# Binary optimal linear codes with various hull dimensions and entanglement-assisted QECC 

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#### Abstract

The hull of a linear code $C$ is the intersection of $C$ with its dual. To the best of our knowledge, there are very few constructions of binary linear codes with the hull dimension $\geq 2$ except for self-orthogonal codes. We propose a building-up construction to obtain a plenty of binary $[n+2, k+1]$ codes with hull dimension $\ell, \ell+1$, or $\ell+2$ from a given binary $[n, k]$ code with hull dimension $\ell$. In particular, with respect to hull dimensions 1 and 2 , we construct all binary optimal $[n, k]$ codes of lengths up to 13 . With respect to hull dimensions 3,4 , and 5 , we construct all binary optimal $[n, k]$ codes of lengths up to 12 and the best possible minimum distances of $[13, k]$ codes for $3 \leq k \leq 10$. As an application, we apply our binary optimal codes with a given hull dimension to construct several entanglement-assisted quantum errorcorrecting codes(EAQECC) with the best known parameters.


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## 1 Introduction

The hull of a linear code $C$ is the intersection of $C$ with its dual. The hull of a linear code was introduced by Assmus, Jr. and Key [1]. The hull determines the complexity of algorithms for checking permutation equivalence of two

[^0]linear codes [25], which are very effective if the dimension of the hull is small, and which are worst if the dimension of the hull is large. The hardness of the Permutation Code Equivalence problem is of great importance when designing cryptographic primitives, such as public-key cryptosystems and identification schemes in the field of code-based cryptography [26].

When the hull contains only the zero vector, that is, the hull dimension is $0, C$ is called a Linear Complementary Dual code (shortly, LCD code). Recently, LCD codes have been actively studied due to its side channel attack. An LCD code was originally constructed by Massey [19], [20] as a reversible code in order to provide an optimum linear coding solution for the two-user binary adder channel. Carlet and Guilley [5] introduced several constructions of LCD codes and investigated an application of LCD codes against sidechannel attacks(SCA) and Fault Injection Attack(FIA).

There are several constructions for binary linear codes with hull dimensions 0 or 1. More precisely, Galvez et al. [8 have constructed all binary optimal LCD $[n, k]$ codes for $1 \leq k \leq n \leq 12$. Harada and Saito [12] have extended this for $1 \leq k \leq n \leq 16$. Li and Zeng [18] have constructed binary linear $[n, k]$ codes with hull dimension 1 for $n=8$ with $k=3,5,7, n=9$ with $k=3,5,6,7$, and $n=10$ with $k=3,4,7$, whose optimality was not discussed.

On the other hand, when the dimension $h$ of the hull of a linear $[n, k]$ code $C$ is equal to $k, C$ is called self-orthogonal, and self-dual if $h=k=n / 2$. Selforthogonal or self-dual codes have been one of the most active research areas in classical coding theory [24] and recently in quantum coding theory [4]. As far as we know, there are few constructions of binary linear codes with the hull dimension $h \geq 2$ except for self-orthogonal codes. It turns out that linear codes with various hull dimensions can be used to construct entanglementassisted quantum error-correcting codes (EAQECC) [7], 9], [27].

Therefore, it is an interesting problem to find a unified method to construct linear codes with various hull dimensions.

In this paper, we give an efficient and systematic method, called a buildingup construction to construct linear codes with various hull dimensions from a given linear code with a fixed hull dimension. More precisely, with respect to hull dimensions 1 and 2 , we construct all binary optimal $[n, k]$ codes of lengths up to 13 . With respect to hull dimensions 3,4 , and 5 , we construct all binary optimal $[n, k]$ codes of lengths up to 12 and the best possible minimum distances of $[13, k]$ codes for $3 \leq k \leq 10$. As a coding theoretical application, given length $2 \leq n \leq 12$ and dimension $k(2 \leq k \leq n)$, by running all values
of the hull dimension $h(0 \leq h \leq[n / 2])$, we can recover all the binary best known linear $[n, k]$ codes in Grassl's table [10] with sometimes more than one inequivalent code. We apply our binary optimal codes with a given hull dimension to the construction of $[[n, k, d ; c]]$ EAQECC with the best known parameters as described in [7], [27].

## 2 Preliminaries

A linear $[n, k, d]$ code $C$ over $G F(q)$ or $\mathbb{F}_{q}$ is a $k$-dimensional subspace of $\mathbb{F}_{2}^{n}$ with minimum distance $d(C)$ or $d$ if there is no confusion. The dual of $C$ is $C^{\perp}=\left\{\mathbf{x} \in \mathbb{F}_{2}^{n} \mid \mathbf{x} \cdot \mathbf{c}=0\right.$ for any $\left.\mathbf{c} \in C\right\}$, where the dot product is the usual inner product. A linear code $C$ is called self-orthogonal if $C \subset C^{\perp}$ and selfdual if $C=C^{\perp}$. A linear code $C$ is called an $L C D$ code (linear complementary dual code) if $C \cap C^{\perp}=\{0\}$. Hence being LCD is the opposite concept of self-orthogonality.

Let $C$ be a linear code over $G F(q)$ with its dual $C^{\perp}$. The Hull of $C$ is defined as $\operatorname{Hull}(C)=C \cap C^{\perp}$. Let $h=$ dimension of $\operatorname{Hull}(C)$.

We call $C h_{i}$-optimal if $d(C)$ is the largest among all the linear $[n, k]$ codes $C$ with $h=i$ for $0 \leq i \leq k$. We call $C$ optimal if $d(C)$ is the largest among all the linear $[n, k]$ codes $C$.

Lemma 1. ([18, Proposition 1]) Let $C$ be a linear $[n, k]$ code over $G F(q)$ with generator matrix $G$. Then $h=k-\operatorname{rank}\left(G G^{T}\right)$.

Hence, if $h=0$, that is, $\operatorname{Hull}(C)=\{0\}$ or $\operatorname{rank}\left(G G^{T}\right)=k$, then $C$ is $L C D$. If $\operatorname{Hull}(C)=C$, then $C$ is self-orthogonal.

Now we also describe entanglement-assisted quantum error-correcting codes(EAQECC). An EAQECC with parameters $[[n, k, d ; c]]$ encodes $k$ logical qubits into $n$ physical qubits with the help of $c$ pre-shared entanglement pairs [23]. If $c=0$, then $[[n, k, d ; c]]$ EAQECC are equivalent to quantum stabilizer codes. Hence, $[[n, k, d ; c]]$ EAQECC are a generalization of $[[n, k, d]]$ QECC.

The following is a useful method to construct $[[n, k, d ; c]]_{q}$ EAQECC from $q$-ary linear $[n, k, d]$ codes.

Proposition 1. ([7], [9, Corollary 3.1], [27, Proposition 8]) Let $C$ be a linear code over $G F(q)$ with parameters $[n, k, d]_{q}$ and $C^{\perp}$ be its dual with parameters
$\left[n, n-k, d^{\prime}\right]_{q}$. Let $\operatorname{dim}(\operatorname{Hull}(C))=h$. Then, there exist an $[[n, k-h, d ; n-$ $k-h]]_{q} E A Q E C C$ and an $\left[\left[n, n-k-h, d^{\prime} ; k-h\right]\right]_{q} E A Q E C C$.
Proposition 2 ([9]). An $[[n, k, d ; c]]_{q} E A Q E C C$ satisfies

$$
n+c-k \geq 2(d-1)
$$

where $0 \leq c \leq n-1$.
An EAQECC attaining this Singleton bound is called an $M D S E A Q E C C$. Chen et al. [6] constructed MDS EAQECC when $q=2^{e}$ with $e$ odd with special values of $n, k, d$, and $c$.

Let $q=2$ and consider the binary Hamming $[7,4,3]$ code $\mathcal{H}_{3}$. Since its dual $\mathcal{H}_{3}^{\perp}$ is the simplex code $\mathcal{S}_{3}$ and is a subcode of $\mathcal{H}_{3}$, we have $h\left(\mathcal{H}_{3}\right)=3$. Hence by Proposition [1 , we obtain a $[[7,1,3 ; 0]]_{2}$ EAQECC which is best known by Grassl's table [10]. Note that $n+c-k=7+0-1=6$ and $2(d-1)=4$. Hence $[[7,1,3 ; 0]]_{2}$ EAQECC is not MDS.

In this paper, we consider $q=2$ and construct various [ $n, k-h, d ; n-$ $k-h]]_{2}$ EAQECC using the building-up constructions.

## 3 Building-up construction for linear codes with various hull dimensions

In the remaining sections, we consider binary codes. We can construct $[n+$ $2, k+1]$ linear codes with hull dimension $\ell+1$ from a given $[n, k]$ linear code with hull dimension $\ell$ as follows.
Theorem 1. Let $C$ be a binary linear $[n, k]$ code. Suppose that the dimension of $\operatorname{Hull}(C)$ is $\ell$, where $0 \leq \ell \leq k$. Let $G$ be a generator matrix for $C$ and $H$ a parity check matrix for $C$.

