



The effects of pre-/post-retirement borrowing constraints on optimal consumption, investment, and retirement

Dayoon Kim¹ · Yong Hyun Shin²

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Abstract

In this paper, we explore an optimization problem in which a working agent with an option to retire wants to maximize her expected lifetime utility. The agent receives labor (or annuity) income before/after retirement and experiences dis-utility due to labor. Additionally, the agent can partially borrow against her future income before/after retirement. We demonstrate that when pre-retirement borrowing constraints are strict, the agent retires early, while strong post-retirement borrowing constraints result in a late retirement. Our numerical results reveal that the retirement threshold level changes with varying interest rates, with an increase in interest rates leading to a general decrease in the retirement threshold level. However, in certain cases with borrowing constraints, the declining effects of pre-/post-retirement discounted income strongly influence the retirement decision as the interest rate increases. Additionally, our findings indicate that the impact of the volatility of the risky asset and the interest rate on the retirement wealth level varies. Specifically, at low interest rates, higher volatility is associated with an increased retirement wealth level, while at high interest rates, lower volatility results in a marginal increase in the retirement wealth level. These findings suggest a diminishing influence of volatility on the retirement wealth level as interest rates rise.

Keywords Portfolio selection · Consumption · Voluntary retirement · Income · Borrowing constraints · Dynamic programming methods

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✉ Yong Hyun Shin
yhshin@sookmyung.ac.kr
Dayoon Kim
dayoon96@sookmyung.ac.kr

¹ Department of Mathematics, Sookmyung Women's University, Seoul 04310, Republic of Korea

² Department of Mathematics and Research Institute of Natural Sciences, Sookmyung Women's University, Seoul 04310, Republic of Korea

1 Introduction

In this article, we discuss the problem of an agent's portfolio selection which was initially studied by Merton (1969, 1971). Since then, various researchers have investigated this problem under different economic conditions. Merton (1969, 1971) and Karatzas et al. (1986) used dynamic programming approaches to solve this problem by deriving the Hamilton-Jacobi-Bellman (HJB) equations. In other studies, Karatzas et al. (1987) and Cox and Huang (1989) designed martingale approaches that use the dual of the problem.

Furthermore, several researchers have studied the problem of optimal consumption and portfolio selection with a (voluntary) retirement option, including (Choi and Shim, 2006; Farhi and Panageas, 2007; Choi et al., 2008; Dybvig and Liu, 2010), and Lim and Shin (2011), etc. In these models, the retirement time is considered as the optimal stopping time (see Karatzas and Wang (2000)), and the agent cannot return to her previous job after retirement. In contrast, in job-switching models, such as those proposed by Shim and Shin (2014), Shim et al. (2018), and Lee et al. (2019), an agent can switch jobs based on their wealth levels.

In this work, we employ a model similar to the one used in Shim et al. (2018). The model in Shim et al. (2018) comprises two types of jobs (or income sources): One with high income and high dis-utility, and the other with low income and low dis-utility. There are two main differences between the model in Shim et al. (2018) and our model. First, Shim et al. (2018) did not account for borrowing constraints, while we did. Second, Shim et al. (2018) focused on job switching rather than (voluntary) retirement.

Another model similar to ours is presented in the research of Ding and Marazzina (2022). In Ding and Marazzina (2022), the authors investigated an optimal retirement problem with varying liquidity constraints and cash flows over different periods. The impact of borrowing constraints in our study is consistent with their results. Specifically, strict pre-retirement borrowing constraints lead to an optimal retirement at a lower wealth level, while strict post-retirement borrowing constraints lead to an optimal retirement at a higher wealth level. There are several differences between the two studies. Firstly, our model considers labor/dis-utility policy based on Choi and Shim (2006), whereas the model in Ding and Marazzina (2022) considers consumption-portfolio-leisure policy based on Choi et al. (2008). Moreover, we derive closed-form solutions to our optimization problems. Secondly we focus on the impact of interest rates and the stock volatility on the retirement threshold level to achieve a more sensitive analysis of the solution in our model.

In this paper, we analyze the optimization problem of an infinitely-lived working agent with an option to retire who aims to maximize her expected lifetime utility. The agent receives labor income before retirement and may earn money, such as from a part-time job or an annuity, after retirement.¹ The agent also incurs dis-utility stemming from her labor or part-time job. Additionally, the agent can partially borrow against her future labor income or annuity income

¹ We choose to consider re-employment as one of the significant issues in the labor market since many countries worldwide are experiencing an aging population. According to a report 'Pensions at a Glance 2021' from OECD (2021), this phenomenon is putting long-term financial pressure on these countries. However, promoting an age-inclusive workforce by giving older workers greater opportunities to work could help increase GDP, as noted in the OECD publication, 'Promoting an Age-Inclusive Workforce' (OECD 2020). Moreover, the Labour and Social Affairs Committee of the OECD recommends that societies should "strengthen incentives for workers to build up longer careers and to continue working at an older age" in 'Recommendation of the Council on Aging and Employment Policies' OECD (2022). Further, according to an article by the World Economic Forum, "many older people left the workforce during the pandemic, but the waning of COVID-19 and the soaring cost of living are reversing this trend. The phenomenon known as the 'Great Unretirement' is seeing older workers (over 50) return to paid employment across advanced economies" (Manktelow 2022). Therefore, we incorporate re-employment as a key aspect of our model.

before/after retirement. Furthermore, our model considers borrowing constraints both before and after retirement simultaneously, distinguishing it from other models that only incorporate borrowing constraints before retirement, such as Park and Jang (2014) and Lee et al. (2017).

One of the significant issues in the labor market is re-employment, especially for older workers. Therefore, we set our problem to consider the impact of pre-/post- retirement borrowing constraints on retirement decisions. Specifically, we examine how different borrowing constraints affect the determination of the threshold wealth level for (voluntary) retirement when the interest rate is fixed, as well as under different interest rate conditions.²

We provide some numerical implications about the effects of the strength of borrowing constraints and the interest rate on the retirement wealth level. Our research found that when pre-retirement borrowing constraints become strong, the agent may retire early to avoid them. Conversely, when post-retirement borrowing constraints become strong, the agent may retire later to avoid them. Additionally, the retirement threshold level generally decreases as the interest rate increases. However, under certain conditions, the retirement threshold level may first increase and then decrease as the interest rate increases because the decreasing effect of post-retirement (resp. pre-retirement) discounted income is stronger than that of pre-retirement (resp. post-retirement) discounted income.

