# Efficient numerical simulations based on an explicit group approach for the time fractional advection-diffusion reaction equation 

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#### Abstract

The time-fractional advection-diffusion reaction equation (TFADRE) is a fundamental mathematical model because of its key role in describing various processes such as oil reservoir simulations, COVID-19 transmission, mass and energy transport, and global weather production. One of the prominent issues with time fractional differential equations is the design of efficient and stable computational schemes for fast and accurate numerical simulations. We construct in this paper, a simple and yet efficient modified fractional explicit group method (MFEGM) for solving the two-dimensional TFADRE with suitable initial and boundary conditions. The proposed method is established using a difference scheme based on L1 discretization in temporal direction and central difference approximations with double spacing in spatial direction. For comparison purposes, the Crank-Nicolson finite difference method (CNFDM) is proposed. The stability and convergence of the presented methods are theoretically proved and numerically affirmed. We illustrate the computational efficiency of the


[^0]MFEGM by comparing it to the CNFDM for four numerical examples including fractional diffusion and fractional advection-diffusion models. The numerical results show that the MFEGM is capable of reducing iteration count and CPU timing effectively compared to the CNFDM, making it well-suited to time fractional diffusion equations.

Keywords Fractional advection-diffusion reaction equation • Explicit group approach •
Finite difference scheme • Stability analysis • Numerical experiments
Mathematics Subject Classification 35R11 • 65N06 • 65N12

## 1 Introduction

Fractional calculus (FC) is considered one of the top topics in applied mathematics, with applications emerging in various scientific and engineering disciplines. FC extends the order of the derivatives and integrals from the set of positive integers to the set of real and even complex numbers. In contrast to classical derivatives, fractional-order derivatives have the non-local property, making them an excellent choice for capturing memory and hereditary properties in a variety of real-world phenomena. Generally speaking, the applications of FC have advanced remarkably in various areas, including but not limited to physics, biology, medicine, signal processing, hydrology, rheology, control systems, robotics, economics, anomalous transport, electronic circuits, inventory control problem and viscoelasticity (Sun et al. 2018; Tarasov 2019; Chávez-Vázquez et al. 2022; Radwan et al. 2021; Rahaman et al. 2020). In line with that, several types of fractional derivatives of different kernels such as Riemann-Liouville, Caputo, Grunwald-Letnikov, Hadamard, Caputo-Fabrizio and Atangana-Baleanu have been proposed for suitable simulation of various complex physical phenomena. A major class of FC is the so-called fractional partial differential equation (FPDE), where the time and/or space derivative in the classical PDE is replaced by its fractional counterpart. In recent years, FPDEs have found a wide range of applications in fluid mechanics (Hamid et al. 2022), financial markets (Taghipour and Aminikhah 2022), material science (Yang et al. 2022), neuronal dynamics (Zou et al. 2022), wave propagation, solar particle transport (Vieira et al. 2022) and quantum mechanics ( Zu and Yu 2022 ), to name a few. For extra information about the analysis and applications of FPDEs, the reader can consult (Ara et al. 2018; Goswami et al. 2019; Nnolim 2022; Chakraverty et al. 2022; Salama et al. 2021; Ali et al. 2022; Khan et al. 2023).

The solutions of mathematical models in terms of FPDEs are mainly investigated by either approximate analytical or numerical methods. However, the analytic treatment of FPDEs is not a straight-forward process due to the complexity and non-local nature of fractional derivatives. Exact analytical solutions to FPDEs are scarce in the literature, and in some cases, they are too complicated to be useful. As a result, numerical methods have become indispensable and are resorted to by many researchers, scholars and practitioners for handling real-life fractional models. In this article, we study the two-dimensional time-fractional advection-diffusion reaction equation (TFADRE) to account for its numerical solutions. The said model problem shall be introduced in the subsequent section. Next, we survey some recent numerical treatments of TFADRE. The authors in Cui (2015) and Ren and Wang (2017) applied, respectively, a compact exponential difference scheme and an extrapolated compact difference method for solving the one-dimensional TFADRE with variable coefficients. Wang and Wen 2020 constructed a compact exponential difference scheme to solve
the general one-dimensional multi-term TFADRE with non-smooth solutions. They extended their approach to the two-dimensional case in the same study. Haq et al. (2020) developed an implicit meshless spectral algorithm for solving the one-dimensional TFADRE with variable coefficients. Hafez et al. (2020) proposed two Galerkin spectral numerical schemes for the solutions of the d-dimensional TFADRE. They pointed out that their schemes are well-suited to time FPDEs since they preserve the singularity of the solution. Toprakseven (2021) utilized the classical L1 discretization in time and a weak Galerkin element in space to establish a weak Galerkin finite element method for the $d$-dimensional ( $d \in\{1,2\}$ ) TFADRE. Kumar and Zeidan (2021) scrutinized the one-dimensional non-linear TFADRE in which the fractional temporal derivative is defined in the Atangana-Baleanu sense. The authors employed a Legendre operational matrix for the fractional derivative along with a Legendre spectral method to account for the numerical solutions to the mentioned problem. Khalighi et al. (2021) established a hybrid algorithm based on the boundary element method combined with the Chebyshev operational matrix approach for handling the two-dimensional multi-order TFADRE. Li and Wang (2021) established a local discontinuous Galerkin (LDG) method for the one-dimensional TFADRE basted on the L1 scheme for uniform and non-uniform temporal meshes and the LDG method for spatial uniform mesh. Recently, Jannelli (2022) extended an adaptive time-stepping numerical scheme proposed in Jannelli (2020) for fractional ordinary differential equations to solve the one-dimensional TFADRE. Afterwards, Zhang and Feng (2022) developed and analyzed a virtual element method to scrutinize the two-dimensional TFADRE with non-smooth solutions. Shortly after, Ngondiep (Ngondiep 2022) constructed an unconditionally stable two-level fourth-order numerical scheme for the one-dimensional TFADRE. More recently, Naeem et al. (2022) generated approximate analytical solutions for the one-dimensional TFADRE based on the combination of the Elzaki transform and the homotopy perturbation method. Other numerical investigations for fractional diffusion models can be found in Roul and Rohil (2022), Hang et al. (2023) and Chen et al. (2020). Most of the aforementioned methods are designed for one-dimensional problems, creating a research gap for developing new efficient computational algorithms for higher-dimensional problems. Herein lies the first motivation for our work.

Unlike classical advection diffusion reaction equations, the numerical solution process of TFADRE is a challenging computational task. The reason for this is the historical dependence and universal mutuality of fractional differential operators. Owing to the non-local property, the solution values at the current time level require the solution information at all previous time levels, which results in more sophisticated computations even for low-dimensional fractional problems (Salama and Ali 2019, 2020). The computational effort may grow dramatically as the mesh size increases and the integration proceeds forward. To surpass such major limitations and reduce the corresponding computational complexity, a number of computational algorithms were suggested, such as the short memory principle (Singh et al. 2021), preconditioning (Sunarto et al. 2022), parallel computing (Wu et al. 2021) and multigrid methods (Pan et al. 2021). The historical dependence feature of fractional derivatives permits accurate and robust modeling of real-life phenomena but leads to severe computational drawbacks in many cases. In fact, the development of efficient numerical methods that produce accurate simulations and maintain low computational effort is one of the open problems in fractional calculus; for example, see (Diethelm et al. 2022). This is the study's second motivation.

Explicit group iterative methods are another viable option for designing efficient numerical schemes for FPDEs. They utilize small, fixed-size groups of points on the standard or rotated grids and can be combined with other numerical methods, including finite difference method, finite element method and collocation method. The main feature of explicit group methods is that they can reduce the spectral radius of the iteration matrix and diminish the computational
cost effectively, making them suitable for non-local fractional problems. In recent years, a number of articles have appeared that dealt successfully with FPDEs using explicit group methods. They have been used for efficient computations of accurate numerical solutions to the two-dimensional fractional diffusion equation (Salama et al. 2022a), two-dimensional fractional cable equation (Salama and Abd Hamid 2020; Khan et al. 2021), two-dimensional fractional reaction diffusion equation (Abdi et al. 2021a) and two-dimensional fractional advection diffusion equation (Salama et al. 2022b; Balasim et al. 2017). For further applications and analysis of explicit group methods, we refer the reader to Salama et al. (2022c), Abdi et al. (2021b), Ali et al. (2021), Khan et al. (2020). In this work, we focus on a general class of time FPDEs (i.e., TFADRE) that constitutes previously researched classes as a special case. This is the third motivation for our work.

Motivated by the previous discussions, the main aim of this paper is to present the modified fractional explicit group method (MFEGM) based on finite difference approximations for handling the more general TFADRE. The MFEGM is an easy-to-implement solution procedure and can be executed on parallel computers. This represents an additional advantage of the selected method. For the sake of comparison, we first simulate the model problem using a fully discrete scheme, namely Crank-Nicolson finite difference method (CNFDM) based on the L1 formula in time and central difference approximations in space. The theoretical investigations of the proposed numerical schemes rely on three factors: stability, consistency and convergence. Different fractional order exponents are utilized to exhibit the dynamics of the model problem. The numerical findings and comparison with the CNFDM show the computational efficiency and reliability of the proposed MFEGM, which can be extended to solve other FPDEs. Moreover, the obtained results are particularly useful for time consuming practical applications in engineering and scientific experimentation. In summary, the interest is directed toward the following three items:

- Detailed description of CNFDM and MFEGM for the TFADRE with Dirichlet-type initial-boundary conditions.
- Illustrations of the stability and convergence of the proposed methods.
- Several numerical examples and discussions that validate our considerations.

The rest of the paper is organized as follows. In Sect. 2, we introduce the two-dimensional TFADRE as a mathematical model of real-world phenomena. The detailed description of the CNFDM and MFEGM is given in Sect. 3 and 4, respectively. In Sect. 5, we analyze the stability of the proposed solution schemes using the Fourier method, followed by convergence in Sect.6. Section7 includes a number of numerical simulations as well as graphical and tabulated results demonstrating the accuracy and efficiency of our numerical schemes. Finally, the paper is concisely concluded in Sect. 8 .

## 2 The mathematical model

The diffusion phenomenon, where molecules or particles are transferred from the region of higher concentration to the region of lower concentration, is one of the basic processes in nature with a wide range of applications in science and engineering. In the context of continuous random walk, the diffusion process can be interpreted by the Brownian motion where the mean square displacement is given by the following relation:

$$
\left\langle(\Delta r)^{2}\right\rangle=2 d D t
$$

where $\Delta r$ is the displacement of the Brownian molecule in some interval $t, d$ and $D$ denote the spatial dimension and the diffusion coefficient, respectively. In many physical complex
phenomena, the mean square displacement revels an anomalous behaviour and does not follow the above linear temporal relation. Such an anomalous diffusion process can be modeled with remarkable success under the framework of fractional derivatives (Oliveira et al. 2019). The diffusion phenomenon can be associated with advection and reaction processes to form a general model in terms of the advection-diffusion reaction equation. Classical advectiondiffusion reaction equations are an important class of PDEs, and many papers on numerical schemes for solving them have been published (Savović et al. 2022; Zhang and Ge 2021; Lin et al. 2020; Chowdhury and Kumar 2020; Singh et al. 2019; Araya et al. 2020). The twodimensional advection-diffusion reaction equation with Dirichlet-type boundary conditions can be written as follows:

$$
\left\{\begin{array}{l}
P_{t}=P_{x x}+P_{y y}-P_{x}-P_{y}-P+f(x, y, t), \quad(x, y, t) \in \Omega \times(0, T], \\
P(x, y, 0)=\theta(x, y), \quad(x, y) \in \Omega \cup \partial \Omega, \\
P(0, y, t)=\theta_{1}(y, t), \quad P(L, y, t)=\theta_{2}(y, t), \\
P(x, 0, t)=\theta_{3}(x, t), \quad P(x, L, t)=\theta_{4}(x, t), \quad(x, y, t) \in \partial \Omega \times(0, T] .
\end{array}\right.
$$

On the other hand, the TFADRE is a variant of the classical advection-diffusion reaction equation in which the integer-order derivative is replaced by the Caputo fractional derivative. The fractional Caputo derivative is one of the most useful differential operators for dealing with real-world phenomena because it allows the implementation of initial and boundary conditions as those of the integer-order case. In recent years, TFADREs have received much attention and have been used to describe various real-world problems. They have been utilized to describe anomalous diffusion phenomena in complex and disordered systems. The usefulness of TFADRE has been verified in modeling various physical processes, such as oil reservoir simulations, transmission of COVID-19, transport of mass and energy and global weather production. Moreover, TFADRE has been employed for modeling option prices with success in financial markets. For these reasons, the solution process of the TFADRE is a hot topic in the field of applications.