Suppose that $\mathbf{x}=\left(x_{1}, x_{2}, \ldots, x_{n}\right) \in G F(2)^{n}$ satisfies $\mathbf{x} \cdot \mathbf{x}=1$. Let $y_{i}=\mathbf{x} \cdot \mathbf{r}_{i}$ for $1 \leq i \leq k$ where $\mathbf{r}_{i}$ is the ith row of $G$ and $z_{j}=\mathbf{x} \cdot \mathbf{s}_{j}$ for $1 \leq j \leq n-k$ where $\mathbf{s}_{j}$ is the $j$ th row of $H$. Then
(a) the following matrix

$$
G_{1}=\left[\begin{array}{cc|ccc}
1 & 0 & x_{1} & \ldots & x_{n} \\
\hline y_{1} & y_{1} & & \mathbf{r}_{1} & \\
y_{2} & y_{2} & & \mathbf{r}_{2} & \\
\vdots & \vdots & & \vdots & \\
y_{k} & y_{k} & & \mathbf{r}_{k} &
\end{array}\right]
$$

generates an $[n+2, k+1]$ linear code $C_{1}$ with $h\left(C_{1}\right)=\ell+1$. This is called Construction I.
(b) A parity check matrix $H_{1}$ for $C_{1}$ is given by

$$
H_{1}=\left[\begin{array}{cc|ccc}
1 & 0 & x_{1} & \ldots & x_{n} \\
\hline z_{1} & z_{1} & & \mathbf{s}_{1} & \\
z_{2} & z_{2} & & \mathbf{s}_{2} & \\
\vdots & \vdots & & \vdots & \\
z_{n-k} & z_{n-k} & & \mathbf{s}_{n-k} &
\end{array}\right]
$$

Proof. We prove (a). Now $\operatorname{rank}\left(G_{1} G_{1}^{T}\right)$ is computed as follows.

$$
G_{1} G_{1}^{T}=\left[\begin{array}{cccc}
0 & 0 & \ldots & 0 \\
0 & & & \\
\vdots & & G G^{T} & \\
0 & & &
\end{array}\right]
$$

Thus $\operatorname{rank}\left(G_{1} G_{1}^{T}\right)=\operatorname{rank}\left(G G^{T}\right)=k-h(C)=k-\ell$ since $h(C)=\ell$. Now $h\left(C_{1}\right)=(k+1)-\operatorname{rank}\left(G_{1} G_{1}^{T}\right)=(k+1)-(k-\ell)=\ell+1$ as desired.

We prove (b) as follows. Notice that

$$
G_{1} H_{1}^{T}=\left[\begin{array}{cccc}
0 & 0 & \ldots & 0 \\
0 & & & \\
\vdots & & G H^{T} & \\
0 & & &
\end{array}\right]=0
$$

Since the top row of $H_{1}$ cannot be a linear combination of the remaining rows of $H_{1}$, the dimension of the row space of $H_{1}$ is $1+(n-k)=(n+2)-(k+1)$ which is the dimension of $\left(C_{1}^{\perp}\right)$. Thus $H_{1}$ is a generator matrix for $C_{1}^{\perp}$.

Theorem 2. Let $C$ be a binary linear $[n, k]$ code. Suppose that the dimension of $\operatorname{Hull}(C)$ is $\ell$, where $0 \leq \ell \leq k$. Let $G$ be a generator matrix for $C$ and $H$ a parity check matrix for $C$.

Suppose that $\mathbf{x}=\left(x_{1}, x_{2}, \ldots, x_{n}\right) \in G F(2)^{n}$ satisfies $\mathbf{x} \cdot \mathbf{x}=0$. Let $y_{i}=\mathbf{x} \cdot \mathbf{r}_{i}$ for $1 \leq i \leq k$ where $\mathbf{r}_{i}$ is the ith row of $G$ and $z_{j}=\mathbf{x} \cdot \mathbf{s}_{j}$ for $1 \leq j \leq n-k$ where $\mathbf{s}_{j}$ is the $j$ th row of $H$. Then
(a) the following matrix

$$
G_{2}=\left[\begin{array}{cc|ccc}
1 & 1 & x_{1} & \ldots & x_{n} \\
\hline y_{1} & 0 & & \mathbf{r}_{1} & \\
y_{2} & 0 & & \mathbf{r}_{2} & \\
\vdots & \vdots & & \vdots & \\
y_{k} & 0 & & \mathbf{r}_{k} &
\end{array}\right]
$$

generates an $[n+2, k+1]$ linear code $C_{2}$ with $h\left(C_{2}\right)=\ell, \ell+1$, or $\ell+2$. More precisely, we characterize them as follows.

- If $y_{i}=0$ for any $1 \leq i \leq k$, then the dimension of $\operatorname{Hull}\left(C_{2}\right)$ is $\ell+1$. This is called Construction II.
- Suppose $y_{i} \neq 0$ for some $1 \leq i \leq k$. So, $G_{2}$ can be rewritten as $G_{2}^{\prime}$ given by

$$
G_{2}^{\prime}=\left[\begin{array}{cc|ccc}
1 & 1 & x_{1} & \ldots & x_{n} \\
\hline 1 & 0 & & & \\
0 & 0 & & & \\
\vdots & \vdots & & G^{\prime} & \\
0 & 0 & & &
\end{array}\right]
$$

where $\left\langle G^{\prime}\right\rangle=\langle G\rangle$. Then the dimension of $\operatorname{Hull}\left(C_{2}\right)$ is $\ell, \ell+1$, or $\ell+2$. This is called Construction III.
(b) A parity check matrix $H_{2}$ for $C_{2}$ is given by

$$
H_{2}=\left[\begin{array}{cc|ccc}
1 & 1 & x_{1} & \ldots & x_{n} \\
\hline 0 & z_{1} & & \mathbf{s}_{1} & \\
0 & z_{2} & & \mathbf{s}_{2} & \\
\vdots & \vdots & & \vdots & \\
0 & z_{n-k} & & \mathbf{s}_{n-k} &
\end{array}\right]
$$

Proof. We prove (a).

- Suppose that $y_{i}=0$ for any $1 \leq i \leq k$. Then we have

$$
G_{2} G_{2}^{T}=\left[\begin{array}{cccc}
0 & 0 & \ldots & 0 \\
0 & & & \\
\vdots & & G G^{T} & \\
0 & & &
\end{array}\right]
$$

Thus $\operatorname{rank}\left(G_{2} G_{2}^{T}\right)=\operatorname{rank}\left(G G^{T}\right)=k-\ell$ since $h(C)=\ell$. The dimension of $\operatorname{Hull}\left(C_{2}\right)=h\left(C_{2}\right)=(k+1)-\operatorname{rank}\left(G_{2} G_{2}^{T}\right)=(k+1)-(k-\ell)=\ell+1$.

- Suppose $y_{i} \neq 0$ for some $1 \leq i \leq k$. By row operations of $G_{2}, G_{2}$ is transformed into $G_{2}^{\prime}$ given by

$$
G_{2}^{\prime}=\left[\begin{array}{cc|ccc}
1 & 1 & x_{1} & \ldots & x_{n} \\
\hline 1 & 0 & & & \\
0 & 0 & & & \\
\vdots & \vdots & & G^{\prime} & \\
0 & 0 & & &
\end{array}\right]
$$

where $\left\langle G^{\prime}\right\rangle=\langle G\rangle$. Furthermore,

$$
G_{2}^{\prime}\left(G_{2}^{\prime}\right)^{T}=\left[\begin{array}{c|ccc}
0 & 0 & \ldots & 0 \\
\hline 0 & & & \\
\vdots & & X & \\
0 & &
\end{array}\right]
$$

where $X=(10 \ldots 0)^{T}(10 \ldots 0)+G^{\prime}\left(G^{\prime}\right)^{T}$. Since $\operatorname{rank}(A+B) \leq \operatorname{rank}(A)$ $+\operatorname{rank}(B)$, we have $\operatorname{rank}(X) \leq \operatorname{rank}\left((10 \ldots 0)^{T}(10 \ldots 0)\right)+\operatorname{rank}\left(G^{\prime}\left(G^{\prime}\right)^{T}\right)$ $=1+\operatorname{rank}\left(G^{\prime}\left(G^{\prime}\right)^{T}\right)$. Noting that $(10 \ldots 0)^{T}(10 \ldots 0)$ affects only the top row of $G^{\prime}\left(G^{\prime}\right)^{T}$, we know that $\operatorname{rank}(X)$ decreases by at most one. So, $\operatorname{rank}(X)$ is $\operatorname{rank}\left(G^{\prime}\left(G^{\prime}\right)^{T}\right), \operatorname{rank}\left(G^{\prime}\left(G^{\prime}\right)^{T}\right)-1$, or $\operatorname{rank}\left(G^{\prime}\left(G^{\prime}\right)^{T}\right)+1$.
Thus $\operatorname{rank}\left(G_{2}\left(G_{2}\right)^{T}\right)=\operatorname{rank}\left(G_{2}^{\prime}\left(G_{2}^{\prime}\right)^{T}\right)=\operatorname{rank}(X)$ is $\operatorname{rank}\left(G^{\prime}\left(G^{\prime}\right)^{T}\right)=$ $k-\ell, \operatorname{rank}\left(G^{\prime}\left(G^{\prime}\right)^{T}\right)-1=k-\ell-1$, or $\operatorname{rank}\left(G^{\prime}\left(G^{\prime}\right)^{T}\right)+1=k-\ell+1$. Hence the dimension of $\operatorname{Hull}\left(C_{2}\right)$ is $k+1-(k-\ell)=\ell+1, k+1-$ $(k-\ell-1)=\ell+2$, or $k+1-(k-\ell+1)=\ell$.

