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How do enterprises determine which breakthrough invention should be commercialized? A multiple attribute group decision-making-based method

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Abstract

The problem of selection of breakthrough inventions to be commercialized (SBIC) is significantly important for innovative enterprises. As the processes of commercializing breakthrough inventions are time-consuming and high-cost, enterprises have to tradeoff among all possible inventions and determine which inventions should be commercialized. SBIC refers to a process in which multiple experts are invited to evaluate the potential of all inventions from multiple aspects and based on their assessments, the breakthrough inventions to be commercialized are chosen. Hence, the essence of SBIC is consistent with multiple attribute group decision-making (MAGDM). This paper proposes a MAGDM-based model to help enterprises conduct SBIC. In our model, probabilistic dual Pythagorean hesitant fuzzy sets (PDPHFSs) are utilized to represent DMs' evaluation information, which can express DMs' evaluation information comprehensively. Then, an attributes' weights determination method is proposed to handle MAGDM with unknown weights information. Afterwards, we extend the classical projection model into PDPHFSs and introduce a novel MAGDM method and its detailed calculation process is illustrated. Finally, our MAGDM method is applied to an SBIC problem to show its effectiveness. Comparison analysis is conducted to show the advantages of our method.

Keywords Breakthrough inventions commercialization · Multiple attribute group decision-making · Probabilistic dual Pythagorean hesitant fuzzy sets · Weights determination method · Projection model

Mathematics Subject Classification 03E72 · 03E75 · 90B50 · 94D05

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1 Introduction

As main productions of breakthrough innovation, breakthrough inventions are a new engine for the development of enterprises in the future market, which has been realized by both entrepreneurs and scholars, and received increasing attention in past decades. The promotion of breakthrough inventions is so difficult, complicated, and uncertain to be carried out, however, owing to the scarcity of innovative resources and funds (Kim et al. 2019). Therefore, more inventions are produced than to be commercialized in most large innovative enterprises. Generally, the process of selection of breakthrough inventions that to be commercialized (SBIC) requires the participation of multiple managers from the entire organization, which is also a rule that managers should follow when allocating scarce resources (Vinokurova and Kapoor 2020). During SBIC process, the main challenge that innovative enterprises have to meet is to determine the promising inventions that can help organizations gain advantages in future competition. SBIC is usually a resource-intensive process, in which, from the very beginning to commercialization, breakthrough inventions must go through multiple stages, involving multiple levels and participants (Vinokurova and Kapoor 2020). For instance, each production proposal should undergo at least 40 or 50 rounds of evaluations before it can be officially accepted by CEO (Herbold 2002). Hence, evidences have shown that not all breakthrough inventions can be commercialized and the problem of SBIC is significantly important for most innovative enterprises.

By focusing on the knowledge-based processes or managerial decision-making processes, scholars in strategic management have discovered many organizational factors that affect SBIC (Tripsas and Gavetti 2000; Christensen et al. 2008). Basically, these factors include expectations from investors and analysts for the future of an enterprise (Noda and Bower 1996), as well as their concern that new inventions may destroy the value of existing advantages or current supplementary assets (Tripsas 1997, Wu et al. 2014), managers' awareness of business models or new trends (Gilbert 2006), demand from important customers (Rosenbloom 2000), inventors' own rules (Vinokurova and Kapoor 2020), etc. In addition, extant studies have revealed difficulties that enterprises have to encounter in SBIC process (Kim et al. 2019; Vinokurova and Kapoor 2020), however, to the best of our knowledge, few researches have been conducted on guiding enterprises to determine appropriate inventions in the SBIC process, up to the present. This situation motivates us to study a method that helps enterprises choose potential inventions in the SBIC process. By reviewing existing literature, we notice that although there is no method appeared in formal academic papers to guide enterprises to make decisions in the SBIC process, enterprises themselves usually have must-be-following rules or principles, which are usually generated from a previous product of the company or popular routines and the experience of decision-making teams (Benner and Tripsas 2012; Tripsas and Gavetti 2000). When enterprises evaluate all candidate inventions in the SBIC process, they mainly focus on market demands, outlook for profits, feasibility and flexibility of an invention. In other words, when an enterprise is conducting SBIC, a group of domain experts are invited to evaluate the performance of each invention under the above-mentioned multiple attributes. Inventions tend to be commercialized are determined according to experts' opinions. Therefore, the SBIC process can be regarded as a multi-attribute group decision-making procedure, in which several decision makers (DMs') are required to express their evaluations over alternatives under a collection of attributes and the final decision results are determined according to DMs' opinions. Based on the above analysis, in this article we attempt to propose a method that can guide enterprise select appropriate inventions in the SBIC process based on the framework of MAGDM.

When dealing with MAGDM problems, one of the most important issues for DMs is how to obtain the ranking order of all candidate alternatives and select the optimal one. Hence, quite a few researchers focused on outranking method that can rank all possible solutions and help DMs to choose the best one. The classical TOPSIS (Chen and Hwang 1992) method is a powerful decision-making method that can assist DMs rank all alternatives based on their evaluation information. In the traditional TOPSIS method, an imaginary positive ideal solution and a negative ideal solution are proposed and the relative importance of each alternative is calculated based on the distance or dissimilarity between the alternative and the positive and negative ideal solutions. It is worth pointing out that the classical TOPSIS was developed for crisp numbers. In another word, the classical TOPSIS can only deal with MAGDM problems in which attribute values are denoted by real numbers. Obviously, the first version of TOPSIS is insufficient and inadequate to handle modern decision-making issues, due to the increased complexity and uncertainties that exist in actual-life MAGDM problems. Publications in the past decades indicate that more and more scholars have begun to deal with fuzzy information in MAGDM problems and DMs' complex cognition from the perspective of fuzzy set theories. Quite a few information representations tools, such as fuzzy sets (Zadeh 1965), intuitionistic fuzzy sets (Atanassov 1999), interval-valued intuitionistic fuzzy sets (Xu and Cai 2009), hesitant fuzzy sets (Torra 2010), dual hesitant fuzzy sets (Zhu et al. 2012), Pythagorean fuzzy sets (Yager 2013), etc., have been widely applied to handling uncertain and vague evaluation information in MAGDM process. Hence, the classical TOPSIS has been extended to different fuzzy sets to accommodate different decision-making environments. For instance, He and Gong (2012) extended the TOPSIS to intuitionistic fuzzy sets and proposed a novel MAGDM method. Muhammad et al. (2019) and Zhan et al. (2010) considered the TOPSIS method under Pythagorean fuzzy sets. Other forms of the classical TOPSIS under hesitant fuzzy sets, hesitant fuzzy linguistic terms sets, dual hesitant fuzzy sets, double hierarchy hesitant fuzzy linguistic term set, etc., can be found in Hu et al. (2016), Qu et al. (2017) and Jordi et al. (2018).

In TOPSIS method, the absolute difference between alternatives is reflected by distance measures. Hence, TOPSIS is still inadequate and insufficient to deal with practical MAGDM problems as it only measures the closeness degree of alternatives from the perspective of numerical sizes. In contrast, the projection model not only considers an alternative's closeness degree in numerical sizes but also takes the direction between two alternatives into consideration when considering the closeness degree of alternatives. In another word, the projection model determines the closeness degree of an alternative from both module and direction aspects. Hence, compared with TOPSIS, the projection model is more suitable and powerful to address realistic MAGDM problems. In addition, it is noted by many researchers and scholars that in most real decision-making problems, decision makers' evaluation information is fuzzy, vague and uncertain (Li et al. 2021; Deng et al. 2022; Wang et al. 2022; Zhan et al. 2021Kang et al. 2020). Hence, like TOPISIS, the classical projection model has also been extended to different fuzzy sets to correspond various decision-making environments. Yue (2013) studied the projection model under the intuitionistic fuzzy circumstance and applied it in a business partner selection problem. Sun et al. (2017) investigated the projection model under hesitant fuzzy linguistic terms sets and applied it in patients' prioritization in Chinese hospitals. Zhang et al. (2018) and Ni et al. (2021) investigated the projection method-based MAGDM method in decision-making problems with probabilistic linguistic and dual hesitant fuzzy information, respectively. In Wan et al. (2018) and Lu et al. (2019), researchers reported new improvements in the projection model in Pythagorean fuzzy and dual hesitant Pythagorean fuzzy contexts, respectively. These publications reveal the effectiveness and merits of the projection model, and indicate that projection model-based MAGDM methods are a compromising research direction.

The recently proposed probabilistic dual Pythagorean hesitant fuzzy set (PDPHFS), introduced by Ji et al. (2021), is an effective tool to depict DMs' complex evaluation values. PDPHFS is extended from a dual Pythagorean hesitant fuzzy set (DPHFS) (Wei and Lu 2017). Compared with DPHFS, PDPHFS is more powerful and flexible, as it considers not only multiple membership degrees (MDs) and non-membership degrees (NMDs), but also take their corresponding probabilistic information into account. In addition, the constraint of PDPHFS is that the square sum of MD and NMD should be less than or equal to one, which also makes it more functional and powerful than DPHFS and the traditional probabilistic dual hesitant fuzzy set (PDHFS) (Hao et al. 2017). Existing publications studied PDPHFSs-based MAGDM methods from the perspective of aggregation operators, however, nothing has been done on the projection model under PDPHFSs, which is worth being conducted by researchers. Given exiting studies on projection model-based MAGDM methods, which illustrates the good performance of the projection model, this article extends the projection model to PDPHFSs to produce a novel MAGDM method. Compared with traditional projection model-based MAGDM methods, our study can extend the application range of the traditional projection model. Compared with some existing PDPHFSs-based MAGDM method, our method can produce more reliable and reasonable decision-making results. (Details are provided in the comparison analysis). At last, the new method is applied to a subsistent decision-making issue. The main contributions of this paper are three-fold. (1) The projection model is investigated in PDPHFSs. The proximities in module and direction of two vectors under a probabilistic dual Pythagorean hesitant fuzzy environment are studied. Some important characteristics of these proximities are studied in detail. (2) A new MAGDM method under PDPHFSs situation based on the projection model is proposed. Main steps of the new MAGDM method are clearly displayed. (3) A new manner for helping enterprises conduct SBIC is presented.

The rest of this paper is organized as follows. Section 2 reviews the basic notions. Section 3 introduces an extension of the projection model under PDPHFSs, and based on which a novel MAGDM method with probabilistic dual Pythagorean hesitant fuzzy evaluation information is provided. Section 4 conducts a numerical experiment, shows the validity of the new method and further proves its advantages over some exiting ones. Conclusion remarks and future research direction are provided in Sect. 5.

2 Preliminaries

This section reviews some basic concepts that will be used in the following sections.

2.1 Probabilistic dual hesitant Pythagorean fuzzy sets

Definition 1 (Ji et al. 2021) Let X be a given ordinary set, then a mathematical expression of a probabilistic dual hesitant Pythagorean fuzzy set (PDHPFS) A defined on X is as follows

$$A = \{\langle x, h_A(x)|p_A(x), g_A(x)|t_A(x)\rangle | x \in X\}, \tag{1}$$

where $h_A(x), g_A(x) \subseteq [0, 1]$ are two sets of some values, representing the possible membership and non-membership degrees of the element $x \in X$ to the set A, respectively. $p_A(x)$ and $t_A(x)$ denote the probabilistic information of the possible MDs and NMDs in $h_A(x)$

and $g_A(x)$, respectively. Additionally, $h_A(x)$, $g_A(x)$, $p_A(x)$ and $t_A(x)$ meet the following constraints, viz.,

$$0 \le \gamma,\ \eta \le 1, \quad \left(\gamma^+\right)^2 + \left(\eta^+\right)^2 \le 1, \tag{2}$$

and

$$\sum_{i=1}^{\#h} p_i = 1, \sum_{j=1}^{\#g} t_j = 1, \quad 0 \le p_i, t_j \le 1, \tag{3}$$

where $\gamma \in h_A(x)$, $\eta \in g_A(x)$, $\gamma^+ = \cup_{\gamma \in h_A(x)} \max\{\gamma\}$, $\eta^+ = \cup_{\eta \in g_A(x)} \max\{\eta\}$, $p_i \in p_A(x)$, $t_j \in t_A(x)$, $\#h$ and $\#g$ denote the numbers of values in $h_A(x)$ and $g_A(x)$, respectively. For convenience, we call the ordered pair $(h_A(x)|p_A(x), g_A(x)|t_A(x))$ probabilistic dual hesitant Pythagorean fuzzy element (PDHPFE), which can be simplified as $d = \left(h|p_h, g|t_g\right)$. For easy description, in the followings, we call $h|p_h$ the probabilistic membership degree elements (PMDEs), and $g|t_g$ the probabilistic non-membership degree elements (PNMDEs). PMDEs are constructed by a series of MDs associated with their probabilities. PNMDEs are compounded by a collection of NMDs associated with probabilistic information. Hence, a PDHPFE is constructed by a series of PMDEs and PNMDEs.

