Computational and Applied Mathematics (2023) 42:74
https://doi.org/10.1007/s40314-023-02216-x

A streamlined numerical method to treat fractional nonlinear terminal value problems with multiple delays appearing in biomathematics

Ömür Kıvanç Kürkçü[1]

Received: 12 October 2022 / Revised: 16 December 2022 / Accepted: 21 January 2023 /
Published online: 6 February 2023

Abstract
In this study, a computational matrix-collocation method based on the Lagrange interpolation polynomial is specifically streamlined to treat the fractional nonlinear terminal value problems with multiple delays, such as the Hutchinson, the Wazewska-Czyzewska and the Lasota models in biomathematics. To do this, the robust nonlinear terms of which are smoothed to be deployed in the method. The uniqueness analysis of the solution is discussed in terms of the Banach contraction principle. An error analysis technique is non-linearly theorized and applied to improve the solutions. A programme for the method is especially developed. Thus, the outcomes of five fractional model problems constrained by terminal conditions are numerically and graphically evaluated in tables and figures. Based on the investigation of the results, one can claim that the method presents a sustainable and effective mathematical procedure for the aforementioned problems.

Keywords Delay differential equation · Error analysis · Fractional derivative · Matrix-collocation method · Terminal condition

Mathematics Subject Classification 34A08 · 65L60 · 92B05

1 Introduction

Recent decades have seen a great deal of interest in the modelling of real-world phenomena by means of fractional differential equations. One of their most applicable subject areas turns out to be biomathematical applications since fractional order derivatives can determine the properties of the hereditary and population dynamics with respect to the temporal aspect. For example, sub-threshold nerve propagation and complex biological systems are modelled

---

Communicated by Agnieszka Malinowska.

✉ Ömür Kıvanç Kürkçü
omurkivanc@outlook.com

1 Department of Engineering Basic Sciences, Konya Technical University, 42250 Konya, Turkey

via the fractional derivative aspect of the mathematical functions (Magin 2004). With the involvement of delay or time lag systems, the fractional delay differential equations (FDDEs) are of great importance in that they can govern the historical behavior of a biological model at previous time, as well as giving details at present time. In that case, some of which are expressed as follows:

(i) The fractional Hutchinson delay model can describe population growth in the resource-limited area, giving a physical detail of gestation in earlier time (Fowler 2005; Pimenov and Hendy 2017). In addition, its behavior can also monitored with aid of the fractional order derivative in terms of the approximation to the population carrying capacity.
(ii) The fractional Wazewska-Czyzewska and Lasota delay model can evaluate the amount of the red blood cells with respect to time t, and its constant delay allows an augmentation time for the red blood cells (Ruan 2006; Caraballo et al. 2005). Besides, the dynamical behavior of the red blood cells is closely chased according to the fractional order aspect.
(iii) The fractional Gompertz delay model can detail the tumour growth of animals with respect to time t and its constant delay phenomenon provides animal cells with a time lag process to cope with tumour growth or augmentation. (Piotrowska and Foryś (2011); Valentim et al. (2020)). The influence of the fractional order derivatives on the tumour growth is generally followed through Gompertzian movement.

It is evident from the studies above that applying an analytical treatment to the biomathematical models especially governing the population growth on a long time process is too hard to obtain their physical response with regard to well-known mathematical functions. That is why the numerical treatments are involved in determining dynamical behavior of such models at extended temporal limits. Some recent numerical methods are pinned from the literature as follows. The multi-step collocation method is attached to find the numerical solutions of fractional retarded differential equations ranging from the life cycle of population lemmings to a model of chronic granulocytic leukaemia (Maleki and Davari 2019). A new method based on Daftardar-Gejji-Jafari technique is applied to solve FDDEs containing four year life cycle of the population of lemmings (Jhinga and Daftardar-Gejji 2019). A generalized finite difference method is extended to treat FDDEs, which are composed of a model of the limit of food on the population dynamics lied on an area, a model of the fluctuations of population of adults in an area, and a model of the propagation of human blood cells (Moghaddam and Mostaghim 2013). A BDF-type method based on the shifted Chebyshev series is approached to FDDEs of functional type, which contains a fractional Hutchinson delay model (Pimenov and Hendy 2017). A numerical oscillation analysis is introduced for Gompertz equation with a delay (Yang and Wang 2020). The reduced differential transforms method is deployed to treat the nonlinear fractional mathematical smoking model in biomathematics (Mahdy et al. 2020). The predictor-corrector regulation method arising from Adams-Bashford-Moulton type is utilized for the numerical solution of a fractional order CoViD-19 virus model expressing the dynamics of reinfection and the importance of quarantine (Maurício de Carvalho and Moreira-Pinto 2021). The optimal treatment problem is solved via the forward-backward iterative method for the combined antiretroviral drug therapy in an HIV infection model (Nath et al. 2023). A cancer-obesity treatment biomathematical model under the control parameters is eliminated for different cases (Dehingia et al. 2022). An algorithm dependent upon the fifth-kind Chebyshev polynomials is applied to solve the variable-order time-fractional diffusion-wave equation with multi-term (Sadri and Aminikhah 2022).

It is the fact that the applications of FDDEs are not limited to biomathematical models. Their other applications can be come across in sea surface temperature distribution (Gande and Madduri 2022), El niño-Southern oscillation in south Africa (Gande and Madduri 2022), mechanical and electro-dynamical systems (Iqbal et al. 2017; Rabiei and Ordokhani 2019), and applied sciences (Kürkçü et al. 2019).

Different from the mentioned studies above, the main aim in this study is concentrated on treating the fractional nonlinear terminal value problems with multiple delays (FNTVPs) appearing in biomathematics, applying the Lagrange interpolation matrix-collocation method. In addition, a general formulation of such problems is extracted and smoothed to be interpreted via matrix expansions provided by the method. Another advantage of the method is that the Lagrange interpolation polynomial allows to approach polynomially to the solutions of FNTVPs, because of the fact that it conveys the standard collocation points, which act like interpolation points. Indeed, the proposed method is streamlined according to the reduced form of the matrix relations derived from the linear and nonlinear terms without requiring additional matrices or their expansions at the collocation points. That is why the proposed method yields the approximate solutions fast. Thanks to this ability, it can also be incorporated with the residual error analysis technique in virtue of the reduced matrix relations of the nonlinear terms. This novelty in a solution form differs from the others in that they contain well-known polynomials. It is also worth expressing that these solutions can be improved by this error analysis technique.

As a new class of FDDEs, the Lagrange interpolation matrix-collocation method treats FNTVPs of the form

$$
{}_{t}^{C}D_{0}^{\alpha} y\left(t\right) + \sum_{j=1}^{3}\sum_{i=0}^{1} P_{ij}(t)\, y^{i}(t)\, y^{j}(\gamma_{ij}t + \beta_{ij}) = g\left(t\right), \;\; 0 < \alpha \leq 2, \;\; 0 \leq t \leq T, \tag{1}
$$

subject to the terminal conditions

$$
y(t) = c_1 \text{ for } t \in [-\beta_{ij}, 0] \text{ and } y(T) = c_2, \tag{2}
$$

where $\gamma_{ij} (\in (0, 1])$ and $\beta_{ij} (\in (0, 1])$ denote the proportional and the constant delays, respectively; $y(t)$ and $P_{ij}(t)$ are continuous functions on $[0, T]$, α can determine both integer and fractional order derivatives, and $g(t)$ is a continuous external force function. In view of Eq. (1), three model problems (i)-(iii) are treated in this study, including a Hutchinson FNTVP with variable coefficients, and a FNTVP made up of a quartic nonlinearity for sake of using higher degree of nonlinearity.

The remaining of this study is planned as follows: Sect. 2 expresses a brief preliminary about fractional calculus and the uniqueness analysis of the Lagrange polynomial solution of Eq. (1). Section 3 informs the conceptual background of the method of solution. Section 4 constitutes an error analysis technique enabling the improvement of the obtained solutions. Section 5 presents five benchmark models and their numerical and graphical results. Section 6 concludes the validity and efficacy of the method, discussing advantages, disadvantages and achievements with respect to outcomes in the previous Sect. 5.

2 Preliminaries

In this section, a brief information about fractional calculus is given to lay out the basement of the proposed method. The uniqueness of the Lagrange polynomial solution of Eq. (1) is also discussed in details.

2.1 Fractional calculus

Let $y(t)$ be a continuous function on $[0, T]$, D stands for a differential operator.

