On the uniqueness of mild solutions to the time-fractional Navier-Stokes equations in $L^N\left(\mathbb{R}^N\right)^N$

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Abstract

In this paper, we present the result of maximum regularity of the mild solution of the fractional Cauchy problem. As our main result, we investigate the uniqueness of mild solutions for time-fractional Navier-Stokes equations in class $C\left([0,\infty); L^N\left(\mathbb{R}^N\right)^N\right)$ by means of the estimates $L^p - L^q$ of Giga-Shor inequality and the Gronwall inequality.

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1 Introduction

Investigating Navier-Stokes equations has always been a challenge for many researchers in the field of partial differential equations, due to their importance and great relevance [21, 22, 23, 5, 9]. For instance, they are fundamental in modeling fluid behavior in physical systems such as sea currents, blood flow and air masses, among others. Navier-Stokes equations form a system of non-linear differential equations which still presents some open problems [5]. In order to the existence, uniqueness, and regularity of solutions of Navier-Stokes equations, we need some specific mathematical tools, which in turn require great effort and dedication [5, 1, 2]. A classic example of this fact is that the existence of a mild global solution to the three-dimensional equations for incompressible fluids remains an open problem.

Fractional calculus is also an important area of mathematics due to its well-founded theoretical basis, as well as its many applications [7, 6, 8, 30]. In recent times, researchers began to investigate the existence, uniqueness and regularity of mild solutions of time-fractional Navier-Stokes equations [3, 4]. The project of unifying fractional calculus and Navier-Stokes equations is in fact something that is growing, and new works with interesting results are certainly to be expected.

In 2015, Neto and Planas [3] wrote a work on mild solutions of time-fractional Navier-Stokes equations, in which they investigated the existence and uniqueness of mild solutions in $\mathbb{R}^N$. Peng et al. [4], in 2017, presented an excellent work on the properties of mild solutions of the time-fractional Navier-Stokes equations in Sobolev space via harmonic analysis. In the same year, Zhou and Peng [14], established the existence and uniqueness of mild solutions (local and global) in $H^{\beta,q}$, for the Navier-Stokes equations with the Caputo fractional derivative of order $\alpha \in (0,1)$. In the same work, the authors investigated the existence and regularity of classical solutions. For a discussion of the results on solutions of Navier-Stokes equations using fractional derivatives, we suggest [15, 16, 17, 18, 19, 20].

In this paper, we consider the N-dimensional time-fractional Navier-Stokes equations in $\mathbb{R}^N$ $(N \geq 3)$, given by

$$\left\{\begin{array}{rcl} {}^C\mathbb{D}_t^\alpha \mathbf{u} & = & \Delta \mathbf{u} - (\mathbf{u}\cdot\nabla)\,\mathbf{u} - \nabla \mathbf{p} \\ \nabla\cdot\mathbf{u} & = & 0 \\ & & (x,t) \in \mathbb{R}^N \times (0,T) \end{array}\right. \tag{1.1}$$

where ${}^C\mathbb{D}_t^\alpha \mathbf{u}(\cdot)$ is a Caputo fractional derivative of order $\alpha \in (0,1)$, $\mathbf{u} = \mathbf{u}(x,t) : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}^N$, $p(x,t) : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}$ is the pressure (unknown), whose role is to maintain the divergence equal to 0, ∇ is the differential operator $(\partial_{x_1}, ..., \partial_{x_N})$, $\nabla\cdot\mathbf{u}$ is the divergence of $\mathbf{u}$, Δ is Laplace operator, while $(\mathbf{u}\cdot\nabla)$ is the derivation operator $\mathbf{u}_1\partial_{x_1} + \mathbf{u}_2\partial_{x_2} + \cdots + \mathbf{u}_N\partial_{x_N}$. We also have: $(\mathbf{u}\cdot\nabla)\,\mathbf{u} = \sum\limits_j \partial_j(\mathbf{u}^j\mathbf{u})$; $p = (-\Delta)^{-1}\sum\limits_{j,k}\partial_j\partial_k\left(\mathbf{u}^j\mathbf{u}^k\right)$; $\mathbb{P} = I_d - \nabla\Delta^{-1}\nabla = I_d + R\otimes R$ where $R = \dfrac{1}{\sqrt{-\Delta}}\nabla$ is the Riesz transform and $R = (R_1, ..., R_N)$, $\widehat{R_j f} = i\dfrac{\xi_j}{|\xi|}\widehat{f}$ and $\mathbb{P} : L^r \to L^r$ is the projector of Helmholtz-Leray.

