Binary optimal linear codes with various hull dimensions and entanglement-assisted QECC

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Abstract

The hull of a linear code C is the intersection of C with its dual. To the best of our knowledge, there are very few constructions of binary linear codes with the hull dimension ≥ 2 except for self-orthogonal codes. We propose a building-up construction to obtain a plenty of binary [n+2, k+1] codes with hull dimension $\ell, \ell+1$, or $\ell+2$ from a given binary [n, k] code with hull dimension ℓ . In particular, with respect to hull dimensions 1 and 2, we construct all binary optimal [n, k]codes of lengths up to 13. With respect to hull dimensions 3, 4, and 5, we construct all binary optimal [n, k] codes of lengths up to 12 and the best possible minimum distances of [13, k] codes for $3 \leq k \leq 10$. As an application, we apply our binary optimal codes with a given hull dimension to construct several entanglement-assisted quantum errorcorrecting codes(EAQECC) with the best known parameters.

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1 Introduction

The hull of a linear code C is the intersection of C with its dual. The hull of a linear code was introduced by Assmus, Jr. and Key [1]. The hull determines the complexity of algorithms for checking permutation equivalence of two

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linear codes [25], which are very effective if the dimension of the hull is small, and which are worst if the dimension of the hull is large. The hardness of the Permutation Code Equivalence problem is of great importance when designing cryptographic primitives, such as public-key cryptosystems and identification schemes in the field of code-based cryptography [26].

When the hull contains only the zero vector, that is, the hull dimension is 0, C is called a Linear Complementary Dual code (shortly, LCD code). Recently, LCD codes have been actively studied due to its side channel attack. An LCD code was originally constructed by Massey [19], [20] as a reversible code in order to provide an optimum linear coding solution for the two-user binary adder channel. Carlet and Guilley [5] introduced several constructions of LCD codes and investigated an application of LCD codes against sidechannel attacks(SCA) and Fault Injection Attack(FIA).

There are several constructions for binary linear codes with hull dimensions 0 or 1. More precisely, Galvez et al. [8] have constructed all binary optimal LCD [n, k] codes for $1 \le k \le n \le 12$. Harada and Saito [12] have extended this for $1 \le k \le n \le 16$. Li and Zeng [18] have constructed binary linear [n, k] codes with hull dimension 1 for n = 8 with k = 3, 5, 7, n = 9with k = 3, 5, 6, 7, and n = 10 with k = 3, 4, 7, whose optimality was not discussed.

On the other hand, when the dimension h of the hull of a linear [n, k] code C is equal to k, C is called self-orthogonal, and self-dual if h = k = n/2. Self-orthogonal or self-dual codes have been one of the most active research areas in classical coding theory [24] and recently in quantum coding theory [4]. As far as we know, there are few constructions of binary linear codes with the hull dimension $h \ge 2$ except for self-orthogonal codes. It turns out that linear codes with various hull dimensions can be used to construct entanglement-assisted quantum error-correcting codes (EAQECC) [7], [9], [27].

Therefore, it is an interesting problem to find a unified method to construct linear codes with various hull dimensions.

In this paper, we give an efficient and systematic method, called a buildingup construction to construct linear codes with various hull dimensions from a given linear code with a fixed hull dimension. More precisely, with respect to hull dimensions 1 and 2, we construct all binary optimal [n, k] codes of lengths up to 13. With respect to hull dimensions 3, 4, and 5, we construct all binary optimal [n, k] codes of lengths up to 12 and the best possible minimum distances of [13, k] codes for $3 \le k \le 10$. As a coding theoretical application, given length $2 \le n \le 12$ and dimension k ($2 \le k \le n$), by running all values of the hull dimension h $(0 \le h \le [n/2])$, we can recover all the binary best known linear [n, k] codes in Grassl's table [10] with sometimes more than one inequivalent code. We apply our binary optimal codes with a given hull dimension to the construction of [[n, k, d; c]] EAQECC with the best known parameters as described in [7], [27].

2 Preliminaries

A linear [n, k, d] code C over GF(q) or \mathbb{F}_q is a k-dimensional subspace of \mathbb{F}_2^n with minimum distance d(C) or d if there is no confusion. The dual of C is $C^{\perp} = \{\mathbf{x} \in \mathbb{F}_2^n \mid \mathbf{x} \cdot \mathbf{c} = 0 \text{ for any } \mathbf{c} \in C\}$, where the dot product is the usual inner product. A linear code C is called *self-orthogonal* if $C \subset C^{\perp}$ and *self*dual if $C = C^{\perp}$. A linear code C is called an LCD code (linear complementary dual code) if $C \cap C^{\perp} = \{0\}$. Hence being LCD is the opposite concept of self-orthogonality.

Let C be a linear code over GF(q) with its dual C^{\perp} . The Hull of C is defined as $Hull(C) = C \cap C^{\perp}$. Let h = dimension of Hull(C).

We call C h_i -optimal if d(C) is the largest among all the linear [n, k] codes C with h = i for $0 \le i \le k$. We call C optimal if d(C) is the largest among all the linear [n, k] codes C.

Lemma 1. ([18, Proposition 1]) Let C be a linear [n, k] code over GF(q) with generator matrix G. Then $h = k - rank(GG^T)$.

Hence, if h = 0, that is, $\operatorname{Hull}(C) = \{0\}$ or $\operatorname{rank}(GG^T) = k$, then C is *LCD*. If $\operatorname{Hull}(C) = C$, then C is self-orthogonal.