In addition, this study examines the effects of both stock volatility and interest rates on retirement wealth levels. The impact of the interest rate on the threshold wealth level remains significant even when different volatility levels are implemented. Specifically, at low interest rates, the threshold wealth level increases as volatility increases. However, at high interest rates, the threshold wealth level increases despite decreasing volatility. Furthermore, the impact of volatility on the threshold wealth level is relatively small at high interest rates. This suggests that at high interest rates, the impact of interest rates on retirement wealth levels is much stronger than that of volatility. In other words, as interest rates increase, the effect of volatility decreases, resulting in threshold wealth levels that tend to have similar values regardless of the volatilities.

The remainder of the paper is structured as follows. In Sect. 2, we present the financial market model. In Sect. 3, we derive an analytic solution to the optimal consumption and portfolio selection problem with a retirement option and borrowing constraints. Section 4 provides numerical implications about the effects of the strength of borrowing constraints, stock volatility and the interest rate on the retirement wealth level. Finally, we conclude the paper in Sect. 5.

2 The financial market

We consider our financial market on an infinite-time horizon. We assume that there are two kinds of assets in the financial market: One risk-free asset with a constant interest rate $r > 0$ and one risky asset. The risky asset (or stock) S_t at time t conforms to the geometric Brownian motion (GBM)

$$dS_t = \mu S_t dt + \sigma S_t dB_t,$$

² It is worth noting that on November 3, 2022, the Federal Reserve announced that interest rates were being raised to 4 percent in response to post-pandemic inflation caused by near-zero ranged interest rates during the COVID-19 pandemic. In addition, according to the U.S. central bank's own projections, this interest rate is expected to increase further to 4.5% or up to 4.75% in 2023. Since the funds rate of the U.S. fed significantly affects the global financial market, we analyze how interest rates affect individual retirement in our study.

where $\{B_t\}_{t \geq 0}$ is a standard Brownian motion on a proper complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$. The rate of return of the risky asset $\mu (> r)$ and the volatility of the risky asset $\sigma > 0$ are both assumed to be constants. Let $\{\mathcal{F}_t\}_{t \geq 0}$ be an augmentation of the filtration generated by the Brownian motion $\{B_t\}_{t \geq 0}$ under the probability measure \mathbb{P} . Moreover, let us define the market price of risk (or Sharpe ratio) as

$$\theta \triangleq \frac{\mu - r}{\sigma} > 0.$$

Next, we denote π_t as the amount of money invested in the risky asset S_t at time t and we denote c_t as the non-negative consumption rate at time t . τ is denoted as the agent's (voluntary) retirement time from labor. We assume that $\{\pi_t\}_{t \geq 0}$ and $\{c_t\}_{t \geq 0}$ are \mathcal{F}_t -progressively measurable satisfying

$$\int_0^t \pi_s^2 ds < \infty \text{ and } \int_0^t c_s ds < \infty, \text{ for all } t \geq 0, \text{ almost surely (a.s.).}$$

The (voluntary) retirement time τ of the agent from labor is considered as the \mathcal{F}_t -stopping time. The agent receives labor income $I_1 > 0$ before retirement. Since the agent works during this period, she experiences stress as a result of her labor. This is expressed as dis-utility $L_1 > 0$ (see Choi and Shim 2006). After retirement, she may make money from (relatively) weaker labor or annuities; by contrast, it may also be impossible for her to make money. Therefore, we assume that the agent receives non-negative labor (or annuity) income $I_2 \geq 0$ with dis-utility $L_2 \geq 0$ after retirement. Obviously we see that $I_1 > I_2$ since, in general, she cannot earn more money from the labor market after retirement. Now we summarize the inequalities derived from income I_i , $i = 1, 2$ and dis-utility L_i , $i = 1, 2$ as follows:

Assumption 1

$$I_1 > I_2 \geq 0, \quad L_1 > L_2 \geq 0.$$

Remark 2 If I_1 is smaller than I_2 , then the agent should retire right away to earn more money, and it is not a case which considers (voluntary) retirement time.

Remark 3 This model is similar to that used by Shim et al. (2018). In the model of Shim et al. (2018), the agent can change her job (between a high income, high dis-utility job and a low income, low dis-utility job) infinitely without any restriction. However in this model, if the agent once decides to retire irreversibly, then she cannot return to her previous job. This is the main difference between two models.

Therefore, the agent's wealth process X_t at time t is as follows:

$$dX_t = [rX_t + \pi_t(\mu - r) - c_t + I_1 \cdot \mathbf{1}_{\{0 \leq t < \tau\}} + I_2 \cdot \mathbf{1}_{\{\tau \leq t\}}]dt + \sigma \pi_t dB_t, \quad X_0 = x, \quad (2.1)$$

where $\mathbf{1}_A$ is an indicator function equal to 1 if $x \in A$, and equal to 0 otherwise.

In the real world, it is possible for people to borrow some amount of money relative to their future income. In this paper, we allow the agent to borrow as much money as a certain portion of her future (labor or annuity) income. It follows that the agent's wealth can be negative. The agent has to abide by the *negative wealth constraint* (or *borrowing constraint*), so her wealth process should satisfy the inequalities below:

$$X_t \geq -v_1 \frac{I_1}{r}, \quad \forall t \in [0, \tau), \quad v_1 \in [0, 1) \quad (2.2)$$

and

$$X_t \geq -v_2 \frac{I_2}{r}, \quad \forall t \in [\tau, \infty), \quad v_2 \in [0, 1). \quad (2.3)$$

There are different borrowing constraints (2.2) and (2.3) before and after retirement, respectively. If $v_1 = v_2 = 1$, there are no borrowing constraints. If $v_1 = v_2 = 0$, there are non-negative wealth constraints, that is, $X_t \geq 0$, for all $t \geq 0$. It is intuitive that, as v_i ($i = 1, 2$) increases, the borrowing constraints become weaker.

Definition 4 A control $(\mathbf{c}, \boldsymbol{\pi}, \tau) \triangleq (\{c_t\}_{t \geq 0}, \{\pi_t\}_{t \geq 0}, \tau)$ is *admissible* at x if the corresponding wealth process with initial wealth x satisfies borrowing constraints (2.2) and (2.3).

3 The optimization problem

Throughout the paper, we assume that the agent has a constant relative risk aversion (CRRA) utility function of consumption, that is, the utility function $u(c)$ is defined as

$$u(c) \triangleq \frac{c^{1-\gamma}}{1-\gamma}, \quad (3.1)$$

where $\gamma > 0$ ($\gamma \neq 1$) is the coefficient of relative risk aversion.