Applying the projector $\mathbb{P}$ on both sides of the Eq.(1.1) and using the condition of divergence, we have $\mathbb{P}\mathbf{u} = \mathbf{u}$, $\mathbb{P}\,{}^C\mathbb{D}_t^\alpha\mathbf{u} = {}^C\mathbb{D}_t^\alpha\mathbf{u}$, $\mathbb{P}\nabla\mathbf{p} = 0$. Substituting the term $(\mathbf{u}\cdot\nabla)\,\mathbf{u}$ by $\nabla\cdot(\mathbf{u}\otimes\mathbf{u}) = (\nabla\cdot\mathbf{u})\,\mathbf{u} + (\mathbf{u}\cdot\nabla)\,\mathbf{u}$ (considered as a distribution), we then have that the Cauchy problem for the incompressible time-fractional Navier-Stokes equations in $\mathbb{R}^N$, can be rewritten as

$$\left\{\begin{array}{rcl} {}^C\mathbb{D}_t^\alpha\mathbf{u} - \Delta\mathbf{u} + \mathbb{P}\nabla\cdot(\mathbf{u}\otimes\mathbf{u})\,\mathbf{u} & = & 0,\ \text{for } t \in [0,T),\ x \in \mathbb{R}^N \\ \nabla\cdot\mathbf{u} & = & 0,\ \text{for } t > 0,\ x \in \mathbb{R}^N \\ \mathbf{u}(0) & = & \mathbf{u}_0. \end{array}\right. \tag{1.2}$$

Throughout the paper, we assume that the speed $\mathbf{u}_0$ satisfies $\nabla\cdot\mathbf{u_0} = 0\cdot T$ with $0 < T \leq \infty$.

The Eq.(1.2), in abstract form, is given by

$$\left\{\begin{array}{rcl} {}^C\mathbb{D}_t^\alpha\mathbf{u} & = & A_r\mathbf{u} + F(\mathbf{u}) \\ \mathbf{u}(0) & = & \mathbf{u}_0 \end{array}\right. \tag{1.3}$$

where $A_r\mathbf{u} = \Delta\mathbf{u}$ with $A_r : D(A_r) \subset L^r_\sigma \to L^r_\sigma$ is the Stokes operator and $F(\mathbf{u}) = -\mathbb{P}\nabla\cdot(\mathbf{u}\otimes\mathbf{u})$.

In what follows we investigate the uniqueness of mild solutions for N-dimensional time-fractional Navier-Stokes equations given by Eq.(1.2) in order to provide new results for this area and strengthen the link between fractional calculus and partial differential equations, especially Navier-Stokes equations. In addition, we demonstrate a result on maximum regularity for the mild solution of the fractional Cauchy problem according to the Lemma 2.4.

The paper is organized as follows. In section 2, we present the definition of fractional Laplacian and the Gagliadro-Niremberg-Sobolev and Gronwall inequalities; in addition, we present the definitions of Riemann-Liouville fractional integral and Caputo fractional derivative. We then show the mild solution for the time-fractional Navier-Stokes equations given by the integral equation; the solution is written in terms of the Mittag-Leffler functions of one and two parameters. We investigate the maximum regularity of the mild solution of the fractional Cauchy problem, that is, Lemma 2.4. To conclude the section, we present the proof of the Lemma 2.5, which is fundamental to the proof of the main result of this paper. In section 3, we investigate the uniqueness of the mild solutions of the time-fractional Navier-Stokes equations written with the Caputo fractional derivative, using the techniques presented in section 2. Concluding remarks close the paper.

2 Preliminary results

Consider the Schwartz class, the class of C^∞ functions on $\mathbb{R}^N$ whose derivatives decay faster than any polynomial.

$$S := \left\{ \mathbf{u} \in C^\infty\left(\mathbb{R}^N\right) : \sup_{x\in\mathbb{R}^N} \left| x^\xi \partial^\delta \mathbf{u}(x) \right| < \infty, \forall \xi, \delta \in \mathbb{N}^N \right\}.$$

Definition 2.1 *Let $s \in (0,1)$. The fractional Laplacian of order s of the function $\mathbf{u} \in S$, in which we denote by $(-\Delta)^s \mathbf{u}$, is defined by* [3]

$$(-\Delta)^s \mathbf{u}(x) := C(N,s)\ \text{P.V} \int_{\mathbb{R}^N} \frac{\mathbf{u}(x) - \mathbf{u}(y)}{|x-y|^{N+2s}} dy \tag{2.1}$$

where $C(N,s) := \dfrac{2^{2s} s\Gamma\left(s+\frac{N}{2}\right)}{\pi^{N/2}\Gamma(1-s)}$ is a normalization constant.