In this study, we present and analyze a simple and yet efficient numerical scheme for the two-dimensional TFADRE reads as,

$$
\left\{\begin{align*}
&{ }_{0}^{C} D_{t}^{\alpha} P= K_{1} P_{x x}+K_{2} P_{y y}-V_{1} P_{x}-V_{2} P_{y}-M P  \tag{1}\\
&+f(x, y, t), \quad(x, y, t) \in \Omega \times(0, T] \\
& P(x, y, 0)=\theta(x, y), \quad(x, y) \in \Omega \cup \partial \Omega \\
& P(0, y, t)=\theta_{1}(y, t), \quad P(L, y, t)=\theta_{2}(y, t), \\
& P(x, 0, t)=\theta_{3}(x, t), \quad P(x, L, t)=\theta_{4}(x, t), \quad(x, y, t) \in \partial \Omega \times(0, T]
\end{align*}\right.
$$

where $\Omega=[0, L]^{2}$ is the solution domain and $\partial \Omega$ is it is boundary. $P(x, y, t)$ is the unknown function representing a concentration, a population size, chemical species or option prices. $K_{1}$ and $K_{2}$ are the diffusion coefficients, $V_{1}$ and $V_{2}$ are the advection coefficients and $M$ is the reaction coefficient. $\alpha \in(0,1), f(x, y, t)$ is the source term, and ${ }_{0}^{C} D_{t}^{\alpha} P$ is the Caputo time fractional derivative of order $\alpha$ defined as,

$$
{ }_{0}^{C} D_{t}^{\alpha} P(x, y, t)=\left\{\begin{array}{l}
\frac{1}{\Gamma(1-\alpha)} \int_{0}^{t}(t-\xi)^{-\alpha} \frac{\partial P(x, y, \xi)}{\partial \xi} d \xi, 0<\alpha<1, \\
\frac{\partial P(x, y, t)}{\partial t}, \alpha=1 .
\end{array}\right.
$$

From the definition of the Caputo fractional derivative, it can be observed that the exact solution of the model problem (1)-(3) may have weak regularity at the initial time. Without loss of this constraint, we assume the considered problem has a unique and sufficiently smooth exact solution.

## 3 Crank Nicolson finite difference method (CNFDM)

In this section, we propose a fully discrete numerical scheme for solving the model problem (1). To this end, the solution domain is discretized as $\Omega_{h \tau}=\Omega_{h} \Omega_{\tau}$, where $\Omega_{h}=\left\{x_{i}=i h_{x}, i=0,1, \ldots, N_{x}, y_{j}=j h_{y}, j=0,1, \ldots, N_{y}\right\}$ and $\Omega_{\tau}=\left\{t_{n}=\right.$ $\left.n \tau, n=0,1, \ldots, N_{t}\right\}$. Here, $h_{x}=L / N_{x}, h_{y}=L / N_{y}$ and $\tau=T / N_{t}$ represent the step sizes in the spatial and temporal directions, respectively. Assume that $P(x, y, t) \in C^{2}(\Omega \times \partial \Omega)$ and $P_{i, j}^{n}=P\left(x_{i}, y_{j}, t_{n}\right)$, the first and second order spatial derivatives can be discretized as follows:

$$
\begin{align*}
\left.\frac{\partial^{2} P}{\partial x^{2}}\right|_{i, j} ^{n+1 / 2}= & \frac{1}{2}\left[\frac{P_{i+1, j}^{n+1}-2 P_{i, j}^{n+1}+P_{i-1, j}^{n+1}}{\left(h_{x}\right)^{2}}+\frac{P_{i+1, j}^{n}-2 P_{i, j}^{n}+P_{i-1, j}^{n}}{\left(h_{x}\right)^{2}}\right] \\
& \left.+O\left(\tau^{2}+\left(h_{x}\right)^{2}\right)+\left(h_{y}\right)^{2}\right),  \tag{4}\\
\left.\frac{\partial^{2} P}{\partial y^{2}}\right|_{i, j} ^{n+1 / 2}= & \frac{1}{2}\left[\frac{P_{i, j+1}^{n+1}-2 P_{i, j}^{n+1}+P_{i, j-1}^{n+1}}{\left(h_{y}\right)^{2}}+\frac{P_{i, j+1}^{n}-2 P_{i, j}^{n}+P_{i, j-1}^{n}}{\left(h_{y}\right)^{2}}\right] \\
& \left.+O\left(\tau^{2}+\left(h_{x}\right)^{2}\right)+\left(h_{y}\right)^{2}\right),  \tag{5}\\
\left.\frac{\partial P}{\partial x}\right|_{i, j} ^{n+1 / 2}= & \left.\frac{1}{2}\left[\frac{P_{i+1, j}^{n+1}-P_{i-1, j}^{n+1}}{2 h_{x}}+\frac{P_{i+1, j}^{n}-P_{i-1, j}^{n}}{2 h_{x}}\right]+O\left(\tau^{2}+\left(h_{x}\right)^{2}\right)+\left(h_{y}\right)^{2}\right),  \tag{6}\\
\left.\frac{\partial P}{\partial y}\right|_{i, j} ^{n+1 / 2}= & \left.\frac{1}{2}\left[\frac{P_{i, j+1}^{n+1}-P_{i, j-1}^{n+1}}{2 h_{y}}+\frac{P_{i, j+1}^{n}-P_{i, j-1}^{n}}{2 h_{x}}\right]+O\left(\tau^{2}+\left(h_{x}\right)^{2}\right)+\left(h_{y}\right)^{2}\right) . \tag{7}
\end{align*}
$$

The discretization of the Caputo time fractional derivative is described by the difference formula with the following form:

$$
\begin{align*}
\left.\frac{\partial^{\alpha} P}{\partial t^{\alpha}}\right|_{i, j} ^{n+1 / 2}= & \sigma\left[W_{1} P_{i, j}^{n}+\sum_{m=1}^{n-1}\left(W_{n-m+1}-W_{n-m}\right) P_{i, j}^{m}-W_{n} P_{i, j}^{0}+\frac{\left(P_{i, j}^{n+1}-P_{i, j}^{n}\right)}{2^{1-\alpha}}\right] \\
& +O\left(\tau^{2-\alpha}\right), \tag{8}
\end{align*}
$$

where

$$
\sigma=\frac{1}{\Gamma(2-\alpha) \tau^{\alpha}}, \quad W_{n}=\left((n+1 / 2)^{1-\alpha}-(n-1 / 2)^{1-\alpha}\right) .
$$

The model problem (1) is now approximated as follows,

$$
\begin{gather*}
\left.\frac{\partial^{\alpha} P}{\partial t^{\alpha}}\right|_{i, j} ^{n+1 / 2}=\left.K_{1} \frac{\partial^{2} P}{\partial x^{2}}\right|_{i, j} ^{n+1 / 2}+\left.K_{2} \frac{\partial^{2} P}{\partial y^{2}}\right|_{i, j} ^{n+1 / 2}-\left.V_{1} \frac{\partial P}{\partial x}\right|_{i, j} ^{n+1 / 2}-\left.V_{2} \frac{\partial P}{\partial y}\right|_{i, j} ^{n+1 / 2} \\
-M P_{i, j}^{n+1 / 2}+f_{i, j}^{k+1 / 2}+O\left(\tau^{2-\alpha}+h_{x}^{2}+h_{y}^{2}\right) . \tag{9}
\end{gather*}
$$

Substituting (4)-(8) into (1), the following CNFDM is obtained:

$$
\begin{aligned}
\mathcal{P}_{i, j}^{n+1}= & \frac{1}{0.5^{1-\alpha}+\kappa+2 A_{1}+2 A_{2}}\left[\left(A_{1}-B_{1}\right)\left(\mathcal{P}_{i+1, j}^{n+1}+\mathcal{P}_{i+1, j}^{n}\right)\right. \\
& +\left(A_{1}+B_{1}\right)\left(\mathcal{P}_{i-1, j}^{n+1}+\mathcal{P}_{i-1, j}^{n}\right)+\left(A_{2}-B_{2}\right)\left(\mathcal{P}_{i, j+1}^{n+1}+\mathcal{P}_{i, j+1}^{n}\right) \\
& +\left(A_{2}+B_{2}\right)\left(\mathcal{P}_{i, j-1}^{n+1}+\mathcal{P}_{i, j-1}^{n}\right)+\left(0.5^{1-\alpha}-\kappa-W_{1}-2 A_{1}-2 A_{2}\right) \mathcal{P}_{i, j}^{n}
\end{aligned}
$$

$$
\begin{align*}
& \left.+\sum_{m=1}^{n-1}\left(W_{n-m}-W_{n-m+1}\right) \mathcal{P}_{i, j}^{m}+W_{n} \mathcal{P}_{i, j}^{0}+m f_{i, j}^{n+1 / 2}\right] \\
& i=1,2, \ldots, N_{x}-1, j=1,2, \ldots, N_{y}-1, n=0,1, \ldots, N_{t}-1 \tag{10}
\end{align*}
$$

with the following initial and boundary conditions

$$
\begin{aligned}
& \mathcal{P}_{i, j}^{0}=\theta\left(x_{i}, y_{j}\right), \quad i=0,1, \ldots, N_{x}, j=0,1, \ldots, N_{y}, \\
& \mathcal{P}_{0, j}^{n}=\theta_{1}\left(y_{j}, t_{n}\right), \quad \mathcal{P}_{N_{x}, j}^{n}=\theta_{2}\left(y_{j}, t_{n}\right), j=0,1, \ldots, N_{y} n=1,2, \ldots, N_{t}, \\
& \mathcal{P}_{i, 0}^{n}=\theta_{3}\left(x_{i}, t_{n}\right), \quad \mathcal{P}_{i, N_{y}}^{n}=\theta_{4}\left(x_{i}, t_{n}\right), i=0,1, \ldots, N_{x} n=1,2, \ldots, N_{t},
\end{aligned}
$$

where

$$
\begin{aligned}
& m=\tau^{\alpha} \Gamma(2-\alpha), \kappa=0.5 M \tau^{\alpha} \Gamma(2-\alpha), \\
& A_{1}=\frac{K_{1} m}{2 h_{x}^{2}}, A_{2}=\frac{K_{2} m}{2 h_{y}^{2}}, B_{1}=\frac{V_{1} m}{4 h_{x}}, B_{2}=\frac{V_{2} m}{4 h_{y}} .
\end{aligned}
$$

Here, $\mathcal{P}_{i, j}^{n}$ denotes the numerical approximation of $P_{i, j}^{n}$ after omitting the higher order terms. Figure 1 highlights the computational molecule of the CNFDM defined by (10). It is well known that the application of Eq. (10) to the points will result in large and sparse linear systems that can be solved using iterative techniques. Considering a point-wise iterative procedure, and at any time level $t_{n}$, Eq. (10) is utilized to iterate solutions at all points until a predetermined convergence criterion is met. Once the converged solutions are attained, they are used as an initial guess for the next time level $t_{n+1}$. The iterative process is terminated when the final time level is reached. In the next section, we propose a new group-wise iterative numerical scheme to simulate the model problem (1) with less computing effort.

