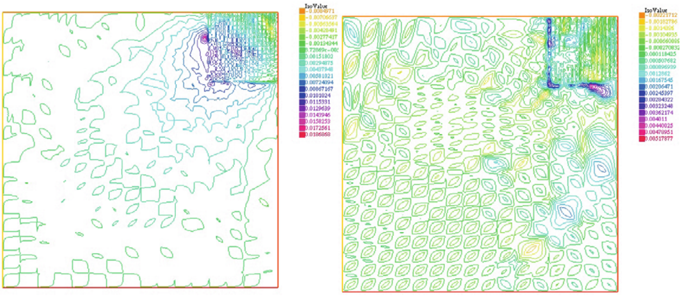
(a) FEM adapted mesh **(b)** MsFEM coarse adapted mesh



(c) MsFEM fine adapted mesh



(d) FEM-Solution **(e)** MsFEM-Solution



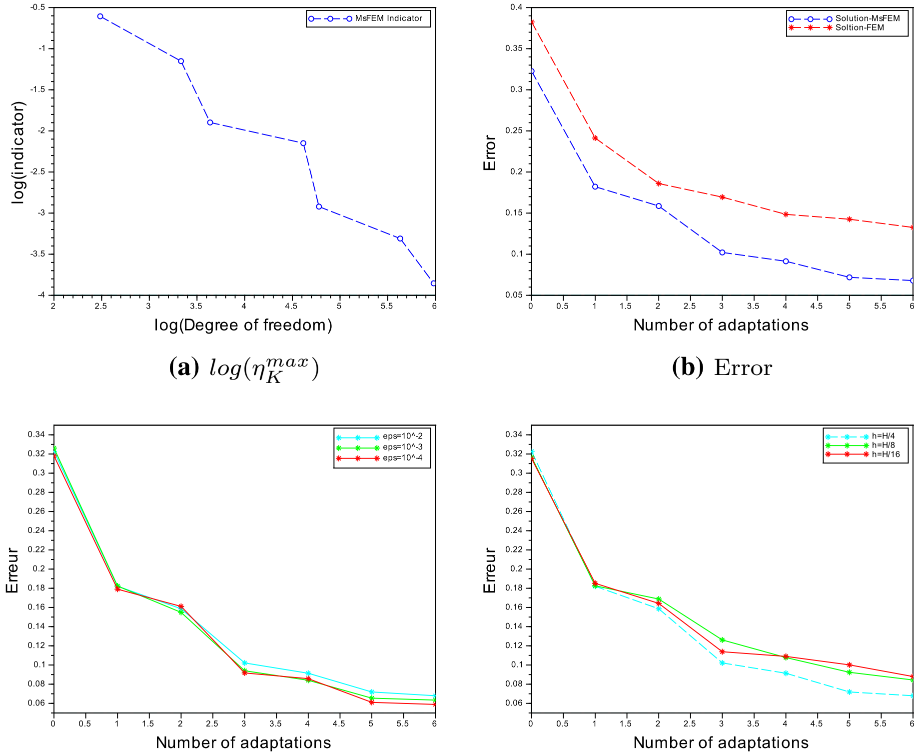
(f) FEM-Error **(g)** MsFEM-Error

Fig. 4 The last mesh adaptation

Table 1 Summary of numerical results

	ITE_0	ITE_1	ITE_2	ITE_3	last ITE
MsFEM					ITE_6
$\eta_K^{\max}$	0.545515	0.316229	0.149957	0.116572	0.02119805
$\eta_K^{\min}$	0.115237	0.017704	0.024604	0.00566127	0.00123631
Gap	0.430278	0.298525	0.125336	0.1109107	0.0199617
$\xi_{E,h}^{\max}$	0.367281	0.20126	0.136833	0.092551	0.0377464
$\xi_{E,h}^{\min}$	0.00992652	0.005439	0.0036982	0.00250138	0.00102017
Gap	0.3573545	0.195821	0.1331348	0.0900496	0.0367262
$\delta_{K,h}^{\max}$	0.215127	0.054444	0.0461517	0.0270767	0.0487454
$\delta_{K,h}^{\min}$	0.00581424	0.00147147	0.00124734	0.000731804	0.00131744
Gap	0.2093128	0.0529729	0.0449044	0.0263449	0.0474280
e_{MsFEM}^1	0.0255982	0.00791987	0.00591898	0.0024664	0.000608505
e_{MsFEM}^2	0.276851	0.152242	0.136405	0.0862585	0.0499367
e_{MsFEM}^3	0.322717	0.182145	0.158629	0.102155	0.0679383
e_{MsFEM}^4	0.082402	0.0326742	0.0286513	0.0144839	0.00498414
Number vertices	12	28	38	101	396
FEM					ITE_8
$I^{\max}$	0.546074	0.22284	0.129121	0.0697954	0.0164632
$I^{\min}$	0.0114615	0.0442334	0.0221407	0.0115651	0.000614228
Gap	0.431459	0.1786066	0.1069803	0.0582303	0.0158521
e_{FEM}^1	0.0328344	0.00978498	0.00575679	0.00458183	0.00282314
e_{FEM}^2	0.314588	0.206161	0.158455	0.140437	0.101871
e_{FEM}^3	0.382601	0.241343	0.185971	0.169446	0.125633
e_{FEM}^4	0.128995	0.0556675	0.0228517	0.0188082	0.0179714
Number vertices	12	26	62	92	644

coarse mesh. In comparison with the standard Finite-Element Method, MsFEM gives better approximation with lower cost.



(a) $log(\eta_K^{max})$ **(b)** Error

(c) Error **(d)** Error

Fig. 5 $\log(\eta_K^{max})$ and error

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