Remark 1 PDHPFS was originated by Ji et al. (2021), which is an extension of DHPFS (Wei and Lu 2017). Compared with DHPFS, PDHPFS is more powerful and flexible, as it considers not only MDs and NMDs but also their probabilistic values. To illustrate the advantage of PDHPFS over DHPFS, we provide the following example. Suppose three experts were invited to evaluate the performance of a tourism destination under the attribute “environmental quality” in a low-carbon tourism destination selection problem. The first expert thinks that the MDs should be $\{0.5, 0.7\}$, while the other two experts would like to use $\{0.4, 0.5, 0.6\}$ and $\{0.6, 0.7, 0.8\}$ to express the MDs of their evaluations, respectively. In addition, the three decision-making experts would like to use $\{0.3\}$, $\{0.6\}$, and $\{0.5, 0.6\}$ to denote the NMDs, respectively. Then the overall opinion of the three experts can be denoted by $a = \{\{0.4, 0.5, 0.6, 0.7, 0.8\}, \{0.3, 0.5, 0.6\}\}$, which is obviously a dual hesitant Pythagorean fuzzy number (DHPFN). However, some information is lost when using DHPFN to express the overall opinion, i.e., the multiple occurrence and appearance of the MDs 0.5, 0.6, and 0.7, and the NMD 0.6 is neglected. Hence, Ji et al. (2021) proposed the concept of PDHPFSs, which consider not only MDs and NMDs but also their probabilistic information. If PDHPFSs are employed to descript DMs’ evaluation information, then it can be denoted by $d = \{\{0.4|0.125, 0.5|0.25, 0.6|0.25, 0.7|0.25, 0.8|0.125\}, \{0.3|0.25, 0.5|0.25, 0.6|0.5\}\}$. It is noted that in d the multiple appearances of some MDs and NMDs are considered. Hence, compared with DHPFSs, PDHPFSs can more accurately depict DMs’ evaluation information.

Remark 2 In real MAGDM problems, DMs usually provide their evaluation values separately and the original probabilistic dual hesitant Pythagorean fuzzy decision matrix can be obtained according to DMs’ individual opinions. For example, in a supplier selection problem, three experts are invited to evaluate the reputation of a supplier. The fist expert would like to use $\{0.5, 0.6\}$ to denote the MDs. The second and third experts would like to use $\{0.6, 0.7, 0.8\}$ and $\{0.5, 0.6, 0.7\}$ to denote the possible MDs, respectively. For NMDs, the three experts would like to use $\{0.1\}$, $\{0.2\}$, and $\{0.2, 0.3\}$ to express their evaluation values. Afterward, a PDHPFE d can be used to describe the overall evaluation values of the three experts. It is noted that the DMs of d are $\{0.5, 0.6, 0.7, 0.8\}$. In addition, the total number of MDs provided by the experts is 8 and the MD 0.5 occurs twice.

Hence, the probabilistic value of the MD 0.5 is $2/8 = 0.25$. Similarly, the probabilistic values of the MDs 0.6, 0.7, 0.8 are 0.375 (3/8), 0.25 (2/8), 0.125 (1/8). Similarly, the probabilistic values of NMDs 0.1, 0.2, and 0.3 are 0.25 (1/4), 0.5 (2/4), and 0.25 (1/4). Hence, the PDHPFE d which describes the overall evaluation values of the three experts are $d = \{\{0.5|0.25, 0.6|0.375, 0.7|0.25, 0.8|0.125\}, \{0.1|0.25, 0.2|0.5, 0.3|0.25\}\}$.

Remark 3 It is noted the members in a PDHPFE are not provided in the order. Hence, for the sake of convenient proceeding, we should arrange members in a PDHPFE in descending order. We use the following method to arrange PMDEs and PNMDEs in a PDHPFE: Let $d = \left(h|p_h, g\middle|t_g\right)$ be a PDHPFE, $\gamma^{(s)}\middle|p_{\gamma^{(s)}}$ and $\gamma^{(k)}\middle|p_{\gamma^{(k)}}$ be two PMDEs of d ($\gamma^{(s)}\middle|p_{\gamma^{(s)}}, \gamma^{(k)}\middle|p_{\gamma^{(k)}} \in h|p_h$), and $\eta^{(l)}\middle|t_{\eta^{(l)}}$ and $\eta^{(m)}\middle|t_{\eta^{(m)}}$ be any two PNMDEs of d ($\eta^{(l)}\middle|t_{\eta^{(l)}}, \eta^{(m)}\middle|t_{\eta^{(m)}} \in g\middle|t_g$), where $s, k = 1, 2, \ldots, \#h$, $l, m = 1, 2, \ldots, \#g$, $\#h$ and $\#g$ denote the numbers of PMDEs and PNMDEs in d, respectively. Then,

(1) if $\gamma^{(s)} p_{\gamma^{(s)}} > \gamma^{(k)} p_{\gamma^{(k)}}$, then $\gamma^{(s)}\middle|p_{\gamma^{(s)}} > \gamma^{(k)}\middle|p_{\gamma^{(k)}}$;
(2) if $\gamma^{(s)} p_{\gamma^{(s)}} = \gamma^{(k)} p_{\gamma^{(k)}}$, then

If $\gamma^{(s)} > \gamma^{(k)}$, then $\gamma^{(s)}\middle|p_{\gamma^{(s)}} > \gamma^{(k)}\middle|p_{\gamma^{(k)}}$;
If $\gamma^{(s)} = \gamma^{(k)}$, then $\gamma^{(s)}\middle|p_{\gamma^{(s)}} = \gamma^{(k)}\middle|p_{\gamma^{(k)}}$.

Similarly, we can compare PNMDEs in the same manner. In addition, for a PDHPFE $d = \left(h|p_h, g\middle|t_g\right)$, if all PMDEs and PNMDEs are in descending order, then we call d an ordered PDHPFE. Then, the ordered PDHPFE d can be written as

$$d = \left\{\left\{\gamma^{(1)}\middle|p_{\gamma^{(1)}}, \gamma^{(2)}\middle|p_{\gamma^{(2)}}, \ldots, \gamma^{(\#h)}\middle|p_{\gamma^{(\#h)}}\right\}, \left\{\eta^{(1)}\middle|t_{\eta^{(1)}}, \eta^{(2)}\middle|t_{\eta^{(2)}}, \ldots, \eta^{(\#g)}\middle|t_{\eta^{(\#g)}}\right\}\right\},$$

where $\gamma^{(\sigma)}\middle|p_{\gamma^{(\sigma)}} \geq \gamma^{(\sigma+1)}\middle|p_{\gamma^{(\sigma+1)}}$, $\eta^{(\rho)}\middle|t_{\eta^{(\rho)}} \geq \eta^{(\rho+1)}\middle|t_{\eta^{(\rho+1)}}$, $\sigma = 1, 2, \ldots, \#h - 1$, and $\rho = 1, 2, \ldots, \#g - 1$.

Basic operational rules were proposed by Ji et al. (2021), which are presented as follows.

Definition 2 (Ji et al. 2021) Let $d_1 = \left(h_1\middle|p_{h_1}, g_1\middle|t_{g_1}\right)$, $d_2 = \left(h_2\middle|p_{h_2}, g_2\middle|t_{g_2}\right)$ and $d = \left(h|p_h, g\middle|t_g\right)$ be three PDHPFEs and λ is a positive real number, then we have

(1) $d_1 \oplus d_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \eta_1 \in g_1, \eta_2 \in g_2}\left\{\left\{\left(\gamma_1^2 + \gamma_2^2 - \gamma_1^2\gamma_2^2\right)^{1/2}\middle|p_{\gamma_1}p_{\gamma_2}\right\}, \left\{\eta_1\eta_2\middle|t_{\eta_1}t_{\eta_2}\right\}\right\}$;
(2) $d_1 \otimes d_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \eta_1 \in g_1, \eta_2 \in g_2}\left\{\left\{\gamma_1\gamma_2\middle|p_{\gamma_1}p_{\gamma_2}\right\}, \left\{\left(\eta_1^2 + \eta_2^2 - \eta_1^2\eta_2^2\right)^{1/2}\middle|t_{\eta_1}t_{\eta_2}\right\}\right\}$;
(3) $\lambda d = \cup_{\gamma \in h, \eta \in g}\left\{\left\{\left(1 - \left(1 - \gamma^2\right)^{\lambda}\right)^{1/2}\middle|p_\gamma\right\}, \left\{\eta^{\lambda}\middle|t_\eta\right\}\right\}$;
(4) $d^{\lambda} = \cup_{\gamma \in h, \eta \in g}\left\{\left\{\gamma^{\lambda}\middle|p_\gamma\right\}, \left\{\left(1 - \left(1 - \eta^2\right)^{\lambda}\right)^{1/2}\middle|t_\eta\right\}\right\}$.

To compare any two PDHFEs, Ji et al. (2021) proposed a method, which is presented as follows.

Definition 3 (Ji et al. 2021) Let $d = \left(h|p_h, g\middle|t_g\right)$ be a PDHPFE, then its score function $S(d)$ is defined as

$$S(d) = \sum_{i=1, \gamma_i \in h}^{\#h} \gamma_i^2 p_{\gamma_i} - \sum_{j=1, \eta_j \in g}^{\#g} \eta_j^2 t_{\eta_i}. \tag{4}$$

And the accuracy function of d is expressed as

$$H(d) = \sum_{i=1,\gamma_i \in h}^{\#h} \gamma_i^2 p_{\gamma_i} + \sum_{j=1,\eta_j \in g}^{\#g} \eta_j^2 t_{\eta_i}. \tag{5}$$

For two PDHPFEs $d_1 = \left(h_1 \middle| p_{h_1}, g_1 \middle| t_{g_1}\right)$ and $d_2 = \left(h_2 \middle| p_{h_2}, g_2 \middle| t_{g_2}\right)$,

(1) If $S(d_1) > S(d_2)$, then $d_1 > d_2$;
(2) If $S(d_1) = S(d_2)$, then

If $H(d_1) > H(d_2)$, then $d_1 > d_2$;
If $H(d_1) = H(d_2)$, then $d_1 = d_2$.

2.2 The notion of projection model

It is noted that when investigating a vector, not only its module but also its direction should be focused. Suppose $a = (a_1, a_2, \ldots, a_n)$ is a vector with n dimensions, where $a_j (j = 1, 2, \ldots, n)$ is a non-negative real number. Then the module of a is defined as $|a| = \sqrt{a_1^2 + a_2^2 + \cdots a_n^2}$. When investigating the direction of a vector, we usually implement cosine value. Let $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ be two vectors with n dimensions, where $a_j, b_j (j = 1, 2, \ldots, n)$ are non-negative real numbers. The cosine value between a and b is expressed as

$$\cos(a, b) = \frac{\sum_{j=1}^n a_j b_j}{\sqrt{\sum_{j=1}^n a_j^2}\sqrt{\sum_{j=1}^n b_j^2}}. \tag{6}$$

From Eq. (6), it is easy to find that the cosine value is effective to measure proximity in the direction between the two vectors a and b. In another word, Eq. (6) only considers the proximity in direction between the two vectors. As mentioned above, when considering a vector, both module and direction should be considered. Similarly, when investigating two vectors, not only the proximity in direction but also the proximity in a module should be taken into account. Hence, based on the assumption, the concept of projection a on b was originated:

$$\mathrm{Prj}_b(a) = \cos(a, b) \times \sqrt{\sum_{j=1}^n a_j^2} = \frac{\sum_{j=1}^n a_j b_j}{\sqrt{\sum_{j=1}^n a_j^2}\sqrt{\sum_{j=1}^n b_j^2}} \sqrt{\sum_{j=1}^n a_j^2} = \frac{\sum_{j=1}^n a_j b_j}{\sqrt{\sum_{j=1}^n b_j^2}}. \tag{7}$$

Therefore, the value $\mathrm{Prj}_b(a)$ can be regarded as the closeness degree between a and b. In MAGDM method process, if b is the positive ideal solution (PIS), then the greater the value of $\mathrm{Prj}_b(a)$, the closer between a and PIS, which indicates the better the corresponding alternative is. If b is the negative ideal solution (NIS), then the smaller the value of $\mathrm{Prj}_b(a)$, the further between a and NIS, which indicates the better the corresponding alternative is.