We prove (b). It is straightforward to check that $G_{2} H_{2}^{T}=0$ by the definition of $y_{i}$ 's and $z_{j}$ 's. Because the rank of $H_{2}$ is $(n-k)+1$ and the dimension of the dual of $C_{2}$ is $(n+2)-(k+1)=n-k+1$, we see that $H_{2}$ is a parity check matrix for $C_{2}$.

We note that Constructions II and III are basically the same construction but we distinguish them in order to guess the hull dimension of the built-up code. We remark that Theorem 11 reproves the original building-up construction of binary self-dual codes [15] where $n$ is even and $k=n / 2=\ell$.

Harada [11] gave a construction of binary LCD $[n+2, k+1]$ codes from a given binary LCD $[n, k]$ code. We generalize this in the following theorem.

By modifying the proof of Theorem [1, we can construct $[n+2, k+1]$ linear codes with the same hull dimension as that of a given $[n, k]$ linear code.

Theorem 3. Let $C$ be a binary linear $[n, k]$ code. Suppose that the dimension of $\operatorname{Hull}(C)$ is $\ell$, where $0 \leq \ell \leq k$. Let $G$ be a generator matrix for $C$ and $H$ a parity check matrix for $C$.

Suppose that $\mathbf{x}=\left(x_{1}, x_{2}, \ldots, x_{n}\right) \in G F(2)^{n}$ satisfies $\mathbf{x} \cdot \mathbf{x}=0$. Let $y_{i}=\mathbf{x} \cdot \mathbf{r}_{i}$ for $1 \leq i \leq k$ where $\mathbf{r}_{i}$ is the ith row of $G$ and $z_{j}=\mathbf{x} \cdot \mathbf{s}_{j}$ for $1 \leq j \leq n-k$ where $\mathbf{s}_{j}$ is the $j$ th row of $H$. The following matrix

$$
G_{3}=\left[\begin{array}{cc|ccc}
1 & 0 & x_{1} & \ldots & x_{n} \\
\hline y_{1} & y_{1} & & \mathbf{r}_{1} & \\
y_{2} & y_{2} & & \mathbf{r}_{2} & \\
\vdots & \vdots & & \vdots & \\
y_{k} & y_{k} & & \mathbf{r}_{k} &
\end{array}\right]
$$

generates an $[n+2, k+1]$ linear code $C_{3}$ with $h\left(C_{3}\right)=\ell$. This is called Construction IV. A parity check matrix $H_{3}$ for $C_{3}$ is given by

$$
H_{3}=\left[\begin{array}{cc|ccc}
0 & 1 & x_{1} & \ldots & x_{n} \\
\hline z_{1} & z_{1} & & \mathbf{s}_{1} & \\
z_{2} & z_{2} & & \mathbf{S}_{2} & \\
\vdots & \vdots & & \vdots & \\
z_{n-k} & z_{n-k} & & \mathbf{s}_{n-k} &
\end{array}\right]
$$

Proof. The proof is almost the same as that of Theorem 1. It is straightforward to see that $G_{3} H_{3}^{T}=0$ and $\operatorname{rank}\left(H_{3}\right)=1+(n-k)$ which implies that $H_{3}$ is a parity check matrix for $C_{3}$. We compute $\operatorname{rank}\left(G_{3} G_{3}^{T}\right)$ as follows.

$$
G_{3} G_{3}^{T}=\left[\begin{array}{cccc}
1 & 0 & \ldots & 0 \\
0 & & & \\
\vdots & & G G^{T} & \\
0 & & &
\end{array}\right]
$$

Thus rank $\left(G_{3} G_{3}^{T}\right)=1+\operatorname{rank}\left(G G^{T}\right)=1+k-h(C)=1+k-\ell$ since $h(C)=\ell$. Now $h\left(C_{3}\right)=(k+1)-\operatorname{rank}\left(G_{3} G_{3}^{T}\right)=(k+1)-(1+k-\ell)=\ell$ as desired.

We can estimate the minimum distance $d\left(C_{i}\right)(i=1,2,3)$ for Constructions I-IV as follows.

Theorem 4. Let $C$ be a binary linear $[n, k]$ code. Let $\mathbf{x}=\left(x_{1}, x_{2}, \ldots, x_{n}\right) \in$ $G F(2)^{n}$. Then we have the following.
(i) The minimum distance $d\left(C_{i}\right)(i=1,3)$ is $\min \{d(C)$, weight $(\mathbf{x}+C)+1\}$ or $\min \{d(C)+2$, weight $(\mathbf{x}+C)+1\}$ for Constructions I and IV.
(ii) $d\left(C_{2}\right)$ is $\min \{d(C)$, weight $(\mathbf{x}+C)+2\}$ for Construction II and $\min \{d(C)+$ 1, weight $(\mathbf{x}+C)+1\}$, or $\min \{d(C)+1$, weight $(\mathbf{x}+C)+2\}$ for Construction III.

Proof. We prove statement (i). Let $i=1,3$. Let $S_{1}$ be the code spanned by all the rows of $G_{i}$ except for the top row. Then $C_{i}$ is the disjoint union of $S_{1}$ and $(10 \mathbf{x})+S_{1}$. Thus $d\left(C_{i}\right)$ is the minimum of $d\left(S_{1}\right)$ and weight $\left((10 \mathbf{x})+S_{1}\right)$. We note that $d\left(S_{1}\right)$ is $d(C)$ or $d(C)+2$ and that weight $\left((10 \mathbf{x})+S_{1}\right)=1+$ weight $(\mathbf{x}+C)$ ). Hence we obtain (i). Similarly, we can prove (ii), whose proof is omitted.

The following is straightforward since $d(C) \leq \rho(C)$, where $\rho(C)$ is the covering radius of $C$.

Corollary 1. Let $C$ be a binary linear $[n, k]$ code with its covering radius $\rho(C)$. Then the minimum distance $d\left(C_{i}\right)(i=1,2,3)$ for Constructions I-IV satisfies $\min \{d(C)$, weight $(\mathbf{x}+C)+1\} \leq d\left(C_{i}\right) \leq \rho(C)+2$.

Remark 1. Although we have considered Constructions I-IV for linear codes over $G F(2)$, it is easy to see that the same Constructions I-IV in Theorems 1-3 hold for linear codes over $G F(q)$, where $q=2^{r}$ for any integer $r \geq 1$. If $q$ is odd, then a slight modification of Constructions I-IV based on the building-up construction for self-dual codes over $G F(q)$ [16],17] will give results similar to Theorems 1-3.

## 4 Some interesting optimal linear codes

We display some interesting optimal linear codes with hull dimensions 2 and 3 from a linear code with hull dimension 1.