With the CRRA utility function (3.1), the agent aims to maximize

$$V^*(x) = \max_{(\mathbf{c}, \boldsymbol{\pi}, \tau)} \mathbb{E} \left[\int_0^\infty e^{-\beta t} \{u(c_t) - L_1 \cdot \mathbf{1}_{\{0 \leq t < \tau\}} - L_2 \cdot \mathbf{1}_{\{\tau \leq t\}}\} dt \right] \quad (3.2)$$

subject to borrowing constraints (2.2) and (2.3) with the wealth process (2.1). Here, $\beta > 0$ is a subjective discount factor.

Assumption 5 We assume that the Merton's constant $K > 0$ is always positive such that the optimization problem (3.2) is well-defined,

$$K \triangleq r + \frac{\beta - r}{\gamma} + \frac{\gamma - 1}{2\gamma^2} \theta^2 > 0.$$

Assumption 6 If the agent's wealth level reaches the minimum wealth level (or borrowing limit) (pre-retirement borrowing limit is $\hat{x}_1 \triangleq -v_1 \cdot I_1/r$ and post-retirement one is $\hat{x}_2 \triangleq -v_2 \cdot I_2/r$), she liquidates all risky assets.

Remark 7 We consider the quadratic equation below for later use

$$g(m) \triangleq \frac{1}{2} \theta^2 m^2 + \left(\beta - r + \frac{1}{2} \theta^2 \right) m - r = 0. \quad (3.3)$$

$m_+ > 0$ and $m_- < -1$ are two real roots of the quadratic equation (3.3). We have the following properties for the roots m_\pm :

$$\frac{r - \frac{1}{2} \theta^2 m_\pm}{\beta} = \frac{m_\pm}{m_\pm + 1}$$

and

$$m_- < -\frac{1}{\gamma} < m_+ \Rightarrow \gamma m_- + 1 < 0$$

since $g(-1/\gamma) = -K < 0$.

Assumption 8 We assume that the below inequality always holds:

$$\kappa \triangleq \frac{\beta m_+(I_1 - I_2)}{r(m_+ + 1)(L_1 - L_2)} < \bar{c}^\gamma, \quad (3.4)$$

where \bar{c} is the consumption level of the retirement time that corresponds to the threshold retirement wealth level, which will be determined shortly (see Theorem 9).

We use the dynamic programming approach (see Merton 1969, 1971 and Karatzas et al. (1986), etc.) to derive the following Bellman equations. For $0 \leq t < \tau$, the Bellman equation induced by the optimization problem (3.2) with the wealth process (2.1) is given by

$$\beta V(x) = \max_{(c, \pi)} \left[\{rx + \pi(\mu - r) - c + I_1\} V'(x) + \frac{1}{2} \pi^2 \sigma^2 V''(x) + \frac{c^{1-\gamma}}{1-\gamma} - L_1 \right] \quad (3.5)$$

and, for $t \geq \tau$, we obtain the Bellman equation similar to the pre-retirement Bellman equation (3.5) as follows:

$$\beta V(x) = \max_{(c, \pi)} \left[\{rx - c + \pi(\mu - r) + I_2\} V'(x) + \frac{1}{2} \pi^2 \sigma^2 V''(x) + \frac{c^{1-\gamma}}{1-\gamma} - L_2 \right]. \quad (3.6)$$

The next theorem presents the solutions to the Bellman equations (3.5) and (3.6) with borrowing constraints (2.2) and (2.3).

Theorem 9 The solutions to the Bellman equations (3.5) and (3.6) are given by

$$V(x) = \begin{cases} V_{pre}(x) = \frac{m_+}{m_+ + 1} A_1 \zeta_1^{-\gamma(m_+ + 1)} + \frac{m_-}{m_- + 1} A_2 \zeta_1^{-\gamma(m_- + 1)} \\ \quad + \frac{1}{(1-\gamma)K} \zeta_1^{1-\gamma} - \frac{L_1}{\beta}, \text{ for } 0 \leq t < \tau, \\ V_{post}(x) = \frac{m_+}{m_+ + 1} B_1 \zeta_2^{-\gamma(m_+ + 1)} + \frac{1}{(1-\gamma)K} \zeta_2^{1-\gamma} - \frac{L_2}{\beta}, \text{ for } t \geq \tau, \end{cases} \quad (3.7)$$

where ζ_1 and ζ_2 are solutions to the algebraic equations

$$x = A_1 \zeta_1^{-\gamma m_+} + A_2 \zeta_1^{-\gamma m_-} + \frac{1}{K} \zeta_1 - \frac{I_1}{r}$$

and

$$x = B_1 \zeta_2^{-\gamma m_+} + \frac{1}{K} \zeta_2 - \frac{I_2}{r},$$

respectively. Here

$$B_1 = \frac{1}{\gamma m_+ K} \hat{c}_2^{\gamma m_+ + 1} > 0, \quad \hat{c}_2 = \frac{\gamma m_+ K}{\gamma m_+ + 1} (1 - v_2) \frac{I_2}{r} > 0.$$

Moreover, if we determine the pair solution (\bar{c}, ξ) , $\xi \in (0, 1)$ to the system of the following algebraic equations

$$0 = \frac{\gamma(m_+ + 1)(m_- + 1)(L_1 - L_2)}{\beta} \bar{c}^\gamma - \frac{\gamma m_- + 1}{K} \xi^{\gamma m_+ + 1} \bar{c} + \frac{\gamma m_- (1 - v_1) I_1}{r} \xi^{\gamma m_+} \\ - \frac{\gamma m_- (m_+ + 1)(I_1 - I_2)}{r} + \frac{m_+ - m_-}{m_+ K} \left[\frac{\gamma m_+ K (1 - v_2) I_2}{r(\gamma m_+ + 1)} \right]^{\gamma m_+ + 1} \bar{c}^{-\gamma m_+}$$

and

$$0 = \frac{\gamma(m_+ + 1)(m_- + 1)(L_1 - L_2)}{\beta} \bar{c}^\gamma - \frac{\gamma m_+ + 1}{K} \xi^{\gamma m_- + 1} \bar{c} + \frac{\gamma m_+ (1 - v_1) I_1}{r} \xi^{\gamma m_-} - \frac{\gamma m_+ (m_- + 1)(I_1 - I_2)}{r},$$