Now we also describe entanglement-assisted quantum error-correcting codes(EAQECC). An *EAQECC with parameters* [[n, k, d; c]] encodes k logical qubits into n physical qubits with the help of c pre-shared entanglement pairs [23]. If c = 0, then [[n, k, d; c]] EAQECC are equivalent to quantum stabilizer codes. Hence, [[n, k, d; c]] EAQECC are a generalization of [[n, k, d]] QECC.

The following is a useful method to construct $[[n, k, d; c]]_q$ EAQECC from q-ary linear [n, k, d] codes.

Proposition 1. ([7], [9, Corollary 3.1], [27, Proposition 8]) Let C be a linear code over GF(q) with parameters $[n, k, d]_q$ and C^{\perp} be its dual with parameters

 $[n, n-k, d']_q$. Let dim(Hull(C)) = h. Then, there exist an $[[n, k-h, d; n-k-h]]_q$ EAQECC and an $[[n, n-k-h, d'; k-h]]_q$ EAQECC.

Proposition 2 ([9]). An $[[n, k, d; c]]_q$ EAQECC satisfies

$$n+c-k \ge 2(d-1)$$

where $0 \leq c \leq n-1$.

An EAQECC attaining this Singleton bound is called an *MDS EAQECC*. Chen et al. [6] constructed MDS EAQECC when $q = 2^e$ with e odd with special values of n, k, d, and c.

Let q = 2 and consider the binary Hamming [7, 4, 3] code \mathcal{H}_3 . Since its dual \mathcal{H}_3^{\perp} is the simplex code \mathcal{S}_3 and is a subcode of \mathcal{H}_3 , we have $h(\mathcal{H}_3) = 3$. Hence by Proposition 1, we obtain a $[[7, 1, 3; 0]]_2$ EAQECC which is best known by Grassl's table [10]. Note that n + c - k = 7 + 0 - 1 = 6 and 2(d-1) = 4. Hence $[[7, 1, 3; 0]]_2$ EAQECC is not MDS.

In this paper, we consider q = 2 and construct various $[[n, k - h, d; n - k - h]]_2$ EAQECC using the building-up constructions.

3 Building-up construction for linear codes with various hull dimensions

In the remaining sections, we consider binary codes. We can construct [n + 2, k+1] linear codes with hull dimension $\ell + 1$ from a given [n, k] linear code with hull dimension ℓ as follows.

Theorem 1. Let C be a binary linear [n, k] code. Suppose that the dimension of Hull(C) is ℓ , where $0 \leq \ell \leq k$. Let G be a generator matrix for C and H a parity check matrix for C.

Suppose that $\mathbf{x} = (x_1, x_2, \dots, x_n) \in GF(2)^n$ satisfies $\mathbf{x} \cdot \mathbf{x} = 1$. Let $y_i = \mathbf{x} \cdot \mathbf{r}_i$ for $1 \leq i \leq k$ where \mathbf{r}_i is the *i*th row of G and $z_j = \mathbf{x} \cdot \mathbf{s}_j$ for $1 \leq j \leq n-k$ where \mathbf{s}_j is the *j*th row of H. Then

(a) the following matrix

$$G_{1} = \begin{bmatrix} 1 & 0 & x_{1} & \dots & x_{n} \\ \hline y_{1} & y_{1} & \mathbf{r}_{1} \\ y_{2} & y_{2} & \mathbf{r}_{2} \\ \vdots & \vdots & \vdots \\ y_{k} & y_{k} & \mathbf{r}_{k} \end{bmatrix}$$

generates an [n + 2, k + 1] linear code C_1 with $h(C_1) = \ell + 1$. This is called **Construction I**.

(b) A parity check matrix H_1 for C_1 is given by

$$H_1 = \begin{bmatrix} 1 & 0 & x_1 & \dots & x_n \\ z_1 & z_1 & \mathbf{s}_1 & \\ z_2 & z_2 & \mathbf{s}_2 & \\ \vdots & \vdots & \vdots & \\ z_{n-k} & z_{n-k} & \mathbf{s}_{n-k} \end{bmatrix}.$$

Proof. We prove (a). Now rank $(G_1G_1^T)$ is computed as follows.

$$G_1 G_1^T = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & G G^T & \\ 0 & & & \end{bmatrix}.$$

Thus rank $(G_1G_1^T)$ = rank $(GG^T) = k - h(C) = k - \ell$ since $h(C) = \ell$. Now $h(C_1) = (k+1) - \text{rank}(G_1G_1^T) = (k+1) - (k-\ell) = \ell + 1$ as desired.

We prove (b) as follows. Notice that

$$G_1 H_1^T = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & G H^T & \\ 0 & & & \end{bmatrix} = 0.$$

Since the top row of H_1 cannot be a linear combination of the remaining rows of H_1 , the dimension of the row space of H_1 is 1 + (n-k) = (n+2) - (k+1)which is the dimension of (C_1^{\perp}) . Thus H_1 is a generator matrix for C_1^{\perp} .

Theorem 2. Let C be a binary linear [n, k] code. Suppose that the dimension of Hull(C) is ℓ , where $0 \leq \ell \leq k$. Let G be a generator matrix for C and H a parity check matrix for C.