Start with an $h_{1}$-optimal $[10,6,3]$ code $C$ whose generator matrix $G$ is given below.

$$
G=\left[\begin{array}{l}
1000000101 \\
0100001001 \\
0010001110 \\
0001000110 \\
0000101010 \\
0000011100
\end{array}\right]
$$

Example 1. Let us take $\mathbf{x}=(0000011000)$ with $G$ by Construction III to get an $h_{2}$-optimal $[12,7,3]$ linear code $C^{\prime}$ with $h=2$. Its generator matrix $G^{\prime}$ is written as follows.

$$
G^{\prime}=\left[\begin{array}{c|c}
11 & 0000011000 \\
\hline 00 & 10000000101 \\
10 & 0100001001 \\
10 & 0010001110 \\
00 & 0001000110 \\
10 & 0000101010 \\
00 & 0000011100
\end{array}\right] \sim\left[\begin{array}{l}
100000101010 \\
010000101110 \\
001000000101 \\
000100100011 \\
000010100100 \\
000001000110 \\
000000011100
\end{array}\right]
$$

By Proposition [1, we can obtain a $[[12,5,3 ; 3]]_{2}$ EAQECC from $C^{\prime}$.
Example 2. Let us take $\mathbf{x}=(1111110011)$ with $G$ above by Construction III to get an optimal $[12,7,4]$ linear code $C^{\prime \prime}$ with $h=3$. Its generator matrix $G^{\prime \prime}$ is written in standard from after row operations.

$$
G^{\prime \prime}=\left[\begin{array}{c|c}
11 & 111110011 \\
\hline 00 & 10000000101 \\
00 & 0100001001 \\
00 & 0010001110 \\
00 & 0001000110 \\
00 & 0000101010 \\
10 & 0000011100
\end{array}\right] \sim\left[\begin{array}{l}
1000000011100 \\
010000001101 \\
001000011001 \\
000100010101 \\
000010001110 \\
00001011010 \\
000000110110
\end{array}\right]
$$

By Proposition [1, we can obtain a $[[12,5,3 ; 3]]_{2}$ EAQECC from $C^{\prime \prime}$.
By exhaustive search, we have checked that there are up to equivalence exactly two optimal $[12,7,4]$ codes. One of them is the above $[12,7,4]$ code $C^{\prime \prime}$ with $h=3$. The other is a $[12,7,4]$ code [14] with $h=1$ whose weight distribution is $[\langle 0,1\rangle,\langle 4,38\rangle,\langle 6,52\rangle,\langle 8,33\rangle,\langle 10,4\rangle]$. This code gives a $[[12,6,4 ; 4]]_{2}$ EAQECC by Proposition 1.

## 5 Optimal linear codes with several hulls and the construction of EAQECC

We construct several optimal linear codes of lengths up to 13 with $h=i$ ( $i=1,2,3,4,5$ ) from a given linear code of a fixed hull dimension $h$. Tables $1,3,5,7,9$ display best possible minimum distances of linear $[n, k]$ codes from hull dimensions 1 to 5 . The upper bounds for the minimum distances in the tables are from Grassl's table [10] by taking not the hull dimension into account and by brute force search. Each cell in each table denotes the
highest minimum distance $d(n, k)$ for given $n, k$, and $h=i$ together with the superscripts referring to Constructions I to IV and o meaning that the codes are optimal. For $n=12$, we apply Constructions I, II, and/or III. For $n=13$, we apply Constructions I, III, and/or IV. All computations were done by Magma [2]. To save the space, we post whole information about the codes in Tables 1,3,5,7,9 in the author's website [14] and list most generator matrices for $n=12$ and 13 in this paper.

Tables $2,4,6,8,10$ display associated $[[n, k, d ; c]]_{2}$ EAQECC based on Proposition 1 and Tables $1,3,5,7,9$. In other words, we obtain $[[n, k-h, d ; n-$ $k-h]]_{2}$ EAQECC from binary $[n, k, d]$ codes with hull dimension $h$.

Example 3. Fix the hull dimension $h=1$. For any $n$ with $k$ such that $1 \leq k \leq n \leq 11$ and $n=12$ with $k(1 \leq k \leq 4)$, we ran exhaustive search to get optimal or $h_{1}$-optimal codes. We note that there is an optimal $[12,5,4]$ code with $h=1$ from Magma database.

For $n=12$ with $k \geq 6$, we apply Constructions I and II to all the LCD codes of length 10 and dimension $k-1$ displayed in [12]. More precisely, we construct optimal $[12,6,4]$ and $[12,9,2]$ codes by Construction I. Similarly, we construct optimal $[12,7,4]$ and $[12,8,3]$ codes by Construction III.

Let $n=13$. We construct optimal $[13,4,6],[13,5,5],[13,6,4],[13,7,4]$, $[13,8,3],[13,10,2],[13,11,2]$ codes by Construction I. We also construct an $h_{1}$-optimal $[13,3,6]$ code by Construction I, which is justified by the nonexistence of $[13,3,7]$ codes with $h=1$ using exhaustive search. Similarly, we construct an $h_{1}$-optimal $[13,9,2]$ code by Construction I, which is justified by the non-existence of $[13,9,3]$ codes with $h=1$ using exhaustive search.

In what follows, $G_{n, k, d}^{i}$ refers to a generator matrix for a binary $[n, k, d]$ code $C_{n, k, d}^{i}$ with $h=i$ and the highest minimum distance $d=d(n, k)$.

- $n=12$ with $h=1$

$$
G_{12,6,4}^{1}=\left[\begin{array}{l}
1011111011100 \\
111000100010 \\
110100100001 \\
110010000011 \\
000001100101 \\
000000001111
\end{array}\right], G_{12,7,4}^{1}=\left[\begin{array}{l}
111111011000 \\
101000100100 \\
100100100010 \\
100010000110 \\
100001100001 \\
100000010101 \\
100000001011
\end{array}\right]
$$

$$
\begin{gathered}
G_{12,8,3}^{1}=\left[\begin{array}{l}
111111000011 \\
101000000011 \\
000100000101 \\
000010000110 \\
10000100011 \\
000000100011 \\
100000010101 \\
000000001111
\end{array}\right], G_{12,9,2}^{1}=\left[\begin{array}{l}
1010000000007 \\
111000000001 \\
000100000001 \\
000001000001 \\
000000100001 \\
00000010001 \\
000000001001 \\
000000000101 \\
000000000011
\end{array}\right] \\
G_{12,10,1}^{1}=\left[\begin{array}{l}
1010000000000 \\
01000000000 \\
000010000000 \\
00000100000 \\
000000100000 \\
00000001000 \\
000000001000 \\
000000000100 \\
000000000000 \\
00000000001
\end{array}\right], G_{12,11,2}^{1}=\left[\begin{array}{l}
10000000001 \\
010000000001 \\
001000000001 \\
000100000001 \\
000010000001 \\
000001000001 \\
000000100001 \\
000000010001 \\
000000001001 \\
000000000101 \\
000000000011
\end{array}\right]
\end{gathered}
$$

- $n=13$ with $h=1$

$$
\begin{gathered}
G_{13,3,6}^{1}=\left[\begin{array}{l}
1010110001100 \\
1111100011001 \\
0000011111001
\end{array}\right], G_{13,4,6}^{1}=\left[\begin{array}{l}
10011101111100 \\
1110011011011 \\
0001011101001 \\
0000111110010
\end{array}\right] \\
G_{13,5,6}^{1}=\left[\begin{array}{l}
1011101111000 \\
1110001100100 \\
1101001010010 \\
1100101010101 \\
0000011100011
\end{array}\right], G_{13,6,4}^{1}=\left[\begin{array}{l}
1010011000000 \\
1110000010111 \\
0001000101010 \\
0000000110011 \\
1100010001110 \\
1100001110101
\end{array}\right] \\
G_{13,7,4}^{1}=\left[\begin{array}{l}
1010011000000 \\
1110000001011 \\
000100010111 \\
0000100101001 \\
1100010001101 \\
1100001101010 \\
000000001110
\end{array}\right], G_{13,8,3}^{1}=\left[\begin{array}{l}
1010011000000 \\
110000001011 \\
0001000001101 \\
000000000110 \\
1100010001110 \\
1100001001111 \\
000000100101 \\
0000000010011
\end{array}\right]
\end{gathered}
$$



$$
G_{13,11,2}^{1}=\left[\begin{array}{l}
10100000000000 \\
1110000000001 \\
0001000000001 \\
000010000001 \\
0000010000001 \\
0000001000001 \\
000000010001 \\
0000000010001 \\
0000000001001 \\
0000000000101 \\
0000000000011
\end{array}\right]
$$

| $n / k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 0 |  |  |  |  |  |  |  |  |  |  |  |
| 2 | 2 | 0 |  |  |  |  |  |  |  |  |  |  |
| 3 | 2 | 1 | 0 |  |  |  |  |  |  |  |  |  |
| 4 | 4 | 1 | 2 | 0 |  |  |  |  |  |  |  |  |
| 5 | 4 | 3 | 2 | 1 | 0 |  |  |  |  |  |  |  |
| 6 | 6 | 3 | 2 | 1 | 2 | 0 |  |  |  |  |  |  |
| 7 | 6 | 4 | 3 | 2 | 2 | 1 | 0 |  |  |  |  |  |
| 8 | 8 | 4 | 4 | 3 | 2 | 1 | 2 | 0 |  |  |  |  |
| 9 | 8 | 5 | 4 | 3 | 3 | 2 | 2 | 1 | 0 |  |  |  |
| 10 | 10 | 5 | 5 | 4 | 4 | 3 | 2 | 1 | 2 | 0 |  |  |
| 11 | 10 | 7 | 6 | 5 | 4 | 3 | 3 | 2 | 2 | 1 | 0 |  |
| 12 | $12^{o}$ | 7 | $6^{o}$ | 5 | $4^{o}$ | $4^{I, o}$ | $4^{I I I, o}$ | $3^{I I I, o}$ | $2^{I, o}$ | 1 | $2^{o}$ | 0 |
| 13 | 12 | $8^{o}$ | $6^{I}$ | $6^{I, o}$ | $5^{I, o}$ | $4^{I, o}$ | $4^{I, o}$ | $3^{I}$ | $2^{I}$ | $2^{I, o}$ | $2^{I, o}$ | 1 |