then we derive

$$\begin{aligned} A_1 &= \frac{(m_+ + 1)(m_- + 1)}{m_+ - m_-} \left(\frac{L_1 - L_2}{\beta} \bar{c}^\gamma - \frac{m_-}{m_- + 1} \frac{I_1 - I_2}{r} \right) \bar{c}^{\gamma m_+}, \\ &\quad + \frac{1}{\gamma m_+ K} \left[\frac{\gamma m_+ K}{\gamma m_+ + 1} (1 - v_2) \frac{I_2}{r} \right]^{\gamma m_+ + 1}, \\ A_2 &= -\frac{(m_+ + 1)(m_- + 1)}{m_+ - m_-} \left(\frac{L_1 - L_2}{\beta} \bar{c}^\gamma - \frac{m_+}{m_+ + 1} \frac{I_1 - I_2}{r} \right) \bar{c}^{\gamma m_-}, \\ \bar{x} &= A_1 \bar{c}^{-\gamma m_+} + A_2 \bar{c}^{-\gamma m_-} + \frac{1}{K} \bar{c} - \frac{I_1}{r} \\ &= B_1 \bar{c}^{-\gamma m_+} + \frac{1}{K} \bar{c} - \frac{I_2}{r}, \end{aligned}$$

where \bar{x} is the threshold wealth level for (voluntary) retirement.

Further, the candidate optimal policies (c^*, π^*, τ^*) are given by

$$\begin{aligned} c_t^* &= \begin{cases} \xi_1, & \text{for } 0 \leq t < \tau \\ \xi_2, & \text{for } t \geq \tau \end{cases} \\ \pi_t^* &= \begin{cases} \frac{\theta}{\sigma \gamma} \left(-\gamma m_+ A_1 \xi_1^{-\gamma m_+} - \gamma m_- A_2 \xi_1^{-\gamma m_-} + \frac{1}{K} \xi_1 \right), & \text{for } 0 \leq t < \tau \\ \frac{\theta}{\sigma \gamma} \left(-\gamma m_+ B_1 \xi_2^{-\gamma m_+} + \frac{1}{K} \xi_2 \right), & \text{for } t \geq \tau \end{cases} \end{aligned}$$

and the optimal stopping time is given by

$$\tau^* = \inf \{t \geq 0 : X_t \geq \bar{x}\},$$

where

$$X_t = A_1 \xi_1^{-\gamma m_+} + A_2 \xi_1^{-\gamma m_-} + \frac{1}{K} \xi_1 - \frac{I_1}{r}, \text{ for } 0 \leq t < \tau$$

and

$$X_t = B_1 \xi_2^{-\gamma m_+} + \frac{1}{K} \xi_2 - \frac{I_2}{r}, \text{ for } t \geq \tau,$$

respectively.

Proof First, we take the pre-retirement case into consideration. For $0 \leq t < \tau$, the first order conditions (FOCs) of the Bellman equation (3.5) yield

$$c^* = (V'(x))^{-\frac{1}{\gamma}}, \quad \pi^* = -\frac{\theta}{\sigma} \frac{V'(x)}{V''(x)}. \quad (3.8)$$

Inserting these FOCs (3.8) into equation (3.5), we obtain

$$-\beta V(x) + rx V'(x) - \frac{1}{2} \theta^2 \frac{(V'(x))^2}{V''(x)} + \frac{\gamma}{1 - \gamma} (V'(x))^{-\frac{1-\gamma}{\gamma}} + I_1 V'(x) - L_1 = 0. \quad (3.9)$$

We assume that the optimal consumption c is a function of wealth. Specifically, we assume that there exists a function $C(\cdot)$ such that $c = C(x)$, and $X(\cdot)$ is the inverse function of $C(\cdot)$, that is, $X(C(x)) = x$. The functions $X(\cdot)$ and $C(\cdot)$ are both increasing functions (as will be shown in Lemma 11). These functions with FOCs (3.8) suggests

$$V'(x) = (C(x))^{-\gamma}, \quad V''(x) = -\gamma(C(x))^{-\gamma-1}C'(x) = -\gamma \frac{(C(x))^{-\gamma-1}}{X'(c)}. \quad (3.10)$$

Substituting (3.10) back into the equation (3.9), we obtain

$$-\beta V(X(c)) + rc^{-\gamma}X(c) + \frac{1}{2\gamma}\theta^2 c^{1-\gamma}X'(c) + \frac{\gamma}{1-\gamma}c^{1-\gamma} + I_1c^{-\gamma} - L_1 = 0. \quad (3.11)$$

Taking the derivative of equation (3.11) with respect to c and dividing it by $c^{-\gamma-1}$, we obtain the second-order ordinary differential equation (ODE)

$$\frac{1}{2\gamma}\theta^2 c^2 X''(c) + \left(r - \beta + \frac{1-\gamma}{2\gamma}\theta^2\right)cX'(c) - r\gamma X(c) + \gamma c - \gamma I_1 = 0. \quad (3.12)$$

If we conjecture a general solution of the form $X_g(c) = c^{-\gamma m}$ to the homogeneous equation of the ODE (3.12), then we obtain the quadratic equation

$$\frac{1}{2}\theta^2 m^2 + \left(\beta - r + \frac{1}{2}\theta^2\right)m - r = 0$$

with two real roots $m_+ > 0$ and $m_- < -1$. We also get the particular solution to the ODE (3.12) of the form

$$X_p(c) = \frac{1}{K}c - \frac{I_1}{r}.$$

Therefore, we have the solution to the ODE (3.12) as

$$X_{pre}(c) = A_1 c^{-\gamma m_+} + A_2 c^{-\gamma m_-} + \frac{1}{K}c - \frac{I_1}{r}, \quad (3.13)$$

where A_1 and A_2 are constants determined later. The solution to Bellman equation (3.5) can be determined from equation (3.11) as follows:

$$\begin{aligned} V(x) = V_{pre}(x) &= \frac{m_+}{m_+ + 1}A_1 \zeta_1^{-\gamma(m_+ + 1)} + \frac{m_-}{m_- + 1}A_2 \zeta_1^{-\gamma(m_- + 1)} \\ &\quad + \frac{1}{(1-\gamma)K}\zeta_1^{1-\gamma} - \frac{L_1}{\beta}, \end{aligned} \quad (3.14)$$

where ζ_1 is a solution to the algebraic equation

$$x = A_1 \zeta_1^{-\gamma m_+} + A_2 \zeta_1^{-\gamma m_-} + \frac{1}{K}\zeta_1 - \frac{I_1}{r}. \quad (3.15)$$