Definition 2.1 Diethelm (2010); Podlubny (1999) Given that $\alpha > 0$ and $t > 0$ $(\alpha, t \in \mathbb{R})$. The Riemann-Liouville fractional integral $\left(J_0^\alpha\right)$ of $y(t)$ with respect to α holds

$$J_0^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds.$$

Definition 2.2 Diethelm (2010); Podlubny (1999); Caputo (1969) Given that $t > a$ and $\alpha, t \in \mathbb{R}$. The Caputo fractional derivative $\left({}_t^C D_0^\alpha\right)$ of order α is then formulated as

$${}_t^C D_0^\alpha y(t) = \begin{cases} \frac{1}{\Gamma(\lceil\alpha\rceil-\alpha)} \int\limits_0^t (t-s)^{\lceil\alpha\rceil-\alpha-1} y^{(\lceil\alpha\rceil)}(s)\, ds, & \lceil\alpha\rceil - 1 < \alpha < \lceil\alpha\rceil \\ y^{(\alpha)}(t), & \alpha \in \mathbb{N}. \end{cases}$$

For foundation of our next uniqueness analysis, it is required to state the following property.

Remark 2.1 Diethelm (2010) A relationship between the Riemann-Liouville fractional integral and the Caputo fractional derivative leads to

$$J_0^\alpha\, {}_t^C D_0^\alpha y(t) = y(t) - \sum_{i=0}^{\lceil\alpha\rceil-1} \frac{y^{(i)}(0)}{i!} t^i.$$

2.2 Uniqueness analysis of Lagrange polynomial solution

The Lagrange interpolation polynomial solution form was previously established for the integral and integer order integro-differential equation (Yüzbaşı and Sezer 2021). In this study, we shall develop it in the solution form of FNTVPs (1) and the matrix relations, as

$$y(t) \cong y_N(t) = \sum_{n=0}^{N} a_n L_n(t), \tag{3}$$

where a_n's are unknown Lagrange interpolation coefficients, which have to be determined, and $L_n(t)$ denotes the Lagrange interpolation polynomial, which possesses an explicit form (see Abramowitz and Stegun 1964)

$$L_n(t) = \prod_{i=0}^{N} \frac{t - t_i}{t_n - t_i} \text{ for } n \neq i,$$

such that t_i represents the standard collocation points $t_i = Ti/N$.
Formerly, the Banach contraction principle was applied to the fractional partial integro-differential equations and the fractional integro-differential equation of Volterra-Fredholm type (Santra and Mohapatra 2022; Hamoud et al. 2018). Motivated by these works, this study aims here to newly adapt the Banach contraction principle to Eq. (1) for $\alpha \in (0, 2]$, indicating the uniqueness of the Lagrange polynomial solution. Let some required fundamental properties are given to establish the uniqueness analysis. Assume that the notation $||\cdot||$ is a standard norm produced by the normed vector space $S = [0, T]$. Also, the following properties are given.

Lemma 2.1 *Santra and Mohapatra (2022); Hamoud et al. (2018) Let* $\mathbb{L} : S \mapsto S$ *be a contraction mapping. Then, for* $\forall x, y \in S$*, it reads*

$$||\mathbb{L}(x) - \mathbb{L}(y)|| < \lambda \, ||x - y|| \, ,$$

from which $\mathbb{L}(x)$ *is a Lipschitz function and* $\lambda \in (0, 1]$ *is a Lipschitz contraction constant.*

Lemma 2.2 *Santra and Mohapatra (2022); Hamoud et al. (2018) Assume that S is complete and it has a contraction mapping* $\mathbb{L}$*. Then S appears as a Banach space, which means that* $\mathbb{L}$ *possesses a unique fixed point* $\tilde{y}$ *in S.*

Lemma 2.3 *Let* $\mathbb{F}_{ij}(y_{1,2}(t))$ *be the Lipschitz continuous function on S, which is derived from*

$$\mathbb{F}_{ij}(y_{1,2}(t)) = y^{i}_{1,2}(t)\, y^{j}_{1,2}(\alpha_{ij} t + \beta_{ij}),$$

in Eq. (1) such that it has a Lipschitz constant $\lambda_{ij} > 0$ *with respect to* $y_1(t)$ *and* $y_2(t)$ *on S. Then, it follows that*

$$||\mathbb{F}_{ij}(y_1(t)) - \mathbb{F}_{ij}(y_2(t))|| < \lambda_{ij} ||y_1(t) - y_2(t)||.$$

Thereby, the following theorem can now be extracted.

Theorem 2.1 *In view of Lemmas 2.1, 2.2 and 2.3, a unique fixed point solution* $y(t)$ *on S yields a Lagrange polynomial solution in case there exists such a condition*

$$\sum_{j=1}^{3} \sum_{i=0}^{1} \frac{||P_{ij}||\lambda_{ij}}{\Gamma(\alpha+1)} < 1.$$

Proof 1 Let $\forall y_{1,2}(t) \in S$, then by incorporating the fractional integral operator $\left(J_0^{\alpha}\right)$ into Eq. (1), it follows from Lemmas 2.1 and 2.3 that

$$\mathbb{L}(y_{1,2}(t)) = c_0(t) + J_0^{\alpha} \sum_{j=1}^{3} \sum_{i=0}^{1} P_{ij}(t)\, \mathbb{F}_{ij}(y_{1,2}(t)) + J_0^{\alpha}\, g\,(t)\, , \tag{4}$$

where

$$c_0(t) = \sum_{i=0}^{\lceil \alpha \rceil - 1} \frac{y^{(i)}_{1,2}(0)}{i!} t^i .$$

Then, Eq. (4) obeys

$$\begin{aligned} ||\mathbb{L}(y_1) - \mathbb{L}(y_2)|| &= ||J_0^{\alpha} \left(\sum_{j=1}^{3} \sum_{i=0}^{1} P_{ij}(t)\, \mathbb{F}_{ij}(y_1(t)) - \mathbb{F}_{ij}(y_2(t)) \right) || \\ &\leq \sum_{j=1}^{3} \sum_{i=0}^{1} J_0^{\alpha} \left(||P_{ij}(t)||\, ||\mathbb{F}_{ij}(y_1(t)) - \mathbb{F}_{ij}(y_2(t))|| \right) \\ &= \sum_{j=1}^{3} \sum_{i=0}^{1} \frac{||P_{ij}||\lambda_{ij}}{\Gamma(\alpha+1)} ||y_1(t) - y_2(t)||, \end{aligned}$$

and if the condition

$$\sum_{j=1}^{3} \sum_{i=0}^{1} \frac{||P_{ij}||\lambda_{ij}}{\Gamma(\alpha+1)} < 1, \;\; ||P_{ij}|| = \max_{t \in S} |P_{ij}(t)|,$$

is satisfied, then there exists a unique fixed point solution $y(t)$ on S according to Lemma 2.2. Thereby, the uniqueness of the solution of Eq. (1) is validated via this theorem □

3 Conceptual background of the method

In this section, the concept of the proposed method is established using the matrix relations of both linear and nonlinear terms in Eq. (1). Thanks to the easiness of the construction, the method conserves only a little conceptual background to treat Eq. (1).
A matrix formation of the Lagrange interpolation polynomial solution (3) is written with the fractional derivative in the Caputo sense, as

$$ {}_t^C D_0^\alpha y\,(t) = {}_t^C D_0^\alpha \boldsymbol{L}\,(t)\,\boldsymbol{A} = \boldsymbol{L}^{(\alpha)}(t)\boldsymbol{A},\ \ 0 < \alpha \le 2, \tag{5}$$

where

$$\begin{aligned}\boldsymbol{L}\,(t) &= \left[\, L_0\,(t)\ L_1\,(t)\ \cdots\ L_N\,(t)\,\right],\\ \boldsymbol{L}^{(\alpha)}(t) &= \left[\, {}_t^C D_0^\alpha(L_0(t))\ {}_t^C D_0^\alpha(L_1(t))\ \cdots\ {}_t^C D_0^\alpha(L_N(t))\,\right],\end{aligned}$$

and

$$\boldsymbol{A} = \left[\, a_0\ a_1\ \cdots\ a_N\,\right]^T.$$

Note here that the matrix relation (5) can be reduced to a main matrix relation for $\alpha = 0$, as

$$y\,(t) = \boldsymbol{L}^{(0)}(t)\boldsymbol{A} = \boldsymbol{L}(t)\boldsymbol{A}, \tag{6}$$

which can be utilized in further matrix relations.
In order to obtain delayed process of the main matrix relation (6), time variable t is lagged by $\alpha_{ij}t + \beta_{ij}$, proportionally and constantly. So, it becomes

$$y\left(\alpha_{ij}t + \beta_{ij}\right) = \boldsymbol{L}(\alpha_{ij}t + \beta_{ij})\boldsymbol{A}, \tag{7}$$

where

$$\boldsymbol{L}(\alpha_{ij}t + \beta_{ij}) = \left[\, L_0(\alpha_{ij}t + \beta_{ij})\ L_1(\alpha_{ij}t + \beta_{ij})\ \cdots\ L_N(\alpha_{ij}t + \beta_{ij})\,\right].$$