Table 1: Each cell refers to the highest minimum distance $d(n, k)$ for $n \leq 13$ when $h=1$, and examples of corresponding generator matrices $G_{12, k, d}^{1}(6 \leq$ $k \leq 11)$ and $G_{13, k, d}^{1}(3 \leq k \leq 11)$

Example 4. Fix the hull dimension $h=2$. For any $n$ with $k$ such that $1 \leq k \leq n \leq 11$ and $n=12$ with $k(1 \leq k \leq 4)$, we ran exhaustive search to get optimal or $h_{2}$-optimal codes.

| $n / k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 2 | $(2 ; 0)$ |  |  |  |  |  |  |  |  |  |  |  |
| 3 | $(2 ; 1)$ | $(1 ; 0)$ |  |  |  |  |  |  |  |  |  |  |
| 4 | $(4 ; 2)$ | $(1 ; 1)$ | $(2 ; 0)$ |  |  |  |  |  |  |  |  |  |
| 5 | $(4 ; 3)$ | $(3 ; 2)$ | $(2 ; 1)$ | $(1 ; 0)$ |  |  |  |  |  |  |  |  |
| 6 | $(6 ; 4)$ | $(3 ; 3)$ | $(2 ; 2)$ | $(1 ; 1)$ | $(2 ; 0)$ |  |  |  |  |  |  |  |
| 7 | $(6 ; 5)$ | $(4 ; 4)$ | $(3 ; 3)$ | $(2 ; 2)$ | $(2 ; 1)$ | $(1 ; 0)$ |  |  |  |  |  |  |
| 8 | $(8 ; 6)$ | $(4 ; 5)$ | $(4 ; 4)$ | $(3 ; 3)$ | $(2 ; 2)$ | $(1 ; 1)$ | $(2 ; 0)$ |  |  |  |  |  |
| 9 | $(8 ; 7)$ | $(5 ; 6)$ | $(4 ; 5)$ | $(3 ; 4)$ | $(3 ; 3)$ | $(2 ; 2)$ | $(2 ; 1)$ | $(1 ; 0)$ |  |  |  |  |
| 10 | $(10 ; 8)$ | $(5 ; 7)$ | $5 ; 6)$ | $(4 ; 5)$ | $(4 ; 4)$ | $(3 ; 3)$ | $(2 ; 2)$ | $(1 ; 1)$ | $(2 ; 0)$ |  |  |  |
| 11 | $(10 ; 9)$ | $(7 ; 8)$ | $6 ; 7)$ | $(5 ; 6)$ | $(4 ; 5)$ | $(3 ; 4)$ | $(3 ; 3)$ | $(2 ; 2)$ | $2 ; 1)$ | $(1 ; 0)$ |  |  |
| 12 | $(12 ; 10)$ | $(7 ; 9)$ | $(6 ; 8)$ | $(5 ; 7)$ | $(4 ; 6)$ | $(4 ; 5)$ | $(4 ; 4)$ | $(3 ; 3)$ | $(2 ; 2)$ | $(1 ; 1)$ | $(2 ; 0)$ |  |
| 13 | $(12 ; 11)$ | $(8 ; 10)$ | $(6 ; 9)$ | $(6 ; 8)$ | $(5 ; 7)$ | $(4 ; 6)(4 ; 5)$ | $(3 ; 4)$ | $(2 ; 3)$ | $(2 ; 2)$ | $(2 ; 1)$ | $(1 ; 0)$ |  |

Table 2: $[[n, k, d ; c]]_{2}$ EAQECC with $(d ; c)$ for $n \leq 13$ when $h=1$ based on Proposition 1 and Table 1

For $n=12$ with $k \geq 5$, we apply Constructions I, II or III to LCD codes or linear codes with $h=1$ of length 10 and dimension $k-1$. More precisely, we construct optimal $[12,5,4],[12,6,4],[12,8,3],[12,9,2],[12,10,2]$ codes from $[10,4,4],[10,5,3],[10,7,2],[10,8,2],[10,9,1]$ codes with $h=0$ respectively by Construction III. On the other hand, we also construct a $[12,7,3]$ code from a $[10,6,3]$ code with $h=0$ by Construction III. By exhaustive search, we check that it is $h_{2}$-optimal.

Let $n=13$. We construct optimal $[13,3,7],[13,4,6],[13,5,5],[13,6,4]$, [13, 8, 4] codes by Construction III from LCD codes of length 11 and dimensions $k=2,3,4,5,7$ respectively. We also construct an optimal [13, 7, 4] code from a linear $[11,6,3]$ code with $h=1$ by Construction I. For $k=2,10,11$, it is easy to construct directly optimal or $h_{2}$-optimal $[13, k]$ codes. For $k=9$, it is known that there exist an optimal $[13,9,3]$ code with $h=2$ by Magma database.

- $n=12$ with $h=2$

$$
G_{12,3,5}^{2}=\left[\begin{array}{l}
100011011010 \\
010100011100 \\
001111100000
\end{array}\right], G_{12,4,6}^{2}=\left[\begin{array}{l}
100010101011 \\
010011010101 \\
001011100110 \\
000111111000
\end{array}\right]
$$

$$
\begin{gathered}
G_{12,5,4}^{2}=\left[\begin{array}{l}
1100001100000 \\
101000111001 \\
000100011101 \\
100010101011 \\
100001001111
\end{array}\right], G_{12,6,4}^{2}=\left[\begin{array}{l}
111111001100 \\
101000100010 \\
100100100001 \\
10000000011 \\
00001100101 \\
000000001111
\end{array}\right] \\
G_{12,7,3}^{2}=\left[\begin{array}{l}
1111011000000 \\
101000001011 \\
100100001101 \\
000010000110 \\
100001000101 \\
100000101111 \\
00000001111
\end{array}\right], G_{12,8,3}^{2}=\left[\begin{array}{l}
11111000110 \\
001000000011 \\
000100000101 \\
100010000110 \\
100001000111 \\
100000100011 \\
10000010101 \\
00000001111
\end{array}\right] \\
G_{12,9,2}^{2}=\left[\begin{array}{l}
110110000000 \\
00100000011 \\
100000000011 \\
100010000010 \\
000000000011 \\
00000100011 \\
000000010011 \\
00000001011 \\
000000000111
\end{array}\right], G_{12,10,2}^{2}=\left[\begin{array}{ll}
111111111111 \\
101000000000 \\
10001000000 \\
100001000000 \\
100000100000 \\
100000010000 \\
100000001000 \\
100000000100 \\
10000000010 \\
100000000001
\end{array}\right]
\end{gathered}
$$

- $n=13$ with $h=2$

$$
\begin{aligned}
& G_{13,3,7}^{2}=\left[\begin{array}{l}
1111000110110 \\
0011100011101 \\
1000011111101
\end{array}\right], G_{13,4,6}^{2}=\left[\begin{array}{l}
1101110111000 \\
1010011011011 \\
0001011101001 \\
0000111110010
\end{array}\right] \\
& G_{13,5,5}^{2}=\left[\begin{array}{l}
11011111011000 \\
1010001100100 \\
1001001010010 \\
1000101010101 \\
0000011100011
\end{array}\right], G_{13,6,4}^{2}=\left[\begin{array}{l}
1111000000000 \\
1010000010111 \\
1001000101010 \\
0000100110011 \\
0000010001110 \\
0000001110101
\end{array}\right] \\
& G_{13,7,4}^{2}=\left[\begin{array}{l}
1011000111011 \\
1110000001010 \\
1101000010010 \\
0000100011100 \\
110001000100 \\
1100001010100 \\
1100000111000
\end{array}\right], G_{13,8,4}^{2}=\left[\begin{array}{l}
1111101000110 \\
001000001011 \\
0001000001101 \\
1000100000110 \\
000000001110 \\
1000001001111 \\
1000000100101 \\
100000010011
\end{array}\right]
\end{aligned}
$$

$$
G_{13,9,3}^{2}=\left[\begin{array}{l}
1000000000011 \\
0100000010100 \\
0010000010111 \\
0001000010001 \\
0000100010010 \\
000010000101 \\
0000001000110 \\
0000000110011 \\
0000000001111
\end{array}\right], G_{13,10,2}^{2}=\left[\begin{array}{l}
1000000000100 \\
0100000000010 \\
0010000000001 \\
000100000001 \\
0000100000001 \\
000001000001 \\
0000000000001 \\
0000000100001 \\
0000000010001 \\
000000001001
\end{array}\right]
$$