For a fixed $T > 0$, we use the notation [5]

$$\|h\|_{p,q,T} = \left(\int_0^T \|h\|^q_{L^p(\mathbb{R}^N)^N} dt \right)^{1/q}, 1 \le p \le \infty,\ 1 \le q \le \infty \tag{2.2}$$

which denotes the standard space $L^q\left((0,T); L^p\left(\mathbb{R}^N\right)^N\right)$ with the obvious modification if $q = \infty$.

We shall use the following inequality [9]:

$$(a+b)^\beta \le 2^{\beta-1}\left(a^\beta + b^\beta\right) \tag{2.3}$$

for $a, b \ge 0$ and $\beta \ge 1$.

Theorem 2.1 [10](Gagliardo-Nirenberg-Sobolev inequality) *Assume that $1 \le p \le N$. Then there exists a constant C dependending only on p and N such that*

$$\|\mathbf{u}\|_{L^{\frac{pN}{N-p}}(\mathbb{R}^N)} \le C \|\nabla \mathbf{u}\|_{L^p(\mathbb{R}^N)}$$

for all $\mathbf{u} \in C_0^1\left(\mathbb{R}^N\right)$.

Theorem 2.2 [6] (Gronwall inequality) *Let u and v be two integrable functions and g a continuous function, with domain $[0,T]$. Let $\psi \in C^1[0,T]$ be an increasing function such that $\psi'(t) \ne 0$, $t \in [0,T]$. Assume that functions u and v are non-negative and g is non-negative and non-decreasing. If*

$$\mathbf{u}(t) \le \mathbf{v}(t) + g(t) \int_0^T \psi'(\tau)(\psi(t) - \psi(\tau))^{\alpha-1} \mathbf{v}(\tau)\, d\tau$$

$t \in [0,T)$, and as v is a non-decreasing function over $[0,T]$, then

$$\mathbf{u}(t) \le \mathbf{v}(t)\, \mathbb{E}_\alpha\left(g(t)\Gamma(\alpha)[\psi(T) - \psi(0)]^\alpha\right), \forall t \in [0,T] \tag{2.4}$$

where $\mathbb{E}_\alpha(\cdot)$ is a Mittag-Leffler function with one parameter, given by $\mathbb{E}_\alpha(t) = \sum_{k=0}^\infty \dfrac{t^k}{\Gamma(\alpha k+1)}$, with $0 < \alpha < 1$.

Let $h:\mathbb{R}^N\times[0,T)\to\mathbb{R}^N$. The Riemann-Liouville fractional integral of order $\alpha\in(0,1]$ of function h is defined as [7, 8, 11]

$$I_t^\alpha h(x,t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}h(x,\tau)\,d\tau,\ t>0.$$

Besides, the Caputo fractional derivative of order α of function q, is given by [7, 8, 11]

$${}^C\mathbb{D}_t^\alpha h(x,t):=\partial_t^\alpha h(x,t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\frac{\partial}{\partial\tau}h(x,\tau)\,d\tau,\ t>0.$$

Let M_α be the Mainardi function, given by [3]

$$M_\alpha(\theta)=\sum_{k=0}^\infty\frac{\theta^n}{n!\Gamma(1-\alpha(1+n))}.$$

This function is a particular case of Wright's function. The following proposition presents a classical result about Mainardi function.

Proposition 2.3 *For $\alpha\in(0,1)$, $-1<r<\infty$ and M_α restricted to positive real line, $M_\alpha(t)\geq 0$ for all $t\geq0$, we have*

$$\int_0^\infty t^rM_\alpha(t)\,dt=\frac{\Gamma(r+1)}{\Gamma(\alpha r+1)}.$$

The mild solution for Eq.(1.2), is given by the following integral equation [4]:

$$\mathbf{u}(t)=\mathbb{E}_\alpha(t^\alpha\Delta)\mathbf{u}_0-\int_0^t(t-\tau)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}\left((t-\tau)^\alpha\Delta\right)\mathbb{P}\nabla\cdot(\mathbf{u}\otimes\mathbf{u})(\tau)\,d\tau \tag{2.5}$$