Suppose that $\mathbf{x} = (x_1, x_2, \dots, x_n) \in GF(2)^n$ satisfies $\mathbf{x} \cdot \mathbf{x} = 0$. Let $y_i = \mathbf{x} \cdot \mathbf{r}_i$ for $1 \leq i \leq k$ where \mathbf{r}_i is the *i*th row of G and $z_j = \mathbf{x} \cdot \mathbf{s}_j$ for $1 \leq j \leq n-k$ where \mathbf{s}_j is the *j*th row of H. Then

(a) the following matrix

$$G_2 = \begin{bmatrix} 1 & 1 & x_1 & \dots & x_n \\ y_1 & 0 & \mathbf{r}_1 \\ y_2 & 0 & \mathbf{r}_2 \\ \vdots & \vdots & \vdots \\ y_k & 0 & \mathbf{r}_k \end{bmatrix}$$

generates an [n+2, k+1] linear code C_2 with $h(C_2) = \ell, \ell+1, \text{ or } \ell+2$. More precisely, we characterize them as follows.

- If $y_i = 0$ for any $1 \le i \le k$, then the dimension of $Hull(C_2)$ is $\ell + 1$. This is called Construction II.
- Suppose $y_i \neq 0$ for some $1 \leq i \leq k$. So, G_2 can be rewritten as G'_2 given by

$$G_2' = \begin{bmatrix} 1 & 1 & x_1 & \dots & x_n \\ 1 & 0 & & & \\ 0 & 0 & & & \\ \vdots & \vdots & & G' & \\ 0 & 0 & & & \end{bmatrix},$$

where $\langle G' \rangle = \langle G \rangle$. Then the dimension of $Hull(C_2)$ is ℓ , $\ell + 1$, or $\ell + 2$. This is called **Construction III**.

(b) A parity check matrix H_2 for C_2 is given by

$$H_2 = \begin{bmatrix} 1 & 1 & x_1 & \dots & x_n \\ 0 & z_1 & & \mathbf{s}_1 & \\ 0 & z_2 & & \mathbf{s}_2 & \\ \vdots & \vdots & & \vdots & \\ 0 & z_{n-k} & & \mathbf{s}_{n-k} & \end{bmatrix}.$$

Proof. We prove (a).

• Suppose that $y_i = 0$ for any $1 \le i \le k$. Then we have

$$G_2 G_2^T = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & G G^T & \\ 0 & & & \end{bmatrix}.$$

Thus $\operatorname{rank}(G_2G_2^T) = \operatorname{rank}(GG^T) = k - \ell$ since $h(C) = \ell$. The dimension of $\operatorname{Hull}(C_2) = h(C_2) = (k+1) - \operatorname{rank}(G_2G_2^T) = (k+1) - (k-\ell) = \ell + 1$.

• Suppose $y_i \neq 0$ for some $1 \leq i \leq k$. By row operations of G_2, G_2 is transformed into G'_2 given by

$$G_2' = \begin{bmatrix} 1 & 1 & x_1 & \dots & x_n \\ 1 & 0 & & & \\ 0 & 0 & & & \\ \vdots & \vdots & & G' & \\ 0 & 0 & & & & \end{bmatrix}$$

where $\langle G' \rangle = \langle G \rangle$. Furthermore,

$$G'_{2}(G'_{2})^{T} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & & \\ \vdots & X & \\ 0 & & \end{bmatrix}$$

where $X = (10...0)^T (10...0) + G'(G')^T$. Since $\operatorname{rank}(A+B) \leq \operatorname{rank}(A) + \operatorname{rank}(B)$, we have $\operatorname{rank}(X) \leq \operatorname{rank}((10...0)^T (10...0)) + \operatorname{rank}(G'(G')^T) = 1 + \operatorname{rank}(G'(G')^T)$. Noting that $(10...0)^T (10...0)$ affects only the top row of $G'(G')^T$, we know that $\operatorname{rank}(X)$ decreases by at most one. So, $\operatorname{rank}(X)$ is $\operatorname{rank}(G'(G')^T)$, $\operatorname{rank}(G'(G')^T) - 1$, or $\operatorname{rank}(G'(G')^T) + 1$. Thus $\operatorname{rank}(G_2(G_2)^T) = \operatorname{rank}(G'_2(G'_2)^T) = \operatorname{rank}(X)$ is $\operatorname{rank}(G'(G')^T) = k - \ell$, $\operatorname{rank}(G'(G')^T) - 1 = k - \ell - 1$, or $\operatorname{rank}(G'(G')^T) + 1 = k - \ell + 1$. Hence the dimension of $Hull(C_2)$ is $k + 1 - (k - \ell) = \ell + 1$, $k + 1 - (k - \ell - 1) = \ell + 2$, or $k + 1 - (k - \ell + 1) = \ell$.

We prove (b). It is straightforward to check that $G_2H_2^T = 0$ by the definition of y_i 's and z_j 's. Because the rank of H_2 is (n-k) + 1 and the dimension of the dual of C_2 is (n+2) - (k+1) = n - k + 1, we see that H_2 is a parity check matrix for C_2 .

We note that Constructions II and III are basically the same construction but we distinguish them in order to guess the hull dimension of the built-up code. We remark that Theorem 1 reproves the original building-up construction of binary self-dual codes [15] where n is even and $k = n/2 = \ell$.

Harada [11] gave a construction of binary LCD [n + 2, k + 1] codes from a given binary LCD [n, k] code. We generalize this in the following theorem.

By modifying the proof of Theorem 1, we can construct [n+2, k+1] linear codes with the same hull dimension as that of a given [n, k] linear code.