3 A new MAGDM framework under PDHPFSs based on projection model

In this paper, we aim to propose a novel MAGDM framework in PDHPFS context based on the projection model. Our MAGDM framework utilizes PDHPFSs to denote DMs' evaluation values, which provides a flexible manner for experts to express their assessment opinion. Afterwards, the traditional projection model is extended into PDHPFSs, which are used to describe the similarities of an alternative to the PIS and NIS. In addition, an attributes' weights determination method is also involved in our framework. To do this, we first give a brief introduction of a typical MAGDM problem with probabilistic dual hesitant Pythagorean fuzzy information. Second, we introduce a new projection model with PDHPFSs. Third, we present a method to objectively determine the weights of attributions in probabilistic dual hesitant Pythagorean fuzzy MAGDM problems. Finally, we illustrate the main steps of our proposed novel MAGDM method.

3.1 Description of a MAGDM problem under PDHPFSs condition

We briefly introduce the typical structure of a MAGDM problem with probabilistic dual hesitant Pythagorean fuzzy decision-making information. We assume that a MAGDM problem involves m alternatives to be assessed, which can be denoted as $\{A_1, A_2, \ldots, A_m\}$. Let $\{G_1, G_2, \ldots, G_n\}$ be a set of attributes, whose weight information is completely unknown. A set of DMs are invited to evaluate the performance of the m alternatives under the n attributes. More specifically, DMs provide MDs and NMDs to express their opinions on the degrees that alternative $A_i (i = 1, 2, \ldots, m)$ satisfies and dissatisfies attribute $G_j (j = 1, 2, \ldots, n)$. Based on the MDs and NMDs provided by DMs, the original probabilistic dual hesitant Pythagorean fuzzy decision matrix can be obtained, which is constructed by a series of PDHPFEs. A PDHPFE $d_{ij} = \left(h_{ij}\middle|p_{h_{ij}}, g_{ij}\middle|t_{g_{ij}}\right)$ from the original decision matrix denotes the evaluation value of an attribute $G_j (j = 1, 2, \ldots, n)$ of alternatives $A_i (i = 1, 2, \ldots, m)$. For convenience, the original decision matrix can be denoted as $D = \left(d_{ij}\right)_{m\times n}$. The detailed structure of D is presented as follows. By solving this MAGDM problem, the final complete ranking order of alternatives is derived:

$$
\begin{array}{ccccc}
 & G_1 & G_2 & \cdots & G_n \\
A_1 & \left(h_{11}\middle|p_{h_{11}}, g_{11}\middle|t_{g_{11}}\right) & \left(h_{12}\middle|p_{h_{12}}, g_{12}\middle|t_{g_{12}}\right) & \cdots & \left(h_{1n}\middle|p_{h_{1n}}, g_{1n}\middle|t_{g_{1n}}\right) \\
A_2 & \left(h_{21}\middle|p_{h_{21}}, g_{21}\middle|t_{g_{21}}\right) & \left(h_{22}\middle|p_{h_{22}}, g_{22}\middle|t_{g_{22}}\right) & \cdots & \left(h_{2n}\middle|p_{h_{2n}}, g_{2n}\middle|t_{g_{2n}}\right) \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
A_m & \left(h_{m1}\middle|p_{h_{m1}}, g_{m1}\middle|t_{g_{m1}}\right) & \left(h_{m2}\middle|p_{h_{m2}}, g_{m2}\middle|t_{g_{m2}}\right) & \cdots & \left(h_{mn}\middle|p_{h_{mn}}, g_{mn}\middle|t_{g_{mn}}\right).
\end{array}
$$

3.2 A PDHPFSs based projection model

In this subsection, we aim to propose a novel projection model under PDHPFSs. In MAGDM process, DMs usually evaluate the performance of alternatives under n attributes. In another word, the evaluation information of an alternative A_i' is constructed by a series of ordered PDHPFEs. Hence, A_i' can be regarded as a vector, constructed by a collection of PDHFEs. Similar to studying proximity in module and direction between two vectors with crisp numbers, when incorporating the projection model into PDHPFSs, we also have to investigate the module of a vector and cosine value between two vectors under PDHPFSs. To do so, we first, investigate the module of a PDHPFE.

Definition 4 Let $d = \left(h|p_h, g\middle|t_g\right)$ be an ordered PDHPFE, then the module of d is defined as

$$
|d| = \sqrt{\sum_{\sigma=1}^{\#h} \left(\gamma^{(\sigma)} p_{\gamma^{(\sigma)}}\right)^2 + \sum_{\rho=1}^{\#g} \left(\eta^{(\rho)} t_{\eta^{(\rho)}}\right)^2}, \tag{8}
$$

where $\gamma^{(\sigma)}\big|p_{\gamma^{(\sigma)}}$ and $\eta^{(\rho)}\big|t_{\eta^{(\rho)}}$ are PMDE and PNMDE in d, respectively.

In addition, in decision-making processes, not only a PDHPFE itself but its weight should be considered. Hence, we propose the concept of a weighted module of a probabilistic dual hesitant fuzzy Pythagorean fuzzy vector A_i', whose mathematical expression is presented as follows:

$$\left|A_i'\right| = \sqrt{\sum_{j=1}^{n} w_j \left|d_{ij}\right|^2}, \tag{9}$$

where $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ is the weight of attributes, and $\left|d_{ij}\right|$ is the module of d_{ij}.

Like investigating the proximity in direction between two vectors with crisp numbers, in the following, we investigate the proximity in direction between two PDHPFEs by introducing the cosine value in PFHPFSs. When studying the cosine value between any two vectors, the two vectors are required to have the same dimensions. Similarly, when investigating the cosine value for two PDHPFEs, they should have the same numbers of PMDEs and PNMDEs. However, this requirement cannot be always met. To proceed effectively, we provide the following method to extend the shorter PDHPFE. We add the smallest PMDE and PNMDE with the probabilistic value being zero until the two PDHPFEs have the same length. Let $d_1 = \left(h_1\big|p_{h_1}, g_1\big|t_{g_1}\right)$ and $d_2 = \left(h_2\big|p_{h_2}, g_2\big|t_{g_2}\right)$ be any two ordered PDHPFEs, with $\#h_1 > \#h_2$ and $\#g_1 < \#g_2$, then we can extend d_1 and d_2 to

$$d_1' = \left\{\begin{matrix} \left\{\gamma_1^{(1)}\Big|p_{\gamma_1^{(1)}}, \gamma_1^{(2)}\Big|p_{\gamma_1^{(2)}}, \ldots, \gamma_1^{(\#h_1)}\Big|p_{\gamma_1^{(\#h_1)}}\right\}, \\ \left\{\eta_1^{(1)}\Big|t_{\eta_1^{(1)}}, \eta_1^{(2)}\Big|t_{\eta_1^{(2)}}, \ldots, \eta_1^{(\#g_1)}\Big|t_{\eta_1^{(\#g_1)}}, \eta_1^{(\#g_1)}|0, \ldots, \eta_1^{(\#g_1)}|0\right\} \end{matrix}\right\}, \tag{10}$$

and

$$d_2' = \left\{\begin{matrix} \left\{\gamma_2^{(1)}\Big|p_{\gamma_2^{(1)}}, \gamma_2^{(2)}\Big|p_{\gamma_2^{(2)}}, \ldots, \gamma_2^{(\#h_2)}\Big|p_{\gamma_2^{(\#h_2)}}, \gamma_2^{(\#h_2)}|0, \ldots \gamma_2^{(\#h_2)}|0\right\}, \\ \left\{\eta_2^{(1)}\Big|t_{\eta_2^{(1)}}, \eta_2^{(2)}\Big|t_{\eta_2^{(2)}}, \ldots, \eta_2^{(\#g_2)}\Big|t_{\eta_1^{(\#g_1)}}\right\} \end{matrix}\right\}, \tag{11}$$

where d_1' has $\#g_2$ PNMDEs and d_2' has $\#h_1$ PMDEs. In MAGDM process, if all the PDHPFEs in a decision matrix are ordered and have the same numbers of PMDEs and PNMDEs, then we call the decision matrix a normalized decision matrix.

Based on the above values, we can define the cosine value between two PDHPFEs.

Definition 5 Let $d_1 = \left(h_1\big|p_{h_1}, g_1\big|t_{g_1}\right)$, and $d_2 = \left(h_2\big|p_{h_2}, g_2\big|t_{g_2}\right)$ be any two ordered PDHPFEs and they have the same numbers of PMDEs and PNMDEs, then the cosine value between d_1 and d_2 is expressed as

$$\begin{aligned} \cos(d_1, d_2) &= \frac{\sum_{\sigma=1}^{\#h} \gamma_1^{(\sigma)} p_{\gamma_1^{(\sigma)}} \gamma_2^{(\sigma)} p_{\gamma_2^{(\sigma)}} + \sum_{\rho=1}^{\#g} \eta_1^{(\rho)} t_{\eta_1^{(\sigma)}} \eta_2^{(\rho)} t_{\eta_2^{(\sigma)}}}{|d_1||d_2|} \\ &= \frac{\sum_{\sigma=1}^{\#h} \gamma_1^{(\sigma)} p_{\gamma_1^{(\sigma)}} \gamma_2^{(\sigma)} p_{\gamma_2^{(\sigma)}} + \sum_{\rho=1}^{\#g} \eta_1^{(\rho)} t_{\eta_1^{(\rho)}} \eta_2^{(\rho)} t_{\eta_2^{(\sigma)}}}{\sqrt{\sum_{\sigma=1}^{\#h} \left(\gamma_1^{(\sigma)} p_{\gamma_1^{(\sigma)}}\right)^2 + \sum_{\rho=1}^{\#g} \left(\eta_1^{(\rho)} t_{\eta_1^{(\sigma)}}\right)^2} \sqrt{\sum_{\sigma=1}^{\#h} \left(\gamma_2^{(\sigma)} p_{\gamma_2^{(\sigma)}}\right)^2 + \sum_{\rho=1}^{\#g} \left(\eta_2^{(\rho)} t_{\eta_2^{(\sigma)}}\right)^2}}, \end{aligned} \tag{12}$$

where $\#h$ denotes the number of PMDEs in d_1 and d_2, and $\#g$ represents the number of PNMDEs in d_1 and d_2.

Remark 4 It should be noted that quite different from dual hesitant fuzzy sets and DHPFS, in PDHPFS not only the MDs and NMDs but also their corresponding probabilistic values are considered. Hence, when calculating the cosine value between any two PDHPFEs, the probabilistic information of MDs and NMDs should be taken into consideration. As it is seen in Eq. (12), when computing cosine value between d_1 and d_2, not only the MDs and NMDs of the two PDHPFEs but also their probabilistic information is considered.

Based on the concept of a module of a PDHPFE and the cosine value between two PDHPFEs, we provide the projection of a PDHPFE on another one.

Definition 6 Let $d_1 = \left(h_1 \middle| p_{h_1}, g_1 \middle| t_{g_1}\right)$, and $d_2 = \left(h_2 \middle| p_{h_2}, g_2 \middle| t_{g_2}\right)$ be any two ordered PDHPFEs and they have the same numbers of PMDEs and PNMDEs, the projection of d_1 on d_2 is defined as

$$\mathrm{Prj}_{d_2}(d_1) = \cos(d_1, d_2)|d_1| \frac{\sum_{\sigma=1}^{\#h} \gamma_1^{(\sigma)} p_{\gamma_1^{(\sigma)}} \gamma_2^{(\sigma)} p_{\gamma_2^{(\sigma)}} + \sum_{\rho=1}^{\#g} \eta_1^{(\rho)} t_{\eta_1^{(\rho)}} \eta_2^{(\rho)} t_{\eta_2^{(\sigma)}}}{\sqrt{\sum_{\sigma=1}^{\#h} \left(\gamma_2^{(\sigma)} p_{\gamma_2^{(\sigma)}}\right)^2 + \sum_{\rho=1}^{\#g} \left(\eta_2^{(\rho)} t_{\eta_2^{(\sigma)}}\right)^2}}. \tag{13}$$

In MAGDM problem, the final chosen alternative should be closest to the PIS and farthest to the NIS, and the projection value can be used to measure the closeness degree to the PIS and NIS. Hence, we should first define the PIS and NIS under a probabilistic dual hesitant Pythagorean fuzzy decision matrix. For convenient depiction, we call them probabilistic dual hesitant Pythagorean fuzzy PIS (PDHPFPIS) and probabilistic dual hesitant Pythagorean fuzzy NIS (PDHPFNIS), respectively.