| $n / k$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 2 | 0 |  |  |  |  |  |  |  |  |  |
| 3 | 0 | 0 |  |  |  |  |  |  |  |  |
| 4 | 2 | 0 | 0 |  |  |  |  |  |  |  |
| 5 | 2 | 1 | 0 | 0 |  |  |  |  |  |  |
| 6 | 4 | 3 | 2 | 0 | 0 |  |  |  |  |  |
| 7 | 4 | 3 | 2 | 1 | 0 | 0 |  |  |  |  |
| 8 | 4 | 4 | 3 | 2 | 2 | 0 | 0 |  |  |  |
| 9 | 4 | 4 | 4 | 3 | 2 | 1 | 0 | 0 |  |  |
| 10 | 6 | 4 | 4 | 3 | 3 | 2 | 2 | 0 | 0 | 0 |
| 11 | 6 | 5 | 4 | 4 | 4 | 3 | 2 | 1 | 0 | 0 |
| 12 | $8^{o}$ | 5 | $6^{o}$ | $4^{I I I, o}$ | $4^{I I I, o}$ | $3^{I I I}$ | $3^{I I I, o}$ | $2^{I I I, o}$ | $2^{I I I, o}$ | 0 |
| 13 | $8^{o}$ | $7^{I I I, o}$ | $6^{I I I, o}$ | $5^{I I I, o}$ | $4^{I I I, o}$ | $4^{I, h_{1}, o}$ | $4^{I I I, o}$ | $3^{o}$ | $2^{o}$ | 1 |

Table 3: Each cell refers to the highest minimum distance $d(n, k)$ for $n \leq 13$ when $h=2$, and examples of corresponding generator matrices $G_{12, k, d}^{2}(3 \leq$ $k \leq 10)$ and $G_{13, k, d}^{2}(3 \leq k \leq 10)$

Example 5. Fix the hull dimension $h=3$.
Since $h=3$, the code length $n$ should be at least 6 . If $n-k \leq 2$, then there does not exist an $[n, k]$ code with $h=3$. If $k=3$, we use the optimal minimum distances of self-orthogonal $[n, 3]$ codes from [3].

For any $n$ with $k$ such that $3 \leq k \leq n \leq 11$ and $n=12$ with $k=3,4$, we ran exhaustive search to obtain optimal or $h_{3}$-optimal codes.

Using Construction III, we construct an optimal [12,5,4] code with $h=3$ from a $[10,4,4]$ code, and an $h_{3}$-optimal $[12,6,4]$ code from a $[10,5,3]$ code with $h=2$. We further construct an optimal $[12,7,4]$ code from a $[10,6,3]$ code with $h=1$ by Construction III.

We construct $h_{3}$-optimal $[12,8,2]$ and $[12,9,2]$ codes from $[10,7,2]$ and $[10,8,2]$ codes with $h=2$, respectively by Construction I. This is justified by exhaustive search that there are no $[12,6,4],[12,8,3]$ codes.

| $n / k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 4 | $(2 ; 0)$ |  |  |  |  |  |  |  |  |  |
| 5 | $(2 ; 1)$ | $(1 ; 0)$ |  |  |  |  |  |  |  |  |
| 6 | $(4 ; 2)$ | $(3 ; 1)$ | $(2 ; 0)$ |  |  |  |  |  |  |  |
| 7 | $(4 ; 3)$ | $(3 ; 2)$ | $(2 ; 1)$ | $(1 ; 0)$ |  |  |  |  |  |  |
| 8 | $(4 ; 4)$ | $(4 ; 3)$ | $(3 ; 2)$ | $(2 ; 1)$ | $(2 ; 0)$ |  |  |  |  |  |
| 9 | $(4 ; 5)(4 ; 4)$ | $(4 ; 3)$ | $(3 ; 2)$ | $(2 ; 1)$ | $(1 ; 0)$ |  |  |  |  |  |
| 10 | $(6 ; 6)$ | $(4 ; 5)$ | $(4 ; 4)$ | $(3 ; 3)$ | $(3 ; 2)$ | $(2 ; 1)$ | $(2 ; 0)$ | 0 | 0 |  |
| 11 | $(6 ; 7)$ | $(5 ; 6)$ | $(4 ; 5)(4 ; 4)$ | $(4 ; 3)$ | $(3 ; 2)$ | $(2 ; 1)$ | $(1 ; 0)$ |  |  |  |
| 12 | $(8 ; 8)(5 ; 7)$ | $(6 ; 6)$ | $(4 ; 5)$ | $(4 ; 4)(3 ; 3)$ | $(3 ; 2)$ | $(2 ; 1)$ | $(2 ; 0)$ |  |  |  |
| 13 | $(8 ; 9)(7 ; 8)$ | $(6 ; 7)$ | $(5 ; 6)$ | $(4 ; 5)$ | $(4 ; 4)$ | $(4 ; 3)$ | $(3 ; 2)$ | $(2 ; 1)(1 ; 0)$ |  |  |

Table 4: $[[n, k, d ; c]]_{2}$ EAQECC with $(d ; c)$ for $n \leq 13$ when $h=2$ based on Proposition 1 and Table 3

For $n=13$, we construct $[13,4,4],[13,5,4],[13,6,4],[13,7,3],[13,8,2]$, $[13,9,2]$ codes with $h=3$ from [11, $k]$ codes with $h=3(3 \leq k \leq 8)$ by Construction IV. Similarly we construct $[13,4,5],[13,5,4],[13,6,4],[13,7,4]$, $[13,8,3],[13,10,2]$ codes with $h=3$ from $[11, k]$ codes with $h=2(3 \leq k \leq$ $7, k=9)$ by Construction I.

- $n=12$ with $h=3$

$$
\begin{gathered}
G_{12,4,4}^{3}=\left[\begin{array}{l}
100001111000 \\
010010110000 \\
001011010000 \\
000111100000
\end{array}\right], G_{12,5,4}^{3}=\left[\begin{array}{l}
100100101100 \\
010100011110 \\
001100110010 \\
000010110100 \\
000001111000
\end{array}\right] \\
G_{12,6,4}^{3}=\left[\begin{array}{l}
111100000000 \\
10100001001 \\
10010001110 \\
000010010110 \\
00000101010 \\
000000111100
\end{array}\right], G_{12,7,4}^{3}=\left[\begin{array}{l}
111111111111 \\
101000000101 \\
100100001001 \\
000010001110 \\
100001000110 \\
100000101010 \\
100000011100
\end{array}\right] \\
G_{12,8,2}^{3}=\left[\begin{array}{l}
111100000000 \\
10100000001 \\
100100000010 \\
000010000100 \\
00000100100 \\
000000100100 \\
000000010100 \\
000000001100
\end{array}\right], G_{12,9,2}^{3}=\left[\begin{array}{l}
1100000000000 \\
001000000001 \\
000100000010 \\
000010000010 \\
000001000010 \\
000000100010 \\
000000010010 \\
00000001010 \\
000000000110
\end{array}\right]
\end{gathered}
$$

- $n=13$ with $h=3$

$$
\begin{gathered}
G_{13,4,5}^{3}=\left[\begin{array}{l}
1011111000000 \\
0010001101101 \\
000101001110 \\
1100111110000
\end{array}\right], G_{13,5,4}^{3}=\left[\begin{array}{l}
10111000000000 \\
1110001100100 \\
1101001011000 \\
1100101101000 \\
0000011110000
\end{array}\right] \\
G_{13,6,4}^{3}=\left[\begin{array}{l}
1001110000000 \\
0010000011101 \\
1101000110010 \\
1100100101100 \\
1100010110100 \\
0000001111000
\end{array}\right], G_{13,7,4}^{3}=\left[\begin{array}{l}
1010110000000 \\
1110000010101 \\
0001000011001 \\
1100100001110 \\
1100010010110 \\
0000001011010 \\
0000000111100
\end{array}\right] \\
G_{13,8,3}^{3}=\left[\begin{array}{l}
1010000110000 \\
1110000001101 \\
000100000101 \\
0000100001001 \\
0000010001110 \\
0000001000110 \\
1100000101010 \\
1100000011100
\end{array}\right], G_{13,9,2}^{3}=\left[\begin{array}{l}
10110000000000 \\
1110000000111 \\
1101000000011 \\
0000100000010000110 \\
0000001000100 \\
0000000100100 \\
0000000010100 \\
0000000001100
\end{array}\right] \\
G_{13,10,2}^{3}=\left[\begin{array}{l}
1011111111100 \\
111000000001 \\
1101000000010 \\
110010000000 \\
1100010000000 \\
110000000000 \\
110000000000 \\
110000000000 \\
110000000000
\end{array}\right]
\end{gathered}
$$

Example 6. Fix the hull dimension $h=4$.
Since $h=4$, the code length $n$ should be at least 8 . If $n-k \leq 3$, then there does not exist a $[n, k]$ code with $h=4$. If $k=4$, we use the optimal minimum distances of self-orthogonal $[n, 4]$ codes from [3].