Theorem 3. Let C be a binary linear [n, k] code. Suppose that the dimension of Hull(C) is ℓ , where $0 \leq \ell \leq k$. Let G be a generator matrix for C and H a parity check matrix for C.

Suppose that $\mathbf{x} = (x_1, x_2, \dots, x_n) \in GF(2)^n$ satisfies $\mathbf{x} \cdot \mathbf{x} = 0$. Let $y_i = \mathbf{x} \cdot \mathbf{r}_i$ for $1 \leq i \leq k$ where \mathbf{r}_i is the *i*th row of G and $z_j = \mathbf{x} \cdot \mathbf{s}_j$ for $1 \leq j \leq n-k$ where \mathbf{s}_j is the *j*th row of H. The following matrix

$$G_{3} = \begin{bmatrix} 1 & 0 & x_{1} & \dots & x_{n} \\ y_{1} & y_{1} & & \mathbf{r}_{1} \\ y_{2} & y_{2} & & \mathbf{r}_{2} \\ \vdots & \vdots & & \vdots \\ y_{k} & y_{k} & & \mathbf{r}_{k} \end{bmatrix}$$

generates an [n + 2, k + 1] linear code C_3 with $h(C_3) = \ell$. This is called Construction IV. A parity check matrix H_3 for C_3 is given by

$$H_3 = \begin{bmatrix} 0 & 1 & x_1 & \dots & x_n \\ \hline z_1 & z_1 & \mathbf{s}_1 & \\ z_2 & z_2 & \mathbf{s}_2 & \\ \vdots & \vdots & & \vdots & \\ z_{n-k} & z_{n-k} & \mathbf{s}_{n-k} & \end{bmatrix}$$

Proof. The proof is almost the same as that of Theorem 1. It is straightforward to see that $G_3H_3^T = 0$ and $\operatorname{rank}(H_3) = 1 + (n - k)$ which implies that H_3 is a parity check matrix for C_3 . We compute $\operatorname{rank}(G_3G_3^T)$ as follows.

$$G_3 G_3^T = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & G G^T & \\ 0 & & & \end{bmatrix}$$

Thus $\operatorname{rank}(G_3G_3^T) = 1 + \operatorname{rank}(GG^T) = 1 + k - h(C) = 1 + k - \ell \text{ since } h(C) = \ell.$ Now $h(C_3) = (k+1) - \operatorname{rank}(G_3G_3^T) = (k+1) - (1+k-\ell) = \ell$ as desired. \Box

We can estimate the minimum distance $d(C_i)$ (i = 1, 2, 3) for Constructions I-IV as follows.

Theorem 4. Let C be a binary linear [n, k] code. Let $\mathbf{x} = (x_1, x_2, \dots, x_n) \in GF(2)^n$. Then we have the following.

(i) The minimum distance $d(C_i)$ (i = 1, 3) is $\min\{d(C), \operatorname{weight}(\mathbf{x}+C)+1\}$ or $\min\{d(C)+2, \operatorname{weight}(\mathbf{x}+C)+1\}$ for Constructions I and IV. (ii) $d(C_2)$ is $\min\{d(C), \operatorname{weight}(\mathbf{x}+C)+2\}$ for Construction II and $\min\{d(C)+1, \operatorname{weight}(\mathbf{x}+C)+1\}$, or $\min\{d(C)+1, \operatorname{weight}(\mathbf{x}+C)+2\}$ for Construction III.

Proof. We prove statement (i). Let i = 1, 3. Let S_1 be the code spanned by all the rows of G_i except for the top row. Then C_i is the disjoint union of S_1 and $(1 \ 0 \ \mathbf{x}) + S_1$. Thus $d(C_i)$ is the minimum of $d(S_1)$ and weight $((1 \ 0 \ \mathbf{x}) + S_1)$. We note that $d(S_1)$ is d(C) or d(C) + 2 and that weight $((1 \ 0 \ \mathbf{x}) + S_1) = 1 +$ weight $(\mathbf{x} + C)$. Hence we obtain (i). Similarly, we can prove (ii), whose proof is omitted.

The following is straightforward since $d(C) \leq \rho(C)$, where $\rho(C)$ is the covering radius of C.

Corollary 1. Let C be a binary linear [n, k] code with its covering radius $\rho(C)$. Then the minimum distance $d(C_i)$ (i = 1, 2, 3) for Constructions I-IV satisfies min $\{d(C), weight(\mathbf{x} + C) + 1\} \leq d(C_i) \leq \rho(C) + 2$.

Remark 1. Although we have considered Constructions I-IV for linear codes over GF(2), it is easy to see that the same Constructions I-IV in Theorems 1-3 hold for linear codes over GF(q), where $q = 2^r$ for any integer $r \ge 1$. If q is odd, then a slight modification of Constructions I-IV based on the building-up construction for self-dual codes over GF(q) [16],[17] will give results similar to Theorems 1-3.

4 Some interesting optimal linear codes

We display some interesting optimal linear codes with hull dimensions 2 and 3 from a linear code with hull dimension 1.

Start with an h_1 -optimal [10, 6, 3] code C whose generator matrix G is given below.

	F1000001017
	0100001001
α	0010001110
G =	0001000110
	0000101010
	0000011100

Example 1. Let us take $\mathbf{x} = (0000011000)$ with G by Construction III to get an h_2 -optimal [12, 7, 3] linear code C' with h = 2. Its generator matrix G' is written as follows.