## 4 Description of the modified fractional explicit group method (MFEGM)

In this section, the MFEGM is proposed for the solution process of the time fractional advection-diffusion-reaction equation (1). To explain the solution method, we consider a new computational mesh $\Omega_{2 h}=\left\{x_{i}=2 i h_{x}, i=0,1, \ldots, N_{x} ; y_{j}=2 j h_{y}, j=0,1, \ldots, N_{y}\right\}$, where $2 h_{x}=\frac{2 L}{N_{x}}$ and $2 h_{y}=\frac{2 L}{N_{y}}$. Applying Crank Nicolson difference scheme on the new mesh for approximating problem (1), the following expression is obtained:

$$
\begin{aligned}
& {\left[W_{1} P_{i, j}^{n}+\sum_{m=1}^{n-1}\left(W_{n-m+1}-W_{n-m}\right) P_{i, j}^{m}-W_{n} P_{i, j}^{0}+\sigma \frac{\left(P_{i, j}^{n+1}-P_{i, j}^{n}\right)}{2^{1-\alpha}}\right]} \\
& \quad=\frac{K_{1}}{2}\left[\frac{P_{i+2, j}^{n+1}-2 P_{i, j}^{n+1}+P_{i-2, j}^{n+1}}{4 h_{x}^{2}}+\frac{P_{i+2, j}^{n}-2 P_{i, j}^{n}+P_{i-2, j}^{n}}{4 h_{x}^{2}}\right] \\
& \quad+\frac{K_{2}}{2}\left[\frac{P_{i, j+2}^{n+1}-2 P_{i, j}^{n+1}+P_{i, j-2}^{n+1}}{4 h_{y}^{2}}+\frac{P_{i, j+2}^{n}-2 P_{i, j}^{n}+P_{i, j-2}^{n}}{4 h_{y}^{2}}\right] \\
& \quad-\frac{V_{1}}{2}\left[\frac{P_{i+2, j}^{n+1}-P_{i-2, j}^{n+1}}{4 h_{x}}+\frac{P_{i+2, j}^{n}-P_{i-2, j}^{n}}{4 h_{x}}\right]
\end{aligned}
$$



Fig. 1 Computational molecule of the CNFDM difference scheme with $N_{x}=N_{y}=10$

$$
\begin{align*}
& -\frac{V_{2}}{2}\left[\frac{P_{i, j+2}^{n+1}-P_{i, j-2}^{n+1}}{4 h_{y}}+\frac{P_{i, j+2}^{n}-P_{i, j-2}^{n}}{4 h_{y}}\right] \\
& -M\left(\frac{P_{i, j}^{n+1}+P_{i, j}^{n}}{2}\right)+f_{i, j}^{n+1 / 2}+O\left(\tau^{2-\alpha}+h_{x}^{2}+h_{y}^{2}\right) \tag{11}
\end{align*}
$$

Disregarding higher order terms and using $\mathcal{P}_{i, j}^{n}$ as an approximation to $P_{i, j}^{n}$, the following $2 h$-spaced difference scheme is attained:

$$
\begin{align*}
\mathcal{P}_{i, j}^{n+1}= & \frac{1}{0.5^{1-\alpha}+\kappa+2 G_{1}+2 G_{2}}\left[\left(G_{1}-H_{1}\right)\left(\mathcal{P}_{i+2, j}^{n+1}+\mathcal{P}_{i+2, j}^{n}\right)\right. \\
& +\left(G_{1}+H_{1}\right)\left(\mathcal{P}_{i-2, j}^{n+1}+\mathcal{P}_{i-2, j}^{n}\right)+\left(G_{2}-H_{2}\right)\left(\mathcal{P}_{i, j+2}^{n+1}+\mathcal{P}_{i, j+2}^{n}\right) \\
& +\left(G_{2}+H_{2}\right)\left(\mathcal{P}_{i, j-2}^{n+1}+\mathcal{P}_{i, j-2}^{n}\right)+\left(0.5^{1-\alpha}-\kappa-W_{1}-2 G_{1}-2 G_{2}\right) \mathcal{P}_{i, j}^{n} \\
& \left.+\sum_{m=1}^{n-1}\left(W_{n-m}-W_{n-m+1}\right) \mathcal{P}_{i, j}^{m}+W_{n} \mathcal{P}_{i, j}^{0}+m f_{i, j}^{n+1 / 2}\right] \\
& \quad i=2,4, \ldots, N_{x}-2, j=2,4, \ldots, N_{y}-2, n=0,1, \ldots, N_{t}-1 \tag{12}
\end{align*}
$$

where

$$
G_{1}=\frac{K_{1} m}{8 h_{x}^{2}}, G_{2}=\frac{K_{2} m}{8 h_{y}^{2}}, H_{1}=\frac{V_{1} m}{8 h_{x}}, H_{2}=\frac{V_{2} m}{8 h_{y}}
$$

Before proceeding to the formulation of the MFEGM, we need a new skewed difference scheme for approximating the model problem (1). Such numerical scheme can be achieved by rotating the standard mesh $45^{\circ}$ clockwise and applying Taylor expansion on the resulted mesh as follows:

$$
\begin{align*}
& {\left[W_{1} P_{i, j}^{n}+\sum_{m=1}^{n-1}\left(W_{n-m+1}-W_{n-m}\right) P_{i, j}^{m}-W_{n} P_{i, j}^{0}+\sigma \frac{\left(P_{i, j}^{n+1}-P_{i, j}^{n}\right)}{2^{1-\alpha}}\right]} \\
& \quad= \\
& \quad \frac{K_{1}}{2}\left[\frac{P_{i+1, j-1}^{n+1}-2 P_{i, j}^{n+1}+P_{i-1, j+1}^{n+1}}{2 h_{x}^{2}}+\frac{P_{i+1, j-1}^{n}-2 P_{i, j}^{n}+P_{i-1, j+1}^{n}}{2 h_{x}^{2}}\right] \\
& \quad+\frac{K_{2}}{2}\left[\frac{P_{i+1, j+1}^{n+1}-2 P_{i, j}^{n+1}+P_{i-1, j-1}^{n+1}}{2 h_{y}^{2}}+\frac{P_{i+1, j+1}^{n}-2 P_{i, j}^{n}+P_{i-1, j-1}^{n}}{2 h_{y}^{2}}\right] \\
& \\
& \quad-\frac{V_{1}}{2}\left[\frac{P_{i+1, j-1}^{n+1}-P_{i-1, j+1}^{n+1}+P_{i+1, j+1}^{n+1}-P_{i-1, j-1}^{n+1}}{4 h_{x}}\right. \\
& \left.\quad+\frac{P_{i+1, j-1}^{n}-P_{i-1, j+1}^{n}+P_{i+1, j+1}^{n}-P_{i-1, j-1}^{n}}{4 h_{x}}\right] \\
&  \tag{13}\\
& \quad-\frac{V_{2}}{2}\left[\frac{P_{i+1, j+1}^{n+1}-P_{i-1, j-1}^{n+1}+P_{i-1, j+1}^{n+1}-P_{i+1, j-1}^{n+1}}{4 h_{y}}\right. \\
& \left.\quad+\frac{P_{i+1, j+1}^{n}-P_{i-1, j-1}^{n}+P_{i-1, j+1}^{n}-P_{i+1, j-1}^{n}}{4 h_{y}}\right] \\
& \\
& \quad-M\left(\frac{P_{i, j}^{n+1}+P_{i, j}^{n}}{2}\right)+f_{i, j}^{n+1 / 2}+O\left(\tau^{2-\alpha}+h_{x}^{2}+h_{y}^{2}\right) .
\end{align*}
$$

After simplification and omitting the small error terms, we obtain the following skewed difference scheme:

$$
\begin{align*}
\mathcal{P}_{i, j}^{n+1}= & \frac{1}{0.5^{1-\alpha}+\kappa+2 S_{1}+2 S_{2}}\left[\left(S_{1}-H_{1}-H_{2}\right)\left(\mathcal{P}_{i+1, j-1}^{n+1}+\mathcal{P}_{i+1, j-1}^{n}\right)\right. \\
& +\left(S_{1}+H_{1}-H_{2}\right)\left(\mathcal{P}_{i-1, j+1}^{n+1}+\mathcal{P}_{i-1, j+1}^{n}\right)+\left(S_{2}-H_{1}-H_{2}\right)\left(\mathcal{P}_{i+1, j+1}^{n+1}+\mathcal{P}_{i+1, j+1}^{n}\right) \\
& +\left(S_{2}+H_{1}+H_{2}\right)\left(\mathcal{P}_{i-1, j-1}^{n+1}+\mathcal{P}_{i-1, j-1}^{n}\right)+\left(0.5^{1-\alpha}-\kappa-W_{1}-2 S_{1}-2 S_{2}\right) \mathcal{P}_{i, j}^{n} \\
& \left.+\sum_{m=1}^{n-1}\left(W_{n-m}-W_{n-m+1}\right) \mathcal{P}_{i, j}^{m}+W_{n} \mathcal{P}_{i, j}^{0}+m f_{i, j}^{n+1 / 2}\right], \\
& i=1,2, \ldots, N_{x}-1, j=1,2, \ldots, N_{y}-1, n=0,1, \ldots, N_{t}-1, \tag{14}
\end{align*}
$$

with

$$
S_{1}=\frac{K_{1} m}{4 h_{x}^{2}}, S_{2}=\frac{K_{2} m}{4 h_{y}^{2}} .
$$

Consider the four-point groups of the discretized solution domain that are associated with the spatial locations $(i, j),(i+2, j),(i+2, j+2)$ and $(i, j+2)$. Applying Eq. (12) to any
group of four points will give the following $(4 \times 4)$ system of equations:

$$
\left(\begin{array}{cccc}
Q_{1} & -Q_{2} & 0 & -Q_{4}  \tag{15}\\
-Q_{3} & Q_{1} & -Q_{4} & 0 \\
0 & -Q_{5} & Q_{1} & -Q_{3} \\
-Q_{5} & 0 & -Q_{2} & Q_{1}
\end{array}\right)\left(\begin{array}{l}
\mathcal{P}_{i, j}^{n+1} \\
\mathcal{P}_{i+2, j}^{n+1} \\
\mathcal{P}_{i+2, j+2}^{n+1} \\
\mathcal{P}_{i, j+2}^{n+1}
\end{array}\right)=\left(\begin{array}{l}
r h s_{i, j} \\
r h s_{i+2, j} \\
r h s_{i+2, j+2} \\
r h s_{i, j+2}
\end{array}\right),
$$

in which

$$
\begin{aligned}
& Q_{1}=0.5^{1-\alpha}+\kappa+2 G_{1}+2 G_{2}, Q_{2}=G_{1}-H_{1}, Q_{3}=G_{1}+H_{1} \\
& Q_{4}=G_{2}-H_{2}, Q_{5}=G_{2}+H_{2}, Q_{6}=0.5^{1-\alpha}-\kappa-W_{1}-2 G_{1}-2 G_{2}
\end{aligned}
$$

and

$$
\begin{aligned}
r h s_{i, j}= & Q_{2} \mathcal{P}_{i+2, j}^{n}+Q_{3}\left(\mathcal{P}_{i-2, j}^{n+1}+\mathcal{P}_{i-2, j}^{n}\right)+Q_{4} \mathcal{P}_{i, j+2}^{n}+Q_{5}\left(\mathcal{P}_{i, j-2}^{n+1}+\mathcal{P}_{i, j-2}^{n}\right) \\
& +Q_{6} \mathcal{P}_{i, j}^{n}+\sum_{m=1}^{n-1}\left(W_{n-m}-W_{n-m+1}\right) \mathcal{P}_{i, j}^{m}+W_{n} \mathcal{P}_{i, j}^{0}+m f_{i, j}^{n+1 / 2}, \\
r h s_{i+2, j}= & Q_{2}\left(\mathcal{P}_{i+4, j}^{n+1}+\mathcal{P}_{i+4, j}^{n}\right)+Q_{3} \mathcal{P}_{i, j}^{n}+Q_{4} \mathcal{P}_{i+2, j+2}^{n}+Q_{5}\left(\mathcal{P}_{i+2, j-2}^{n+1}+\mathcal{P}_{i+2, j-2}^{n}\right) \\
& +Q_{6} \mathcal{P}_{i+2, j}^{n}+\sum_{m=1}^{n-1}\left(W_{n-m}-W_{n-m+1}\right) \mathcal{P}_{i+2, j}^{m}+W_{n} \mathcal{P}_{i+2, j}^{0}+m f_{i+2, j}^{n+1 / 2}, \\
r h s_{i+2, j+2}= & Q_{2}\left(\mathcal{P}_{i+4, j+2}^{n+1}+\mathcal{P}_{i+4, j+2}^{n}\right)+Q_{3} \mathcal{P}_{i, j+2}^{n}+Q_{4}\left(\mathcal{P}_{i+2, j+4}^{n+1}+\mathcal{P}_{i+2, j+4}^{n}\right) \\
& +Q_{5} \mathcal{P}_{i+2, j}^{n}+Q_{6} \mathcal{P}_{i+2, j+2}^{n}+\sum_{m=1}^{n-1}\left(W_{n-m}-W_{n-m+1}\right) \mathcal{P}_{i+2, j+2}^{m} \\
& +W_{n} \mathcal{P}_{i+2, j+2}^{0}+m f_{i+2, j+2}^{n+1 / 2}, \\
r h s_{i, j+2}= & Q_{2} \mathcal{P}_{i+2, j+2}^{n}+Q_{3}\left(\mathcal{P}_{i-2, j+2}^{n+1}+\mathcal{P}_{i-2, j+2}^{n}\right)+Q_{4}\left(\mathcal{P}_{i, j+4}^{n+1}+\mathcal{P}_{i, j+4}^{n}\right)+Q_{5} \mathcal{P}_{i, j}^{n} \\
& +Q_{6} \mathcal{P}_{i, j+2}^{n}+\sum_{m=1}^{n-1}\left(W_{n-m}-W_{n-m+1}\right) \mathcal{P}_{i, j+2}^{m}+W_{n} \mathcal{P}_{i, j+2}^{0}+m f_{i, j+2}^{n+1 / 2} .
\end{aligned}
$$