Using now the matrix relation (7), the matrix relation of the nonlinear terms is operated under a unique formulation

$$\begin{aligned}&\left[\sum_{j=1}^{3}\sum_{i=0}^{1} P_{ij}(t)\, y^i(t)\, y^j(\alpha_{ij}t + \beta_{ij})\right]\\ &\quad = \sum_{j=1}^{3}\sum_{i=0}^{1} \boldsymbol{P}_{ij}(t)\,(\boldsymbol{L}(t))^i \left(\prod_{k=1}^{j} \boldsymbol{L}^{*k+i-1}(\alpha_{ij}t + \beta_{ij})\right)\boldsymbol{A}^{*i+j-1},\end{aligned} \tag{8}$$

where

$$\begin{aligned}&\boldsymbol{L}^{*0}(\alpha_{ij}t + \beta_{ij}) = \boldsymbol{L}(\alpha_{ij}t + \beta_{ij}),\ \ \boldsymbol{L}^{*1}(\alpha_{ij}t + \beta_{ij})\\ &\quad = \begin{bmatrix} \boldsymbol{L}\left(\alpha_{ij}t + \beta_{ij}\right) & 0 & \cdots & 0\\ 0 & \boldsymbol{L}\left(\alpha_{ij}t + \beta_{ij}\right) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \boldsymbol{L}\left(\alpha_{ij}t + \beta_{ij}\right)\end{bmatrix}_{(N+1)\times(N+1)^2},\end{aligned}$$

$$\boldsymbol{L}^{*k}(\alpha_{ij}t+\beta_{ij}) = \operatorname{diag}\left[\boldsymbol{L}^{*k-1}(\alpha_{ij}t+\beta_{ij})\right]_{(N+1)^k\times(N+1)^{k+1}},\ k>0,$$

$$\boldsymbol{P}_{ij}(t) = \begin{bmatrix} P_{ij}(t) & 0 & \cdots & 0 \\ 0 & P_{ij}(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & P_{ij}(t) \end{bmatrix} \text{and}$$

$$\boldsymbol{A}^{*i+j-1} = \left[a_0\boldsymbol{A}^{*i+j-2}\ a_1\boldsymbol{A}^{*i+j-2}\ \cdots\ a_N\boldsymbol{A}^{*i+j-2}\right]^T,$$

such that $\boldsymbol{A}^{*0}$ represents a traditional $\boldsymbol{A}$.

By the matrix relations (5) and (8), a substantial matrix equation at the standard collocation points leads to

$$\boldsymbol{L}^{(\alpha)}\boldsymbol{A} + \sum_{j=1}^{3}\sum_{i=0}^{1}\boldsymbol{P}_{ij}\boldsymbol{L}^{i}\left(\prod_{k=1}^{j}\boldsymbol{L}^{*k+i-1}(\alpha_{ij},\beta_{ij})\right)\boldsymbol{A}^{*i+j-1} = \boldsymbol{G}, \tag{9}$$

where

$$\boldsymbol{L}^{(\alpha)} = \begin{bmatrix} \boldsymbol{L}^{(\alpha)}(t_0) \\ \boldsymbol{L}^{(\alpha)}(t_1) \\ \vdots \\ \boldsymbol{L}^{(\alpha)}(t_N) \end{bmatrix} = \begin{bmatrix} {}_t^C D_0^{\alpha}(L_0(t_0)) & {}_t^C D_0^{\alpha}(L_1(t_0)) & \cdots & {}_t^C D_0^{\alpha}(L_N(t_0)) \\ {}_t^C D_0^{\alpha}(L_0(t_1)) & {}_t^C D_0^{\alpha}(L_1(t_1)) & \cdots & {}_t^C D_0^{\alpha}(L_N(t_1)) \\ \vdots & \vdots & \ddots & \vdots \\ {}_t^C D_0^{\alpha}(L_0(t_N)) & {}_t^C D_0^{\alpha}(L_1(t_N)) & \cdots & {}_t^C D_0^{\alpha}(L_N(t_N)) \end{bmatrix},$$

$$\boldsymbol{P}_{ij} = \begin{bmatrix} P_{ij}(t_0) & 0 & \cdots & 0 \\ 0 & P_{ij}(t_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & P_{ij}(t_N) \end{bmatrix},\ \boldsymbol{G} = \left[g(t_0)\ g(t_1)\ \cdots\ g(t_N)\right]^T,$$

$$\boldsymbol{L}^{*1}(\alpha_{ij},\beta_{ij}) = \begin{bmatrix} \boldsymbol{L}\left(\alpha_{ij}t_0+\beta_{ij}\right) & 0 & \cdots & 0 \\ 0 & \boldsymbol{L}\left(\alpha_{ij}t_1+\beta_{ij}\right) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \boldsymbol{L}\left(\alpha_{ij}t_N+\beta_{ij}\right) \end{bmatrix}_{(N+1)\times(N+1)^2},$$

and

$$\boldsymbol{L}^{*0}(\alpha_{ij},\beta_{ij}) = \boldsymbol{L}(\alpha_{ij},\beta_{ij}) = \boldsymbol{L}(\alpha_{ij}t+\beta_{ij}),\ \boldsymbol{L}^{*k}(\alpha_{ij},\beta_{ij}) = \operatorname{diag}\left[\boldsymbol{L}^{*k-1}(\alpha_{ij},\beta_{ij})\right]_{(N+1)^k\times(N+1)^{k+1}},\ k>0.$$

For sake of brevity, the matrix equation (9) can be written as

$$\boldsymbol{W}\boldsymbol{A} + \sum_{j=1}^{3}\sum_{i=0}^{1}\boldsymbol{Y}_{ij} = \boldsymbol{G} \Rightarrow [\boldsymbol{W};\ \boldsymbol{Y}_{01};\ \boldsymbol{Y}_{02};\ \boldsymbol{Y}_{03};\ \boldsymbol{Y}_{11};\ \boldsymbol{Y}_{12};\ \boldsymbol{Y}_{13}\ :\ \boldsymbol{G}], \tag{10}$$

where

$$\boldsymbol{W} = \boldsymbol{L}^{(\alpha)} \text{ and } \boldsymbol{Y}_{ij} = \boldsymbol{P}_{ij}\boldsymbol{L}^{i}\left(\prod_{k=1}^{j}\boldsymbol{L}^{*k+i-1}(\alpha_{ij},\beta_{ij})\right)\boldsymbol{A}^{*i+j-1}.$$

On the other hand, the matrix relations of the terminal conditions in Eq. (2) can be created using the main matrix relation (6) as

$$\left.\begin{array}{r} y\,(t) = c_1 \Rightarrow \left[\, L_0(t)\ L_1(t)\ \cdots\ L_N(t) : c_1 \,\right],\ \ t \in [-\beta_{ij}, 0], \\ y\,(T) = c_2 \Rightarrow \left[\, L_0(T)\ L_1(T)\ \cdots\ L_N(T) : c_2 \,\right]. \end{array}\right\} \tag{11}$$

Then, the terminal matrix conditions (11) are replaced by the last row(s) of the substantial matrix equation (10), which are determined according to the number of the terminal conditions (2). Also, zero conditional matrices are added into the last row(s) of the nonlinear matrix forms $\boldsymbol{Y}_{ij}$. The augmented matrix system is, thereby, established as

$$\left[\widetilde{\boldsymbol{W}};\ \widetilde{\boldsymbol{Y}_{01}};\ \widetilde{\boldsymbol{Y}_{02}};\ \widetilde{\boldsymbol{Y}_{03}};\ \widetilde{\boldsymbol{Y}_{11}};\ \widetilde{\boldsymbol{Y}_{12}};\ \widetilde{\boldsymbol{Y}_{13}}\ :\ \widetilde{\boldsymbol{G}}\right],$$

which can be solved by Solve command on Mathematica and later the Lagrange interpolation coefficients are obtained to be inserted into Eq. (3) and this is the final process by which the Lagrange polynomial solution arises.

4 An error analysis technique: improvement of a solution by nonlinear treatment of residual function

In the previous papers, the improvement of the polynomial solutions was enabled by an error analysis consisting of linear and nonlinear operator (Yüzbaşı and Yıldırım 2021; Gümgüm et al. 2020; Kürkçü and Sezer 2022). In this study, an error analysis technique is established by nonlinear and linear operators of the residual function to improve the obtained solution (3). As a first step, after inserting the obtained solution (3) into Eq. (1), a residual function $R_N(t)$ appears as

$$R_N(t) = {}_t^C D_0^{\alpha} y_N\,(t) + \sum_{j=1}^{3}\sum_{i=0}^{1} P_{ij}(t)\, y_N^i(t)\, y_N^j(\alpha_{ij}t + \beta_{ij}) - g\,(t)\,,$$

which corresponds to N and is on $[0, T]$.