where

$$\mathbb{E}_\alpha(t^\alpha\Delta)\mathbf{v}(x)=\left((4\pi t^\alpha)^{-\frac{N}{2}}\int_0^\infty\theta^{-\frac{N}{2}}M_\alpha(\theta)\exp\left(\frac{-|\cdot|^2}{4\theta t^2}\right)d\theta*\mathbf{v}\right)(x)$$

and

$$\mathbb{E}_{\alpha,\alpha}(t^\alpha\Delta)\mathbf{v}(x)=\left((4\pi t^\alpha)^{-\frac{N}{2}}\int_0^\infty\alpha\theta^{1-\frac{N}{2}}M_\alpha(\theta)\exp\left(\frac{-|\cdot|^2}{4\theta t^2}\right)d\theta*\mathbf{v}\right)(x),$$

with $\mathbb{E}_{\alpha,\beta}(t)=\sum_{k=0}^\infty\frac{t^k}{\Gamma(\alpha k+\beta)}$, $0<\alpha<1$ and $0<\beta<1$.

The mild solution $u\in C\left([0,T),L^N\left(\mathbb{R}^N\right)^N\right)$ is associated with the initial condition $\mathbf{u}_0\in L^N\left(\mathbb{R}^N\right)^N$ as $\nabla\cdot\mathbf{u}_0=0$.

Before investigating our main result, we need the results presented in Lemma 2.4 and Lemma 2.5, below.

Lemma 2.4 *Let $1<p,q<\infty$, $0<T<\infty$. If $h\in L^q\left((0,T);L^p\left(\mathbb{R}^N\right)^N\right)$, the function*

$$\mathbf{u}(t)=\mathbb{E}_\alpha(t^\alpha\Delta)\mathbf{u}_0+\int_0^t(t-\tau)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}\left((t-\tau)^\alpha\Delta\right)\mathbb{P}h(\tau)\,d\tau \tag{2.6}$$

belongs to $L^q\left((0,T);L^p\left(\mathbb{R}^N\right)^N\right)$ and solves the following Cauchy problem:

$$\begin{cases} {}^C\mathbb{D}_t^\alpha\mathbf{u}-\Delta\mathbf{u} &= \mathbb{P}h \text{ for almost everywhere } t\in(0,T)\\ \mathbf{u}(0) &= 0\end{cases} \tag{2.7}$$

In addition, the solution u satisfies the estimate

$$\|\Delta \mathbf{u}\|_{p,q,T} \leq C \|h\|_{p,q,T} \tag{2.8}$$

with $C = C(p, N, q) > 0$ *independent of* h *and* T.

In the proof of Lemma 2.4, we will use the following definitions:

1. $\Omega_1 = \mathbb{R}^N$;

2. $\Omega_2 =$ limited domain;

3. $\Omega_3 =$ half space;

4. $\Omega_4 =$ external domain of $\mathbb{R}^N$.

The proof shall be adapted from the proof of Theorem 2.7 [24]. The result ensures that, if $\Omega \subseteq \mathbb{R}^N$ satisfies one of the definitions Ω_1-Ω_4, then the solution u of the Navier-Stokes equation is unique.

Proof: Indeed, as we have seen earlier, we have been able to rewrite the N-dimensional time-fractional Navier-Stokes equation Eq.(1.1) in the form of Eq.(1.2). On the other hand, Eq.(2.7) can be written as follows:

$$\left\{ \begin{array}{rcl} {}^C\mathbb{D}_t^\alpha \mathbf{u} - \Delta \mathbf{u} + \nabla \mathbf{p} & = & h \\ \nabla \cdot \mathbf{u} & = & 0 \\ \mathbf{u}(x, 0) & = & \mathbf{u}_0. \end{array} \right. \tag{2.9}$$

Next, we will use the embedding property for the second-order derivative $\Delta \mathbf{u} = \nabla^2 \mathbf{u} = (\partial_j \partial_j \mathbf{u})$, $j = 1, ..., m$,

$$\|\Delta \mathbf{u}\|_{p,q,T} = \left\|\nabla^2 \mathbf{u}\right\|_{p,q,T} \leq C \|A_q\|_{p,q,T}, \ \mathbf{v} \in D(A_q) \tag{2.10}$$

where $A_q = -\Delta \mathbf{u}$.