Through the reversal of the coefficient matrix in (15), the four-point MFEGM is obtained as follows:

$$
\left(\begin{array}{l}
\mathcal{P}_{i, j}^{n+1}  \tag{16}\\
\mathcal{P}_{i+2, j}^{n+1} \\
\mathcal{P}_{i+2, j+2}^{n+1} \\
\mathcal{P}_{i, j+2}^{n+1}
\end{array}\right)=\frac{1}{Z}\left(\begin{array}{llll}
Z_{1} & Z_{2} & Z_{3} & Z_{4} \\
Z_{5} & Z_{1} & Z_{4} & Z_{6} \\
Z_{7} & Z_{8} & Z_{1} & Z_{5} \\
Z_{8} & Z_{9} & Z_{2} & Z_{1}
\end{array}\right)\left(\begin{array}{c} 
\\
r h s_{i, j} \\
r h s_{i+2, j} \\
r h s_{i+2, j+2} \\
r h s_{i, j+2}
\end{array}\right)
$$

where

$$
\begin{aligned}
Z & =Q_{1}^{4}-2 Q_{1}^{2} Q_{2} Q_{3}-2 Q_{1}^{2} Q_{4} Q_{5}+Q_{2}^{2} Q_{3}^{2}-2 Q_{2} Q_{3} Q_{4} Q_{5}+Q_{4}^{2} Q 5^{2} \\
Z_{1} & =Q_{1}\left(Q_{1}^{2}-Q_{2} Q_{3}-Q_{4} Q_{5}\right), Z_{2}=Q_{2}\left(Q_{1}^{2}-Q_{2} Q_{3}+Q_{4} Q_{5}\right) \\
Z_{3} & =2 Q_{1} Q_{2} Q_{4}, Z_{4}=Q_{4}\left(Q_{1}^{2}+Q_{2} Q_{3}-Q_{4} Q_{5}\right) \\
Z_{5} & =Q_{3}\left(Q_{1}^{2}-Q_{2} Q_{3}+Q_{4} Q_{5}\right), Z_{6}=2 Q_{1} Q_{3} Q_{4}
\end{aligned}
$$



Fig. 2 Computational molecule of the MFEGM with $N_{x}=N_{y}=10$

$$
\begin{aligned}
& Z_{7}=2 Q_{1} Q_{3} Q_{5}, Z_{8}=Q_{5}\left(Q_{1}^{2}+Q_{2} Q_{3}-Q_{4} Q_{5}\right) \\
& Z_{9}=2 Q_{1} Q_{2} Q_{5}
\end{aligned}
$$

As shown in Fig. 2, the mesh points of the discretized solution domain using the MFEGM are branched into three types of points, i.e. $\downarrow$, $\square$ and $\bigcirc$. It can be verified that the implementation of the MFEGM (16) requires only points of type $\downarrow$, makes it independent from the remaining points of type $\square$ and $\bigcirc$. Therefore, and at any time level $t_{n}$, the MFEGM proceeds by iterating the solutions at the points until a predefined convergence criterion is achieved. After convergence, the solutions at the residual points of type $\square$ and $\bigcirc$ are computed directly once using Eqs. (14) and (10), respectively. Compared to the CNFDM presented in the previous section, the MFEGM comprise only quarter of the mesh points in the iterative process which is expected to accelerate the rate of convergence, especially when dealing with large scale linear systems. In this work, the four-point MFEGM is combined with the Gauss-Seidel iterative scheme and is illustrated through the next algorithm.

## 5 Stability analysis

In this section, we analyze the stability of the proposed numerical schemes with the help of Fourier analysis approach. To facilitate our theoretical analysis, the following lemma is introduced.

## Algorithm 1 The MFEGM computational algorithm

Branch the mesh points of the computational domain into two categories:
(a) Iterative points of type
(b) Direct points of type $\square$ and $\bigcirc$.
2. Assemble all the points into four-point groups.
3. set up an initial guess and error tolerance $\epsilon=10^{-5}$.
4. Iterate the intermediate solutions at the
points using the following formula:

$$
\left(\begin{array}{c}
\hat{\mathcal{P}}_{i, j}^{n+1, k+1} \\
\hat{\mathcal{P}}_{i+2, j}^{n+1, k+1} \\
\hat{\mathcal{P}}_{i+2, j+2}^{n+1, k+1} \\
\hat{\mathcal{P}}_{i, j+2}^{n+1, k+1}
\end{array}\right)=\frac{1}{Z}\left(\begin{array}{llll}
Z_{1} & Z_{2} & Z_{3} & Z_{4} \\
Z_{5} & Z_{1} & Z_{4} & Z_{6} \\
Z_{7} & Z_{8} & Z_{1} & Z_{5} \\
Z_{8} & Z_{9} & Z_{2} & Z_{1}
\end{array}\right)\left(\begin{array}{c}
r h s_{i, j} \\
r h s_{i+2, j} \\
r h s_{i+2, j+2} \\
r h s_{i, j+2}
\end{array}\right),
$$

and carry out the Gauss-Seidel iterative procedure

$$
\left(\begin{array}{l}
\mathcal{P}_{i, j}^{n+1, k+1} \\
\mathcal{P}_{i+2, j}^{n+1, k+1} \\
\mathcal{P}_{i+2, j+2}^{n+1, k+1} \\
\mathcal{P}_{i, j+2}^{n+1, k+1}
\end{array}\right)=\omega\left(\begin{array}{l}
\hat{\mathcal{P}}_{i, j}^{n+1, k+1} \\
\hat{\mathcal{P}}_{i+2, j}^{n+1, k+1} \\
\hat{\mathcal{P}}_{i+2, j+2}^{n+1, k+1} \\
\hat{\mathcal{P}}_{i, j+2}^{n+1, k+1}
\end{array}\right)+(1-\omega)\left(\begin{array}{l}
\mathcal{P}_{i, j}^{n+1, k} \\
\mathcal{P}_{i+2, j}^{n+1, k} \\
\mathcal{P}_{i+2, j+2}^{n+1, k} \\
\mathcal{P}_{i, j+2}^{n+1, k}
\end{array}\right),
$$

where $\omega$ is the relaxation factor and $k$ is the iteration number.
5. If the infinity error norm is less than $\epsilon$, go to to step 6 . Otherwise, repeat step 4 .
6. Evaluate the remaining solutions directly once at $\square$ points using Eq. (14) followed by $\bigcirc$ points utilizing Eq. (10).
7. Carry out the previous steps at each time level until the final time step is reached.
8. Display numerical results.

Lemma 1 The coefficients $W_{s},(s=0,1, \ldots)$ in (8) satisfy

1. $W_{n-m}>W_{n-m+1}, m=0,1,2, \ldots, n-1$.
2. $\sum_{m=1}^{n-1}\left(W_{n-m}-W_{n-m+1}\right)=W_{1}-W_{n}$.

### 5.1 Stability of the $h$-spaced numerical scheme

Suppose that $\mathfrak{P}_{i, j}^{n}$ is the approximate solution of (10), and define

$$
\rho_{i, j}^{n}=\mathcal{P}_{i, j}^{n}-\mathfrak{P}_{i, j}^{n}, i=0,1, \ldots, N_{x}, j=0,1, \ldots, N_{y}, n=0,1, \ldots, N_{t},
$$

and

$$
\begin{gathered}
\rho^{n}=\left(\rho_{1,1}^{n}, \rho_{1,2}^{n}, \ldots, \rho_{1, N_{y}-1}^{n}, \rho_{2,1}^{n}, \rho_{2,2}^{n}, \ldots, \rho_{2, N_{y}-1}^{n}, \ldots, \rho_{N_{x}-1,1}^{n},\right. \\
\left.\rho_{N_{x}-1,2}^{n}, \ldots, \rho_{N_{x}-1, N_{y}-1}^{n}\right)^{T}, n=1,2, \ldots, N_{t} .
\end{gathered}
$$

The roundoff error equation can be obtained as follows,

$$
\begin{aligned}
& \left(0.5^{1 \alpha}+\kappa+2 A_{1}+2 A_{2}\right) \rho_{i, j}^{n+1}-\left(A_{1}-B_{1}\right) \rho_{i+1, j}^{n+1}-\left(A_{1}+B_{1}\right) \rho_{i-1, j}^{n+1} \\
& \quad-\left(A_{2}-B_{2}\right) \rho_{i, j+1}^{n+1}-\left(A_{2}+B_{2}\right) \rho_{i, j-1}^{n+1}=\left(A_{1}-B_{1}\right) \rho_{i+1, j}^{n} \\
& \quad+\left(A_{1}+B_{1}\right) \rho_{i-1, j}^{n}+\left(A_{2}-B_{2}\right) \rho_{i, j+1}^{n}
\end{aligned}
$$

$$
\begin{align*}
& +\left(A_{2}+B_{2}\right) \rho_{i, j-1}^{n}+\left(0.5^{1-\alpha}-\kappa-W_{1}-2 A_{1}-2 A_{2}\right) \rho_{i, j}^{n} \\
& +\sum_{m=1}^{n-1}\left(W_{n-m}-W_{n-m+1}\right) \rho_{i, j}^{m}+W_{n} \rho_{i, j}^{0}, \tag{17}
\end{align*}
$$

with the following initial and boundary conditions

$$
\begin{aligned}
\rho_{i, j}^{0} & =0, \quad i=0,1, \ldots, N_{x}, \quad j=0,1, \ldots, N_{y}, \\
\rho_{0, j}^{n} & =\rho_{N_{x}, j}^{n}=0, \quad j=0,1, \ldots, N_{y}, n=0,1, \ldots, N_{t}, \\
\rho_{i, 0}^{n} & =\rho_{i, N_{y}}^{n}=0, \quad i=0,1, \ldots, N_{x}, n=0,1, \ldots, N_{t} .
\end{aligned}
$$

Next, we define the following grid function

$$
\rho^{n}(x, y)= \begin{cases}\rho_{i, j}^{n}, & x_{i-\frac{h_{x}}{2}}<x \leq x_{i+\frac{h_{x}}{2}}, y_{j-\frac{h_{y}}{2}}<y \leq y_{j+\frac{h_{y}}{2}}, \\ 0, & 0 \leq x \leq \frac{h_{x}}{2} \text { or } L-\frac{h_{x}}{2} \leq x \leq L, \\ 0, & 0 \leq y \leq \frac{h_{y}}{2} \text { or } L-\frac{h_{y}}{2} \leq y \leq L,\end{cases}
$$

where $\rho^{n}(x, y)$ has the Fourier expansion given by

$$
\rho^{n}(x, y)=\sum_{l_{1}=-\infty}^{\infty} \sum_{l_{2}=-\infty}^{\infty} \Upsilon^{n}\left(l_{1}, l_{2}\right) e^{2 \pi I\left(l_{1} x / L+l_{2} y / L\right)}
$$

in which $I=\sqrt{-1}$ and

$$
\begin{equation*}
\Upsilon^{n}\left(l_{1}, l_{2}\right)=\frac{1}{L^{2}} \int_{0}^{L} \int_{0}^{L} \rho^{n}(x, y) e^{-2 \pi I\left(l_{1} x / L+l_{2} y / L\right)} \mathrm{d} x \mathrm{~d} y . \tag{18}
\end{equation*}
$$