Definition 7 Let $D = \left(d_{ij}\right)_{m \times n}$ be a normalized probabilistic dual hesitant fuzzy Pythagorean fuzzy decision matrix, and h and g are the numbers of PMDEs and PNMDEs of each PDHPFE, then the PDHPFPIS is expressed as

$$A^+ = \left(d_1^+, d_2^+, \ldots, d_n^+\right), \tag{14}$$

where

$$d_j^+ = \left\{\left\{\left(\gamma_j^{(\sigma)} p_{\gamma_j^{(\sigma)}}\right)^+ | \sigma = 1, 2, \ldots, h\right\}, \left\{\left(\eta_j^{(\rho)} t_{\eta_j^{(\rho)}}\right)^- | \rho = 1, 2, \ldots, g\right\}\right\}, \tag{15}$$

and

$$\left(\gamma_j^{(\sigma)} p_{\gamma_j^{(\sigma)}}\right)^+ = \max_i \left\{\gamma_{ij}^{(\sigma)} p_{\gamma_{ij}^{(\sigma)}}\right\}, \left(\eta_j^{(\rho)} t_{\eta_j^{(\rho)}}\right)^- = \min_i \left\{\eta_{ij}^{(\rho)} t_{\eta_{ij}^{(\rho)}}\right\}, \tag{16}$$

among which $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

The PDHPFNIS is expressed as

$$A^- = \left(d_1^-, d_2^-, \ldots, d_n^-\right), \tag{17}$$

where

$$d_j^- = \left\{\left\{\left(\gamma_j^{(\sigma)} p_{\gamma_j^{(\sigma)}}\right)^- | \sigma = 1, 2, \ldots, h\right\}, \left\{\left(\eta_j^{(\rho)} t_{\eta_j^{(\rho)}}\right)^+ | \rho = 1, 2, \ldots, g\right\}\right\}, \tag{18}$$

and

$$\left(\gamma_j^{(\sigma)} p_{\gamma_j^{(\sigma)}}\right)^- = \min_i \left\{\gamma_{ij}^{(\sigma)} p_{\gamma_{ij}^{(\sigma)}}\right\}, \left(\eta_j^{(\rho)} t_{\eta_j^{(\rho)}}\right)^+ = \max_i \left\{\eta_{ij}^{(\rho)} t_{\eta_{ij}^{(\rho)}}\right\}, \tag{19}$$

among which $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

In addition, when considering the modules of PDHPFPIS and PDHPFNIS, their weights should be also taken into account. Hence, the weighted modules of A^+ and A^- are

$$\left|A^+\right| = \sqrt{\sum_{j=1}^{n} w_j \left|d_j^+\right|^2}, \quad \left|A^-\right| = \sqrt{\sum_{j=1}^{n} w_j \left|d_j^-\right|^2}, \tag{20}$$

Based on the cosine value between any two PDHPFEs (see Eq. (12)), we give the following concept.

Definition 8 Let A_i' be the vector of all attribute values of ith alternative, and A^+ and A^- be the PDHPFPIS and PDHPFNIS, respectively. The cosine of the included angle between A_i' and A^+ is expressed as

$$\cos\left(A_i', A^+\right) = \frac{\sum_{j=1}^{n}\left(w_j\sqrt{\sum_{\sigma=1}^{\#h}\gamma_{ij}^{(\sigma)}p_{\gamma_{ij}^{(\sigma)}}\left(\gamma_j^{(\sigma)}p_{\gamma_j^{(\sigma)}}\right)^{+} + \sum_{\rho=1}^{\#g}\eta_{ij}^{(\rho)}t_{\eta_{ij}^{(\rho)}}\left(\eta_j^{(\rho)}t_{\eta_j^{(\rho)}}\right)^{-}}\right)}{\left|A_i'\right|\left|A^+\right|}. \tag{21}$$

Similarly, the cosine of the included angle between A_i' and A^- is expressed as

$$\cos\left(A_i', A^-\right) = \frac{\sum_{j=1}^{n}\left(w_j\sqrt{\sum_{\sigma=1}^{\#h}\gamma_{ij}^{(\sigma)}p_{\gamma_{ij}^{(\sigma)}}\left(\gamma_j^{(\sigma)}p_{\gamma_j^{(\sigma)}}\right)^{-} + \sum_{\rho=1}^{\#g}\eta_{ij}^{(\rho)}t_{\eta_{ij}^{(\rho)}}\left(\eta_j^{(\rho)}t_{\eta_j^{(\rho)}}\right)^{+}}\right)}{\left|A_i'\right|\left|A^-\right|}. \tag{22}$$

As mentioned above, the cosine value represents the proximity in direction of two PDHPFE vectors. To simultaneously consider the proximity degrees in module and direction, we propose the projection between any two PDHPFE vectors.

Definition 9 Let A_i' be the vector of all attribute values of ith alternative, and A^+ and A^- be the PDHPFPIS and PDHPFNIS, respectively. Then the projection of A_i' on A^+ is expressed as

$$\begin{aligned}\mathrm{Prj}_{A^+}\left(A_i'\right) &= \left|A_i'\right|\cos\left(A_i', A^+\right)\\ &= \frac{\sum_{j=1}^{n}\left(w_j\sqrt{\sum_{\sigma=1}^{\#h}\gamma_{ij}^{(\sigma)}p_{\gamma_{ij}^{(\sigma)}}\left(\gamma_j^{(\sigma)}p_{\gamma_j^{(\sigma)}}\right)^{+} + \sum_{\rho=1}^{\#g}\eta_{ij}^{(\rho)}t_{\eta_{ij}^{(\rho)}}\left(\eta_j^{(\rho)}t_{\eta_j^{(\rho)}}\right)^{-}}\right)}{\left|A^+\right|},\end{aligned} \tag{23}$$

and the projection of A_i' on A^- is expressed as

$$\begin{aligned}\mathrm{Prj}_{A^-}\left(A_i'\right) &= \left|A_i'\right|\cos\left(A_i', A^-\right)\\ &= \frac{\sum_{j=1}^{n}\left(w_j\sqrt{\sum_{\sigma=1}^{\#h}\gamma_{ij}^{(\sigma)}p_{\gamma_{ij}^{(\sigma)}}\left(\gamma_j^{(\sigma)}p_{\gamma_j^{(\sigma)}}\right)^{-} + \sum_{\rho=1}^{\#g}\eta_{ij}^{(\rho)}t_{\eta_{ij}^{(\rho)}}\left(\eta_j^{(\rho)}t_{\eta_j^{(\rho)}}\right)^{+}}\right)}{\left|A^-\right|}.\end{aligned} \tag{24}$$

It is noted that the greater the value $\text{Prj}_{A^+}\left(A_i'\right)$, the closer A_i' to the PDHPFPIS A^+, which indicates the better the alternative A_i'. In addition, the smaller the value $\text{Prj}_{A^-}\left(A_i'\right)$, the further A_i' to the PDHPFNIS A^-, which indicates the better the alternative A_i'. Hence, based on $\text{Prj}_{A^+}\left(A_i'\right)$ and $\text{Prj}_{A^-}\left(A_i'\right)$, we provide the following concept of relative importance.

Definition 10 Assume $\text{Prj}_{A^+}\left(A_i'\right)$ and $\text{Prj}_{A^-}\left(A_i'\right)$ are the projections of alternative A_i' on the PDHPFPIS A^+ and PDHPFNIS A^-, respectively. The relative importance of alternative A_i' is defined as

$$\text{RI}_i = \frac{\text{Prj}_{A^+}\left(A_i'\right)}{\text{Prj}_{A^+}\left(A_i'\right) + \text{Prj}_{A^-}\left(A_i'\right)}. \tag{25}$$

Afterward, the bigger RI_i, the better alternative A_i is. Hence, we can rank all alternatives according to RI_i

3.3 An attributes' weights determination model under PDHPFSs

In MAGDM problems, there are two methods to determine the weight vector of attributes. One method to determine attributes' weight vector is that DMs provide the weights directly and subjectively. The other method is to objectively determine the weight vector according to DMs' matrices. Due to many reasons, such as time shortage, lacking enough expertise and prejudice, it is inappropriate for DMs to provide attributes' weight vector directly. Hence, it is necessary to introduce a method to objectively determine the weight vector of attributes in MAGDM under PDHPFSs. There are some methods to determine weight information of attributes in MAGDM problems. For instance, Liu and Wang (2021) introduced a best–worst-method to determine attribute weights of attributes in MAGDM problems with 2-dimensional uncertain linguistic decision-making environment. Biswas and Sarkar (2019) introduced an entropy measure for PFSs and based on which an approach to determine attributes' weight information was developed. Similarly, Verma (2020) and Farhadinia (2016) also investigated methods to determine weight information of attributes from the aspect of entropy measures under q-rung orthopair fuzzy and hesitant fuzzy linguistic term sets, respectively. Additionally, there are some publications which studied weights determining methods on the basis of maximizing deviation method. These weights determining methods can be found in Şahin and Liu (2016), Farrokhizadeh et al. (2021) and Song and Chen (2021). In this paper, we attempt to introduce a method to appropriately determine the weights of attributes in MAGDM process under PDHPFSs. Before doing so, we should first standardize the original decision matrix $D = \left(d_{ij}\right)_{m\times n}$, which is constructed by a series of PDHPFEs and the element d_{ij} denotes DMs' provided attribute value of attribute $G_j\,(j = 1, 2, \ldots m)$ of alternative $A_i\,(i = 1, 2, \ldots, n)$. The process of standardizing the original probabilistic dual hesitant fuzzy decision matrix is presented as follows:

Step 1. In general, there are usually two types of attributes, i.e., benefit type and cost type. Hence, to effectively handle different kinds of attributes, the following transformation should be carried out, i.e.

$$d_{ij} = \begin{cases} \left(h_{ij}\middle|p_{h_{ij}}, g_{ij}\middle|t_{g_{ij}}\right) G_j \in I_1 \\ \left(g_{ij}\middle|t_{g_{ij}}, h_{ij}\middle|p_{h_{ij}}\right) G_j \in I_2 \end{cases}, \tag{26}$$

wherein I_1 and I_2 denote the benefit type of attributes and cost type attributes, respectively.

Step 2. Arranges elements in each PDHPFEs of the decision matrix in descending order. The method to arrange PMDEs and PNMDEs is shown in Remark 3. After this step, all PDHPFEs are ordered.

Step 3. Extend the shorter PDHPFEs until all evaluation values with respect to attribute $G_j (j = 1, 2, \ldots, n)$ have the same length. The approach to extend the shorter PDHPFEs can be found in Sect. 3.2.

By carrying out the above four steps, the original probabilistic dual hesitant Pythagorean fuzzy decision matrix is transformed into a normalized one. For the sake of convenient description, we still use $D = \left(d_{ij}\right)_{m\times n}$ to denote the obtained standardized decision matrix. In the followings, we attempt to propose two principles which should be followed when determining attributes' weights. Before doing so, we first transform the probabilistic dual hesitant Pythagorean fuzzy decision matrix into a new matrix wherein elements are crisp number by introducing a new concept, called average value.

Definition 11 Let $d_i = \left(h_i \middle| p_{h_i}, g_i \middle| t_{g_i}\right) (i = 1, 2, \ldots, n)$ be a set of standardized PDHPFEs, and the numbers PMDEs and PNMEs in each PDHPFE are denoted as $\#h$ and $\#g$, respectively, then the expected value of d_i is calculated by

$$a_i = \frac{1}{\#h} \sum_{k=1}^{\#h} ah_i^{(k)} - \frac{1}{\#g} \sum_{s=1}^{\#g} ag_i^{(s)}, \tag{27}$$

where

$$ah_i^{(k)} = \frac{\gamma_i^{(k)} p_i^{(k)} - \min_i\left(\gamma_i^{(k)} p_i^{(k)}\right)}{\max_i\left(\gamma_i^{(k)} p_i^{(k)}\right) - \min_i\left(\gamma_i^{(k)} p_i^{(k)}\right)}, \tag{28}$$

and

$$ag_i^{(k)} = \frac{\eta_i^{(k)} t_i^{(k)} - \min_i\left(\eta_i^{(k)} t_i^{(k)}\right)}{\max_i\left(\eta_i^{(k)} t_i^{(k)}\right) - \min_i\left(\eta_i^{(k)} t_i^{(k)}\right)}, \tag{29}$$

where $\gamma_i^{(k)} \middle| p_i^{(k)}$ and $\eta_i^{(k)} \middle| t_i^{(k)}$ denote the PMDE and PNME of d_i, respectively.