This applies to $C = C(p, q, N) > 0$ for $1 < p, q < \infty$ if $\Omega \subseteq \mathbb{R}^N$ satisfies one of the definitions Ω_1-Ω_3 and for $1 < p, q < \frac{N}{2}$ in Ω_4.

In fact, the result for the case Ω_1 follows from Lemma 3.1 [25]; for Ω_2, it follows from Lemma 2.4 [26, 27]. The uniqueness in case Ω_3 follows from Theorem 3.6 [29]. For the case Ω_4, see [28].

In this sense, applying Eq.(2.10) in Eq.(2.9) and using $\nabla \mathbf{p} = g - {}^C\mathbb{D}_t^\alpha + \Delta \mathbf{u}$, we obtain the following result:

$$\begin{aligned} \|\Delta \mathbf{u}\|_{p,q,T} & \leq \left\|{}^C\mathbb{D}_t^\alpha \mathbf{u}\right\|_{p,q,T} + \|Ph\|_{p,q,T} \\ & \leq C \|h\|_{p,q,T}. \end{aligned} \tag{2.11}$$

This completes the demonstration. □

In the proof of Lemma 1 the authors Giga and Sohr assumed that Ω has an external domain, that is, a domain whose complement in $\mathbb{R}^N$ is a non-empty compact set. But since $\Omega = \mathbb{R}^N$ is all space, Lemma 2.4 has been proved following the same steps as Theorem 2.7 [24].

Lemma 2.5 *Let* $g \in L^q\left((0,T);L^p\left(\mathbb{R}^N\right)^{N^2}\right)$ *where* $1<p,q<\infty$, $0<T<\infty$. *Then, there exists a unique solution* $\mathbf{v}=(-\Delta)^{-1/2}\mathbf{u}$ *belonging to* $L^q\left((0,T);L^p\left(\mathbb{R}^N\right)^N\right)$ *which solves the Cauchy problem*

$$\left\{\begin{array}{rcl} {}^C\mathbb{D}_t^\alpha \mathbf{v}-\Delta\mathbf{v} &=& \mathbb{P}(-\Delta)^{-1/2}\nabla\cdot h,\ \textit{almost everywhere}\ t\in(0,T)\\ \mathbf{v}(0) &=& 0,\end{array}\right. \tag{2.12}$$

and satisfies the following estimates:

$$\|\nabla\mathbf{u}\|_{p,q,T}\le C\|h\|_{p,q,T}$$

and

$$\|\mathbf{u}\|_{\frac{pN}{N-p},q,T}\le C\|h\|_{p,q,T},\ 1<p<N, \tag{2.13}$$

with $C=C(p,N,q)>0$ *independent of* h *and* T.

Proof: Applying $(-\Delta)^{-1/2}$ in Eq.(2.7), we have

$${}^C\mathbb{D}_t^\alpha\mathbf{v}-\Delta\mathbf{v}=P(-\Delta)^{-1/2}\nabla\cdot h. \tag{2.14}$$

Thus, by the maximum regularity theorem in the fractional sense (Lemma 2.4), we know that there is a unique solution $\mathbf{v}\in L^q\left((0,T);L^p\left(\mathbb{R}^N\right)^N\right)$ of Eq.(2.12) for all $T>0$.

Moreover, from the Calderon-Zygmund theorem on singular integrals [31, 32] and inequality (2.7), we get

$$\begin{aligned}\|\nabla\mathbf{u}\|_{p,q,T} &= \left\|\nabla(-\Delta)^{1/2}\mathbf{v}\right\|_{p,q,T}\\ &= \|\Delta\mathbf{v}\|_{p,q,T}\\ &\le C\|h\|_{p,q,T}\end{aligned} \tag{2.15}$$

using inequality (2.7).

Using the inequality of Gagliardo-Nirenberg-Sobolev (Theorem 2.1) and inequality (2.15), we have that, for every $t\in[0,T]$,

$$\begin{aligned}\int_0^T\|\mathbf{u}(\tau)\|_{L^{\frac{pN}{N-p}}(\mathbb{R}^N)}d\tau &\le C\int_0^T\|\nabla\mathbf{u}(\tau)\|^q_{L^p(\mathbb{R}^N)}d\tau\le\\ &\le \widetilde{C}\int_0^T\|g(\tau)\|^q_{L^p(\mathbb{R}^N)}d\tau.\end{aligned} \tag{2.16}$$