The 2-norm can be defined as

$$
\left\|\rho^{n}\right\|_{2}=\left(\sum_{j=1}^{N_{y}-1} \sum_{i=1}^{N_{x}-1} h_{y} h_{x}\left|\rho_{i, j}^{n}\right|^{2}\right)^{1 / 2}=\left(\int_{0}^{L} \int_{0}^{L}\left|\rho_{i, j}^{n}\right|^{2} \mathrm{~d} x \mathrm{~d} y\right)^{1 / 2}
$$

Introducing the Parseval equality

$$
\int_{0}^{L} \int_{0}^{L}\left|\rho_{i, j}^{n}\right|^{2} \mathrm{~d} x \mathrm{~d} y=\sum_{l_{2}=-\infty}^{\infty} \sum_{l_{1}=-\infty}^{\infty}\left|\Upsilon^{n}\left(l_{1}, l_{2}\right)\right|^{2}
$$

we obtain

$$
\begin{equation*}
\left\|\rho^{n}\right\|_{2}=\left(\sum_{l_{2}=-\infty}^{\infty} \sum_{l_{1}=-\infty}^{\infty}\left|\Upsilon^{n}\left(l_{1}, l_{2}\right)\right|^{2}\right)^{1 / 2} . \tag{19}
\end{equation*}
$$

Suppose the solution of (17) has the following form

$$
\begin{equation*}
\rho_{i, j}^{n}=\Upsilon^{n} e^{I\left(\beta_{1} i h_{x}+\beta_{2} j h_{y}\right)}, \tag{20}
\end{equation*}
$$

where $\beta_{1}=2 \pi l_{1} / L$ and $\beta_{2}=2 \pi l_{2} / L$ are the real spatial wave numbers. Setting (20) into (17), yields

$$
\Upsilon^{n+1}=\frac{0.5^{1-\alpha}-\kappa-W_{1}-\mu_{1}-I \mu_{2}}{0.5^{1-\alpha}+\kappa+\mu_{1}+I \mu_{2}} \Upsilon^{n}
$$

$$
\begin{equation*}
+\frac{1}{0.5^{1-\alpha}+\kappa+\mu_{1}+I \mu_{2}}\left[\sum_{m=1}^{n-1}\left(W_{n-m}-W_{n-m+1}\right) \Upsilon^{m}+W_{n} \Upsilon^{0}\right] \tag{21}
\end{equation*}
$$

where

$$
\begin{aligned}
& \mu_{1}=4 A_{1} \sin ^{2}\left(\frac{\beta_{1} h_{x}}{2}\right)+4 A_{2} \sin ^{2}\left(\frac{\beta_{2} h_{y}}{2}\right), \\
& \mu_{2}=2\left(B_{1} \sin \left(\beta_{1} h_{x}\right)+B_{2} \sin \left(\beta_{2} h_{y}\right)\right) .
\end{aligned}
$$

Lemma 2 Let $\Upsilon^{n+1}, n=0,1, \ldots, N_{t}-1$ be the solution of (21), given that $3^{1-\alpha} \leq 2$, then it holds that

$$
\left|\Upsilon^{n+1}\right| \leq\left|\Upsilon^{0}\right|, n=0,1, \ldots, N_{t}-1 .
$$

Proof Here, mathematical induction shall be used to complete the proof. For $n=0$, we have

$$
\left|\Upsilon^{1}\right|=\left|\frac{0.5^{1-\alpha}-\kappa-\mu_{1}-I \mu_{2}}{0.5^{1-\alpha}+\kappa+\mu_{1}+I \mu_{2}}\right|\left|\Upsilon^{0}\right| \leq\left|\Upsilon^{0}\right| .
$$

Now, we assume that

$$
\left|\Upsilon^{s+1}\right| \leq\left|\Upsilon^{0}\right|, s=0,1, \ldots, n-1 .
$$

For $s=n$, utilizing Eq. (21) along with lemma 1 , we have

$$
\begin{aligned}
\left|\Upsilon^{n+1}\right| \leq & \left|\frac{0.5^{1-\alpha}-\kappa-W_{1}-\mu_{1}-I \mu_{2}}{0.5^{1-\alpha}+\kappa+\mu_{1}+I \mu_{2}}\right|\left|\Upsilon^{n}\right| \\
& +\left|\frac{1}{0.5^{1-\alpha}+\kappa+\mu_{1}+I \mu_{2}}\right|\left[\sum_{m=1}^{n-1}\left|\left(W_{n-m}-W_{n-m+1}\right)\right|\left|\Upsilon^{n}\right|+\left|W_{n}\right|\left|\Upsilon^{0}\right|\right] \\
\leq & \left|\frac{0.5^{1-\alpha}-\kappa-W_{1}-\mu_{1}-I \mu_{2}}{0.5^{1-\alpha}+\kappa+\mu_{1}+I \mu_{2}}\right|\left|\Upsilon^{0}\right| \\
& +\left|\frac{1}{0.5^{1-\alpha}+\kappa+\mu_{1}+I \mu_{2}}\right|\left[\sum_{m=1}^{n-1}\left|\left(W_{n-m}-W_{n-m+1}\right)\right|\left|\Upsilon^{0}\right|+\left|W_{n}\right|\left|\Upsilon^{0}\right|\right] \\
& =\frac{\left|0.5^{1-\alpha}-\kappa-W_{1}-\mu_{1}-I \mu_{2}\right|+W_{1}}{0.5^{1-\alpha}+\kappa+\mu_{1}+I \mu_{2}}\left|\Upsilon^{0}\right| .
\end{aligned}
$$

As $n \rightarrow \infty, \tau, \kappa, \mu_{1}$ and $\mu_{2}$ approach to zero and

$$
\left|\Upsilon^{n+1}\right| \leq \frac{\left|0.5^{1-\alpha}-W_{1}\right|+W_{1}}{0.5^{1-\alpha}}\left|\Upsilon^{0}\right| .
$$

This leads to the following cases:
Case I. If $0.5^{1-\alpha}-W_{1} \geq 0$, then

$$
\left|\Upsilon^{n+1} \leq\left|\Upsilon^{0}\right| .\right.
$$

Case II. If $0.5^{1-\alpha}-W_{1}<0$, then

$$
\left|\Upsilon^{n+1}\right| \leq \frac{2 W_{1}-0.5^{1-\alpha}}{5^{1-\alpha}}\left|\Upsilon^{0}\right|
$$

Here,

$$
\left|\Upsilon^{n+1}\right| \leq\left|\Upsilon^{0}\right|
$$

$$
\begin{aligned}
& \Leftrightarrow 2 W_{1}-0.5^{1-\alpha} \leq 0.5^{1-\alpha} \\
& \Leftrightarrow 3^{1-\alpha} \leq 2
\end{aligned}
$$

which ends the proof.
Theorem 1 Given that $3^{1-\alpha}<2$, the $h$-spaced difference scheme (10) is stable.
Proof On the basis of lemma 2 and Parseval equality, it follows that

$$
\begin{aligned}
\left\|\rho^{n}\right\|_{2} & =\sum_{j=1}^{N_{y}-1} \sum_{i=1}^{N_{x}-1} h_{y} h_{x}\left|\rho_{i, j}^{k}\right|^{2}=h_{y} h_{x} \sum_{j=1}^{N_{y}-1} \sum_{i=1}^{N_{x}-1}\left|\Upsilon^{n} e^{I\left(\beta_{1} h_{x}+\beta_{2} j h_{y}\right)}\right|^{2} \\
& =h_{y} h_{x} \sum_{j=1}^{N_{y}-1} \sum_{i=1}^{N_{x}-1}\left|\Upsilon^{n}\right|^{2} \leq h_{y} h_{x} \sum_{j=1}^{N_{y}-1} \sum_{i=1}^{N_{x}-1}\left|\Upsilon^{0}\right|^{2} \\
& =h_{y} h_{x} \sum_{j=1}^{N_{y}-1} \sum_{i=1}^{N_{x}-1}\left|\Upsilon^{0} e^{I\left(\beta_{1} i h_{x}+\beta_{2} j h_{y}\right)}\right|^{2}=\left\|\rho^{0}\right\|_{2}
\end{aligned}
$$

This means that the difference scheme (10) is unconditionally stable.

### 5.2 Stability of the $\mathbf{2 h}$-spaced numerical scheme

Let $\hat{\mathcal{P}}_{i, j}^{n}$ is the approximate solution of (12), and define

$$
\mathcal{E}_{i, j}^{n}=\mathcal{P}_{i, j}^{n}-\hat{\mathcal{P}}_{i, j}^{n}, i=0,2, \ldots, N_{x}, j=0,2, \ldots, N_{y}, n=0,1, \ldots, N_{t},
$$

and

$$
\begin{gathered}
\mathcal{E}^{n}=\left(\mathcal{E}_{2,2}^{n}, \mathcal{E}_{2,4}^{n}, \ldots, \mathcal{E}_{2, N_{y}-2}^{n}, \mathcal{E}_{4,2}^{n}, \mathcal{E}_{4,4}^{n}, \ldots, \mathcal{E}_{4, N_{y}-2}^{n}, \ldots, \mathcal{E}_{N_{x}-2,2}^{n},\right. \\
\\
\left.\mathcal{E}_{N_{x}-2,4}^{n}, \ldots, \mathcal{E}_{N_{x}-2, N_{y}-2}^{n}\right)^{T}, n=1,2, \ldots, N_{t} .
\end{gathered}
$$

The roundoff error equation can be easily obtained as,

$$
\begin{align*}
& \left(0.5^{1 \alpha}+\kappa+2 G_{1}+2 G_{2}\right) \mathcal{E}_{i, j}^{n+1}-\left(G_{1}-H_{1}\right) \mathcal{E}_{i+1, j}^{n+1}-\left(G_{1}+H_{1}\right) \mathcal{E}_{i-1, j}^{n+1} \\
& \quad-\left(G_{2}-H_{2}\right) \mathcal{E}_{i, j+1}^{n+1}-\left(H_{2}+H_{2}\right) \mathcal{E}_{i, j-1}^{n+1}=\left(G_{1}-H_{1}\right) \mathcal{E}_{i+1, j}^{n} \\
& \quad+\left(G_{1}+H_{1}\right) \mathcal{E}_{i-1, j}^{n}+\left(G_{2}-H_{2}\right) \mathcal{E}_{i, j+1}^{n} \\
& \quad+\left(G_{2}+H_{2}\right) \mathcal{E}_{i, j-1}^{n}+\left(0.5^{1-\alpha}-\kappa-W_{1}-2 G_{1}-2 G_{2}\right) \mathcal{E}_{i, j}^{n} \\
& \quad+\sum_{m=1}^{n-1}\left(W_{n-m}-W_{n-m+1}\right) \mathcal{E}_{i, j}^{m}+W_{n} \mathcal{E}_{i, j}^{0} \tag{22}
\end{align*}
$$

with the following initial and boundary conditions

$$
\begin{aligned}
\mathcal{E}_{i, j}^{0} & =0, \quad i=0,2, \ldots, N_{x}, j=0,2, \ldots, N_{y}, \\
\mathcal{E}_{0, j}^{n} & =\mathcal{E}_{N_{x}, j}^{n}=0, \quad j=0,2, \ldots, N_{y}, n=0,1, \ldots, N_{t}, \\
\mathcal{E}_{i, 0}^{n} & =\mathcal{E}_{i, N_{y}}^{n}=0, \quad i=0,2, \ldots, N_{x}, n=0,1, \ldots, N_{t} .
\end{aligned}
$$