In addition, let $D = \left(d_{ij}\right)_{m\times n}$ be a standardized decision matrix, then its corresponding average value matrix $A = \left(a_{ij}\right)_{m\times n}$ can be determined by the above two equations. Based on the average value matrix A, in the following, we should propose a method to determine the weight information of attributes. For convenient description, in the following, we use a_j^* to denote the maximum value of a_{ij} with respect to i. Zhang et al. (2018) proposed a method to determine the weights of attributes in decision-making situations under a hesitant fuzzy linguistic environment. By incorporating Zhang et al.'s (2018) ideas into our study, we can propose an approach to objectively determine the weight vector of attributes in MAGDM problems with probabilistic dual Pythagorean hesitant fuzzy information. Motivated by Zhang et al.'s (2018) study, we notice that when determining the weight vector of attributes, the following two principles should be considered.

Principle I: The weight vector should make the total deviation between all the alternatives and the PIS to be a minimum.

Principle II: The weight vector should make the Shannon information entropy be maximum.

When taking Principle I into account, we notice that for alternative A_i, we can use

$$\Delta_j(w) = w_j \sqrt{\sum_{i=1}^{m} \left(a_j^* - a_{ij}\right)^2}, \tag{30}$$

to denote the deviation between $A_i (i = 1, 2 \ldots, m)$ and the PIS. Therefore, the overall deviation is derived as follows

$$\Delta(w) = \sum_{j=1}^{n} w_j \sqrt{\sum_{i=1}^{m} \left(a_j^* - a_{ij}\right)^2}. \tag{31}$$

When determining the weights of attributes, one objective is to make the overall deviation $\Delta(w)$ to be as small as possible. In addition, when considering Principle II, we can use the following equation

$$H = -k \sum_{j=1}^{n} w_j \ln w_j, \tag{32}$$

to represent the Shannon information entropy of attributes' weights. When calculating attributes' weights, the second objective to make the Shannon information entropy of attributes to be as big as possible. Hence, to take both Principles I and II into consideration, and achieve the above two mentioned objectives, we establish the following optimization model for calculating the attributes' weights, i.e.

$$\min\left\{\theta \sum_{j=1}^{n} w_j \sqrt{\sum_{i=1}^{n} \left(a_j^* - a_{ij}\right)^2} + (1-\theta) \sum_{j=1}^{n} w_j \ln w_j\right\}, \tag{33}$$
$$\text{s.t. } w_j \geq 0; \quad j = 1, 2, \ldots, n, \ \sum_{j=1}^{n} w_j = 1,$$

in which $0 < \theta < 1$, denoting the balance coefficient between the above two objectives. Generally, we usually use $\theta = 0.5$, which indicates that the two objectives have the same importance degrees. By solving this model, we can derive the following result,

$$w_j = \frac{\exp\left(-\frac{\theta}{1-\theta}\sqrt{\sum_{i=1}^{n}\left(a_j^* - a_{ij}\right)^2} - 1\right)}{\sum_{j=1}^{n} \exp\left(-\frac{\theta}{1-\theta}\sqrt{\sum_{i=1}^{n}\left(a_j^* - a_{ij}\right)^2} - 1\right)} \quad (j = 1, 2, \ldots, n). \tag{34}$$

Remark 5 It is noted that Batool et al. (2020) proposed a method to calculate the weight information of attributes in MAGDM problems under PDHPFSs by considering to maximize entropy of DMs' evaluation information. This weight determination method has some drawbacks. This is because, as analyzed above, when determining the weight information of attributes, two principles should be taken into consideration (see Principle I and Principle II). In another word, the weight vector of attributes should not only make the Shannon information entropy of DMs' evaluation values to be maximum, but also make the total deviation between all the alternatives and the positive ideal solution to be a minimum. Our proposed weight determination method takes the two factors into consideration. Hence, our proposed weight determination method is more reliable and reasonable than Batool et al.'s (2020) method.

3.4 Steps of the novel MAGDM method with PDHPFSs

Based on the novel projection model under PDHPFSs and the proposed weight vector determination method, in the following we introduce a novel MAGDM method under PDHPFSs circumstance. For convenience, we separate our decision-making method into four phases, and they are: Phase I (The evaluation process); Phase II (Original decision matrix preprocessing); Phase III (The calculation process), and Phase IV (The ranking and selection process of alternatives). In Phase I, DMs are required to express their evaluation information by using MDs and NMDs. Based on DMs' opinions, the original decision matrix is established, which is constructed by a series of PDHPFEs. Phase II preprocesses the original decision matrix and at the end of this phase, these PDHPFEs with regard to G_j of A_i have the numbers of PMDEs and PNMDEs. In addition, PMDEs and PNMDEs in each PDHPFEs are in ascending order. Phase III is the core of our proposed method and in this phase, weights of attributes are determined by using the optimization model presented in Sect. 3.3. Afterwards, PDHPFPIS and PDHPFNIS are selected. Then, the projection values of each alternative to PDHPFPIS and PDHPFNIS are computed and based on which, relative importance degree of each alternative is determined. Based on the above three phases, the last phase (Phase IV) ranks all candidate alternatives and selects the optimal one. The main detailed steps of our MAGDM method are shown as follows. A flowchart (Fig. 1) is also provided at the end of this subsection to better demonstrate the new decision-making method.

Phase I: The evaluation process

Step 1. DMs are required to evaluate the performance of all candidates and for attribute $G_j (j = 1, 2, \ldots, n)$ of alternative $A_i (i = 1, 2, \ldots, m)$, each DM uses a several MDs and NMDs to express his/her evaluation opinions.

Step 2. Based on the MDs and NMDs that provided by DMs, the original probabilistic dual hesitant Pythagorean fuzzy decision matrix is established, which can be denoted as $D = \left(d_{ij}\right)_{m\times n} = \left(h_{ij}\middle|p_{h_{ij}}, g_{ij}\middle|t_{g_{ij}}\right)_{m\times n}$. Each member $d_{ij} = \left(h_{ij}\middle|p_{h_{ij}}, g_{ij}\middle|t_{g_{ij}}\right)$ of D is a PDHPFE, which is constructed by a series PMDEs $(h_{ij}|p_{h_{ij}})$ and PNMDEs $(g_{ij}|t_{g_{ij}})$.

Phase II: Original decision matrix preprocessing

Step 3. Standardize the original decision matrix according to

$$d_{ij} = \begin{cases} \left(h_{ij}\middle|p_{h_{ij}}, g_{ij}\middle|t_{g_{ij}}\right) G_j \in I_1 \\ \left(g_{ij}\middle|t_{g_{ij}}, h_{ij}\middle|p_{h_{ij}}\right) G_j \in I_2 \end{cases}, \tag{35}$$

wherein I_1 and I_2 denote the benefit type of attributes and cost type attributes, respectively.

Step 4. Normalize the decision matrix. After this step, all PDHFEs in the decision matrix have the numbers of PMDEs and PNMDEs, and all PMDs and PNMDEs are arranged in descending order. At this stage, the decision matrix provided by DMs becomes a standard probabilistic dual hesitant Pythagorean fuzzy decision matrix.

Phase III: The calculation process

Step 5. Determine the weight vector of attributes according to Eq. (34).

Step 6. Determine the PDHPFPIS A^+ and PDHPFNIS A^- according to Eqs. (15) and (18).

Step 7. For alternative $A_i (i = 1, 2, \ldots, n)$, calculate the projections of A_i on the PDHPFPIS A^+ and PDHPFNIS A^-, respectively.

Step 8. Compute the closeness degree of each alternative according to Eq. (25)

Phase IV: The ranking and selection process of alternatives

Step 9. Rank alternatives according to their closeness degrees.

Step 10. Select the optimal alternative(s).

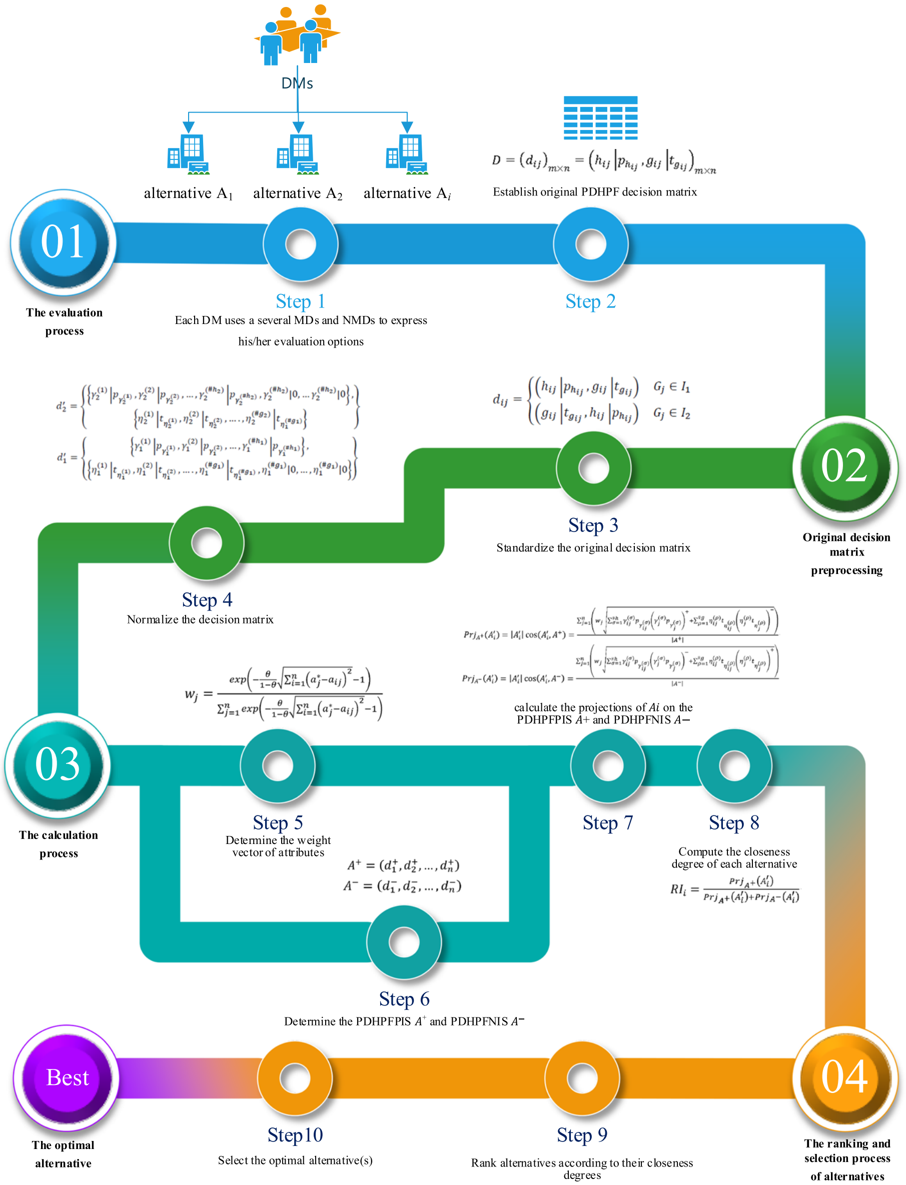

Fig. 1 The flowchart of our proposed MAGDM method

4 An application of our method in SBIC

Basically, SBIC refers to the process in which multiple domain experts are invited to evaluate all possible inventions and determine which inventions should be commercialized. As enterprises usually make more inventions than to be commercialized, and the process of commercialization is complicated and time-consuming, SBIC is significantly important before the final inventions to be commercialized are chosen. Hence, many innovative enterprises carry out SBIC to determine the most suitable inventions that are to be commercialized. For instance, based on the successful experience of its popular product 914 laser printer, Xerox has formed a set of internal evaluation criteria to determine the commercialization of the company's new ideas and breakthrough inventions, to find the next possible successful product (Vinokurova and Kapoor 2020). As analyzed above, the essence of SBIC is consistent with MAGDM. In other word, MAGDM methods can be applied in SBIC to help experts choose the most suitable inventions. Section 3 proposes a new MAGDM framework under uncertainty, which helps DMs to select optimal alternatives. Therefore, our proposed novel decision-making framework is suitable to be applied in SBIC. In addition, obtained from past experience or related successful cases, there are usually several principles that enterprises should follow when choosing potential inventions. That is to say, in SBIC decision experts usually evaluate the capacity of each alternative under several attributes. In this article, we assume that experts evaluate all inventions under four attributes and they are: Market demand (G_1), Profit prospect (G_2), Rationality of R&D process (G_3), and the Ability for increasing enterprise's competitiveness (G_4). In the following, we briefly introduce these attributes.