Thus, raising both sides of this inequality to $1/q$, we conclude that

$$\|\mathbf{u}\|_{\frac{pN}{N-p},q,T}=\left(\int_0^T\|\mathbf{u}(\tau)\|_{L^{\frac{pN}{N-p}}(\mathbb{R}^N)}d\tau\right)^{1/q}\le C\left(\int_0^T\|g(\tau)\|^q_{L^p(\mathbb{R}^N)}d\tau\right)^{1/q}=C\|g\|_{p,q,T}. \tag{2.17}$$

□

3 Uniqueness of mild solution

In this section, we demonstrate the main result of this paper, namely, the uniqueness of mild solution for time-fractional Navier-Stokes equations Eq.(1.2), by means of the estimates in Lemma 2.4 and Lemma 2.5 and the Gronwall inequality (Theorem 2.2).

Theorem 3.1 *Let $0 < T \leq \infty$ and let $\mathbf{u}, \mathbf{v} \in C\left([0,T); L^N\left(\mathbb{R}^N\right)^N\right)$ be two solutions of the time-fractional Navier-Stokes equation on $(0,T) \times \mathbb{R}^N$ with the same initial condition $\mathbf{u}_0$. Then, $\mathbf{u} = \mathbf{v} \in C[0,T)$.*

Proof: For $\mathbf{u}, \mathbf{v} \in C\left([0,T); L^N\left(\mathbb{R}^N\right)^N\right)$ and an $\varepsilon > 0$, there are two decomposition $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$ and $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$ such that, for every $T > 0$,

$$\|\mathbf{u}_1\|_{C\left([0,T); L^N\left(\mathbb{R}^N\right)^N\right)} \leq \varepsilon \quad ; \quad \sup_{(x,t) \in \mathbb{R}^N \times (0,T)} |\mathbf{u}_2(x,t)| < K(\varepsilon) \tag{3.1}$$

and

$$\|\mathbf{v}_1\|_{C\left([0,T); L^N\left(\mathbb{R}^N\right)^N\right)} \leq \varepsilon \quad ; \quad \sup_{(x,t) \in \mathbb{R}^N \times (0,T)} |\mathbf{v}_2(x,t)| < K(\varepsilon). \tag{3.2}$$

We can consider

$$\mathbf{u}_2(x,t) = \begin{cases} \mathbf{u}(x,t), & \text{for} \quad |\mathbf{u}(x,t)| < K \\ 0, & \text{for} \quad |\mathbf{u}(x,t)| \geq K \end{cases}$$

and

$$\mathbf{v}_2(x,t) = \begin{cases} \mathbf{v}(x,t), & \text{for} \quad |\mathbf{v}(x,t)| < K \\ 0, & \text{for} \quad |\mathbf{v}(x,t)| \geq K \end{cases}$$

for a large enough K.

Now, assume that $\mathbf{u}$ and $\mathbf{v}$ are solutions in $C\left([0,T); L^N\left(\mathbb{R}^N\right)^N\right)$ with the same initial conditions, for instance $\mathbf{u}(0) = \mathbf{v}(0) = \mu$. Then, the difference $\xi = \mathbf{u} - \mathbf{v}$ is a solution of the integral equation

$$\xi(t) = -\int_0^T (t-\tau)^{\alpha-1} \mathbb{E}_{\alpha,\alpha}\left((t-\tau)^\alpha \Delta\right) \mathbb{P}\nabla \cdot (\xi \otimes \mathbf{u} + \mathbf{v} \otimes \xi)(\tau)\, d\tau.$$

Now, consider the functions

$$\xi_1(t) = -\int_0^T (t-\tau)^{\alpha-1} \mathbb{E}_{\alpha,\alpha}\left((t-\tau)^\alpha \Delta\right) \mathbb{P}\nabla \cdot (\xi \otimes \mathbf{u}_1 + \mathbf{v}_1 \otimes \xi)(\tau)\, ds$$

and

$$\xi_2(t) = -\int_0^T (t-\tau)^{\alpha-1} \mathbb{E}_{\alpha,\alpha}\left((t-\tau)^\alpha \Delta\right) \mathbb{P}\nabla \cdot (\xi \otimes \mathbf{u}_2 + \mathbf{v}_2 \otimes \xi)(\tau)\, ds.$$