Next, we define the following grid function

$$
\mathcal{E}^{n}(x, y)= \begin{cases}\mathcal{E}_{i, j}^{n}, & x_{i-\frac{h_{x}}{4}}<x \leq x_{i+\frac{h_{x}}{4}}, y_{j-\frac{h_{y}}{4}}<y \leq y_{j+\frac{h_{y}}{4}}, \\ 0, & 0 \leq x \leq \frac{h_{x}}{4} \text { or } L-\frac{h_{x}}{4} \leq x \leq L, \\ 0, & 0 \leq y \leq \frac{h_{y}}{2} \text { or } L-\frac{h_{y}}{2} \leq y \leq L,\end{cases}
$$

where $\mathcal{E}^{n}(x, y)$ has the Fourier expansion given by

$$
\mathcal{E}^{n}(x, y)=\sum_{l_{1}=-\infty}^{\infty} \sum_{l_{2}=-\infty}^{\infty} \Theta^{n}\left(l_{1}, l_{2}\right) e^{2 \pi I\left(l_{1} x / L+l_{2} y / L\right)}
$$

such that

$$
\begin{equation*}
\Theta^{n}\left(l_{1}, l_{2}\right)=\frac{1}{L^{2}} \int_{0}^{L} \int_{0}^{L} \mathcal{E}^{n}(x, y) e^{-2 \pi I\left(l_{1} x / L+l_{2} y / L\right)} \mathrm{d} x \mathrm{~d} y . \tag{23}
\end{equation*}
$$

The 2-norm is given by

$$
\left\|\mathcal{E}^{n}\right\|_{2}=\left(\sum_{j=1}^{N_{y}-1} \sum_{i=1}^{N_{x}-1} h_{y} h_{x}\left|\mathcal{E}_{i, j}^{n}\right|^{2}\right)^{1 / 2}=\left(\int_{0}^{L} \int_{0}^{L}\left|\mathcal{E}_{i, j}^{n}\right|^{2} \mathrm{~d} x \mathrm{~d} y\right)^{1 / 2} .
$$

With the help of the Parseval equality

$$
\int_{0}^{L} \int_{0}^{L}\left|\mathcal{E}_{i, j}^{n}\right|^{2} \mathrm{~d} x \mathrm{~d} y=\sum_{l_{2}=-\infty}^{\infty} \sum_{l_{1}=-\infty}^{\infty}\left|\Theta^{n}\left(l_{1}, l_{2}\right)\right|^{2}
$$

we get

$$
\begin{equation*}
\left\|\mathcal{E}^{n}\right\|_{2}=\left(\sum_{l_{2}=-\infty}^{\infty} \sum_{l_{1}=-\infty}^{\infty}\left|\Theta^{n}\left(l_{1}, l_{2}\right)\right|^{2}\right)^{1 / 2} . \tag{24}
\end{equation*}
$$

Suppose the solution of (22) has the following form

$$
\begin{equation*}
\mathcal{E}_{i, j}^{n}=\Theta^{n} e^{I\left(\beta_{1} i h_{x}+\beta_{2} j h_{y}\right)}, \tag{25}
\end{equation*}
$$

Substituting (25) into (22) leads to

$$
\begin{align*}
\Theta^{n+1}= & \frac{0.5^{1-\alpha}-\kappa-W_{1}-\eta_{1}-I \eta_{2}}{0.5^{1-\alpha}+\kappa+\eta_{1}+I \eta_{2}} \Theta^{n} \\
& +\frac{1}{0.5^{1-\alpha}+\kappa+\eta_{1}+I \eta_{2}}\left[\sum_{m=1}^{n-1}\left(W_{n-m}-W_{n-m+1}\right) \Theta^{m}+W_{n} \Theta^{0}\right], \tag{26}
\end{align*}
$$

where

$$
\begin{aligned}
& \eta_{1}=4 G_{1} \sin ^{2}\left(\beta_{1} h_{x}\right)+4 G_{2} \sin ^{2}\left(\beta_{2} h_{y}\right), \\
& \eta_{2}=2\left(H_{1} \sin \left(2 \beta_{1} h_{x}\right)+H_{2} \sin \left(2 \beta_{2} h_{y}\right)\right) .
\end{aligned}
$$

Lemma 3 Let $\Theta^{n+1}, n=0,1, \ldots, N_{t}-1$ be the solution of (26), given that $3^{1-\alpha} \leq 2$, then it holds that

$$
\left|\Theta^{n+1}\right| \leq\left|\Theta^{0}\right|, n=0,1, \ldots, N_{t}-1 .
$$

Proof Again, mathematical induction is utilized for the completion of the proof. For $n=0$, we have

$$
\left|\Theta^{1}\right|=\left|\frac{0.5^{1-\alpha}-\kappa-\eta_{1}-I \eta_{2}}{0.5^{1-\alpha}+\kappa+\eta_{1}+I \eta_{2}}\right| \Theta^{0}\left|\leq\left|\Theta^{0}\right| .\right.
$$

Now, we suppose that

$$
\left|\Theta^{s+1}\right| \leq\left|\Theta^{0}\right|, s=0,1, \ldots, n-1 .
$$

For $s=n$, using Eq. (26) together with lemma 1, we have

$$
\begin{aligned}
\left|\Theta^{n+1}\right| \leq & \left|\frac{0.5^{1-\alpha}-\kappa-W_{1}-\eta_{1}-I \eta_{2}}{0.5^{1-\alpha}+\kappa+\eta_{1}+I \eta_{2}}\right|\left|\Theta^{n}\right| \\
& +\left|\frac{1}{0.5^{1-\alpha}+\kappa+\eta_{1}+I \eta_{2}}\right|\left[\sum_{m=1}^{n-1}\left|\left(W_{n-m}-W_{n-m+1}\right) \| \Theta^{n}\right|+\left|W_{n}\right|\left|\Theta^{0}\right|\right] \\
\leq & \left|\frac{0.5^{1-\alpha}-\kappa-W_{1}-\eta_{1}-I \eta_{2}}{0.5^{1-\alpha}+\kappa+\eta_{1}+I \eta_{2}}\right|\left|\Theta^{0}\right| \\
& +\left|\frac{1}{0.5^{1-\alpha}+\kappa+\eta_{1}+I \eta_{2}}\right|\left[\sum_{m=1}^{n-1}\left|\left(W_{n-m}-W_{n-m+1}\right) \| \Theta^{0}\right|+\left|W_{n}\right|\left|\Theta^{0}\right|\right] \\
= & \frac{\left|0.5^{1-\alpha}-\kappa-W_{1}-\eta_{1}-I \eta_{2}\right|+W_{1}}{0.5^{1-\alpha}+\kappa+\eta_{1}+I \eta_{2}}\left|\Theta^{0}\right| .
\end{aligned}
$$

As $n \rightarrow \infty, \tau, \kappa, \eta_{1}$ and $\eta_{2}$ approach to zero and

$$
\left|\Theta^{n+1}\right| \leq \frac{\left|0.5^{1-\alpha}-W_{1}\right|+W_{1}}{0.5^{1-\alpha}}\left|\Theta^{0}\right| .
$$

From the previous subsection, we know that

$$
\begin{aligned}
\frac{\left|0.5^{1-\alpha}-W_{1}\right|+W_{1}}{0.5^{1-\alpha}} & \leq 1 \\
& \Leftrightarrow 3^{1-\alpha} \leq 2 .
\end{aligned}
$$

This completes the proof.
Theorem 2 Given that $3^{1-\alpha} \leq 2$, the $2 h$-spaced difference scheme (12) is stable.
Proof According to lemma 3 and Parseval equality, we obtain

$$
\begin{aligned}
\left\|\mathcal{E}^{n}\right\|_{2} & =\sum_{j=1}^{N_{y}-1} \sum_{i=1}^{N_{x}-1} h_{y} h_{x}\left|\mathcal{E}_{i, j}^{k}\right|^{2}=h_{y} h_{x} \sum_{j=1}^{N_{y}-1} \sum_{i=1}^{N_{x}-1}\left|\Theta^{n} e^{I\left(\beta_{1} h_{x}+\beta_{2} j h_{y}\right)}\right|^{2} \\
& =h_{y} h_{x} \sum_{j=1}^{N_{y}-1} \sum_{i=1}^{N_{x}-1}\left|\Theta^{n}\right|^{2} \leq h_{y} h_{x} \sum_{j=1}^{N_{y}-1} \sum_{i=1}^{N_{x}-1}\left|\Theta^{0}\right|^{2} \\
& =h_{y} h_{x} \sum_{j=1}^{N_{y}-1} \sum_{i=1}^{N_{x}-1}\left|\Theta^{0} e^{I\left(\beta_{1} i h_{x}+\beta_{2} j h_{y}\right)}\right|^{2}=\left\|\mathcal{E}^{0}\right\|_{2} .
\end{aligned}
$$

Hence, the difference scheme (12) is unconditionally stable.

## 6 Convergence analysis

In this part, we describe the convergence analysis of the proposed numerical schemes. For this purpose, the following theorem is required.

Theorem 3 (Lax-Richertmyer theorem) if the differential equation (1) is well-posed and the finite difference scheme is consistent, then the finite difference scheme is convergent if and only if, it is stable

The next theorem discuss the convergence of the $h$-spaced difference scheme defined by (10).

Theorem 4 The $h$-spaced difference scheme (10) is consistent, and hence is convergent.

Proof Let $R_{i, j}^{n}$ be the truncation error at the point $\left(x_{i}, y_{j}, t_{n}\right)$. According to Eqs. (4-8), the local truncation error of the numerical scheme (10) is

$$
\begin{aligned}
R_{i, j}^{n+1 / 2}= & {\left[W_{1} P_{i, j}^{n}+\sum_{m=1}^{n-1}\left(W_{n-m+1}-W_{n-m}\right) P_{i, j}^{m}-W_{n} P_{i, j}^{0}+\sigma \frac{\left(P_{i, j}^{n+1}-P_{i, j}^{n}\right)}{2^{1-\alpha}}\right] } \\
& -\frac{K_{1}}{2}\left[\frac{P_{i+1, j}^{n+1}-2 P_{i, j}^{n+1}+P_{i-1, j}^{n+1}}{h_{x}^{2}}+\frac{P_{i+1, j}^{n}-2 P_{i, j}^{n}+P_{i-1, j}^{n}}{h_{x}^{2}}\right] \\
& -\frac{K_{2}}{2}\left[\frac{P_{i, j+1}^{n+1}-2 P_{i, j}^{n+1}+P_{i, j-1}^{n+1}}{h_{y}^{2}}+\frac{P_{i, j+1}^{n}-2 P_{i, j}^{n}+P_{i, j-1}^{n}}{h_{y}^{2}}\right] \\
& +\frac{V_{1}}{2}\left[\frac{P_{i+1, j}^{n+1}-P_{i-1, j}^{n+1}}{2 h_{x}}+\frac{P_{i+1, j}^{n}-P_{i-1, j}^{n}}{2 h_{x}}\right] \\
& +\frac{V_{2}}{2}\left[\frac{P_{i, j+1}^{n+1}-P_{i, j-1}^{n+1}}{2 h_{y}}+\frac{P_{i, j+1}^{n}-P_{i, j-1}^{n}}{2 h_{y}}\right] \\
& +M\left(\frac{P_{i, j}^{n+1}+P_{i, j}^{n}}{2}\right)-f_{i, j}^{n+1 / 2}, \\
= & {\left[W_{1} \mathcal{P}_{i, j}^{n}+\sum_{m=1}^{n-1}\left(W_{n-m+1}-W_{n-m}\right) \mathcal{P}_{i, j}^{m}-W_{n} \mathcal{P}_{i, j}^{0}+\sigma \frac{\left(\mathcal{P}_{i, j}^{n+1}-\mathcal{P}_{i, j}^{n}\right)}{2^{1-\alpha}}\right] } \\
& -\frac{\partial^{\alpha} P\left(x_{i}, y_{j}, t_{n+1 / 2}\right)}{\partial t^{\alpha}} \\
& -\frac{K_{1}}{2}\left[\frac{\mathcal{P}_{i+1, j}^{n+1}-2 \mathcal{P}_{i, j}^{n+1}+\mathcal{P}_{i-1, j}^{n+1}}{h_{x}^{2}}+\frac{\mathcal{P}_{i+1, j}^{n}-2 \mathcal{P}_{i, j}^{n}+\mathcal{P}_{i-1, j}^{n}}{h_{x}^{2}}\right] \\
& -\frac{\partial^{2} P\left(x_{i}, y_{j}, t_{n+1 / 2}\right)}{\partial x^{2}} \\
& -\frac{K_{2}}{2}\left[\frac{\mathcal{P}_{i, j+1}^{n+1}-2 \mathcal{P}_{i, j}^{n+1}+\mathcal{P}_{i, j-1}^{n+1}}{h_{y}^{2}}+\frac{\mathcal{P}_{i, j+1}^{n}-2 \mathcal{P}_{i, j}^{n}+\mathcal{P}_{i, j-1}^{n}}{h_{y}^{2}}\right] \\
& -\frac{\partial^{2} P\left(x_{i}, y_{j}, t_{n+1 / 2}\right)}{\partial y^{2}}
\end{aligned}
$$

$$
\begin{align*}
& \quad+\frac{V_{1}}{2}\left[\frac{\mathcal{P}_{i+1, j}^{n+1}-\mathcal{P}_{i-1, j}^{n+1}}{2 h_{x}}+\frac{\mathcal{P}_{i+1, j}^{n}-\mathcal{P}_{i-1, j}^{n}}{2 h_{x}}\right]-\frac{\partial P\left(x_{i}, y_{j}, t_{n+1 / 2}\right)}{\partial x} \\
& \quad+\frac{V_{2}}{2}\left[\frac{\mathcal{P}_{i, j+1}^{n+1}-\mathcal{P}_{i, j-1}^{n+1}}{2 h_{y}}+\frac{\mathcal{P}_{i, j+1}^{n}-\mathcal{P}_{i, j-1}^{n}}{2 h_{y}}\right]-\frac{\partial P\left(x_{i}, y_{j}, t_{n+1 / 2}\right)}{\partial y} \\
& M\left(\frac{\mathcal{P}_{i, j}^{n+1}+\mathcal{P}_{i, j}^{n}}{2}\right)-P\left(x_{i}, y_{j}, t_{n+1 / 2}\right)=O\left(\tau^{2-\alpha}+h_{x}^{2}+h_{y}^{2}\right) . \tag{27}
\end{align*}
$$