Market demand (G_1): This attribute is incorporated to measure the prospect of the market of a certain breakthrough invention, which is significant for invention commercialization decision. High market demand indicates that commercial products meet market expectations. In addition, high market demand will improve sales performance and enterprise may achieve considerable sales growth. Moreover, good sales performance can enable an enterprise to further improve its market share.

Profit prospect (G_2): This attribute is incorporated to measure the ability of breakthrough invention in improving an enterprise's benefit with the increase of market share after its commercialization. Profit brings income growth to enterprises, which is not only the guarantee of enterprise survival but also the direct source of value distribution to shareholders and managers. Products with high profits not only benefit enterprises but also increase creditors and investors' confidence. In addition, they can also promote enterprises' long-term and stable development. In a nutshell, high profits provide a firmer foundation and guarantee for enterprises.

Rationality of R&D process (G_3): As upstream links of enterprise R&D management, the value of R&D process and R&D quality management is hidden and easy to be ignored. However, it is of strategic importance. Breakthrough inventions are characterized by high risk, high investment, and high challenges, which require enterprises to not only pay attention to quality but also to grasp efficiency and speed. Through a good R&D process, enterprises can grasp research progress, make rational resource allocation plan, and catch opportunities for business competition.

Ability for increasing enterprise's competitiveness (G_4): Competitive advantage is the ability of an enterprise to surpass its opponents in market competition. With such an ability, an enterprise can accumulate reputation, transfer more value to customers, and obtain higher customer satisfaction as well as brand values. If breakthrough inventions can help an enterprise obtain a good competitive attitude and advantage, then this enterprise will be in a leading position in the market competition.

An enterprise is now undergoing its SBIC and to select the most suitable breakthrough inventions, five domain experts are invited to evaluate the performance of all the candidates under the above-mentioned four attributes. Suppose that there are four breakthrough inventions to be evaluated, which can be denoted as $A = \{A_1, A_1, A_3, A_4\}$. The weight vector of attributes is completely unknown. In the following, we apply our proposed method in this problem and help the enterprise select the optimal inventions.

4.1 The process of determining the optimal breakthrough inventions

Step 1. DMs are required to express their evaluations of the four candidates under the four attributes, i.e., G_1, G_2, G_3 and G_4. Each DM provides the MDs and NMDs of the alternatives under the four attributes, which are listed in Tables 1, 2, 3 and 4.

Step 2. Based on DMs' evaluation opinions, the original probabilistic dual Pythagorean hesitant fuzzy decision matrix is established, which is listed in Table 5. Attribute values are determined by the MDs and NMDs that provided by the five DMs. As analyzed in Remark 2, the MDs and NMDs as well as their probabilistic values are determined by DMs'

Table 1 The MDs and NMDs provide by each DM for alternative A_1

DMs	G_1		G_2		G_3		G_4	
	MDs	NMDs	MDs	NMDs	MDs	NMDs	MDs	NMDs
$DM^{(1)}$	{0.6}	{0.2}	{0.7}	{0.25}	{0.2}	{0.2}	{0.6}	{0.3}
$DM^{(2)}$	{0.5, 0.6}	{0.2}	{0.7}	{0.25}	{0.2}	{0.2}	{0.7}	{0.3}
$DM^{(3)}$	{0.5,0.7}	{0.2}	{0.7}	{0.25}	{0.2}	{0.2}	{0.6, 0.7}	{0.3}
$DM^{(4)}$	{0.5, 0.7}	{0.2}	{0.7}	{0.25}	{0.2}	{0.2}	{0.6, 0.7}	{0.3}
$DM^{(5)}$	{0.5, 0.6, 0.7}	{0.2}	{0.7}	{0.25}	{0.2}	{0.2}	{0.6, 0.7}	{0.3}

Table 2 The MDs and NMDs provide by each DM for alternative A_2

DMs	G_1		G_2		G_3		G_4	
	MDs	NMDs	MDs	NMDs	MDs	NMDs	MDs	NMDs
$DM^{(1)}$	{0.1}	{0.4}	{0.3}	{0.7}	{0.7}	{0.2}	{0.3}	{0.3}
$DM^{(2)}$	{0.1}	{0.4}	{0.3}	{0.7}	{0.7}	{0.3}	{0.3}	{0.3}
$DM^{(3)}$	{0.1}	{0.4}	{0.3}	{0.7}	{0.7}	{0.3}	{0.3}	{0.3}
$DM^{(4)}$	{0.1}	{0.4}	{0.3}	{0.7}	{0.7}	{0.2}	{0.3}	{0.3}
$DM^{(5)}$	{0.1}	{0.4}	{0.3}	{0.7}	{0.7}	{0.2, 0.3}	{0.3}	{0.3}

Table 3 The MDs and NMDs provide by each DM for alternative A_3

DMs	G_1		G_2		G_3		G_4	
	MDs	NMDs	MDs	NMDs	MDs	NMDs	MDs	NMDs
$DM^{(1)}$	{0.6}	{0.35}	{0.56}	{0.2}	{0.1}	{0.7}	{0.2}	{0.4}
$DM^{(2)}$	{0.6}	{0.35}	{0.56}	{0.2}	{0.1}	{0.7}	{0.2}	{0.4}
$DM^{(3)}$	{0.6}	{0.35}	{0.56}	{0.2}	{0.1}	{0.7}	{0.2}	{0.4}
$DM^{(4)}$	{0.6}	{0.35}	{0.56}	{0.2}	{0.1}	{0.7}	{0.4}	{0.4}
$DM^{(5)}$	{0.6}	{0.35}	{0.56}	{0.2}	{0.1}	{0.7}	{0.4}	{0.4}

Table 4 The MDs and NMDs provide by each DM for alternative A_4

DMs	G_1		G_2		G_3		G_4	
	MDs	NMDs	MDs	NMDs	MDs	NMDs	MDs	NMDs
$DM^{(1)}$	{0.1}	{0.5}	{0.3, 0.5}	{0.6}	{0.8}	{0.15}	{0.2}	{0.6}
$DM^{(2)}$	{0.2}	{0.5}	{0.5}	{0.6}	{0.8}	{0.15}	{0.2}	{0.6}
$DM^{(3)}$	{0.1}	{0.5}	{0.3}	{0.5, 0.6}	{0.8}	{0.15}	{0.2}	{0.6}
$DM^{(4)}$	{0.2}	{0.5}	{0.3, 0.5}	{0.5}	{0.8}	{0.15}	{0.2}	{0.6}
$DM^{(5)}$	{0.1}	{0.5}	{0.3, 0.5}	{0.5}	{0.8}	{0.15}	{0.2}	{0.6}

Table 5 The original probabilistic dual Pythagorean hesitant fuzzy decision matrix

	G_1	G_2
A_1	$\{\{0.7\|0.3, 0.6\|0.3, 0.5\|0.4\}, \{0.2\|1\}\}$	$\{\{0.7\|1\}, \{0.25\|1\}\}$
A_2	$\{\{0.1\|1\}, \{0.4\|1\}\}$	$\{\{0.3\|1\}, \{0.7\|1\}\}$
A_3	$\{\{0.6\|1\}, \{0.35\|1\}\}$	$\{\{0.56\|1\}, \{0.2\|1\}\}$
A_4	$\{\{0.1\|0.6, 0.2\|0.4\}, \{0.5\|1\}\}$	$\{\{0.3\|0.5, 0.2\|0.5\}, \{0.6\|0.5, 0.5\|0.5\}\}$
	G_3	G_4
A_1	$\{\{0.2\|1\}, \{0.2\|1\}\}$	$\{\{0.7\|0.5, 0.6\|0.5\}, \{0.3\|1\}\}$
A_2	$\{\{0.7\|1\}, \{0.3\|0.5, 0.2\|0.5\}\}$	$\{\{0.3\|1\}, \{0.3\|1\}\}$
A_3	$\{\{0.1\|1\}, \{0.7\|1\}\}$	$\{\{0.2\|0.6, 0.4\|0.4\}, \{0.4\|1\}\}$
A_4	$\{\{0.8\|1\}, \{0.15\|1\}\}$	$\{\{0.2\|1\}, \{0.6\|1\}\}$

evaluations. Take the attribute value of G_1 of alternative A_1 as an example. The MDs proved by the five DMs are $\{0.6\}$, $\{0.5, 0.6\}$, $\{0.5, 0.7\}$, $\{0.5, 0.7\}$, and $\{0.5, 0.6, 0.7\}$, respectively. There are 10 MDs among which the MD 0.5 appears 4 times and hence its probabilistic value is 0.4. Similarly, the probabilistic values of 0.6 and 0.7 are 0.3 and 0.3, respectively. Similarly, the MD is 0.2 and its probabilistic value is 1. Hence, we use a PDHPFE $d = \{\{0.7|0.3, 0.6|0.3, 0.5|0.4\}, \{0.2|1\}\}$ to denote the value of an attribute g_1 of alternative A_1. Other attribute values can be determined in a similar manner and a probabilistic dual Pythagorean hesitant fuzzy decision matrix is obtained, which is listed in Table 5.

Step 3. Standardize the original decision matrix. It is noted that all attributes are benefit type, and hence the original decision matrix does not need to be standardized.

Step 4. Normalize the decision matrix and we can determine the following results (see Table 6).

Step 5. Determine the weight vector of attributes according to Eq. (34). Without loss of generality, we assume $\theta = 0.5$, and the obtained weight vector of attributes is $w = (0.7414, 0.1546, 0.0744, 0.0296)^T$.

Table 6 The normalized decision matrix

	G_1	G_2
A_1	$\{\{0.6\|0.3, 0.5\|0.4, 0.7\|0.3\}, \{0.2\|1\}\}$	$\{\{0.7\|1, 0.7\|0\}, \{0.25\|1, 0.25\|0\}\}$
A_2	$\{\{0.1\|1, 0.1\|0, 0.1\|0\}, \{0.4\|1\}\}$	$\{\{0.3\|1, 0.3\|0\}, \{0.7\|1, 0.7\|0\}\}$
A_3	$\{\{0.6\|1, 0.6\|0, 0.6\|0\}, \{0.35\|1\}\}$	$\{\{0.56\|1, 0.56\|0\}, \{0.2\|1, 0.2\|0\}\}$
A_4	$\{\{0.1\|0.6, 0.2\|0.4, 0.2\|0\}, \{0.5\|1\}\}$	$\{\{0.2\|0.5, 0.3\|0.5\}, \{0.5\|0.5, 0.6\|0.5\}\}$

	G_3	G_4
A_1	$\{\{0.2\|1\}, \{0.2\|1, 0.2\|0\}\}$	$\{\{0.6\|0.5, 0.7\|0.5\}, \{0.3\|1\}\}$
A_2	$\{\{0.7\|1\}, \{0.2\|0.5, 0.3\|0.5\}\}$	$\{\{0.3\|1, 0.3\|0\}, \{0.3\|1\}\}$
A_3	$\{\{0.1\|1\}, \{0.7\|1, 0.7\|0\}\}$	$\{\{0.2\|0.6, 0.4\|0.4\}, \{0.4\|1\}\}$
A_4	$\{\{0.8\|1\}, \{0.15\|1, 0.15\|0\}\}$	$\{\{0.2\|1, 0.2\|0\}, \{0.6\|1\}\}$

Step 6. Determine the PDHPFPIS A^+ and PDHPFNIS A^- according to Eqs. (15) and (18), and we have

$$A^+ = \begin{pmatrix} \{\{0.6|1, 0.5|0.4, 0.7|0.3\}, \{0.5|1\}\}, \{\{0.7|1, 0.3|0.5\}, \{0.7|1, 0.6|0.5\}\}, \\ \{\{0.8|1\}, \{0.7|1, 0.3|0.5\}\}, \{\{0.6|0.5, 0.7|0.5\}, \{0.6|1\}\} \end{pmatrix}$$

and

$$A^- = \begin{pmatrix} \{\{0.1|0.6, 0.1|0, 0.1|0\}, \{0.2|1\}\}, \{\{0.2|0.5, 0.7|0\}, \{0.2|1, 0.25|0\}\}, \\ \{\{0.1|1\}, \{0.2|0.5, 0.2|0\}\}, \{\{0.2|0.6, 0.3|0\}, \{0.3|1\}\} \end{pmatrix}.$$

Step 7. Calculate the projections of A_i on the PDHPFPIS A^+ and PDHPFNIS A^-, respectively. Hence, we can obtain the following results (see Table 7).