By Proposition 1, we can obtain a $[[12, 5, 3; 3]]_2$ EAQECC from C'.

Example 2. Let us take $\mathbf{x} = (1111110011)$ with G above by Construction III to get an optimal [12, 7, 4] linear code C'' with h = 3. Its generator matrix G'' is written in standard from after row operations.

By Proposition 1, we can obtain a $[[12, 5, 3; 3]]_2$ EAQECC from C''.

By exhaustive search, we have checked that there are up to equivalence exactly two optimal [12, 7, 4] codes. One of them is the above [12, 7, 4] code C'' with h = 3. The other is a [12, 7, 4] code [14] with h = 1 whose weight distribution is $[\langle 0, 1 \rangle, \langle 4, 38 \rangle, \langle 6, 52 \rangle, \langle 8, 33 \rangle, \langle 10, 4 \rangle]$. This code gives a [[12, 6, 4; 4]]₂ EAQECC by Proposition 1.

5 Optimal linear codes with several hulls and the construction of EAQECC

We construct several optimal linear codes of lengths up to 13 with h = i(i = 1, 2, 3, 4, 5) from a given linear code of a fixed hull dimension h. Tables 1,3,5,7,9 display best possible minimum distances of linear [n, k] codes from hull dimensions 1 to 5. The upper bounds for the minimum distances in the tables are from Grassl's table [10] by taking not the hull dimension into account and by brute force search. Each cell in each table denotes the highest minimum distance d(n, k) for given n, k, and h = i together with the superscripts referring to Constructions I to IV and o meaning that the codes are optimal. For n = 12, we apply Constructions I, II, and/or III. For n = 13, we apply Constructions I, III, and/or IV. All computations were done by Magma [2]. To save the space, we post whole information about the codes in Tables 1,3,5,7,9 in the author's website [14] and list most generator matrices for n = 12 and 13 in this paper.

Tables 2,4,6,8,10 display associated $[[n, k, d; c]]_2$ EAQECC based on Proposition 1 and Tables 1, 3, 5, 7, 9. In other words, we obtain $[[n, k - h, d; n - k - h]]_2$ EAQECC from binary [n, k, d] codes with hull dimension h.

Example 3. Fix the hull dimension h = 1. For any n with k such that $1 \le k \le n \le 11$ and n = 12 with k $(1 \le k \le 4)$, we ran exhaustive search to get optimal or h_1 -optimal codes. We note that there is an optimal [12, 5, 4] code with h = 1 from Magma database.

For n = 12 with $k \ge 6$, we apply Constructions I and II to all the LCD codes of length 10 and dimension k - 1 displayed in [12]. More precisely, we construct optimal [12, 6, 4] and [12, 9, 2] codes by Construction I. Similarly, we construct optimal [12, 7, 4] and [12, 8, 3] codes by Construction III.

Let n = 13. We construct optimal [13, 4, 6], [13, 5, 5], [13, 6, 4], [13, 7, 4], [13, 8, 3], [13, 10, 2], [13, 11, 2] codes by Construction I. We also construct an h_1 -optimal [13, 3, 6] code by Construction I, which is justified by the nonexistence of [13, 3, 7] codes with h = 1 using exhaustive search. Similarly, we construct an h_1 -optimal [13, 9, 2] code by Construction I, which is justified by the non-existence of [13, 9, 3] codes with h = 1 using exhaustive search.

In what follows, $G_{n,k,d}^i$ refers to a generator matrix for a binary [n, k, d] code $C_{n,k,d}^i$ with h = i and the highest minimum distance d = d(n, k).

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$$n = 12$$
 with $h = 1$



Table 1: Each cell refers to the highest minimum distance d(n, k) for $n \leq 13$ when h = 1, and examples of corresponding generator matrices $G_{12,k,d}^1$ ($6 \leq k \leq 11$) and $G_{13,k,d}^1$ ($3 \leq k \leq 11$)

Example 4. Fix the hull dimension h = 2. For any n with k such that $1 \le k \le n \le 11$ and n = 12 with k $(1 \le k \le 4)$, we ran exhaustive search to get optimal or h_2 -optimal codes.

n/k	0	1	2	3	4	5	6	7	8	9	10	11
2	(2;0)											
3	(2;1)	(1;0)										
4	(4;2)	(1;1)	(2;0)									
5	(4;3)	(3;2)	(2;1)	(1;0)								
6	(6;4)	(3;3)	(2;2)	(1;1)	(2;0)							
7	(6;5)	(4;4)	(3;3)	(2;2)	(2;1)	(1;0)						
8	(8;6)	(4;5)	(4;4)	(3;3)	(2;2)	(1;1)	(2;0)					
9	(8;7)	(5;6)	(4;5)	(3;4)	(3;3)	(2;2)	(2;1)	(1; 0)				
10	(10; 8)	(5;7)	(5;6)	(4;5)	(4; 4)	(3;3)	(2;2)	(1;1)	(2;0)			
11	(10;9)	(7;8)	(6;7)	(5;6)	(4;5)	(3;4)	(3;3)	(2;2)	(2;1)	(1;0)		
$\overline{12}$	(12; 10)	(7;9)	(6; 8)	(5;7)	(4; 6)	(4;5)	(4; 4)	(3; 3)	(2;2)	(1;1)	(2; 0)	
13	(12; 11)	(8;10)	(6; 9)	(6; 8)	(5;7)	(4; 6)	(4; 5)	(3; 4)	(2;3)	(2;2)	(2;1)	(1; 0)