In addition, $e_N(t) = y(t) - y_N(t)$ stands for an error function on $[0, T]$. Then, as a theoretical aspect, the following theorem can be stated to improve the obtained solution (3).

Theorem 4.1 *Let Eq. (1) conserves an error problem subjected to the homogeneous terminal conditions of the form*

$$\begin{cases} L[e_N(t)] + N[y_N(t) + e_N(t)] - N[y_N(t)] = -R_N(t), \\ \qquad e_N(t) = 0, \quad t \in [-\beta_{ij}, 0] \text{ and } e_N(T) = 0, \end{cases}$$

which is solved by the same procedure with computation limit M as in Sect. 3 and whose solution provides an estimated error function $e_{N,M}(t)$ to be merged with the Lagrange polynomial solution (3).

Proof 2 Let Eq. (1) is presented as a summarized form with linear and nonlinear operators

$$L[y(t)] + N[y(t)] = g(t), \tag{12}$$

where

$$L[y(t)] = {}_t^C D_0^{\alpha} y\,(t) \ \text{ and } N[y(t)] = \sum_{j=1}^{3}\sum_{i=0}^{1} P_{ij}(t)\, y^i\,(t)\, y^j(\alpha_{ij}t + \beta_{ij}).$$

In that case, the residual function can be rewritten with regard to Eq. (12), as

$$R_N(t) = L\left[y_N\left(t\right)\right] + N\left[y_N\left(t\right)\right] - g(t),$$

or its more regularized form

$$L\left[y_N\left(t\right)\right] + N\left[y_N\left(t\right)\right] = R_N(t) + g(t). \tag{13}$$

By subtracting Eq. (13) from Eq. (12), it follows that

$$L[y(t) - y_N\left(t\right)] + N[y(t)] - N\left[y_N\left(t\right)\right] = -R_N(t), \tag{14}$$

and adding the error function into Eq. (14), it is transformed into the error equation

$$L[e_N\left(t\right)] + N[y_N(t) + e_N\left(t\right)] - N[y_N(t)] = -R_N(t), \tag{15}$$

under the homogeneous terminal conditions

$$e_N(t) = 0, \quad t \in [-\beta_{ij}, 0] \text{ and } e_N(T) = 0.$$

After solving the error problem above by means of the same procedure, which is limited up to M, the estimated error function $e_{N,M}\left(t\right)$ is, then, acquired as

$$e_{N,M}\left(t\right) = \sum_{i=0}^{M} \tilde{a}_i L_i\left(t\right), \ \ M > N,$$

and then it is merged with the Lagrange polynomial solution (3), as

$$y_{N,M}(t) = y_N(t) + e_{N,M}\left(t\right) \quad \text{and} \quad \left|E_{N,M}(t)\right| = \left|y(t) - y_{N,M}(t)\right|,$$

which state the corrected Lagrange polynomial solution and the corrected absolute error function, respectively.

Hence, the Lagrange polynomial solution (3) is improved by the similar procedure to the main method in Sect. 3. This completes the proof. □

5 Model problems

In this section, five model problems are treated by the proposed method and error analysis technique. In doing so, a hybrid programme containing these two methods is devised on Mathematica 11.3 settled on a personal computer equipped with 6 GB memory and 2.50 GHz central processing unit. Thereby, absolute sensitivity is aimed in calculations. Indeed, the CPU time is evaluated peculiarly after wiping off the programming kernel. On the other hand, for sake of making comparison between the proposed method with existing ones, the root mean squared error RMSE_N is employed as (see Dehghan and Lakestani 2009)

$$\text{RMSE}_N = \left(\frac{1}{N}\sum_{i=0}^{N}\left(y(t_i) - y_N(t_i)\right)^2\right)^{\frac{1}{2}}.$$

In addition, since the most of the model problems of fractional order have no exact solution, the stepwise absolute error $\text{SWE}_N = |y_{N+1}(t) - y_N(t)|$ is computed. For the model problems of integer order, the Mathematica solution is also involved to directly monitor the approximation of the Lagrange interpolation polynomial solution.

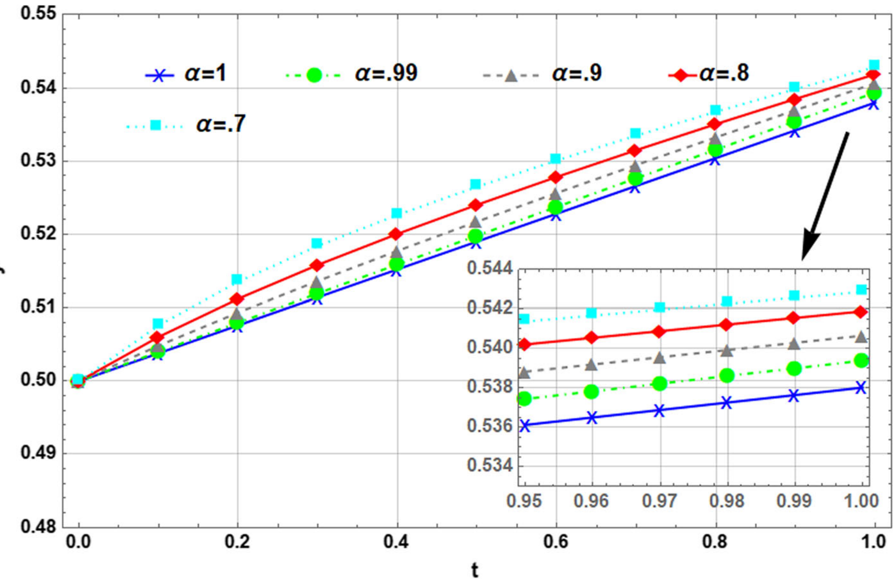


Fig. 1 Dynamical behavior of the Lagrange polynomial solution $y_5(t)$ with regard to α and $T = 1$ for Model 5.1

Model 5.1. Pimenov and Hendy (2017) Consider the fractional Hutchinson model

$${}_{t}^{C}D_{0}^{(\alpha)}y\,(t) - 0.15y\,(t) + 0.15y\,(t)\,y(t-0.1) = 0,\;\; 0 < \alpha \le 1,\;\; t \in [0, T],$$

subject to an initial condition $y(t) = 0.5$ for $t \in [-0.1, 0]$. The exact solution of this model is not determined. For sake of overseeing the efficiency of the method, the model can be solved applying NDSolve command on Mathematica, as follows:

Remark 5.1 Let $\alpha = 1$ and arbitrary $T(> 0)$,

$$\text{Input} := \text{NDSolve}\Big[\Big\{y'[t] - 0.15 * y[t] + 0.15 * y[t] * y[t-0.1]] == 0,$$
$$y[t/; t \le 0] == 0.5\Big\}, y[t], \{t, 0, T\}\Big][[1, 1, 2]];$$

which returns a Mathematica solution. As soon as the model is solved by the proposed method for different α, $T = 1$ and a long time limit $T = 30$, the dynamical behavior of the Lagrange polynomial solution is illustrated in Fig. 1 with respect to α's and $T = 1$. Figure 2 also stresses that a long time profile of this solution is very consistent with the Mathematica solution for $\alpha = 1$ and $T = 30$. They reach the population capacity, which tends to stabilize towards the straight line $y = 1$. Figure 3 points out that the error values in terms of RMSE_N and SWE_N are decayed, despite the increment of both $N = 2(1)10$ and α. As a numerical investigation via Table 1, the stepwise absolute error values take four decimal places on average even for fractional order derivative, and the accuracy of the method is sharpen with the implementation of the residual error analysis, causing to obtain low timing complexity.