Step 8. Compute the closeness degree of alternative $A_i (i = 1, 2, \ldots, m)$ by Eq. (25), and we can obtain

$$\mathrm{RI}_1 = 0.3574, \mathrm{RI}_2 = 0.3032, \mathrm{RI}_3 = 0.3458, \mathrm{RI}_4 = 0.3035.$$

Step 9. Rank alternatives according to their closeness degrees and we can get $A_1 \succ A_3 \succ A_4 \succ A_2$.

Step 10. Select the optimal alternative A_1 is the best alternative.

Table 7 The projection values of each alternative to PDHPFPIS and PDHPFNIS

	A_1	A_2	A_3	A_4
$\mathrm{Prj}_{A^+}(A_i')$	0.3830	0.3768	0.4694	0.3837
$\mathrm{Prj}_{A^-}(A_i')$	0.6887	0.8659	0.8880	0.8805

4.2 Comparison analysis

In this subsection, we conduct a comparison analysis to the advantages of our proposed method. We use our method and some extant decision-making methods to solve the above-mentioned SBIC problem and compare their results. Batool et al. (2021) proposed the probabilistic dual Pythagorean hesitant fuzzy weighted average (PDPHFWA) operator to handle MAGDM problems under PDPHFS situations. In another paper authored by Batool et al. (2020), a novel decision-making method, called probabilistic dual Pythagorean hesitant fuzzy TOPSIS (PDHF-TOPSIS) was developed, by extending the classical TOPSIS into PDPHFSs. We use these two MAGDM methods to solve the above SBIC problem, and the corresponding decision-making results are listed in Tables 8 and 9, respectively. In addition, to better compare the ranking orders derived by the three methods, we list the final ranking results of alternatives determined by the three methods in Table 10.

As we can see from Table 10, the ranking orders derived by the three methods mentioned above are different. However, the three MAGDM method produce the same optimal alternative, i.e., A_1, which also proves the validity of our proposed method. In addition, our method has significant advantages over Batool et al.'s (2020, 2021) methods. First, the

Table 8 The aggregation results and corresponding score values by using PDPHFWA

	Aggregation result	Score values
A_1	$\left\{\left\{\begin{array}{c}0.6822\vert 0.15, 0.6796\vert 0.15,\\ 0.6061\vert 0.15\vert 0.6026\vert 0.15,\\ 0.5371\vert 0.2, 0.5326\vert 0.2\end{array}\right\}, \{0.2095\vert 1\}\right\}$	0.3192
A_2	$\{\{0.2687\vert 1\}, \{0.4233\vert 0.5, 0.4107\vert 0.5\}\}$	– 0.1017
A_3	$\{\{0.5689\vert 0.6, 0.5712\vert 0.4\}, \{0.3393\vert 1\}\}$	0.2095
A_4	$\left\{\left\{\begin{array}{c}0.3074\vert 0.3, 0.2949\vert 0.3,\\ 0.3390\vert 0.2, 0.3280\vert 0.2\end{array}\right\}, \{0.4728\vert 0.5, 0.4596\vert 0.5\}\right\}$	– 0.1184

Table 9 The deviations and closeness index of each alternative by using PDHF-TOPSIS

	$d(A_i, A^+)$	$d(A_i, A^-)$	$CD(A_i)(CD(A_i) = d(A_i, A^-)/(d(A_i, A^+) + d(A_i, A^-)))$
A_1	0.0010	0.0214	0.9546
A_2	0.0524	0.0711	0.5756
A_3	0.0537	0.0707	0.5682
A_4	0.0300	0.0643	0.6815

Table 10 The ranking orders derived by the three decision-making methods

	Ranking order of alternatives	Best alternative	Worst alternative
The PDPHFWA based method presented in Batool et al. (2021)	$A_1 \succ A_3 \succ A_2 \succ A_4$	A_1	A_4
The PDHF-TOPSIS method introduced in Batool et al. (2020)	$A_1 \succ A_4 \succ A_2 \succ A_3$	A_1	A_3
Our projection model based method in this article	$A_1 \succ A_3 \succ A_4 \succ A_2$	A_1	A_2

PDPHFWA-based method only considers the overall evaluation values of alternatives. Second, the PDHF-TOPSIS can only consider the distance between an alternative to the positive and negative ideal solutions. Hence, our proposed method is more powerful than them.

4.3 Further discussion

In the above subsection, some MAGDM methods are employed to solve the SBIC in the present study and the advantages of our proposed method is explained through comparison analysis. To further illustrate the effectiveness and advantages of our proposed method, this subsection provides a more comparative analysis. Details are presented as follows.

4.3.1 Compared with Batool et al.'s (2021) method based on the PDPHFWA operator

In this subsection, we compare our proposed method with that by Batool et al. (2021) based on the PDPHFWA operator. We use these two decision-making methods to solve the following example, which is adopted and revised from Batool et al. (2022), compare the decision results and analyze the advantages of our method.

Example 1 (Revised from Batool et al. 2021) Let's consider an emergency decision-making problem for the emergency situation of coronavirus disease 2019 (COVID-19). There are four alternatives that can be considered to respond to the emergency situation of COVID-19, which are given by: make high-tech wards in hospitals (A_1), vaccinate the individuals (A_2), lock down the borders (A_3), and screen and treat tactics (A_4). When considering to determine the ranking and superiority of the four alternatives, the following three attributes are taken into consideration, i.e., establish to rescue individuals influenced by the pandemic (G_1), control the situation immediately (G_2), and recover and prevent others from viruses (G_3). Weight vector of the three attributes is $w = (0.314, 0.355, 0.331)^{\mathrm{T}}$. A group of experts are invited and evaluate the four possible alternatives and they use PDHPFEs to express their evaluation opinions. The original probabilistic dual hesitant Pythagorean fuzzy decision matrix is listed in Table 11. We use Batool et al.'s (2021) method based on the PDPHFWA operator and our proposed method based on projection model to solve this example and present the decision results in Table 12.

As it is seen from Table 12, both our proposed method and Batool et al.'s (2021) method can solve this example and the ranking results derived by the two decision-making methods are slightly. The ranking order of alternatives that derived by Batool et al.'s (2021) method

Table 11 The original probabilistic dual Pythagorean hesitant fuzzy decision matrix of Example 1

	G_1	G_2	G_3
A_1	$\{\{0.2\vert 0.6, 0.3\vert 0.4\}, \{0.3\vert 1\}\}$	$\{\{0.45\vert 1\}, \{0.2\vert 0.6, 0.8\vert 0.4\}\}$	$\{\{0.7\vert 0.9, 0.6\vert 0.1\}, \{0.6\vert 0.7, 0.7\vert 0.3\}\}$
A_2	$\{\{0.8\vert 0.3, 0.1\vert 0.7\}, \{0.1\vert 1\}\}$	$\{\{0.5\vert 1\}, \{0.3\vert 0.7, 0.4\vert 0.3\}\}$	$\{\{0.9\vert 1\}, \{0.3\vert 0.6, 0.2\vert 0.4\}\}$
A_3	$\{\{0.05\vert 0.5, 0.2\vert 0.5\}, \{0.1\vert 1\}\}$	$\{\{0.1\vert 1\}, \{0.3\vert 0.4, 0.4\vert 0.6\}\}$	$\{\{0.5\vert 0.5, 0.6\vert 0.5\}, \{0.3\vert 0.9, 0.1\vert 0.1\}\}$
A_4	$\{\{0.4\vert 0.4, 0.6\vert 0.6\}, \{0.5\vert 1\}\}$	$\{\{0.7\vert 1\}, \{0.1\vert 0.5, 0.1\vert 0.5\}\}$	$\{\{0.2\vert 1\}, \{0.3\vert 0.2, 0.6\vert 0.8\}\}$

Table 12 The ranking orders of Example 1 derived by the two decision-making methods

	Ranking order of alternatives	Best alternative	Worst alternative
Batool et al.'s (2021) method based on the PDPHFWA operator	$A_2 \succ A_4 \succ A_1 \succ A_3$	A_2	A_3
Our projection model based method in this article	$A_2 \succ A_1 \succ A_3 \succ A_4$	A_2	A_4

is $A_2 \succ A_4 \succ A_1 \succ A_3$, and A_2 is the best alternative. The ranking result derived by our method is $A_2 \succ A_1 \succ A_3 \succ A_4$, which also indicates that A_2 is the optimal alternative. This also implies the effectiveness of our proposed method in solving real MAGDM problems. However, our method is still more powerful than the one introduced by Batool et al. (2021). First of all, Batool et al.'s (2021) method is based on the simple weighted average operator, which our method is based on the projection model, which considers the closeness degrees of an alternatives to not only the PIS and NIS. Hence, the decision results produced by our method are more reliable and explainable. Second, Batool et al.'s (2021) method is based on the assumption that the weight information of attributes is completely known. However, in most real decision-making problems weight information of attributes is known and it is difficult for DMs to determine the weights of attributes subjectively. Our proposed method provides a manner to calculate the weight information objectively, which makes it more suitable and sufficient to handle MAGDM problems in reality. Hence, our method is more powerful than Batool et al.'s (2021) method and the decision results derived from our method are more reliable and reasonable than those obtained by the method proposed by Batool et al. (2021)

4.3.2 Compared with Batool et al.'s (2020) method based on PDHF-TOPSIS

In this subsection, we compare our proposed method with that by Batool et al. (2020) based on PDHF-TOPSIS. We use these two decision-making methods to solve Example 2 and the advantages of our method are analyzed.

Example 2 (Revised from Batool et al. 2020). Let's consider a fog-haze factor assessment problem, which can be also regarded as a MAGDM problem. As it is known, the main influencing factors of fog-haze weather are PM 10 concentration (A_1), PM2.5 concentration (A_2), geographical conditions (A_3), meteorological condition (A_4), and PM1.7 concentration (A_5). A group of experts is invited to assess the five factors under four attributes (G_1, G_2, G_3 and G_4). DMs are required to use PDHPFEs to express their evaluation values and the original probabilistic dual hesitant Pythagorean fuzzy decision matrix is listed in Table 13. We use Batool et al.'s (2020) method and the proposed method in this study to solve Example 2 and the decision results are presented in Table 14.