For any $n$ with $k$ such that $4 \leq k \leq n \leq 11$ and $n=12$ with $k=4$, we ran exhaustive search to obtain optimal or $h_{4}$-optimal codes.

We construct optimal $[12,5,4]$ and $[12,6,4]$ codes with $h=4$ from [10, 4, 4] and $[10,5,3]$ codes with $h=3$ respectively by Construction I.

We construct $h_{4}$-optimal $[12,7,3],[12,8,2]$ codes with $h=4$ from $[10,6,1]$, [10, 7, 2] codes respectively with $h=3$ by Construction I. This is justified by exhaustive checking that there are no $[12,7,4]$ and $[12,8,3]$ codes with $h=4$.

| $n / k$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 6 | 2 | 0 | 0 | 0 |  |  |  |  |
| 7 | 4 | 3 | 0 | 0 | 0 |  |  |  |
| 8 | 4 | 3 | 2 | 0 | 0 | 0 |  |  |
| 9 | 4 | 4 | 3 | 2 | 0 | 0 |  |  |
| 10 | 4 | 4 | 4 | 2 | 2 | 0 | 0 |  |
| 11 | 4 | 4 | 4 | 3 | 2 | 2 | 0 |  |
| 12 | $6^{o}$ | 4 | $4^{I I I, o}$ | $4^{I I I, o}$ | $4^{I I I, h_{1}, o}$ | $2^{I I I}$ | $2^{I I, o}$ |  |
| 13 | 6 | $\geq 5^{I}$ | $\geq 4^{I, I V}$ | $4^{I, I V, o}$ | $4^{I, o}$ | $\geq 3^{I}$ | $\geq 2^{I V}$ | $2^{I, o}$ |

Table 5: Each cell refers to the highest minimum distance $d(n, k)$ for $n \leq 13$ when $h=3$, and examples of corresponding generator matrices $G_{12, k, d}^{3}(4 \leq$ $k \leq 9)$ and $G_{13, k, d}^{3}(4 \leq k \leq 10)$

| $n / k \mid$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 6 | $(2 ; 0)$ |  |  |  |  |  |  |  |
| 7 | $(4 ; 1)$ | $(3 ; 0)$ |  |  |  |  |  |  |
| 8 | $(4 ; 2)$ | $(3 ; 1)$ | $(2 ; 0)$ |  |  |  |  |  |
| 9 | $(4 ; 3)$ | $(4 ; 2)$ | $(3 ; 1)$ | $(2 ; 0)$ |  |  |  |  |
| 10 | $(4 ; 4)$ | $(4 ; 3)$ | $(4 ; 2)$ | $(2 ; 1)$ | $(2 ; 0)$ |  |  |  |
| 11 | $(4 ; 5)$ | $(4 ; 4)$ | $(4 ; 3)$ | $(3 ; 2)$ | $(2 ; 1)$ | $(2 ; 0)$ |  |  |
| 12 | $(6 ; 6)$ | $(4 ; 5)$ | $(4 ; 4)$ | $(4 ; 3)$ | $(4 ; 2)$ | $(2 ; 1)$ | $(2 ; 0)$ |  |
| 13 | $(6 ; 7)$ | $(\geq 5 ; 6)$ | $(\geq 4 ; 5)$ | $(4 ; 4)$ | $(4 ; 3)$ | $(\geq 3 ; 2)$ | $(\geq 2 ; 1)$ | $(2 ; 0)$ |

Table 6: $[[n, k, d ; c]]_{2}$ EAQECC with $(d ; c)$ for $n \leq 13$ when $h=3$ based on Proposition 1 and Table 5

For $n=13$, we obtain $[13,5,4],[13,6,4],[13,7,3],[13,8,2]$ codes with $h=$ 4 from [11, $k$ ] codes with $h=4(4 \leq k \leq 7)$ by Construction IV. Similarly, we construct $[13,4,4],[13,5,4],[13,6,4],[13,7,3],[13,8,2],[13,9,2]$ codes with $h=4$ from $[11, k]$ codes with $h=3(3 \leq k \leq 8)$ by Construction I.

- $n=12$ with $h=4$

$$
G_{12,5,4}^{4}=\left[\begin{array}{l}
101000000011 \\
001000011110 \\
000100101100 \\
000010110100 \\
000001111000
\end{array}\right], G_{12,6,4}^{4}=\left[\begin{array}{l}
101011000000 \\
111000011001 \\
000100001110 \\
110010010110 \\
110001011010 \\
000000111100
\end{array}\right]
$$

$$
G_{12,7,3}^{4}=\left[\begin{array}{l}
100001111001 \\
11000110100 \\
001001110101 \\
000100110010 \\
000011100010 \\
000001011111 \\
000000100011
\end{array}\right], G_{12,8,2}^{4}=\left[\begin{array}{l}
1010000000000 \\
11000000001 \\
000100000010 \\
000010000100 \\
000001000100 \\
000000100100 \\
000000010100 \\
000000001100
\end{array}\right]
$$

- $n=13$ with $h=4$

$$
\begin{gathered}
G_{13,5,4}^{4}=\left[\begin{array}{l}
0010000111100 \\
0001001011000 \\
0000101101000 \\
000001110000
\end{array}\right], G_{13,6,4}^{4}=\left[\begin{array}{l}
1010110000000 \\
110000110010 \\
0001000011100 \\
1100100101100 \\
100010110100 \\
0000001111000
\end{array}\right] \\
G_{13,7,3}^{4}=\left[\begin{array}{l}
1011100000000 \\
1110000010001 \\
1101000010010 \\
1100100011100 \\
0000010001100 \\
0000001010100 \\
0000000111000
\end{array}\right], G_{13,8,2}^{4}=\left[\begin{array}{l}
1010000000000 \\
1110000000010 \\
0001000000100 \\
0000100001000 \\
0000010001000 \\
0000001001000 \\
000000101000 \\
0000000011000
\end{array}\right]
\end{gathered}
$$

$$
G_{13,9,2}^{4}=\left[\begin{array}{l}
1010000000000 \\
1110000000111 \\
000100000011 \\
0000100000101 \\
0000010000110 \\
0000001000100 \\
0000000100100 \\
000000010100 \\
0000000001100
\end{array}\right]
$$

| $n / k \mid$ | 4 | 5 | 6 | 7 | 8 | 9 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 8 | 4 | 0 | 0 | 0 | 0 | 0 |
| 9 | 4 | 2 | 0 | 0 | 0 | 0 |
| 10 | 4 | 4 | 2 | 0 | 0 | 0 |
| 11 | 4 | 4 | 3 | 2 | 0 | 0 |
| 12 | 4 | $4^{I, o}$ | $4^{I, o}$ | $3^{I}$ | $2^{I}$ | 0 |
| 13 | 4 | $\geq 4^{I, I T}$ | $4^{I, I V, o} \geq 3^{I, I T}$ | $\geq 2^{I, I T}$ | $\geq 2^{I}$ |  |

Table 7: Each cell refers to the highest minimum distance $d(n, k)$ for $n \leq 13$ when $h=4$, and examples of corresponding generator matrices $G_{12, k, d}^{4}(5 \leq$ $k \leq 8)$ and $G_{13, k, d}^{4}(5 \leq k \leq 9)$

| $n / k$ | 0 | 1 | 2 | 3 | 4 | 5 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 8 | $(4 ; 0)$ |  |  |  |  |  |
| 9 | $(4 ; 1)$ | $(2 ; 0)$ |  |  |  |  |
| 10 | $(4 ; 2)$ | $(4 ; 1)$ | $(2 ; 0)$ |  |  |  |
| 11 | $(4 ; 3)$ | $(4 ; 2)$ | $(3 ; 1)$ | $(2 ; 0)$ |  |  |
| 12 | $(4 ; 4)$ | $(4 ; 3)$ | $(4 ; 2)$ | $(3 ; 1)$ | $(2 ; 0)$ |  |
| 13 | $(4 ; 5)$ | $(\geq 4 ; 4)$ | $(4 ; 3)$ | $(\geq 3 ; 2)$ | $(\geq 2 ; 1)$ | $(\geq 2 ; 0)$ |

Table 8: [[ $n, k, d ; c]]_{2}$ EAQECC with $(d ; c)$ for $n \leq 13$ when $h=4$ based on Proposition 1 and Table 7

Example 7. Fix the hull dimension $h=5$.
Since $h=5$, the code length $n$ should be at least 10. If $n-k \leq 4$, then there does not exist a $[n, k]$ code with $h=5$. If $k=5$, we use the optimal minimum distances of self-orthogonal $[n, 5]$ codes from [3].