FOCs (3.8) and (3.10) imply that the optimal consumption c^* is ζ_1 —which is the unique solution to algebraic equation (3.15)—and that the optimal portfolio π^* is given by

$$\pi^* = \frac{\theta}{\sigma\gamma}\zeta_1 X'(\zeta_1) = \frac{\theta}{\sigma\gamma} \left(-\gamma m_+ A_1 \zeta_1^{-\gamma m_+} - \gamma m_- A_2 \zeta_1^{-\gamma m_-} + \frac{1}{K}\zeta_1 \right). \quad (3.16)$$

For $t \geq \tau$, we obtain Bellman equation (3.6) in a manner similar to the pre-retirement Bellman equation (3.5),

and the solution to Bellman equation (3.6) is

$$V(x) = V_{post}(x) = \frac{m_+}{m_+ + 1} B_1 \zeta_2^{-\gamma(m_+ + 1)} + \frac{1}{(1 - \gamma)K} \zeta_2^{1 - \gamma} - \frac{L_2}{\beta} \quad (3.17)$$

and

$$X_{post}(c) = B_1 c^{-\gamma m_+} + \frac{1}{K} c - \frac{I_2}{r}. \quad (3.18)$$

Here, B_1 is a constant determined later (see 3.30), and ζ_2 is a solution to the algebraic equation

$$x = B_1 \zeta_2^{-\gamma m_+} + \frac{1}{K} \zeta_2 - \frac{I_2}{r}. \quad (3.19)$$

Similarly, the FOCs to Bellman equation (3.6) imply that the optimal consumption c^* is ζ_2 —which is the unique solution to algebraic equation (3.19)—and that the optimal portfolio π^* is given by

$$\pi^* = \frac{\theta}{\sigma \gamma} \zeta_2 X'(\zeta_2) = \frac{\theta}{\sigma \gamma} \left(-\gamma m_+ B_1 \zeta_2^{-\gamma m_+} + \frac{1}{K} \zeta_2 \right) \quad (3.20)$$

(indeed, the FOCs to Bellman equation (3.6) are the same as FOCs (3.8) and (3.10)). We now consider the boundary conditions of this problem. We denote \bar{x} as the threshold wealth level for (voluntary) retirement. This implies that the agent chooses to retire from labor when her wealth level exceeds the critical wealth level \bar{x} . Indeed, for $-v_1 \cdot I_1/r \leq x < \bar{x}$, the solution to Bellman equation (3.5) is $V_{pre}(x)$ in (3.14) and for $x \geq \bar{x}$, the solution to the Bellman equation (3.6) is $V_{post}(x)$ in (3.17). Along with \bar{x} , there exists a consumption level $\bar{c} > 0$ such that

$$X_{pre}(\bar{c}) = \bar{x} = X_{post}(\bar{c}), \quad (3.21)$$

that is

$$\bar{x} = X_{pre}(\bar{c}) = A_1 \bar{c}^{-\gamma m_+} + A_2 \bar{c}^{-\gamma m_-} + \frac{1}{K} \bar{c} - \frac{I_1}{r}, \quad (3.22)$$

$$\bar{x} = X_{post}(\bar{c}) = B_1 \bar{c}^{-\gamma m_+} + \frac{1}{K} \bar{c} - \frac{I_2}{r}. \quad (3.23)$$

Simplifying the two equations (3.22) and (3.23) by equating \bar{x} , we get

$$(A_1 - B_1) \bar{c}^{-\gamma m_+} + A_2 \bar{c}^{-\gamma m_-} - \frac{I_1 - I_2}{r} = 0.$$

Moreover, Assumption 6 assures that there is zero risky investment ($\pi^* \equiv 0$) when the wealth level reaches the minimum levels \hat{x}_1 and \hat{x}_2 of pre-/post-retirement, respectively. This is because the agent should stop investing in the risky asset due to borrowing constraints (2.2) and (2.3). Thus we have minimum wealth levels \hat{x}_1 and \hat{x}_2 with corresponding minimum consumption levels \hat{c}_1 and \hat{c}_2 , respectively, as follows:

$$\hat{x}_1 = X_{pre}(\hat{c}_1) = A_1 \hat{c}_1^{-\gamma m_+} + A_2 \hat{c}_1^{-\gamma m_-} + \frac{1}{K} \hat{c}_1 - \frac{I_1}{r} = -v_1 \frac{I_1}{r}, \quad (3.24)$$

$$\hat{x}_2 = X_{post}(\hat{c}_2) = B_1 \hat{c}_2^{-\gamma m_+} + \frac{1}{K} \hat{c}_2 - \frac{I_2}{r} = -v_2 \frac{I_2}{r}. \quad (3.25)$$

Moreover, from (3.16) and (3.20), we see that $X'_{pre}(\widehat{c}_1) = 0$ and $X'_{post}(\widehat{c}_2) = 0$ guarantee $\pi^* = 0$, that is,

$$X'_{pre}(\widehat{c}_1) = -\gamma m_+ A_1 \widehat{c}_1^{-\gamma m_+ - 1} - \gamma m_- A_2 \widehat{c}_1^{-\gamma m_- - 1} + \frac{1}{K} = 0, \quad (3.26)$$

$$X'_{post}(\widehat{c}_2) = -\gamma m_+ B_1 \widehat{c}_2^{-\gamma m_+ - 1} + \frac{1}{K} = 0. \quad (3.27)$$

From (3.24) and (3.26), we get

$$A_1 = \frac{(\gamma m_- + 1) \frac{1}{K} \widehat{c}_1 - \gamma m_- (1 - \nu_1) \frac{I_1}{r}}{\gamma(m_+ - m_-)} \widehat{c}_1^{\gamma m_+}, \quad (3.28)$$

$$A_2 = -\frac{(\gamma m_+ + 1) \frac{1}{K} \widehat{c}_1 - \gamma m_+ (1 - \nu_1) \frac{I_1}{r}}{\gamma(m_+ - m_-)} \widehat{c}_1^{\gamma m_-}. \quad (3.29)$$

From (3.25) and (3.27), we also get

$$\begin{aligned} \widehat{c}_2 &= \frac{\gamma m_+ K}{\gamma m_+ + 1} (1 - \nu_2) \frac{I_2}{r} > 0, \\ B_1 &= \frac{1}{\gamma m_+ K} \widehat{c}_2^{\gamma m_+ + 1} > 0. \end{aligned} \quad (3.30)$$