The convolution operator $\mathbb{E}_{\alpha,\alpha}\left((t-\tau)^\alpha \Delta\right) \mathbb{P}\nabla$ has an integrable core whose standard is $O\left((t-\tau)^{-\alpha/2}\right)$ in L_1. From this property and using the estimates Eq.(3.1) and Eq.(3.2) and Hölder's inequality repeatedly in time, we have that

$$\|\xi_2(t)\|_{L^N} \leq C \int_0^T (t-\tau)^{\frac{\alpha}{2}-1} \|\xi(\tau)\|_{L^N} \left(\|\mathbf{u}_2(\tau)\|_{L^\infty} + \|\mathbf{v}_2(\tau)\|_{L^\infty}\right) d\tau$$

$$
\begin{aligned}
&\leq 2CK(\varepsilon)\left(\int_0^T (t-\tau)^{\frac{2}{3}(\alpha-2)}\,d\tau\right)^{3/4}\left(\int_0^T \|\xi(\tau)\|_{L^N}^4\,d\tau\right)^{1/4}\\
&\leq 2CK(\varepsilon)\,t^{\frac{2\alpha-1}{4}}\left(\int_0^T \|\xi(\tau)\|_{L^N}^4\,d\tau\right)^{1/4}, \qquad (3.3)
\end{aligned}
$$

where C denotes a constant independent of ξ, t.

Now, raising both sides of inequality (3.3) to the fourth power and taking the integral with respect to $\tau \in (0,T)$, we have

$$
\int_0^T \|\xi_2(\tau)\|_{L^N}^4\,d\tau \leq 2C^4(K(\varepsilon))^4\,T^{2\alpha-1}\int_0^T\left(\int_0^\tau \|\xi(s)\|_{L^N}^4\,ds\right)d\tau. \qquad (3.4)
$$

On the other hand, by estimate Eq.(2.13) of Lemma 2.5, estimates Eq.(3.1) and Eq.(3.2), Hölder's inequality and inequality Eq.(2.3), we obtain

$$
\begin{aligned}
\int_0^T \|\xi_1(\tau)\|_{L^N}^4\,d\tau &\leq C\int_0^T \|(\xi\otimes(\mathbf{u}_1+\mathbf{v}_1))(\tau)\|_{L^{\frac{N}{2}}}^4\,d\tau\\
&\leq C\left(\|\mathbf{u}_1(\tau)\|_{C\left([0,T);L^N(\mathbb{R}^N)^N\right)}+\|\mathbf{v}_1(\tau)\|_{C\left([0,T);L^N(\mathbb{R}^N)^N\right)}\right)\int_0^T \|\xi(\tau)\|_{L^N}^4\,d\tau\\
&\leq 2\varepsilon C\int_0^T \|\xi(\tau)\|_{L^N}^4\,d\tau. \qquad (3.5)
\end{aligned}
$$

Taking ε small, we have from inequalities Eq.(3.4) and Eq.(3.5) that

$$
\int_0^T \|\xi(\tau)\|_{L^N}^4\,d\tau \leq \widetilde{C_\alpha}(K(\varepsilon))^4\,T^{2\alpha-1}\int_0^T (t-\tau)^{\alpha-1}\left(\int_0^\tau \|\xi(s)\|_{L^N}^4\,ds\right)d\tau
$$

$0 \leq \tau \leq T$.

Using the Gronwall inequality (Theorem 2.2), we finally have

$$
\int_0^T \|\xi(\tau)\|_{L^N}^4\,d\tau \leq 0.\mathbb{E}_\alpha\left(\widetilde{C_\alpha}(K(\varepsilon))^4\,T^{3\alpha-1}\Gamma(\alpha)\right)=0,
$$

which implies that $\int_0^T \|\xi(\tau)\|_{L^N}^4\,d\tau = 0 \Longleftrightarrow \xi = 0$. Therefore, $\mathbf{u} = \mathbf{v}$. □

4 Concluding remarks

We investigated the uniqueness of mild solution for time-fractional Navier-Stokes equations in $L^N\left(\mathbb{R}^N\right)^N$ by means of estimates (Lemma 2.4 and Lemma 2.5) and the Gronwall inequality. A direct consequence of the results obtained here is that when $\alpha = 1$, we recover the result valid for the classical Navier-Stokes equation. It is worth mentioning that it remains an open problem the investigation of the existence, uniqueness and regularity of mild solutions for time-fractional Navier-Stokes equations introduced by ψ-Caputo fractional derivative [7]. It seems that a possible way to approach this open problem would be to introduce a new Laplace transform involving the derivative of a function taken in relation to another function.

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