Table 2: $[[n, k, d; c]]_2$ EAQECC with (d; c) for $n \leq 13$ when h = 1 based on Proposition 1 and Table 1

For n = 12 with $k \ge 5$, we apply Constructions I, II or III to LCD codes or linear codes with h = 1 of length 10 and dimension k - 1. More precisely, we construct optimal [12, 5, 4], [12, 6, 4], [12, 8, 3], [12, 9, 2], [12, 10, 2] codes from [10, 4, 4], [10, 5, 3], [10, 7, 2], [10, 8, 2], [10, 9, 1] codes with h = 0 respectively by Construction III. On the other hand, we also construct a [12, 7, 3] code from a [10, 6, 3] code with h = 0 by Construction III. By exhaustive search, we check that it is h_2 -optimal.

Let n = 13. We construct optimal [13, 3, 7], [13, 4, 6], [13, 5, 5], [13, 6, 4], [13, 8, 4] codes by Construction III from LCD codes of length 11 and dimensions k = 2, 3, 4, 5, 7 respectively. We also construct an optimal [13, 7, 4] code from a linear [11, 6, 3] code with h = 1 by Construction I. For k = 2, 10, 11, it is easy to construct directly optimal or h_2 -optimal [13, k] codes. For k = 9, it is known that there exist an optimal [13, 9, 3] code with h = 2 by Magma database.

•
$$n = 12$$
 with $h = 2$

•
$$n = 13$$
 with $h = 2$

$G^2_{13,9,}$	3 =	$\begin{bmatrix} 100\\ 010\\ 001\\ 000\\ 000\\ 000\\ 000\\ 000$	000000 000000 00000 010000 00100 000100 000010 000001 000000	000011 01010(010111) 010001 01001(000101) 00011(0001111)	L]) L L L L L L L L L	12 13,10,2	$= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\$	20000 10000 21000 20100 20010 20001 20000 20000 20000 20000	00001 00000 00000 00000 00000 00000 10000 01000 00100 00010	00- 10 01 01 01 01 01 01 01_ 01_
n/k	2	3	4	5	6	7	8	9	10	11
2	0									
3	0	0								
4	2	0	0	0						
5 6	$\frac{2}{4}$	1	0	0	0					
7	$\frac{4}{1}$	<u>्</u> य	$\frac{2}{2}$	1	0	0				
8	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0			
$\overline{9}$	4	4	4	$\frac{1}{3}$	$\frac{1}{2}$	Ĩ	ŏ	0		
10	6	4	4	- Ĵ	3	2	$\tilde{2}$	Ő	0	
11	6	5	4	4	4	3	2	1	0	0
12	8^{o}	5	6^{o}	$4^{III,o}$	$4^{III,o}$	3^{III}	$3^{III,o}$	$2^{III,o}$	$2^{III,o}$	0
13	8^{o}	$7^{III,o}$	$6^{III,o}$	$5^{III,o}$	$4^{III,o}$	$4^{I,h_1,o}$	$4^{III,o}$	3^o	2^{o}	1

Table 3: Each cell refers to the highest minimum distance d(n, k) for $n \leq 13$ when h = 2, and examples of corresponding generator matrices $G_{12,k,d}^2$ $(3 \leq k \leq 10)$ and $G_{13,k,d}^2$ $(3 \leq k \leq 10)$

Example 5. Fix the hull dimension h = 3.

Since h = 3, the code length n should be at least 6. If $n - k \leq 2$, then there does not exist an [n, k] code with h = 3. If k = 3, we use the optimal minimum distances of self-orthogonal [n, 3] codes from [3].

For any n with k such that $3 \le k \le n \le 11$ and n = 12 with k = 3, 4, we ran exhaustive search to obtain optimal or h_3 -optimal codes.

Using Construction III, we construct an optimal [12, 5, 4] code with h = 3 from a [10, 4, 4] code, and an h_3 -optimal [12, 6, 4] code from a [10, 5, 3] code with h = 2. We further construct an optimal [12, 7, 4] code from a [10, 6, 3] code with h = 1 by Construction III.

We construct h_3 -optimal [12, 8, 2] and [12, 9, 2] codes from [10, 7, 2] and [10, 8, 2] codes with h = 2, respectively by Construction I. This is justified by exhaustive search that there are no [12, 6, 4], [12, 8, 3] codes.

n/k	0	1	2	3	4	5	6	7	8	9
4	(2;0)									
5	(2;1)	(1;0)								
6	(4;2)	(3;1)	(2;0)							
7	(4;3)	(3;2)	(2;1)	(1;0)						
8	(4;4)	(4;3)	(3;2)	(2;1)	(2;0)					
9	(4;5)	(4;4)	(4;3)	(3;2)	(2;1)	(1;0)				
10	(6;6)	(4;5)	(4;4)	(3;3)	(3;2)	(2;1)	(2;0)	0	0	
11	(6;7)	(5;6)	(4;5)	(4;4)	(4;3)	(3;2)	(2;1)	(1;0)		
12	(8;8)	(5;7)	(6; 6)	(4;5)	(4; 4)	(3;3)	(3;2)	(2;1)	(2;0)	
13	(8;9)	(7; 8)	(6;7)	(5; 6)	(4; 5)	(4; 4)	(4;3)	(3; 2)	(2;1)	(1;0)

Table 4: $[[n, k, d; c]]_2$ EAQECC with (d; c) for $n \leq 13$ when h = 2 based on Proposition 1 and Table 3

For n = 13, we construct [13, 4, 4], [13, 5, 4], [13, 6, 4], [13, 7, 3], [13, 8, 2], [13, 9, 2] codes with h = 3 from [11, k] codes with h = 3 ($3 \le k \le 8$) by Construction IV. Similarly we construct [13, 4, 5], [13, 5, 4], [13, 6, 4], [13, 7, 4], [13, 8, 3], [13, 10, 2] codes with h = 3 from [11, k] codes with h = 2 ($3 \le k \le 7, k = 9$) by Construction I.