From the above paragraph, we see that the domain of function $X_{pre}(c)$ is $[\widehat{c}_1, \bar{c})$ whereas that of $X_{post}(c)$ is $[\widehat{c}_2, \infty)$. We can also see that the relationship between \widehat{c}_i ($i = 1, 2$) and \bar{c} is

$$0 < \widehat{c}_i < \bar{c}, \quad i = 1, 2.$$

Now, we define

$$\xi \triangleq \frac{\widehat{c}_1}{\bar{c}} \in (0, 1). \quad (3.31)$$

By using $\widehat{c}_1 = \xi \bar{c}$ in (3.31), A_1 in (3.28) and A_2 in (3.29) are rewritten as follows:

$$A_1 = \frac{(\gamma m_- + 1) \frac{1}{K} \xi \bar{c} - \gamma m_- (1 - \nu_1) \frac{I_1}{r}}{\gamma(m_+ - m_-)} (\xi \bar{c})^{\gamma m_+}, \quad (3.32)$$

$$A_2 = -\frac{(\gamma m_+ + 1) \frac{1}{K} \xi \bar{c} - \gamma m_+ (1 - \nu_1) \frac{I_1}{r}}{\gamma(m_+ - m_-)} (\xi \bar{c})^{\gamma m_-}. \quad (3.33)$$

Further, the smooth-pasting condition of $V(x)$ at $x = \bar{x}$ implies

$$V_{pre}(\bar{x}) = V_{post}(\bar{x}), \quad V'_{pre}(\bar{x}) = V'_{post}(\bar{x}).$$

From $V_{pre}(\bar{x}) = V_{post}(\bar{x})$, we get

$$\frac{m_+}{m_+ + 1} (A_1 - B_1) \bar{c}^{-\gamma(m_+ + 1)} + \frac{m_-}{m_- + 1} A_2 \bar{c}^{-\gamma(m_- + 1)} - \frac{L_1 - L_2}{\beta} = 0. \quad (3.34)$$

Further, from $V'_{pre}(\bar{x}) = V'_{post}(\bar{x})$ and (3.10), we get

$$V'_{pre}(\bar{x}) = V'_{post}(\bar{x}) = \bar{c}^{-\gamma} \Rightarrow X_{pre}(\bar{c}) = \bar{x} = X_{post}(\bar{c}),$$

that is,

$$(A_1 - B_1) \bar{c}^{-\gamma m_+} + A_2 \bar{c}^{-\gamma m_-} - \frac{I_1 - I_2}{r} = 0. \quad (3.35)$$

From (3.34) and (3.35), we get

$$A_1 = \frac{(m_+ + 1)(m_- + 1)}{m_+ - m_-} \left(\frac{L_1 - L_2}{\beta} \bar{c}^\gamma - \frac{m_-}{m_- + 1} \frac{I_1 - I_2}{r} \right) \bar{c}^{\gamma m_+} + \frac{1}{\gamma m_+ K} \left[\frac{\gamma m_+ K}{\gamma m_+ + 1} (1 - v_2) \frac{I_2}{r} \right]^{\gamma m_+ + 1}, \quad (3.36)$$

$$A_2 = -\frac{(m_+ + 1)(m_- + 1)}{m_+ - m_-} \left(\frac{L_1 - L_2}{\beta} \bar{c}^\gamma - \frac{m_+}{m_+ + 1} \frac{I_1 - I_2}{r} \right) \bar{c}^{\gamma m_-}. \quad (3.37)$$

We now have two versions of each A_1 and A_2 . Setting (3.32) equal to (3.36), we get

$$0 = \frac{\gamma(m_+ + 1)(m_- + 1)(L_1 - L_2)}{\beta} \bar{c}^\gamma - \frac{\gamma m_- + 1}{K} \xi^{\gamma m_+ + 1} \bar{c} + \frac{\gamma m_- (1 - v_1) I_1}{r} \xi^{\gamma m_+} - \frac{\gamma m_- (m_+ + 1)(I_1 - I_2)}{r} + \frac{m_+ - m_-}{m_+ K} \left[\frac{\gamma m_+ K (1 - v_2) I_2}{r(\gamma m_+ + 1)} \right]^{\gamma m_+ + 1} \bar{c}^{-\gamma m_+} \quad (3.38)$$

and setting (3.33) equal to (3.37), we get

$$0 = \frac{\gamma(m_+ + 1)(m_- + 1)(L_1 - L_2)}{\beta} \bar{c}^\gamma - \frac{\gamma m_+ + 1}{K} \xi^{\gamma m_- + 1} \bar{c} + \frac{\gamma m_+ (1 - v_1) I_1}{r} \xi^{\gamma m_-} - \frac{\gamma m_+ (m_- + 1)(I_1 - I_2)}{r} \quad (3.39)$$

(we will show the existence and uniqueness of the solution to the system of algebraic equations (3.38) and (3.39) in Lemma 14 with the specific case of $I_2 = L_2 = 0$). Hence, if we can solve the system of algebraic equations (3.38) and (3.39), we can decide ξ and \bar{c} and consequently obtain A_1 , A_2 and \bar{x} . \square

Lemma 10 *We see that*

$$A_1 > 0, \quad \text{and} \quad A_2 > 0.$$

Proof From A_2 in (3.37), the below term is always positive

$$\frac{L_1 - L_2}{\beta} \bar{c}^\gamma - \frac{m_+}{m_+ + 1} \frac{I_1 - I_2}{r} > 0$$

according to the inequality (3.4) in Assumption 8. Thus we see that $A_2 > 0$. From A_2 in (3.29), $A_2 > 0$ implies

$$\widehat{c}_1 < \frac{\gamma m_+ K (1 - v_1) I_1}{r(\gamma m_+ + 1)} < \frac{K (1 - v_1) I_1}{r}. \quad (3.40)$$

From A_1 in (3.28), the below term is always positive

$$\begin{aligned} (\gamma m_- + 1) \frac{1}{K} \widehat{c}_1 - \gamma m_- (1 - v_1) \frac{I_1}{r} &> (\gamma m_- + 1) \frac{(1 - v_1) I_1}{r} - \gamma m_- (1 - v_1) \frac{I_1}{r} \\ &= \frac{(1 - v_1) I_1}{r} [(\gamma m_- + 1) - \gamma m_-] = \frac{(1 - v_1) I_1}{r} \\ &> 0, \end{aligned}$$

where the first inequality is derived from $\gamma m_- + 1 < 0$ and the inequality (3.40). Thus we see that $A_1 > 0$. \square

Lemma 11 Functions $X_{pre}(c)$ for $c \in [\widehat{c}_1, \bar{c})$ in (3.13) and $X_{post}(c)$ for $c \in [\widehat{c}_2, \infty)$ in (3.18) are both increasing functions.