Model 5.2. Shi et al. (2020); Nemati and Kalansara (2022) Consider the fractional Hutchinson-type model with variable coefficients

$${}_{t}^{C}D_{0}^{1.5}y\,(t) - (t+1)y\,(t) - \frac{t-1}{2}y\,(t)\,y(t-0.1) = g(t),\;\; t \in [0, 1],$$

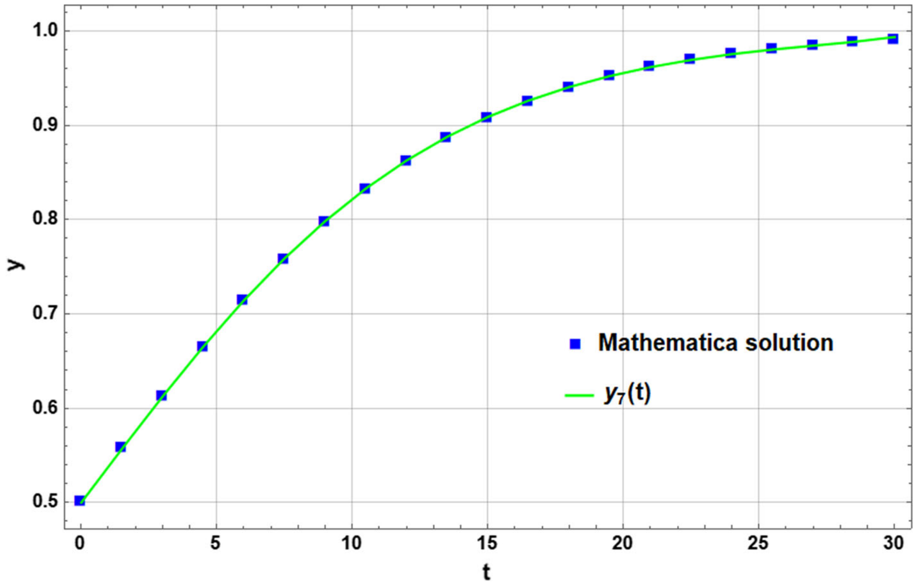


Fig. 2 A long-time behavior of the solutions for Model 5.1 with $T = 30$

Table 1 Numerical profile of the absolute and the corrected absolute error values for Model 5.1 with $\alpha = 0.9$ and $\alpha = 1$

t_i	$\lvert e_2(t_i)\rvert, \alpha = 1$	$\lvert E_{2,3}(t_i)\rvert, \alpha = 1$	$\mathrm{SWE}_2, \alpha = 0.9$
0.2	7.0241e–05	1.4305e–05	2.2330e–04
0.4	1.2610e–04	1.4287e–05	1.5194e–04
0.6	1.8533e–04	1.4258e–05	1.0187e–04
0.8	2.5137e–04	1.4211e–05	4.2593e–04
1.0	3.2764e–04	1.4121e–05	7.0806e–04
Timing	0.046129 ($N = 2$)	0.075054 ($N = 3$)	1.34375 ($N = 3$)

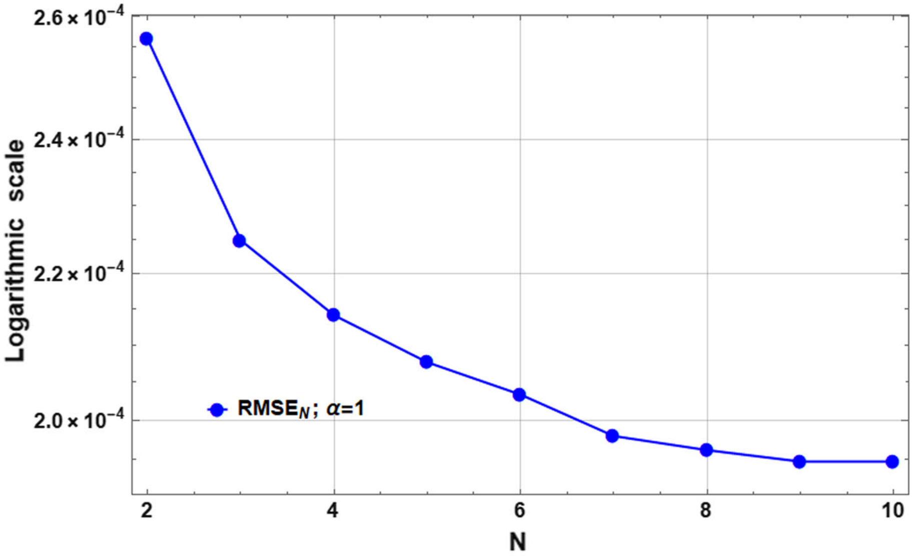
subject to the terminal conditions $y(t) = 0$ for $t \in [-0.1, 0]$ and $y(1) = 1$. Here, the exact solution of this problem is $y(t) = t^{2.3}$ or $y(t) = t^{4.3}$ such that $g(t)$ can be determined, respectively, as

$$g(t) = \frac{1}{2}\left(5.76226t^{0.8} + \left(-2 + (-0.1 + t)^{2.3}\right)t^{2.3} - \left(2 + (-0.1 + t)^{2.3}\right)t^{3.3}\right),$$

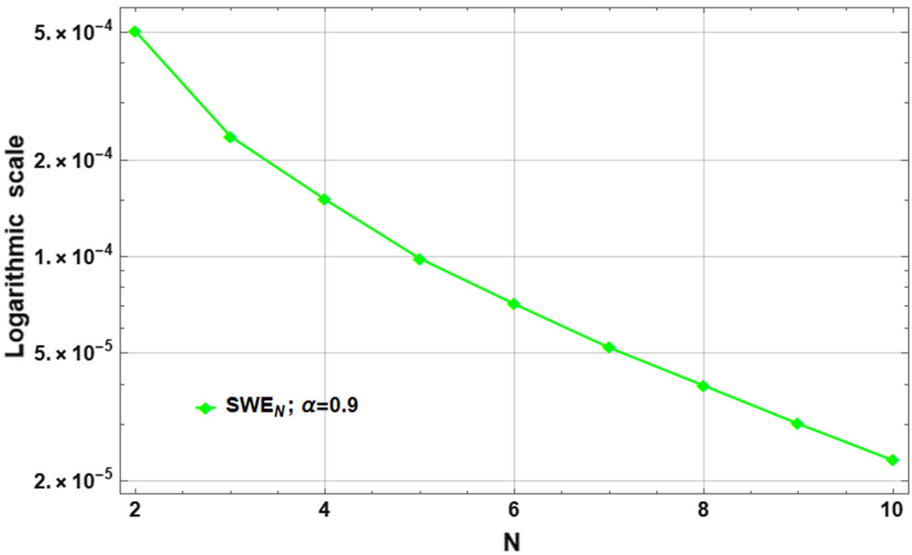
and

$$g(t) = \frac{1}{2}\left(16.2235t^{2.8} + \left(-2 + (-0.1 + t)^{4.3}\right)t^{4.3} - \left(2 + (-0.1 + t)^{4.3}\right)t^{5.3}\right).$$

When the problem is solved by the proposed method with $N = 5, 6$ and the residual error analysis technique with $M = 7(1)10$ for both exact solution forms, such present results as the infinity normed error and timing complexity are compared in Tables 2 and 3 with those of the stable collocation method (SCM) Shi et al. (2020) and the spectral method based on the modified hat functions (SM) Nemati and Kalansara (2022). The proposed method is capable



(a) The root mean squared error for $\alpha = 1$.



(b) The stepwise absolute error for $\alpha = 0.9$.

Fig. 3 Decaying profiles of the error values versus N on the logarithmic scale for Model 5.1 with $T = 1$

Table 2 Numerical evaluations in terms of the proposed method ($N = 5, 6$), SCM ($n = 9, 17$) and SM ($n = 8, 16$) for Model 5.2 with $y(t) = t^{2.3}$

(N, M)	L_∞	SCM L_∞ Shi et al. (2020)	SM L_∞ Nemati and Kalansara (2022)	Timing
(5,n.a)	1.4398e–03	1.2299e–03 (n=9)	3.0969e–03 (n=8)	6.0625
(6,n.a)	8.4522e–04	3.5591e–04 (n=17)	8.7023e–04 (n=16)	12.6719
(6,7)	3.6100e–04	n.a	n.a	4.0938
(6,8)	3.0491e–04	n.a	n.a	6.3750
(6,9)	2.0326e–04	n.a	n.a	7.8281

Table 3 Numerical evaluations in terms of the proposed method ($N = 9$) and SCM ($n = 9$) for Model 5.2 with $y(t) = t^{4.3}$

(N, M)	L_∞	SCM L_∞ Shi et al. (2020)	Timing
(9,n.a)	4.8479e–04	1.7226e–05 (n=9)	11.6875
(9,10)	1.8064e–06	n.a	16.5781

of yielding better results than SCM and SM, applying the residual error analysis for low or equivalent computation limit.