It can be easily shown that the local truncation error tends to zero as $\tau, h_{x}$ and $h_{y}$ go to zero. This means that the difference scheme (10) is consistent, so that by Theorem 3, the difference scheme (10) is convergent.

Theorem 5 The $2 h$-spaced difference scheme (12) is consistent, and hence is convergent.
Proof The proof is similar to Theorem 4.

## 7 Numerical experiments and discussion of results

In this part, four numerical examples are presented to demonstrate the computational efficiency of the MFEGM by comparing to the CNFDM. All programming codes are written in MATLAB and run on a PC with i7-8550U CPU, 8 GB RAM and Windows 10 (64-bit) operating system. The obtained numerical results are represented in terms of several illustrative figures and tables. In these tables, "Sec" denotes the amount of computational time required for solving the problem under consideration. "Iter" means the number of iterations needed to achieve convergence at the last time level. And, "MAE" stands for the maximum absolute error between the numerical and exact solutions. In all numerical examples, we let $N_{x}=N_{y}=N$, and give the source terms by the exact solutions.

Example 1 Consider the model problem with the exact solution $u(x, y, t)=e^{t}\left(1-x^{2}\right)^{2}(1-$ $\left.y^{2}\right)^{2}$,

$$
\left\{\begin{array}{l}
{ }_{0}^{C} D_{t}^{\alpha} P=P_{x x}+P_{y y}-P_{x}-P_{y}-P+f(x, y, t), 0 \leq x, y \leq 1,0<t \leq T, \\
P(x, y, 0)=\left(1-x^{2}\right)^{2}\left(1-y^{2}\right)^{2}, \quad(x, y) \in \Omega \cup \partial \Omega, \\
P(0, y, t)=e^{t}\left(1-y^{2}\right)^{2}, \quad P(1, y, t)=0, \\
P(x, 0, t)=e^{t}\left(1-x^{2}\right)^{2}, \quad P(x, 1, t)=0, \quad(x, y, t) \in \partial \Omega \times(0, T] .
\end{array}\right.
$$

Table 1 presents a comparison between the elapsed time in seconds, number of iterations and maximum absolute errors of the MFEGM and CNFDM at $T=1, \tau=0.1,0.01$ and various values of $\alpha$. From this table, we can observe that the approximate solution of the MFEGM converges to the exact solution with less computational time and iteration count in comparison to the CNFDM. For instance, at $\alpha=0.9$ and $\tau=0.01$, the CNFDM reaches convergence after 135 iterations with 165.41 computing time, while the MFEGM terminates after only 27 iterations with 20.10 computing time. The numerical solutions of the proposed methods and the exact solution at $y=0.5, N=70, N_{t}=100, \alpha=0.5$ and different final times $T=1,1.5$ and 2 are drawn in Fig. 3. In addition, Fig. 4 depicts the plot of maximum errors for the CNFDM and the MFEGM at $N=62, N_{t}=100, \alpha=0.5$ and $T=1$. From these figures, we can see that the numerical solutions of our proposed methods are in good agreement with the exact solution.
Table 1 Numerical results of the CNFDM and the MFEGM for Example 1 at $N=50$ and $T=1$

| $\alpha$ | $\tau=0.1$ |  |  |  |  |  | $\underline{\tau}=0.01$ |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | CNFDM |  |  | MFEGM |  |  | CNFDM |  |  | MFEGM |  |  |
|  | Sec | Iter | MAE | Sec | Iter | MAE | Sec | Iter | MAE | Sec | Iter | MAE |
| 0.1 | 17.28 | 884 | $1.0782 \mathrm{E}-02$ | 2.70 | 166 | $1.2017 \mathrm{E}-02$ | 566.30 | 418 | $3.0966 \mathrm{E}-03$ | 40.75 | 104 | $4.3711 \mathrm{E}-03$ |
| 0.2 | 13.23 | 855 | $2.1010 \mathrm{E}-02$ | 2.17 | 161 | $2.3871 \mathrm{E}-02$ | 543.45 | 418 | $4.3301 \mathrm{E}-03$ | 37.44 | 102 | $4.8689 \mathrm{E}-03$ |
| 0.3 | 17.32 | 840 | $2.9084 \mathrm{E}-02$ | 2.37 | 158 | $3.1990 \mathrm{E}-02$ | 543.14 | 397 | $4.1834 \mathrm{E}-03$ | 39.26 | 95 | $4.0513 \mathrm{E}-03$ |
| 0.4 | 13.79 | 824 | $3.2339 \mathrm{E}-02$ | 2.22 | 154 | $3.4044 \mathrm{E}-02$ | 442.97 | 366 | $3.4144 \mathrm{E}-03$ | 35.71 | 86 | $3.2324 \mathrm{E}-03$ |
| 0.5 | 19.40 | 800 | $3.0330 \mathrm{E}-02$ | 2.50 | 149 | $3.0919 \mathrm{E}-02$ | 367.05 | 326 | $2.6888 \mathrm{E}-03$ | 34.80 | 74 | $2.4596 \mathrm{E}-03$ |
| 0.6 | 12.19 | 766 | $2.5040 \mathrm{E}-02$ | 2.14 | 141 | $2.5015 \mathrm{E}-02$ | 365.45 | 277 | $1.9892 \mathrm{E}-03$ | 23.85 | 61 | $2.1790 \mathrm{E}-03$ |
| 0.7 | 14.26 | 722 | $1.8403 \mathrm{E}-02$ | 2.06 | 133 | $1.8218 \mathrm{E}-02$ | 263.25 | 226 | $1.3541 \mathrm{E}-03$ | 25.25 | 48 | $1.3997 \mathrm{E}-03$ |
| 0.8 | 10.55 | 672 | $1.1840 \mathrm{E}-02$ | 1.92 | 123 | $1.1645 \mathrm{E}-02$ | 154.95 | 177 | $9.9507 \mathrm{E}-04$ | 16.34 | 36 | $4.9416 \mathrm{E}-04$ |
| 0.9 | 12.45 | 619 | $5.6289 \mathrm{E}-03$ | 1.89 | 113 | $5.4907 \mathrm{E}-03$ | 165.41 | 135 | $9.2038 \mathrm{E}-04$ | 20.10 | 27 | $1.2801 \mathrm{E}-04$ |



Fig. 3 Numerical solutions and exact solutions for Example 1 at $y=0.5, N=70, N_{t}=100, \alpha=0.5$ and $T=1,1.5,2$


Fig. 4 The 3D error plot of absolute errors using a CNFDM and $\mathbf{b}$ MFEGM for Example 1 at $N=62$, $N_{t}=100, \alpha=0.5$ and $T=1$


Fig. 5 Numerical solutions and exact solutions for Example 2 at $y=0.5, N=70, N_{t}=100, \alpha=0.5$ and $T=1,1.5,2$

Example 2 Consider the following model problem,

$$
\left\{\begin{array}{l}
{ }_{0}^{C} D_{t}^{\alpha} P=P_{x x}+P_{y y}-P_{x}-P_{y}-P+f(x, y, t), \quad 0 \leq x, y \leq 1,0<t \leq T, \\
P(x, y, 0)=0, \quad(x, y) \in \Omega \cup \partial \Omega, \\
P(0, y, t)=0, \quad P(1, y, t)=20 t^{1+a} y \sin (y) e^{y}, \\
P(x, 0, t)=0, \quad P(x, 1, t)=20 t^{1+a} x \sin (x) e^{x}, \quad(x, y, t) \in \partial \Omega \times(0, T] .
\end{array}\right.
$$

The exact solution of the above problem is $u(x, y, t)=20 t^{1+\alpha} x y \sin (x y) e^{x y}$. Table 2 enumerates the numerical results of the CNFDM and the MFEGM in solving Example 2 at $N=62, N_{t}=62$, several values of $\alpha$ and diverse final times $T=1,1.5$ and 2 . From which we notice that the CNFDM and the MFEGM simulate the considered problem accurately where the later method results in much faster simulations that the former one for different values of $T$. The graphical representations of the numerical and exact solutions of Example 2 at $y=0.5, N=70, N_{t}=100, \alpha=0.5$ and $T=1,1.5$ and 2 are highlighted in Fig. 5. We can observe that the numerical solutions agree well with the exact solutions. One more time, we plot the execution time of the proposed methods versus several mesh sizes at fixed $T=1$, $\tau=0.01$ and $\alpha=0.5$ in Fig. 6. One can see that the MFEGM requires smaller amount of running time compared to the CNFDM, especially for larger values of $N$. This is very useful when solving large-scale linear systems.
Table 2 Numerical results of the CNFDM and the MFEGM for Example 2 at $N=62$ and $N_{t}=62$

| $\alpha$ | $T=1$ |  |  |  | $T=1.5$ |  |  |  | $T=2$ |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | CNFDM |  | MFEG |  | CNFDM |  | MFE |  | CNFDM |  | MFEG |  |
|  | Sec | MAE | Sec | MAE | Sec | MAE | Sec | MAE | Sec | MAE | Sec | MAE |
| 0.1 | 133.22 | $1.5943 \mathrm{E}-03$ | 12.68 | $1.3083 \mathrm{E}-03$ | 100.6 | $1.9003 \mathrm{E}-03$ | 9.94 | $2.0143 \mathrm{E}-03$ | 112.82 | $2.2659 \mathrm{E}-03$ | 10.50 | $2.7664 \mathrm{E}-03$ |
| 0.2 | 112.43 | $1.7413 \mathrm{E}-03$ | 13.31 | $9.6864 \mathrm{E}-04$ | 102.16 | $1.8789 \mathrm{E}-03$ | 9.58 | $1.5404 \mathrm{E}-03$ | 107.85 | $1.9814 \mathrm{E}-03$ | 10.07 | $2.1743 \mathrm{E}-03$ |
| 0.3 | 109.36 | $1.8278 \mathrm{E}-03$ | 11.65 | $8.8172 \mathrm{E}-04$ | 98.25 | $1.8901 \mathrm{E}-03$ | 9.34 | $1.3920 \mathrm{E}-03$ | 115.32 | $1.9863 \mathrm{E}-03$ | 10.16 | 1.9842E-03 |
| 0.4 | 101.55 | $1.8252 \mathrm{E}-03$ | 13.30 | $8.8153 \mathrm{E}-04$ | 94.11 | $1.9395 \mathrm{E}-03$ | 9.05 | $1.4450 \mathrm{E}-03$ | 109.81 | $2.0515 \mathrm{E}-03$ | 9.72 | $2.0880 \mathrm{E}-03$ |
| 0.5 | 90.67 | $1.8307 \mathrm{E}-03$ | 9.71 | $8.7798 \mathrm{E}-04$ | 86.66 | $1.9484 \mathrm{E}-03$ | 8.47 | $1.4874 \mathrm{E}-03$ | 139.63 | $2.0669 \mathrm{E}-03$ | 9.11 | $2.2245 \mathrm{E}-03$ |
| 0.6 | 76.73 | $1.8155 \mathrm{E}-03$ | 7.80 | $8.8178 \mathrm{E}-04$ | 77.86 | $1.9540 \mathrm{E}-03$ | 7.44 | $1.5506 \mathrm{E}-03$ | 136.04 | $2.0865 \mathrm{E}-03$ | 8.07 | $2.3762 \mathrm{E}-03$ |
| 0.7 | 78.25 | $1.7958 \mathrm{E}-03$ | 6.71 | $8.9973 \mathrm{E}-04$ | 67.61 | $1.9636 \mathrm{E}-03$ | 6.25 | $1.6276 \mathrm{E}-03$ | 119.01 | $2.1080 \mathrm{E}-03$ | 7.26 | $2.5527 \mathrm{E}-03$ |
| 0.8 | 65.13 | $1.7657 \mathrm{E}-03$ | 5.44 | $9.2198 \mathrm{E}-04$ | 56.64 | $1.9648 \mathrm{E}-03$ | 5.22 | $1.7104 \mathrm{E}-03$ | 91.22 | $2.1286 \mathrm{E}-03$ | 8.94 | $2.7346 \mathrm{E}-03$ |
| 0.9 | 50.95 | $1.6745 \mathrm{E}-03$ | 4.27 | $9.0524 \mathrm{E}-04$ | 46.88 | $1.9026 \mathrm{E}-03$ | 4.32 | $1.7353 \mathrm{E}-03$ | 85.62 | $2.0729 \mathrm{E}-03$ | 5.32 | $2.8430 \mathrm{E}-03$ |