Proof First, let us consider the function

$$X_{pre}(c) = A_1 c^{-\gamma m_+} + A_2 c^{-\gamma m_-} + \frac{1}{K} c - \frac{I_1}{r} \text{ for } c \in [\widehat{c}_1, \bar{c}).$$

We derive the first and second derivatives of $X_{pre}(c)$ as follows:

$$\begin{aligned} X'_{pre}(c) &= -\gamma m_+ A_1 c^{-\gamma m_+ - 1} - \gamma m_- A_2 c^{-\gamma m_- - 1} + \frac{1}{K}, \\ X''_{pre}(c) &= \gamma m_+ (\gamma m_+ + 1) A_1 c^{-\gamma m_+ - 2} + \gamma m_- (\gamma m_- + 1) A_2 c^{-\gamma m_- - 2}. \end{aligned}$$

Since $m_- < -1$ and $\gamma m_- + 1 < 0$ from Remark 7, and $A_1 > 0$ and $A_2 > 0$ from Lemma 10, $X''_{pre}(c) > 0$ for $c \in (\widehat{c}_1, \bar{c})$. Moreover, since $X'_{pre}(\widehat{c}_1) = 0$ in (3.26) and $X'_{pre}(c)$ is increasing for $c \in (\widehat{c}_1, \bar{c})$, $X'_{pre}(c) > 0$ for $c \in (\widehat{c}_1, \bar{c})$, and consequently, $X_{pre}(c)$ is increasing for $c \in [\widehat{c}_1, \bar{c})$.

Let us consider the function

$$X_{post}(c) = B_1 c^{-\gamma m_+} + \frac{1}{K} c - \frac{I_2}{r}, \text{ for } c \in [\widehat{c}_2, \infty).$$

We derive the first and second derivatives of $X_{post}(c)$ as follows:

$$\begin{aligned} X'_{post}(c) &= -\gamma m_+ B_1 c^{-\gamma m_+ - 1} + \frac{1}{K}, \\ X''_{post}(c) &= \gamma m_+ (\gamma m_+ + 1) B_1 c^{-\gamma m_+ - 2}. \end{aligned}$$

Since $B_1 > 0$ in (3.30), $X''_{post}(c) > 0$ for $c \in (\widehat{c}_2, \infty)$. Since $X'_{post}(\widehat{c}_2) = 0$ in (3.27) and $X'_{post}(c)$ is increasing for $c \in (\widehat{c}_2, \infty)$, $X'_{post}(c) > 0$ for $c \in (\widehat{c}_2, \infty)$, and consequently, $X_{post}(c)$ is increasing for $c \in [\widehat{c}_2, \infty)$. \square

To demonstrate the existence and uniqueness of the solution to the system of algebraic equations (3.38) and (3.39) in the case of $I_2 = L_2 = 0$, we need the two lemmas below.

Lemma 12 Let

$$f_1(\xi) \triangleq (1 - v_1) (m_+ \xi^{\gamma m_-} - m_- \xi^{\gamma m_+}) - (m_+ - m_-), \text{ for } \xi \in (0, 1).$$

The function $f_1(\xi)$ is decreasing for $0 < \xi < 1$ and $f_1(\xi) > 0$ for $0 < \xi < \widehat{\xi}$, where $\widehat{\xi} (0 < \widehat{\xi} < 1)$ satisfies $f_1(\widehat{\xi}) = 0$.

Proof First we see that

$$f_1(0+) = +\infty, \quad f_1(1-) = -v_1(m_+ - m_-) < 0.$$

Now we consider the derivative of $f_1(\xi)$ as follows:

$$f'_1(\xi) = \gamma m_+ m_- (1 - v_1) (\xi^{\gamma m_-} - \xi^{\gamma m_+}) \xi^{-1} < 0, \text{ for } \xi \in (0, 1)$$

since $m_- < 0$ and $\xi^{\gamma m_-} > \xi^{\gamma m_+}$ for $0 < \xi < 1$. Thus $f_1(\xi)$ is decreasing for $0 < \xi < 1$. Since $f_1(0+) > 0$ and $f_1(1-) < 0$, there exists a unique $\widehat{\xi} > 0$ ($0 < \widehat{\xi} < 1$) such that $f_1(\widehat{\xi}) = 0$. Therefore, $f_1(\xi) > 0$ for $0 < \xi < \widehat{\xi}$. \square

Lemma 13 *Let*

$$f_2(\xi) \triangleq (\gamma m_+ + 1)\xi^{\gamma m_- + 1} - (\gamma m_- + 1)\xi^{\gamma m_+ + 1}, \text{ for } \xi \in (0, 1).$$

Then $f_2(\xi) > 0$ *for* $0 < \xi < 1$.

Proof First, we see that

$$f_2(0+) = +\infty, \quad f_2(1-) = \gamma(m_+ - m_-) > 0$$

since $\gamma m_- + 1 < 0$. Now we consider the derivative of $f_2(\xi)$ as follows:

$$f_2'(\xi) = (\gamma m_+ + 1)(\gamma m_- + 1)(\xi^{\gamma m_-} - \xi^{\gamma m_+}) < 0, \text{ for } \xi \in (0, 1)$$

since $\gamma m_- + 1 < 0$ and $\xi^{\gamma m_-} > \xi^{\gamma m_+}$ for $0 < \xi < 1$. Thus, $f_2(\xi)$ is decreasing for $0 < \xi < 1$, and consequently, $f_2(\xi) > 0$ for $0 < \xi < 1$, since $f_2(0+) > 0$ and $f_2(1-) > 0$. \square

Using Lemmas 12 and 13, we will demonstrate the existence and uniqueness of the solution to the system of algebraic equations (3.38) and (3.39) in the case of $I_2 = L_2 = 0$ in the following lemma:

Lemma 14 ($I_2 = L_2 = 0$) *The system of algebraic equations (3.38) and (3.39) has a unique solution.*

Proof We present a proof in the case of $I_2 = L_2 = 0$. In this case, (3.38) is given by

$$\begin{aligned} \frac{\gamma(m_+ + 1)(m_- + 1)L_1}{\beta} \bar{c}^\gamma &= \frac{\gamma m_- + 1}{K} \xi^{\gamma m_+ + 1} \bar{c} - \frac{\gamma m_- (1 - v_1) I_1}{r} \xi^{\gamma m_+} \\ &\quad + \frac{\gamma m_- (m_+ + 1) I_1}{r} \end{aligned} \quad (3.41)$$

and (3.39) is given by

$$\begin{aligned} \frac{\gamma(m_+ + 1)(m_- + 1)L_1}{\beta} \bar{c}^\gamma &= \frac{\gamma m_+ + 1}{K} \xi^{\gamma m_- + 1} \bar{c} - \frac{\gamma m_+ (1 - v_1) I_1}{r} \xi^{\gamma m_-} \\ &\quad + \frac{\gamma m_+ (m_- + 1) I_1}{r}. \end{aligned} \quad (3.42)$$