•
$$n = 12$$
 with $h = 3$



Example 6. Fix the hull dimension h = 4.

Since h = 4, the code length n should be at least 8. If $n - k \leq 3$, then there does not exist a [n, k] code with h = 4. If k = 4, we use the optimal minimum distances of self-orthogonal [n, 4] codes from [3].

For any n with k such that $4 \le k \le n \le 11$ and n = 12 with k = 4, we ran exhaustive search to obtain optimal or h_4 -optimal codes.

We construct optimal [12, 5, 4] and [12, 6, 4] codes with h = 4 from [10, 4, 4] and [10, 5, 3] codes with h = 3 respectively by Construction I.

We construct h_4 -optimal [12, 7, 3], [12, 8, 2] codes with h = 4 from [10, 6, 1], [10, 7, 2] codes respectively with h = 3 by Construction I. This is justified by exhaustive checking that there are no [12, 7, 4] and [12, 8, 3] codes with h = 4.

n/k	3	4	5	6	7	8	9	10
6	2	0	0	0				
7	4	3	0	0	0			
8	4	3	2	0	0	0		
9	4	4	3	2	0	0		
10	4	4	4	2	2	0	0	
11	4	4	4	3	2	2	0	
12	6^{o}	4	$4^{III,o}$	$4^{III,o}$	$4^{III,h_1,o}$	2^{III}	$2^{II,o}$	
13	6	$\geq 5^{I}$	$\geq 4^{I,IV}$	$4^{I,IV,o}$	$4^{I,o}$	$\geq 3^{I}$	$\geq 2^{IV}$	$2^{I,o}$

Table 5: Each cell refers to the highest minimum distance d(n, k) for $n \leq 13$ when h = 3, and examples of corresponding generator matrices $G^3_{12,k,d}$ $(4 \leq k \leq 9)$ and $G^3_{13,k,d}$ $(4 \leq k \leq 10)$

n/k	0	1	2	3	4	5	6	7
6	(2;0)							
7	(4;1)	(3;0)						
8	(4;2)	(3;1)	(2;0)					
9	(4;3)	(4;2)	(3;1)	(2;0)				
10	(4;4)	(4;3)	(4;2)	(2;1)	(2;0)			
11	(4;5)	(4;4)	(4;3)	(3;2)	(2;1)	(2;0)		
$\overline{12}$	(6; 6)	(4;5)	(4; 4)	(4; 3)	(4; 2)	(2;1)	(2;0)	
13	(6;7)	$(\geq 5; 6)$	$(\geq 4;5)$	(4; 4)	(4;3)	$(\geq 3; 2)$	$(\geq 2; 1)$	(2; 0)

Table 6: $[[n, k, d; c]]_2$ EAQECC with (d; c) for $n \leq 13$ when h = 3 based on Proposition 1 and Table 5

For n = 13, we obtain [13, 5, 4], [13, 6, 4], [13, 7, 3], [13, 8, 2] codes with h = 4 from [11, k] codes with h = 4 ($4 \le k \le 7$) by Construction IV. Similarly, we construct [13, 4, 4], [13, 5, 4], [13, 6, 4], [13, 7, 3], [13, 8, 2], [13, 9, 2] codes with h = 4 from [11, k] codes with h = 3 ($3 \le k \le 8$) by Construction I.

• n = 12 with h = 4

10/10	-	0	•	•	0	0
8	4	0	0	0	0	0
9	4	2	0	0	0	0
10	4	4	2	0	0	0
11	4	4	3	2	0	0
$\overline{12}$	4	$4^{I,o}$	$4^{I,o}$	3^{I}	2^{I}	0
13	4	$\geq 4^{I,IV}$	$4^{I,IV,o}$	$\geq 3^{I,IV}$	$\geq 2^{I,IV}$	$\geq 2^I$

Table 7: Each cell refers to the highest minimum distance d(n,k) for $n \leq 13$ when h = 4, and examples of corresponding generator matrices $G^4_{12,k,d}$ ($5 \leq k \leq 8$) and $G^4_{13,k,d}$ ($5 \leq k \leq 9$)

n/k	0	1	2	3	4	5
8	(4;0)					
9	(4;1)	(2;0)				
10	(4;2)	(4;1)	(2;0)			
11	(4;3)	(4;2)	(3;1)	(2;0)		
12	(4;4)	(4;3)	(4; 2)	(3;1)	(2;0)	
13	(4;5)	$(\geq 4; 4)$	(4;3)	$(\geq 3;2)$	$(\geq 2;1)$	$(\geq 2; 0)$

Table 8: $[[n, k, d; c]]_2$ EAQECC with (d; c) for $n \leq 13$ when h = 4 based on Proposition 1 and Table 7

Example 7. Fix the hull dimension h = 5.