Fig. 6 The plot of elapsed time in (seconds) against $N$ for Example 2 using CNFDM (red line) and MFEGM (green line)

Example 3 Consider the following time fractional diffusion equation,

$$
\left\{\begin{array}{l}
{ }_{0}^{C} D_{t}^{\alpha} P=P_{x x}+P_{y y}+f(x, y, t), 0 \leq x, y \leq 1,0<t \leq T, \\
P(x, y, 0)=0, \quad(x, y) \in \Omega \cup \partial \Omega, \\
P(0, y, t)=t^{2} e^{y}, \quad P(1, y, t)=t^{2} e^{1+y}, \\
P(x, 0, t)=t^{2} e^{x}, \quad P(x, 1, t)=t^{2} e^{x+1}, \quad(x, y, t) \in \partial \Omega \times(0, T] .
\end{array}\right.
$$

The exact solution of Example 3 is given by $u(x, y, t)=t^{2} e^{x+y}$. Table 3 shows the performance of the proposed methods in solving the considered problem. The results in this table are achieved at $T=4, \tau=0.1,0.01$ and different fractional orders. To show the accuracy of the solution methods, Fig. 7 portrays absolute errors of the CNFDM (left frame) and MFEGM (right frame) at $N=62, N_{t}=100, \alpha=0.5$ and $T=4$.

Example 4 Consider the following time fractional advection-diffusion equation,

$$
\left\{\begin{array}{l}
{ }_{0}^{C} D_{t}^{\alpha} P=P_{x x}+P_{y y}-P_{x}-P_{y}+f(x, y, t), 0 \leq x, y \leq 1,0<t \leq T, \\
P(x, y, 0)=0, \quad(x, y) \in \Omega \cup \partial \Omega, \\
P(0, y, t)=t^{2}\left(y-y^{2}\right), \quad P(1, y, t)=t^{2}\left(y-y^{2}\right), \\
P(x, 0, t)=t^{2}\left(x-x^{2}\right), \quad P(x, 1, t)=t^{2}\left(x-x^{2}\right), \quad(x, y, t) \in \partial \Omega \times(0, T] .
\end{array}\right.
$$

The exact solution of Example 4 reads as $u(x, y, t)=t^{2}\left(x-x^{2}+y-y^{2}\right)$. The results of solving this problem are shown in Table 4. Numerical simulations have been implemented at
Table 3 Numerical results of the CNFDM and the MFEGM for Example 3 at $N=50$ and $T=4$

| $\alpha$ | $\tau=0.1$ |  |  |  |  |  | $\tau=0.01$ |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | CNFDM |  |  | MFEGM |  |  | CNFDM |  |  | MFEGM |  |  |
|  | Sec | Iter | MAE | Sec | Iter | MAE | Sec | Iter | MAE | Sec | Iter | MAE |
| 0.1 | 3.65 | 2078 | $2.9837 \mathrm{E}-03$ | 0.75 | 324 | $3.7882 \mathrm{E}-03$ | 206.49 | 1521 | $7.5728 \mathrm{E}-04$ | 15.02 | 254 | $8.2790 \mathrm{E}-04$ |
| 0.2 | 3.65 | 2055 | $5.1809 \mathrm{E}-03$ | 0.68 | 321 | $6.1574 \mathrm{E}-03$ | 196.18 | 1474 | $9.5808 \mathrm{E}-04$ | 14.31 | 245 | $8.0758 \mathrm{E}-04$ |
| 0.3 | 3.63 | 2030 | $6.5976 \mathrm{E}-03$ | 0.61 | 317 | $7.7327 \mathrm{E}-03$ | 214.07 | 1409 | $9.9538 \mathrm{E}-04$ | 14.04 | 234 | $7.7698 \mathrm{E}-04$ |
| 0.4 | 3.55 | 2002 | $7.2350 \mathrm{E}-03$ | 0.59 | 312 | $8.5070 \mathrm{E}-03$ | 207.90 | 1324 | $1.0374 \mathrm{E}-03$ | 12.75 | 220 | $7.3687 \mathrm{E}-04$ |
| 0.5 | 3.63 | 1970 | $7.3058 \mathrm{E}-03$ | 0.62 | 307 | $8.7204 \mathrm{E}-03$ | 197.59 | 1217 | $1.1046 \mathrm{E}-03$ | 12.06 | 202 | $6.8067 \mathrm{E}-04$ |
| 0.6 | 3.53 | 1937 | $7.2468 \mathrm{E}-03$ | 0.57 | 302 | $8.7552 \mathrm{E}-03$ | 214.19 | 1091 | $1.1822 \mathrm{E}-03$ | 11.04 | 180 | $5.9799 \mathrm{E}-04$ |
| 0.7 | 3.69 | 1902 | $7.5522 \mathrm{E}-03$ | 0.60 | 296 | $9.1450 \mathrm{E}-03$ | 126.54 | 953 | $1.2579 \mathrm{E}-03$ | 9.36 | 157 | $5.2511 \mathrm{E}-04$ |
| 0.8 | 3.53 | 1865 | $8.8204 \mathrm{E}-03$ | 0.56 | 291 | $1.0468 \mathrm{E}-02$ | 105.35 | 811 | $1.3019 \mathrm{E}-03$ | 8.27 | 134 | $4.9347 \mathrm{E}-04$ |
| 0.9 | 3.47 | 1829 | $1.1662 \mathrm{E}-02$ | 0.58 | 285 | $1.3342 \mathrm{E}-02$ | 88.47 | 676 | $1.2187 \mathrm{E}-03$ | 7.32 | 11 | $5.7431 \mathrm{E}-04$ |



Fig. 7 The 3D error plot of absolute errors using a CNFDM and b MFEGM for Example 3 at $N=62$, $N_{t}=100, \alpha=0.5$ and $T=4$
$T=8, \tau=0.1,0.01$ and different fractional orders. Figure 8 draws absolute errors of the CNFDM (left frame) and MFEGM (right frame) at $N=62, N_{t}=100, \alpha=0.5$ and $T=8$.

## 8 Conclusion

Due to the non-local property of fractional derivatives, the numerical treatment of time FPDEs usually requires substantial computer resources, which may result in some computational challenges in practice. In this paper, we have proposed the MFEGM for solving a class of the general two-dimensional TFADRE involving source terms and Caputo-type temporal derivative. It has been constructed utilizing a difference scheme based on L1 discretization in the temporal direction and central difference approximations with double spacing in the spatial direction. We have illustrated the reliability of the MFEGM by comparing it to the CNFDM derived in Sect. 3. The stability and convergence of the proposed methods have been proven theoretically and comprehended by several numerical examples. The comparison of the numerical results obtained by the MFEGM with the numerical results obtained by the CNFDM shows the accuracy, computational efficiency and simple implementation of the former. We have also discussed the application of the methods to the fractional diffusion and fractional advection-diffusion models, from which we deduce the potential capability of the MFEGM to solve time fractional diffusion models and other FPDEs that arise in science and engineering disciplines, providing a new line of future research.
Table 4 Numerical results of the CNFDM and the MFEGM for Example 4 at $N=50$ and $T=8$

| $\alpha$ | $\tau=0.1$ |  |  |  |  |  | $\tau=0.01$ |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | CNFDM |  |  | MFEGM |  |  | CNFDM |  |  | MFEGM |  |  |
|  | Sec | Iter | MAE | Sec | Iter | MAE | Sec | Iter | MAE | Sec | Iter | MAE |
| 0.1 | 3.84 | 1910 | $6.2849 \mathrm{E}-03$ | 0.69 | 300 | $6.1501 \mathrm{E}-03$ | 200.02 | 1364 | $1.6307 \mathrm{E}-03$ | 22.07 | 231 | $5.5616 \mathrm{E}-04$ |
| 0.2 | 3.83 | 1897 | $1.3224 \mathrm{E}-02$ | 0.73 | 298 | $1.2951 \mathrm{E}-02$ | 189.04 | 1333 | $1.8731 \mathrm{E}-03$ | 13.45 | 226 | $6.5897 \mathrm{E}-04$ |
| 0.3 | 3.46 | 1885 | $2.0344 \mathrm{E}-02$ | 0.73 | 296 | $1.9924 \mathrm{E}-02$ | 190.39 | 1292 | $1.7839 \mathrm{E}-03$ | 14.57 | 219 | $6.7402 \mathrm{E}-04$ |
| 0.4 | 3.74 | 1872 | $2.7215 \mathrm{E}-02$ | 0.70 | 294 | $2.6673 \mathrm{E}-02$ | 209.78 | 1240 | $1.7533 \mathrm{E}-03$ | 13.51 | 209 | $6.9541 \mathrm{E}-04$ |
| 0.5 | 3.53 | 1859 | $3.3463 \mathrm{E}-02$ | 0.83 | 292 | $3.2781 \mathrm{E}-02$ | 219.93 | 1176 | $1.7842 \mathrm{E}-03$ | 12.53 | 198 | 7.1742E-04 |
| 0.6 | 3.60 | 1846 | $3.8725 \mathrm{E}-02$ | 0.62 | 290 | $3.7938 \mathrm{E}-02$ | 287.81 | 1101 | $1.8109 \mathrm{E}-03$ | 11.45 | 185 | $7.3930 \mathrm{E}-04$ |
| 0.7 | 3.53 | 1834 | $4.2779 \mathrm{E}-02$ | 0.72 | 288 | $4.1914 \mathrm{E}-02$ | 279.52 | 1015 | $1.8412 \mathrm{E}-03$ | 10.41 | 170 | $7.6253 \mathrm{E}-04$ |
| 0.8 | 3.46 | 1823 | $4.5552 \mathrm{E}-02$ | 0.74 | 287 | $4.4600 \mathrm{E}-02$ | 241.38 | 923 | $1.8450 \mathrm{E}-03$ | 8.79 | 155 | $7.5872 \mathrm{E}-04$ |
| 0.9 | 3.64 | 1814 | $4.6994 \mathrm{E}-02$ | 0.74 | 285 | $4.6002 \mathrm{E}-02$ | 201.77 | 828 | $1.8115 \mathrm{E}-03$ | 8.57 | 138 | $7.3205 \mathrm{E}-04$ |



Fig. 8 The 3D error plot of absolute errors using a CNFDM and $\mathbf{b}$ MFEGM for Example 4 at $N=62$, $N_{t}=100, \alpha=0.5$ and $T=8$

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## Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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