Model 5.3. Consider the fractional nonlinear delay differential equation with a variable coefficient

$$ {}_t^C D_0^{1.9} y(t) + t^2 y(0.9t - 0.9) - 0.1 y(t) y^3 (0.1t - 0.01) = g(t), \quad t \in [0, T], $$

subject to the terminal conditions $y(t) = 0$ for $t \in [-0.9, 0]$ and $y(T) = e^{-T}\sin(T)$. The exact solution leads to $y(t) = e^{-t}\sin(t)$ such that $g(t)$ is of the form

$$ \begin{aligned} g(t) = &-2.10227\, t^{0.1}\, {}_P F_Q \left(1;\ 1.025, 0.775, 0.525, 0.275;\ -\frac{t^4}{64}\right) \\ &+ 1.91116\, t^{1.1}\, {}_P F_Q \left(1;\ 1.275, 1.025, 0.775, 0.525;\ -\frac{t^4}{64}\right) \\ &- 0.587145\, t^{3.1}\, {}_P F_Q \left(1;\ 1.775, 1.525, 1.275, 1.025;\ -\frac{t^4}{64}\right) \\ &+ 2.4596\, e^{-0.9t}\, t^2 \sin(0.9t - 0.9) \\ &- 0.103045\, e^{-1.3t} \sin^3(0.1t - 0.01)\sin(t), \end{aligned} $$

where ${}_P F_Q$ is the generalized hypergeometric function Abramowitz and Stegun (1964). Solving this special problem via the proposed method for $T = \{1, 3\}$, the Lagrange polynomial solutions are immediately acquired. Thereby, Fig. 4 depicts that the Lagrange polynomial solution is consistent with the exact solution on a long time scale like $T = 3$. Table 4 releases that the accuracy of the method enhances as N is increased. This situation can also be noticed in Fig. 5, thanks to RMSE_N.

Model 5.4. Ruan (2006); Caraballo et al. (2005) Consider the fractional Wazewska-Czyzewska and Lasota model

$$ {}_t^C D_0^{\alpha} y(t) + 0.2 y(t) - 0.1 \exp(-0.01 y(t - 0.1)) = 0, \quad t \in [0, T], \tag{16} $$

subject to an initial condition $y(t) = 0.2$ for $t \in [-0.1, 0]$. Here, the exact solution to the problem is unknown. In order to adapt it to the proposed method, it is beneficial to use Taylor series expansion of the exponential function. That is, the functional nonlinearity can

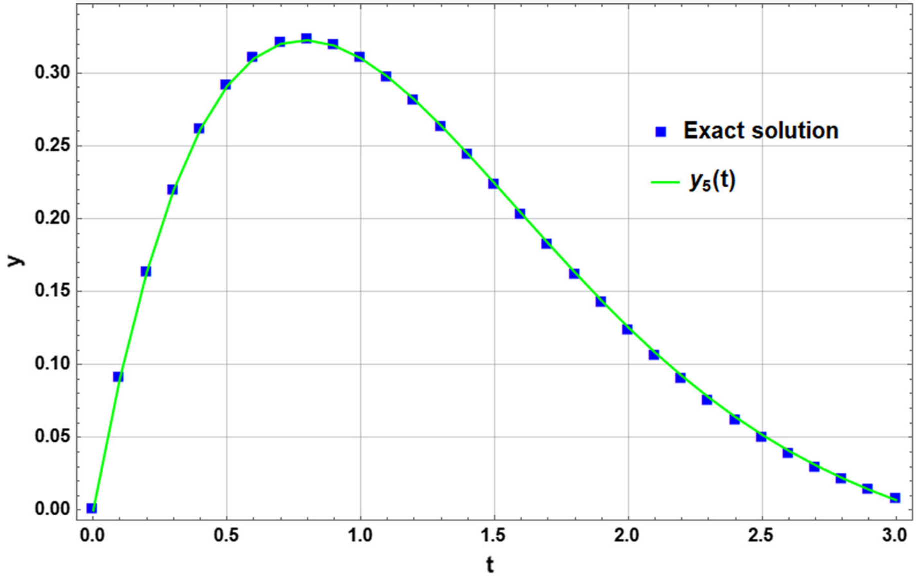


Fig. 4 A long-time behavior of the solutions for Model 5.3 with $T = 3$

Table 4 Numerical profile of the absolute error values in terms of N for Model 5.3 with $T = 1$

t_i	$\lvert e_3(t_i)\rvert$	$\lvert e_6(t_i)\rvert$
0.2	4.8832e–04	1.4649e–05
0.4	1.5922e–04	2.0834e–05
0.6	4.2752e–04	1.7033e–05
0.8	1.0903e–03	9.2449e–06

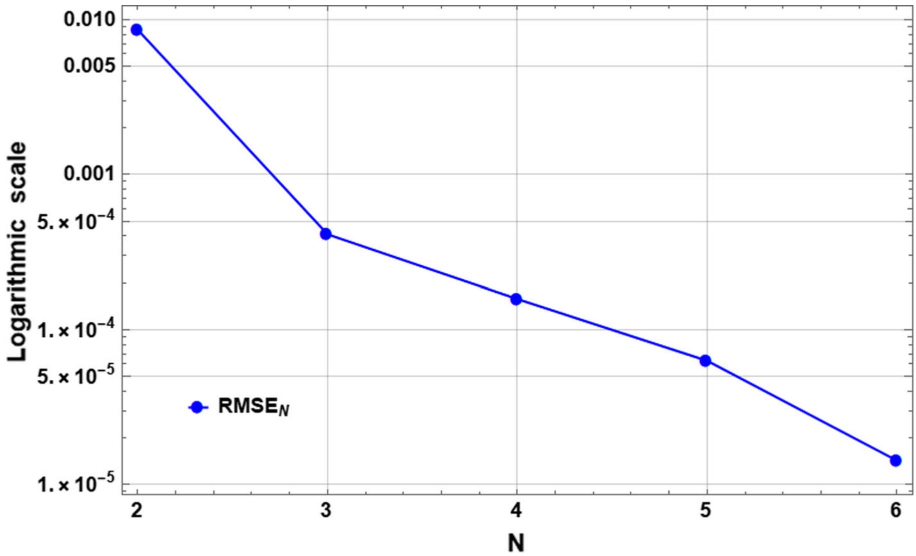


Fig. 5 Decaying profile of the error values versus N on the logarithmic scale for Model 5.3 with $T = 1$

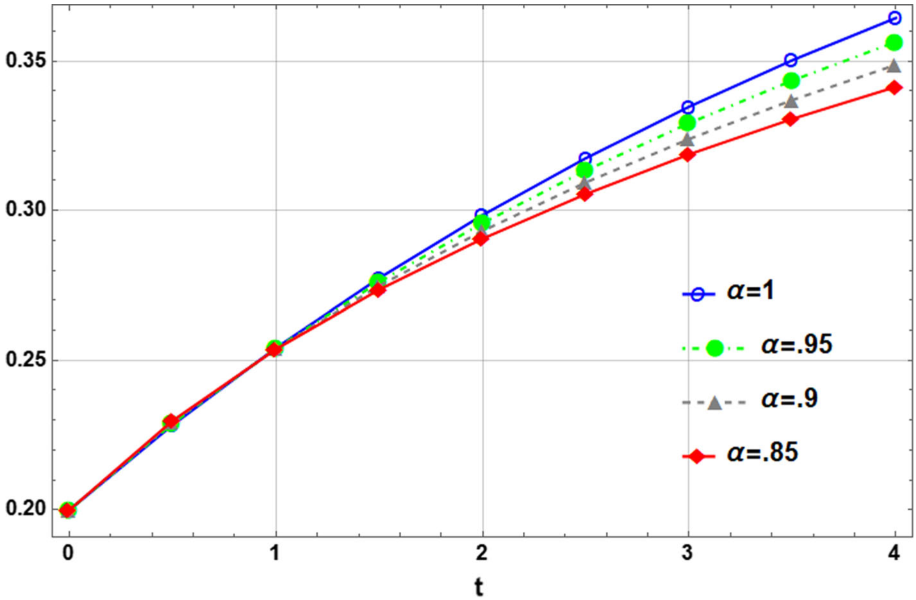


Fig. 6 Dynamical behavior of the Lagrange polynomial solution $y_5(t)$ with regard to α and $T = 4$ for Model 5.4

be expanded by its Taylor series around $x_0 = 0.1$ up to third degree since the functional nonlinear term contains a constant delay. It is, thus, follows that

$$\begin{aligned}\exp(-0.01y(t-0.1)) &= \sum_{n=0}^{\infty} \frac{(-1)^n (0.01)^n y^n(t-0.1)}{n!} \\ &= 1 - 0.01y(t-0.1) + 5\times 10^{-5} y(t-0.1)^2 \\ &\quad - 1.67\times 10^{-7} y(t-0.1)^3 + O(y(t-0.1)^4).\end{aligned}$$

By embedding this expansion into the functional nonlinear term of the problem (16), it can now be solved via the proposed method for different α and T. For sake of comparison of the results, the Mathematica can also solve the problem (16) as the following:

Remark 5.2 Let $\alpha = 1$ and arbitrary $T(> 0)$,

$$\begin{aligned}\text{Input} := \text{NDSolve}[\{y'[t] == -0.2 * y[t] + 0.1 * Exp\,[-0.01 * y[t-0.1]]\,,\\ y[t/; t <= 0] == 0.2\}, y[t], \{t, 0, T\}][[1, 1, 2]].\end{aligned}$$

The effect of the fractional derivative on the amount of red blood cells, which is governed by the Lagrange polynomial solution, can be frankly monitored in Fig. 6, whose time interval is limited up to $T = 4$. In addition, Fig. 7 simulates that when $N = 7$, a sustainable diagram of the red blood cells overlaps with that of Mathematica throughout a long time interval for $T = 30$. It is worth indicating that both reach a level of equilibrium limit $y = 0.5$ as T increases. In addition, in Table 5, the proposed method provides seven decimal places for integer order derivative and also, four decimal places for fractional order derivative according to SWE_N.