For $n=10,11$ with $k=4,5$, we ran exhaustive search to obtain optimal or $h_{5}$-optimal codes.

It is well known that there is a self-orthogonal $[12,5,4]$ code [3], which is optimal. We construct $h_{5}$-optimal $[12,6,3]$ and $[12,7,3]$ codes from $[10,5,4]$ and $[10,6,2]$ codes with $h=4$ by Construction I. This is justified by exhaustive checking that there are no $[12,6,4]$ and $[12,7,4]$ codes with $h=5$.

For $n=13$, we obtain $[13,6,3]$, $[13,7,3]$ codes with $h=5$ from $[11, k]$ codes with $h=5(5 \leq k \leq 6)$ by Construction IV. Similarly we construct $[13,6,4],[13,7,3],[13,8,2]$ codes from $[11, k]$ codes with $h=4(5 \leq k \leq 7)$ by Construction I.

Although we cannot construct a $[13,7,4]$ code with $h=5$ from Constructions I and IV, we observe the following. Using the unique self-dual $[12,6,4]$ code $B_{12}$ [22] with the below generator matrix $G_{12}$, we obtain an optimal $[13,7,4]$ code $C_{13,7,4}$ with the below generator matrix $G_{13,7,4}$ by augmenting a coset leader $\mathbf{v}=(000000010101)$ to $B_{12}$ because the covering radius of $B_{12}$ is 3 . We show that $h\left(C_{13,7,4}\right)=5$ in what follows. The top row $\mathbf{r}_{1}=(1 \mid \mathbf{v})$ of $G_{13,7,4}$ is orthogonal to only five rows $\mathbf{r}_{2}, \mathbf{r}_{3}, \mathbf{r}_{5}, \mathbf{r}_{6}, \mathbf{r}_{7}$ of $G_{13}$. Therefore, the hull of $C_{13,7,4}$ consists of these five rows, resulting in $h\left(C_{13,7,4}\right)=5$.

$$
G_{12}=\left[\begin{array}{l}
111100000000 \\
001111000000 \\
000011110000 \\
000000111100 \\
000000001111 \\
010101010101
\end{array}\right], \quad G_{13,7,4}=\left[\begin{array}{c|c}
1 & 000000010101 \\
0 & 1111000000000 \\
0 & 001111000000 \\
0 & 000011110000 \\
0 & 000000111100 \\
0 & 000000001111 \\
0 & 010101010101
\end{array}\right]
$$

- $n=12$ with $h=5$

$$
\begin{gathered}
G_{12,5,4}^{5}=\left[\begin{array}{l}
101000000110 \\
00100001100 \\
110100101100 \\
110010110100 \\
000001111000
\end{array}\right], G_{12,6,3}^{5}=\left[\begin{array}{l}
1010110000000 \\
111000001111 \\
000100010111 \\
110010011001 \\
110001011010 \\
000000111100
\end{array}\right] \\
G_{12,7,3}^{5}=\left[\begin{array}{l}
100001011101 \\
1100000000011 \\
001001110000 \\
000100000111 \\
000010111000 \\
000001011011 \\
000000110011
\end{array}\right]
\end{gathered}
$$

- $n=13$ with $h=5$

$$
\begin{gathered}
G_{13,5,4}^{5}=\left[\begin{array}{l}
10000000110100 \\
0100101011100 \\
001000011000 \\
0001100110000 \\
0000011110000
\end{array}\right], G_{13,6,4}^{5}=\left[\begin{array}{l}
1011100000000 \\
1110000011101 \\
1101000011110 \\
1100100101100 \\
0000010110100 \\
0000001111000
\end{array}\right] \\
G_{13,7,4}^{5}=\left[\begin{array}{l}
10000000010101 \\
0111100000000 \\
000111100000 \\
000001110000 \\
000000011100 \\
0000000001111 \\
0010101010101
\end{array}\right], G_{13,8,2}^{5}=\left[\begin{array}{l}
1010000000000 \\
1110000000111 \\
0001000000011 \\
0000100000101 \\
0000010000110 \\
0000001001000 \\
000000101000 \\
0000000011000
\end{array}\right]
\end{gathered}
$$

There are not many known $[[n, k, d ; c]$ ] EAQECC when $n \leq 13$. We compare our results with some known EAQECC in Table 11. In fact, the parameters in boldface in the third column of the table are better than the currently known parameters from [21, [27].

| $n / k$ | 5 | 6 | 7 | 8 |
| :---: | :---: | :---: | :---: | :---: |
| 10 | 2 | 0 | 0 | 0 |
| 11 | 4 | 3 | 0 | 0 |
| 12 | $4^{o}$ | $3^{I}$ | $3^{I}$ | 0 |
| 13 | 4 | $4^{I, o}$ | $4^{a}$ | $\geq 2^{I}$ |

Table 9: Each cell refers to the highest minimum distance $d(n, k)$ for $n \leq 13$ when $h=5$, and examples of corresponding generator matrices $G_{12, k, d}^{5}(5 \leq$ $k \leq 7)$ and $G_{13, k, d}^{5}(5 \leq k \leq 8)$

| $n / k$ | 0 | 1 | 2 | 3 |
| :---: | :---: | :---: | :---: | :---: |
| 10 | $(2 ; 0)$ |  |  |  |
| 11 | $(4 ; 1)$ | $(3 ; 0)$ |  |  |
| 12 | $(4 ; 2)$ | $(3 ; 1)$ | $(3 ; 0)$ |  |
| 13 | $(4 ; 3)$ | $(4 ; 2)$ | $(4 ; 1)$ | $(\geq 2 ; 0)$ |

Table 10: $[[n, k, d ; c]]_{2}$ EAQECC with $(d ; c)$ for $n \leq 13$ when $h=5$ based on Proposition 1 and Table 9

## 6 Conclusion

This paper has introduced a systematic and efficient method to construct binary optimal or possibly optimal $[n, k]$ codes of lengths up to 13 with respect to hull dimensions 1-5. These codes are used to construct EAQECC with the best known parameters.

The complexity of Constructions I-IV mainly depends on the binary vectors $\mathbf{x}$ of length $n$, whose cardinality is at most $2^{n-1}$ due to the parity of $\mathbf{x}$. This complexity can be reduced if we consider the standard generator matrix $G$ in Theorems 1,2 , and 3 . Since $n \leq 13$ we need at most $2^{12}=4,096$ vectors for $\mathbf{x}$. As we prefer to keep a non-standard generator matrix to distinguish Constructions I-IV, we have run all possibilities for $\mathbf{x}$ and have checked the equivalence of codes by Magma. Using our linux machine Intel ( $R$ ) Xeon ( $R$ ) CPU E3-1225 V2 @ 3.20 GHz , calculations for Theorems $1-3$ were performed within ten minutes while some exhaustive search took more than two weeks. As future work, it is worth considering similar constructions for other finite fields and rings.

| currently known EAQECC | Ref | our related EAQECC | Tables |
| :---: | :---: | :---: | :---: |
| $[[9,1,3 ; 1]]_{2}$ | 21] | $[[\mathbf{9}, \mathbf{2}, \mathbf{3} ; \mathbf{1}]]_{2}$ | Table 6 |
| $[[12,1,7 ; 9]]_{2}$ | 27) | $[[12,1,7 ; 9]]_{2}$ | Table 2 |
| $[[12,3,5 ; 7]]_{2}$ | 27] | $[[12,3,5 ; 7]]_{2}$ | Table 2 |
| $[[12,4,4 ; 6]]_{2}$ | 27) | $[[12,4,4 ; 6]]_{2},[[12,2,6 ; 6]]_{2}$ | Table 2, Table 4 |
| $[[12,5,3 ; 5]]_{2}$ | (27) | $[[\mathbf{1 2}, \mathbf{5}, \mathbf{4} ; \mathbf{5}]]_{2},[[12,3,4 ; 5]]_{2}$ | Table 2, Table 4 |
| $[[13,7,3 ; 4]]_{2}$ | (27) | $[[13,7,3 ; 4]]_{2},[[13,5,4 ; 4]]_{2}$ | Table 2, Table 4 |
| $[[13,3,5 ; 8]]_{2}$ | [27] | $[[\mathbf{1 3}, \mathbf{3}, \mathbf{6} ; \mathbf{8}]]_{2},[[13,1,7 ; 8]]_{2}$ | Table 2, Table 4 |

Table 11: Comparison with some known EAQECC

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## Declarations

Conflict of interest The author declares that he has no conflict of interest regarding the publication of this paper.

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