Since h = 5, the code length n should be at least 10. If $n - k \leq 4$, then there does not exist a [n, k] code with h = 5. If k = 5, we use the optimal minimum distances of self-orthogonal [n, 5] codes from [3].

For n = 10, 11 with k = 4, 5, we ran exhaustive search to obtain optimal or h_5 -optimal codes.

It is well known that there is a self-orthogonal [12, 5, 4] code [3], which is optimal. We construct h_5 -optimal [12, 6, 3] and [12, 7, 3] codes from [10, 5, 4] and [10, 6, 2] codes with h = 4 by Construction I. This is justified by exhaustive checking that there are no [12, 6, 4] and [12, 7, 4] codes with h = 5.

For n = 13, we obtain [13, 6, 3], [13, 7, 3] codes with h = 5 from [11, k] codes with h = 5 ($5 \le k \le 6$) by Construction IV. Similarly we construct [13, 6, 4], [13, 7, 3], [13, 8, 2] codes from [11, k] codes with h = 4 ($5 \le k \le 7$) by Construction I.

Although we cannot construct a [13, 7, 4] code with h = 5 from Constructions I and IV, we observe the following. Using the unique self-dual [12, 6, 4] code B_{12} [22] with the below generator matrix G_{12} , we obtain an optimal [13, 7, 4] code $C_{13,7,4}$ with the below generator matrix $G_{13,7,4}$ by augmenting a coset leader $\mathbf{v} = (000000010101)$ to B_{12} because the covering radius of B_{12} is 3. We show that $h(C_{13,7,4}) = 5$ in what follows. The top row $\mathbf{r}_1 = (1 | \mathbf{v})$ of $G_{13,7,4}$ is orthogonal to only five rows $\mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_5, \mathbf{r}_6, \mathbf{r}_7$ of G_{13} . Therefore, the hull of $C_{13,7,4}$ consists of these five rows, resulting in $h(C_{13,7,4}) = 5$.

•
$$n = 13$$
 with $h = 5$

There are not many known [[n, k, d; c]] EAQECC when $n \leq 13$. We compare our results with some known EAQECC in Table 11. In fact, the parameters in boldface in the third column of the table are better than the currently known parameters from [21], [27].

n/k	5	6	7	8
10	2	0	0	0
11	4	3	0	0
12	4^{o}	3^{I}	3^{I}	0
13	4	$4^{I,o}$	4^{o}	$\geq 2^{I}$

Table 9: Each cell refers to the highest minimum distance d(n, k) for $n \leq 13$ when h = 5, and examples of corresponding generator matrices $G_{12,k,d}^5$ ($5 \leq k \leq 7$) and $G_{13,k,d}^5$ ($5 \leq k \leq 8$)

n/k	0	1	2	3
10	(2;0)			
11	(4;1)	(3;0)		
12	(4; 2)	(3;1)	(3;0)	
13	(4;3)	(4; 2)	(4;1)	$(\geq 2; 0)$

Table 10: $[[n, k, d; c]]_2$ EAQECC with (d; c) for $n \leq 13$ when h = 5 based on Proposition 1 and Table 9

6 Conclusion

This paper has introduced a systematic and efficient method to construct binary optimal or possibly optimal [n, k] codes of lengths up to 13 with respect to hull dimensions 1-5. These codes are used to construct EAQECC with the best known parameters.

The complexity of Constructions I-IV mainly depends on the binary vectors \mathbf{x} of length n, whose cardinality is at most 2^{n-1} due to the parity of \mathbf{x} . This complexity can be reduced if we consider the standard generator matrix G in Theorems 1, 2, and 3. Since $n \leq 13$ we need at most $2^{12} = 4,096$ vectors for \mathbf{x} . As we prefer to keep a non-standard generator matrix to distinguish Constructions I-IV, we have run all possibilities for \mathbf{x} and have checked the equivalence of codes by Magma. Using our linux machine Intel(R) Xeon(R) CPU E3-1225 V2 @ 3.20GHz, calculations for Theorems 1-3 were performed within ten minutes while some exhaustive search took more than two weeks. As future work, it is worth considering similar constructions for other finite fields and rings.

currently known EAQECC	Ref	our related EAQECC	Tables
$[[9, 1, 3; 1]]_2$	[21]	$[[{f 9},{f 2},{f 3};{f 1}]]_2$	Table 6
$[[12, 1, 7; 9]]_2$	[27]	$[[12, 1, 7; 9]]_2$	Table 2
$[[12, 3, 5; 7]]_2$	[27]	$[[12, 3, 5; 7]]_2$	Table 2
$[[12, 4, 4; 6]]_2$	[27]	$[[12, 4, 4; 6]]_2, [[12, 2, 6; 6]]_2$	Table 2, Table 4
$[[12, 5, 3; 5]]_2$	[27]	$[[12, 5, 4; 5]]_2, [[12, 3, 4; 5]]_2$	Table 2, Table 4
$[[13, 7, 3; 4]]_2$	[27]	$[[13, 7, 3; 4]]_2, [[13, 5, 4; 4]]_2$	Table 2, Table 4
$[[13, 3, 5; 8]]_2$	[27]	$[[13, 3, 6; 8]]_2, [[13, 1, 7; 8]]_2$	Table 2, Table 4

Table 11: Comparison with some known EAQECC

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Declarations

Conflict of interest The author declares that he has no conflict of interest regarding the publication of this paper.

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