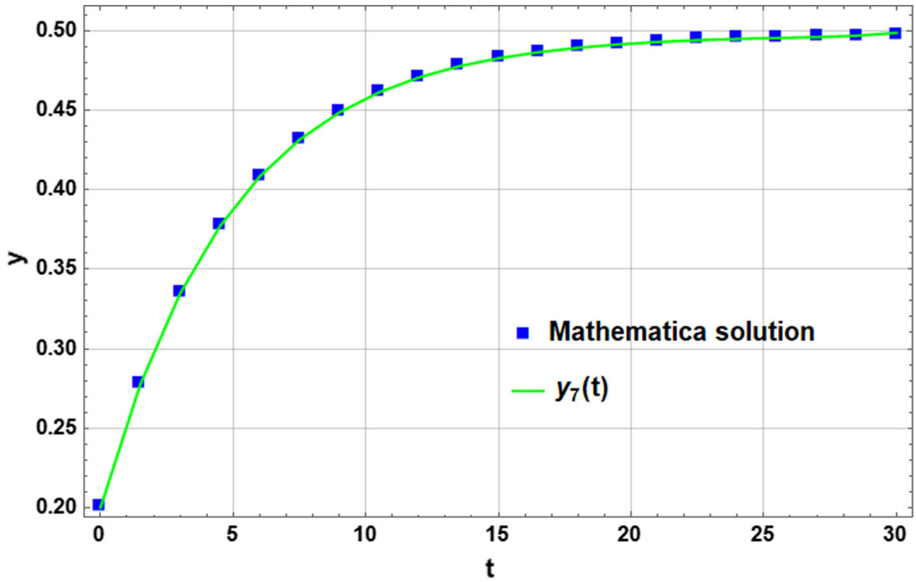


Fig. 7 A long-time behavior of the solutions for Model 5.4 with $T = 30$

Table 5 Numerical profile of the absolute error values and time complexity in terms of α and N for Model 5.4 with $T = 1$

t_i	$\lvert e_6(t_i)\rvert$ $\alpha = 1$	SWE_5 $\alpha = 0.85$
0.2	2.7693e–07	1.9973e–04
0.4	2.6482e–07	1.5268e–04
0.6	2.5316e–07	1.4151e–04
0.8	2.4197e–07	1.2384e–04
1.0	2.3120e–07	1.2690e–04
Timing	30.8102 ($N = 6$)	47.8679 ($N = 6$)

Model 5.5. Piotrowska and Foryś (2011); Valentim et al. (2020) Consider the fractional Gompertz equation with a delay force

$$ {}_{t}^{C}D_0^{\alpha} y\,(t) - 0.4y(t)\ln\left(\frac{2}{y(t-\beta)}\right) = 0,\ \ t \in [0, T], \tag{17}$$

subject to an initial condition $y(t) = 0.5$ for $t \in [-\beta, 0]$. Here, its exact solution is unknown. For the easiness of the calculations, by using the transformation $y(t) = \exp(u(t))$, the problem can be reduced to a fractional order differential delay equation in the form

$$ {}_{t}^{C}D_0^{\alpha} u(t) + 0.4u(t-\beta) = 0.4\ln(2),\ \ u(t) = \ln(0.5),\ \ t \in [-\beta, 0],$$

which can be solved linearly by the proposed method. Additionally, Mathematica returns the solution to the problem (17) in view of the following module.

Remark 5.3 Let $\alpha = 1$, $\beta \in (0, 1]$ and arbitrary $T(> 0)$,

$$\begin{aligned} \text{Input} := \text{NDSolve}[\{y'[t] - 0.4 * y[t] * Log[2/y[t-\beta]] == 0,\\ y[t/; t <= 0] == 0.5\}, y[t], \{t, 0, T\}][[1, 1, 2]]. \end{aligned}$$

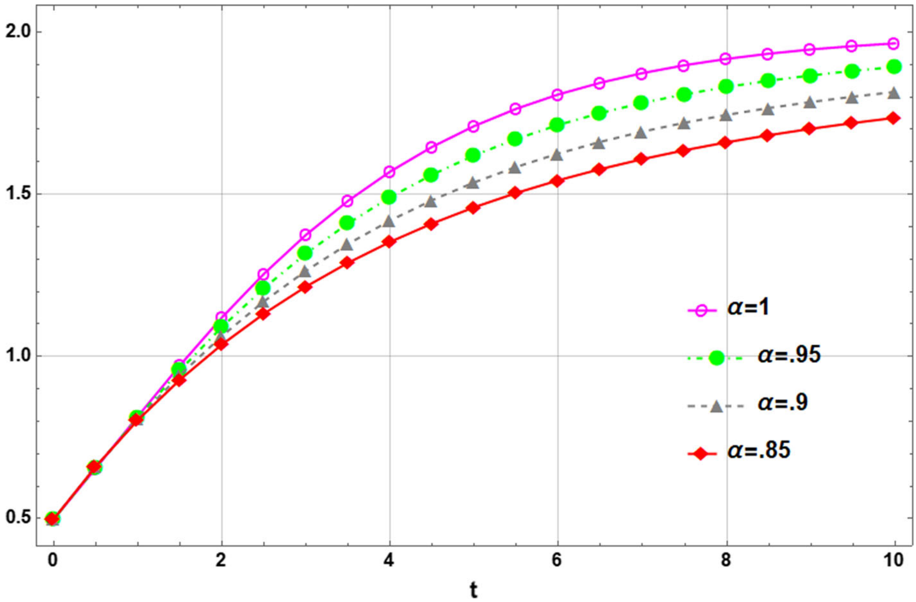


Fig. 8 Dynamical behavior of the Lagrange polynomial solution $y_{10}(t)$ with regard to the fractional derivative force α for Model 5.5 with $T = 10$

For different α, the Lagrange polynomial solution in Fig. 8 simulates the tumour growth propagation on a long time scale up to $T = 10$. In that case, the physical profile of this solution is also monitored in Fig. 9, from which it is exposed to the different levels of delay forces $\beta = 0.1(0.1)0.6$ and integer order derivative $\alpha = 1$. One can state that the tumour growth propagation reaches at the carrying capacity $K = 2$ as $t \to \infty$. Table 6 emphasizes that the accuracy is enhanced according to the increment of N for different derivatives, consuming the low amount of time in seconds.

6 Conclusions

A streamlined computational method based on the Lagrange interpolation polynomial has been efficiently proposed to treat fractional terminal value problems with multiple delays appearing in biomathematics. A matrix-collocation method has newly streamlined in peculiar to the linear and the nonlinear terms with proportional and constant delays. The method has transformed them into a reduced matrix expansion (10). Due to this eligibility, the method has maintained a good approach along with the residual error analysis technique for different models, which makes it efficient and accurate with regard to the other methods, such as SCM (Shi et al. 2020) and SM (Nemati and Kalansara 2022). This novelty along with the time complexity values can be overseen in Tables 1, 2, 3, 4, 5 and 6. Therefore, one of the main aim has been accomplished. For another aim, an error analysis technique has been suitably deployed in the model problems, decreasing the error values as seen in Tables 1, 2 and 3. Although some model problems have no exact solutions, the accuracy and efficiency of the proposed method have been tested in both traditional and long time intervals, which are laid on Figs. 1, 2, 4 and 6, 7, 8 and 9.