As it is seen from Table 14, both Batool et al.'s (2020) method and our proposed method can solve this example and the ranking orders derived by the two methods are slightly different. The ranking order of alternatives derived from Batool et al.'s (2020) method is $A_2 \succ A_5 \succ A_4 \succ A_1 \succ A_3$, and A_2 and A_3 are the best and worst alternatives, respectively. The ranking order of alternatives derived by our proposed method in this study is $A_2 \succ A_3 \succ A_1 \succ A_4 \succ A_5$, and A_2 and A_5 are the best and worst alternatives, respectively. This reveals

Table 13 The original probabilistic dual Pythagorean hesitant fuzzy decision matrix of Example 2

	G_1	G_2
A_1	{{0.4|0.6 , 0.6|0.4 }, {0.2|0.6 , 0.3|0.4 }}	{{0.1|1 }, {0.6|0.6 , 0.8|0.4 }}
A_2	{{0.1|0.3 .0.3|0.7 }, {0.5|1 }}	{{0.5|0.4 , 0.6|0.6 }, {0.3|0.7 , 0.4|0.3 }}
A_3	{{0.5|0.5 , 0.7|0.5 }, {0.1|0.5 , 0.2|0.5 }}	{{0.1|1 }, {0.3|0.4 , 0.4|0.6 }}
A_4	{{0.4|0.4 , 0.6|0.6 }, {0.3|1 }}	{{0.7|0.5 , 0.9|0.5 }, {0.1|0.5 , 0.1|0.5 }}
A_5	{{0.3|0.3 , 0.6|0.7 }, {0.2|0.6 , 0.3|0.4 }}	{{0.3|1 }, {0.6|0.8 , 0.3|0.2 }}
	G_3	G_4
A_1	{{0.1|0.9, 0.3|0.1 }, {0.6|0.7 , 0.7|0.3 }}	{{0.6|0.4 , 0.5|0.6 }, {0.1|1 }}
A_2	{{0.4|1 }, {0.3|0.6 , 0.2|0.4 }}	{{0.3|0.2 , 0.2|0.8 }, {0.6|0.5 , 0.3|0.5 }}
A_3	{{0.5|0.5 , 0.6|0.5 }, {0.3|0.9 , 0.1|0.1 }}	{{0.1|0.1 , 0.1|0.9 }, {0.7|1 }}
A_4	{{0.2|1 }, {0.3|0.2 , 0.6|0.8 }}	{{0.4|0.5 , 0.6|0.5 }, {0.3|0.9 , 0.1|0.1 }}
A_5	{{0.7|0.4 , 0.6|0.6 }, {0.1|0.1 , 0.3|0.9 }}	{{0.7|0.3 , 0.4|0.7 }, {0.2|1 }}

Table 14 The ranking orders of Example 2 derived by different decision-making methods

	Ranking order of alternatives	Best alternative	Worst alternative
Batool et al.'s (2020) method based on PDHF-TOPSIS	$A_2 \succ A_5 \succ A_4 \succ A_1 \succ A_3$	A_2	A_3
Our projection model based method in this article	$A_2 \succ A_3 \succ A_1 \succ A_4 \succ A_5$	A_2	A_5

that A_2 is the best alternative for both Batool et al.'s (2020) method and our method, which indicates the correctness of our proposed method. However, our proposed method is still more powerful than the developed by Batool et al.'s (2020). This is because crossaisle et al.'s (2020) are based on the classical TOPSIS and our method is based on the projection model. As analyzed above, the TOPSIS method can only consider the proximity of two alternatives from module, while the projection model can consider the proximities of two alternatives from both module and directions. In other words, compared with classical TOPSIS, the projection model can absorb more information from the original decision matrix. Hence, the projection model is more powerful than the classical TOPSIS, which also makes our method more powerful than Batool et al.'s (2020) method. Second, the weights determination method proposed by Batool et al.'s (2020) only considers maximizing the entropy of DMs' evaluation information. However, our proposal not only considers maximizing the entropy of DMs' evaluation information but also makes the total deviation between all the alternatives and the positive ideal solution to be a minimum. Hence, the weights determination method proposed by our approach is also more reasonable and flexible than that developed by Batool et al.'s (2020). The above two reasons illustrate that our proposed method is more powerful and more suitable to handle realistic MAGDM problems than that developed by Batool et al.'s (2020).

4.3.3 Compared with Ning et al. (2022) method based on PDHFSs

In this subsection, we compare our proposed method with that developed by Ning et al. (2022) based on PDHFSs. We used these two decision-making methods to solve the following example, compare the decision results and analyze the advantages of our method.

Example 3 (Revised from Ning et al. 2022) Let's consider a supplier selection problem, which is regarded as a MAGDM problem. Supplier selection plays a very important role in supply chain management. The selection of sustainable suppliers is even more important due to its direct operation effects on the whole sustainable supply chain. In increasingly complex economic activities, it becomes more difficult for a manager in deciding which supplier to choose. Suppose that a company plans to select a sustainable supplier from four alternatives $A_i (i = 1, 2, 3, 4)$. After some discussion, three attributes are employed as the selection criteria, the economical factor (G_1), the social factor (G_2) and the environmental factor (G_3), whose weight vector is given as $w = (0.2, 0.3, 0.5)^{\mathrm{T}}$. Due to the complex decision environment and insufficient information, the manager decides to utilize PDHFS to represent the evaluation and the original decision matrix is listed in Table 15. It is noted that in this example DMs' evaluation information is represented by PDHFSs. As mentioned above, PDHFS is a special case of PDHPFS. Hence, our method can also solve MAGDM problems where DMs' evaluation values are in the form of PDHPFS. We use both Ning et al.'s (2022) method and our proposed method to solve Example 3 and present the decision results in Table 16.

As it is seen from Table 16, both Ning et al.'s (2022) decision-making method and our proposed MAGDM method can solve Example 3. In addition, the ranking orders produced by the two methods are slightly different. Ranking order derived by Ning et al.'s (2022) method is $A_2 \succ A_4 \succ A_3 \succ A_1$, and A_2 is the best alternative. Our method produces the ranking result $A_2 \succ A_4 \succ A_1 \succ A_3$, and A_2 is also the best alternative. This result indicates the correctness and effectiveness of our proposed method. However, our method is still more powerful and flexible than Ning et al.'s (2022) method. First, Ning et al.'s (2022) method is

Table 15 The original decision matrix of Example 3

	G_1	G_2	G_3
A_1	$\left\{\left\{0.4\mid 0.2, 0.3\mid 0.1, 0.2\mid 0.3, 0.3\mid 0.5\right\}, \{0.4\mid 1\}\right\}$	$\left\{\{0.5\mid 0.5, 0.6\mid 0.5\}, \{0.3\mid 0.6, 0.4\mid 0.4\}\right\}$	$\left\{\{0.1\mid 0.4, 0.2\mid 0.6\}, \{0.3\mid 1\}\right\}$
A_2	$\left\{\{0.7\mid 0.1\}, \left\{0.4\mid 0.3, 0.3\mid 0.2, 0.2\mid 0.4, 0.4\mid 0.1\right\}\right\}$	$\left\{\{0.4\mid 0.4, 0.3\mid 0.6\}, \{0.2\mid 1\}\right\}$	$\left\{\left\{0.6\mid 0.1, 0.5\mid 0.1, 0.4\mid 0.5, 0.3\mid 0.3\right\}, \{0.3\mid 0.4, 0.2\mid 0.6\}\right\}$
A_3	$\left\{\{0.5\mid 0.4, 0.3\mid 0.6\}, \{0.4\mid 1\}\right\}$	$\left\{\{0.4\mid 0.6, 0.3\mid 0.4\}, \left\{0.3\mid 0.5, 0.2\mid 0.2, 0.2\mid 0.2, 0.1\mid 0.1\right\}\right\}$	$\left\{\{0.4\mid 0.5, 0.5\mid 0.5\}, \{0.5\mid 0.6, 0.4\mid 0.4\}\right\}$
A_4	$\left\{\{0.3\mid 0.7, 0.4\mid 0.3\}, \{0.3\mid 0.4, 0.3\mid 0.6\}\right\}$	$\left\{\{0.4\mid 0.3, 0.3\mid 0.7\}, \{0.5\mid 1\}\right\}$	$\left\{\{0.3\mid 0.6, 0.4\mid 0.4\}, \{0.6\mid 0.3, 0.4\mid 0.7\}\right\}$

Table 16 The ranking orders of Example 3 derived by different decision-making methods

	Ranking order of alternatives	Best alternative	Worst alternative
Ning et al.'s (2022) method based on PDHFSs	$A_2 \succ A_4 \succ A_3 \succ A_1$	A_2	A_1
Our projection model based method in this article	$A_2 \succ A_4 \succ A_1 \succ A_3$	A_2	A_3

based on an ggregation model, while our method is based on the projection model, which considers the closeness degrees of alternatives to not only the PIS and NIS. Hence, compared with the method proposed by Ning et al. (2022), our method can produce more reliable and explainable decision results. Second, Ning et al.'s (2022) method is based on PDHFSs which our method is based on PDHPFS. As analyzed above, PDHPFS is more powerful than PDHFS and PDHFS is a special case of PDHPFS. In other words, our method can solve all the MAGDM problems where Ning et al.'s (2022) method is applicable. Hence, our method is more powerful and has a larger range of applications than that introduced by Ning et al. (2022).

4.4 Summary

To better demonstrate the advantages of our proposed method, we summarize the advantages and superiorities of our proposed method as follows.

(1) Our proposed method is more suitable, effective and flexible to denote DMs' evaluation values. Our method uses PDHPFSs to denote DMs' evaluations in MAGDM procedure. As mentioned above, PDHPFSs describe not only MDs and NMDs but also their corresponding probabilistic information. In addition, PDHPFSs have a relatively lax constraint, which provides DM enough freedom to express their assessments. For instance, Yue (2020) proposed an intuitionistic fuzzy sets-based projection model, which ignores DMs' hesitation in decision-making processes. Ni et al. (2021) presented a dual hesitant fuzzy sets-based projection model, which has a rigorous constraint and DMs cannot comprehensively express their evaluation values. Hence, our proposed method provides DMs a useful manner to express their evaluation values, making it more suitable to handle complicated realistic MAGDM problems.

(2) Our proposed method can fully consider DMs' provided assessment information, compared with Batool et al.'s (2021) PDPHFWA-based MAGDM method. Batool et al.'s (2020) method determine the final ranking order of alternatives based on aggregation results of alternatives. Compared with this method, our method ranks alternatives by considering the closeness degrees of alternatives to not only the PIS and NIS. Hence, ranking results of alternatives produced by our method are more reliable and dependable.

(3) Our projection-based method is more powerful and explainable than that developed by Batool et al. (2021), which is based on the traditional TOPSIS. As analyzed in Sect. 2.2, projection model considers proximities of two alternatives from both module and directions, making it more powerful than the classical TOPSIS method. Hence, our method is more useful and reliable than that developed by Batool et al. (2021).

(4) Our method is based on PDHPFS, which is more powerful to describe DMs' evaluation information under PDHPFSs, DMs have more freedom and can more comprehensively express their evaluation information. This characteristic makes our method more powerful and flexible than the MAGDM approach proposed by Ning et al. (2022).
(5) In our proposed MAGDM method, an approach for determining weights information of attributes is put forwarded. The weights determination method is based on minimizing the overall deviation of all alternatives to the PIS and maximizing Shannon's entropy of weight information, simultaneously. In Batool et al.'s (2020) PDPHFWA-based MAGDM method, weight information of attributes is assumed to be completely known, which is somewhat inconsistent with reality. In Batool et al.'s (2020) TOPSIS-based MAGDM method, authors proposed a weight determination method by solely considering maximizing the entropy of DMs' evaluation information. As analyzed above, our method can calculate the weight information of attributes more comprehensively and reasonably by considering Principle I and Principle II. Hence, compared with Batool et al.'s (2020) determination method, our method is more powerful and reliable.

5 Conclusion remarks

The problem of SBIC is vital for most innovative enterprises. This article provides a new method to solve SBIC problems from the angle of MAGDM. To do this, we first introduced a novel MAGDM framework in which DMs use PDPHFSs to express their evaluation values over alternatives under a set of attributes. Second, considering it is usually difficult for DMs to provide the weight information of attributes, a novel optimization model was developed to objectively calculate attributes' weight information. The proposed model can not only minimize the overall deviations between all alternatives to the PIS but also maximize the Shannon information entropy of attributes, simultaneously. Third, we extended the classical projection model into PDPHFSs. The projection between two PDPHFE vectors was originated and based on which a rule for ranking alternatives in a MAGDM problem with probabilistic dual Pythagorean fuzzy decision-making information was put forwarded. Finally, our proposed MAGDM framework was applied in an SBIC problem. Comparison analysis showed the advantages of our proposed method. In future works, we shall continue our research from the following aspects. First, our proposed method in this article does not consider DMs' consensus. In other words, we did not consider whether the final decision results are accepted by all DMs. As a matter of fact, to improve the quality of the final decision results and make them more acceptable, MAGDM methods based on consensus have become one of the most important and promising research topics in modern decision sciences (Ren et al. 2022; Cao et al. 2022; Zheng et al. 2022). Hence, we shall investigate consensus-driven MAGDM methods under PDHPFSs, which can eliminate DMs' disagreements and make the final decision results more reliable. Second, we assume DMs are independent when evaluating the performance of alternatives, which is somewhat inconsistent with reality. In most real decision-making situations, a DM is affected by others and he/she also affects others when making decisions. Recently, more and more scholars have focused on the social network-based decision-making methods, which can take the interaction and relationship among DMs into consideration (Liu et al. 2022; Li et al. 2022; Lu et al. 2022). Therefore, in the next we shall study social network-based MAGDM methods, which take the trust relationship among DMs into consideration. Third, three-way decision-making has received much interest and interests and quite a few interesting findings have been reported (Huang et al. 2022a, b; Zhu et al. 2022; Zhan et al. 2020a, b). Hence, in future works we will also study three-way decision-making methods under PDHPFSs.

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Declarations

Conflict of interest The authors declare that they have no conflict of interests.

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