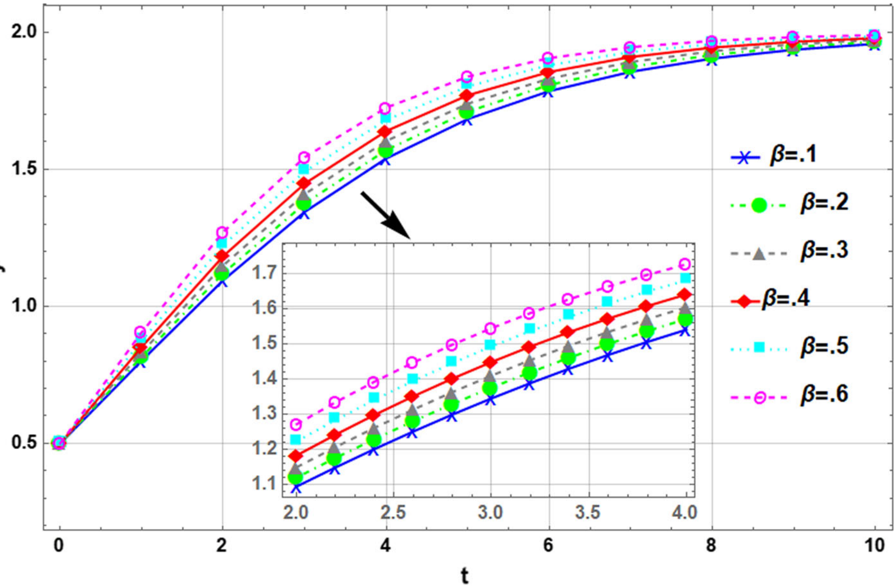


Fig. 9 Dynamical behavior of the Lagrange polynomial solution $y_{10}(t)$ with regard to the delay force β for Model 5.5 with $\alpha = 1$ and $T = 10$

Table 6 Numerical profile of the absolute error values and time complexity in terms of α and N for Model 5.5 with $\beta = 0.01$ and $T = 1$

t_i	$\|e_3(t_i)\|$ $\alpha = 1$	$\|e_5(t_i)\|$ $\alpha = 1$	SWE_3 $\alpha = 0.95$	SWE_5 $\alpha = 0.95$
0.2	1.1938e–05	5.7576e–06	1.0442e–03	3.1902e–04
0.4	1.4151e–05	5.8632e–06	1.1700e–03	2.9320e–04
0.6	5.9517e–06	5.9284e–06	1.0283e–03	2.9516e–04
0.8	1.5331e–05	5.9465e–06	1.0338e–03	2.8231e–04
1.0	1.1612e–05	6.0114e–06	1.1299e–03	2.9638e–04
Timing	0.0092339 ($N = 3$)	0.0198319 ($N = 5$)	15.2248 ($N = 3$)	29.4452 ($N = 5$)

On the other hand, the proposed method may have a disadvantage in terms of reaching far higher computation limits since it takes relatively a bit of time. Based on these deductions, one can draw attention to fact that the proposed method is highly suitable and applicable tool to treat the fractional terminal value problems encountered in biomathematics. For its a future implementation, it can be applied to the time-space fractional-partial order nonlinear terminal value problems arising in applied sciences. Thereby, this study exceeds a milestone in terms of matrix-collocation methods found in the literature.

Acknowledgements The author would like to thank anonymous reviewers for their constructive and valuable comments, which led to the improvement of the paper.

Data Availability Statement Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest The author states that there is no conflict of interest.

References

Abramowitz M, Stegun IA (1964) Handbook of Mathematical Functions, with Formulas, Graphs, and Mathematical Tables, Nat. Bureau of Standards, Appl Math Series, 55

Caputo M (1969) Elasticitàe Dissipazione. Zanichelli, Bologna

Caraballo T, Marin-Rubio P, Valero J (2005) Autonomous and non-autonomous attractors for differential equations with delays. J. Differ. Equ. 208:9–41

Dehghan M, Lakestani M (2009) The use of Chebyshev cardinal functions for solution of the second-order onedimensional telegraph equation. Numer. Methods Partial Differ. Equ. 25(4):931–938

Dehingia K, Yao S-W, Sadri K, Das A, Sarmah HK, Zeb A, Inc M (2022) A study on cancer-obesity-treatment model with quadratic optimal control approach for better outcomes. Results Phys. 42:105963

Diethelm K (2010) The analysis of fractional differential equations. Springer-Verlag, Berlin Heidelberg

Fowler AC (2005) Asymptotic methods for delay equations. J. Eng. Math. 53:271–290

Gande NR, Madduri H (2022) Higher order numerical schemes for the solution of fractional delay differential equations. J. Comput. Appl. Math. 402:113810

Gümgüm S, Baykuş Savaşaneril N, Kürkçü ÖK, Sezer M (2020) Lucas polynomial solution for neutral differential equations with proportional delays. TWMS J Appl Eng Math 10(1):259–269

Hamoud AA, Ghadle KP, Issa MB, Giniswamy G (2018) Existence and uniqueness theorems for fractional Volterra-Fredholm integro-differential equations. Int. J. Appl. Math. 31(3):333–348

Iqbal MA, Shakeel M, Mohyud-Din ST, Rafiq M (2017) Modified wavelets-based algorithm for nonlinear delay differential equations of fractional order. Adv. Mech. Eng. 9(4):1–8

Jhinga A, Daftardar-Gejji V (2019) A new numerical method for solving fractional delay differential equations. Comput. Appl. Math. 38:166

Kürkçü ÖK, Sezer M (2022) A new characteristic numerical approach with evolutionary residual error analysis to nonlinear boundary value problems occurring in heat and mass transfer via combinatoric Mittag-Leffler polynomial. Numer. Heat Transf. A. 82(9):561–575

Kürkçü ÖK, Aslan E, Sezer M (2019) An integrated numerical method with error analysis for solving fractional differential equations of quintic nonlinear type arising in applied sciences. Math. Methods. Appl. Sci. 42:6114–6130

Magin RL (2004) Fractional calculus in bioengineering. Crit. Rev. Biomed. Eng. 32(1):1–104

Mahdy AMS, Mohamed MS, Gepreel KA, Al-Amiri A, Higazy M (2020) Dynamical characteristics and signal flow graph of nonlinear fractional smoking mathematical model. Chaos Solitons Fractals 141:110308

Maleki M, Davari A (2019) Fractional retarded differential equations and their numerical solution via a multistep collocation method. Appl. Numer. Math. 143:203–222

Maurício de Carvalho JPS, Moreira-Pinto B (2021) A fractional-order model for CoViD-19 dynamics with reinfection and the importance of quarantine. Chaos Solitons Fractals 151:111275

Moghaddam BP, Mostaghim ZS (2013) A numerical method based on finite difference for solving fractional delay differential equations. J. Taibah Univ. Sci. 7(3):120–127

Nath BJ, Dehingia K, Sadri K, Sarmah HK, Hosseini K, Park C (2023) Optimal control of combined antiretroviral therapies in an HIV infection model with cure rate and fusion effect. Int. J. Biomath. 16(01):2250062

Nemati S, Kalansara ZR (2022) A low-cost computational method for solving nonlinear fractional delay differential equations. Commun. Nonlinear Sci. Numer. Simul. 114:106650

Pimenov VG, Hendy AS (2017) BDF-type shifted Chebyshev approximation scheme for fractional functional differential equations with delay and itserror analysis. Appl. Numer. Math. 118:266–276

Piotrowska MJ, Foryś U (2011) The nature of Hopf bifurcation for the Gompertz model with delays. Math. Comput. Model. 54:2183–2198

Podlubny I (1999) Fractional Differential Equations. Academic Press, New York

Rabiei K, Ordokhani Y (2019) Solving fractional pantograph delay differential equations via fractional-order Boubaker polynomials. Eng. Comput. 35:1431–1441

Ruan S (2006) Delay Differential Equations in Single Species Dynamics. In: Arino O, Hbid M, Dads EA (eds) Delay Differential Equations and Applications. NATO Science Series, vol. 205. Springer, Dordrecht
Sadri K, Aminikhah H (2022) A new efficient algorithm based on fifth-kind Chebyshev polynomials for solving multi-term variable-order time-fractional diffusion-wave equation. Int. J. Comput. Math. 99(5):966–992
Santra S, Mohapatra J (2022) A novel finite difference technique with error estimate for time fractional partial integro-differential equation of Volterra type. J. Comput. Appl. Math. 400:113746
Shi L, Chen Z, Ding X, Ma Q (2020) A new stable collocation method for solving a class of nonlinear fractional delay differential equations. Numer. Algorithms 85:1123–1153
Valentim CA Jr, Oliveira NA, Rabi JA, David SA (2020) Can fractional calculus help improve tumor growth models? J. Comput. Appl. Math. 379:112964
Yang Q, Wang Q (2020) Numerical oscillation analysis for Gompertz equation with one delay. Fund. J. Math. Appl. 3:1–7
Yüzbaşı Ş, Sezer M (2021) On solutions of linear functional integral and integro-differential equations via Lagrange polynomials. J. Sci. Arts 3(56):707–720
Yüzbaşı Ş, Yıldırım G (2021) A Laguerre approach for solving of the systems of linear differential equations and residual improvement. Comput. Meth. Differ. Equ. 9(2):553–576