Here, setting (3.41) equal to (3.42), we get

$$\begin{aligned} \bar{c} &= f_3(\xi) \triangleq \frac{\gamma K I_1}{r} \cdot \frac{(1 - v_1)(m_+ \xi^{\gamma m_-} - m_- \xi^{\gamma m_+}) - (m_+ - m_-)}{(\gamma m_+ + 1)\xi^{\gamma m_- + 1} - (\gamma m_- + 1)\xi^{\gamma m_+ + 1}} \\ &= \frac{\gamma K I_1}{r} \cdot \frac{f_1(\xi)}{f_2(\xi)}, \end{aligned} \quad (3.43)$$

where functions $f_1(\xi)$ and $f_2(\xi)$ are defined in Lemmas 12 and 13, respectively. Then, we see that

$$f_3(0+) = \frac{\gamma K I_1}{r} \lim_{\xi \rightarrow 0+} \left[\frac{(1 - v_1)m_+}{(\gamma m_+ + 1)\xi} - \frac{m_+ - m_-}{\gamma m_+ + 1} \xi^{-(\gamma m_- + 1)} \right] = +\infty$$

since $\gamma m_- + 1 < 0$ and

$$f_3(\widehat{\xi}) = \frac{\gamma K I_1}{r} \cdot \frac{f_1(\widehat{\xi})}{f_2(\widehat{\xi})} = 0,$$

where $\widehat{\xi}$ ($0 < \widehat{\xi} < 1$) satisfying $f_1(\widehat{\xi}) = 0$ is given in Lemma 12. Thus, we see that

$$\bar{c} = f_3(\xi) > 0, \text{ for } 0 < \xi < \widehat{\xi}$$

(since we need $\bar{c} > 0$, we consider this proof process for $0 < \xi < \widehat{\xi}$).

Now, let us consider the function $h(\xi) \triangleq f'_1(\xi)f_2(\xi) - f_1(\xi)f'_2(\xi)$ for $0 < \xi < \widehat{\xi}$ as follows:

$$\begin{aligned} h(\xi) &= (1 - v_1) (\xi^{\gamma m_-} - \xi^{\gamma m_+}) \left[\gamma m_+ m_- \{ (\gamma m_+ + 1) \xi^{\gamma m_-} - (\gamma m_- + 1) \xi^{\gamma m_+} \} \right. \\ &\quad \left. - (\gamma m_+ + 1)(\gamma m_- + 1) (m_+ \xi^{\gamma m_-} - m_- \xi^{\gamma m_+}) \right] \\ &\quad + (m_+ - m_-)(\gamma m_+ + 1)(\gamma m_- + 1) (\xi^{\gamma m_-} - \xi^{\gamma m_+}) \\ &= (1 - v_1) (\xi^{\gamma m_-} - \xi^{\gamma m_+}) \left[m_- (\gamma m_- + 1) \xi^{\gamma m_+} - m_+ (\gamma m_+ + 1) \xi^{\gamma m_-} \right] \\ &\quad + (m_+ - m_-)(\gamma m_+ + 1)(\gamma m_- + 1) (\xi^{\gamma m_-} - \xi^{\gamma m_+}) \\ &< (1 - v_1) (\xi^{\gamma m_-} - \xi^{\gamma m_+}) \left[m_- (\gamma m_- + 1) - m_+ (\gamma m_+ + 1) \right] \\ &\quad + (m_+ - m_-)(\gamma m_+ + 1)(\gamma m_- + 1) (\xi^{\gamma m_-} - \xi^{\gamma m_+}) \\ &= (m_+ - m_-) (\xi^{\gamma m_-} - \xi^{\gamma m_+}) \\ &\quad \times \left[- (1 - v_1)(\gamma m_+ + \gamma m_- + 1) + (\gamma m_+ + 1)(\gamma m_- + 1) \right] \\ &< 0, \end{aligned} \tag{3.44}$$

where the first inequality is derived from $\xi^{\gamma m_+} < 1 < \xi^{\gamma m_-}$ for $0 < \xi < \widehat{\xi}$, and the second inequality holds if $\gamma m_+ + \gamma m_- + 1 \geq 0$. If $\gamma m_+ + \gamma m_- + 1 < 0$, then from (3.44), we obtain

$$\begin{aligned} h(\xi) &< (m_+ - m_-) (\xi^{\gamma m_-} - \xi^{\gamma m_+}) \left[- (1 - v_1)(\gamma m_+ + \gamma m_- + 1) + (\gamma m_+ + 1)(\gamma m_- + 1) \right] \\ &\leq (m_+ - m_-) (\xi^{\gamma m_-} - \xi^{\gamma m_+}) \left[- (\gamma m_+ + \gamma m_- + 1) + (\gamma^2 m_+ m_- + \gamma m_+ + \gamma m_- + 1) \right] \\ &= \gamma^2 m_+ m_- (m_+ - m_-) (\xi^{\gamma m_-} - \xi^{\gamma m_+}) \\ &< 0, \end{aligned}$$

where the second inequality is derived from $(1 - v_1) \leq 1$. Thus, we see that $h(\xi) < 0$ for $0 < \xi < \widehat{\xi}$ regardless of the sign of $\gamma m_+ + \gamma m_- + 1$. This implies that $f_3(\xi)$ is a decreasing function for $0 < \xi < \widehat{\xi}$.

Now, plugging (3.43) into (3.41), we obtain the algebraic equation with respect to ξ as follows:

$$\begin{aligned} &\frac{\gamma(m_+ + 1)(m_- + 1)L_1}{\beta} f_3(\xi)^\gamma - \frac{\gamma m_- + 1}{K} \xi^{\gamma m_+ + 1} f_3(\xi) + \frac{\gamma m_- (1 - v_1) I_1}{r} \xi^{\gamma m_+} \\ &\quad - \frac{\gamma m_- (m_+ + 1) I_1}{r} = 0. \end{aligned}$$

Let us define a function $F(\xi)$ as follows: For $\xi \in (0, \widehat{\xi})$,

$$F(\xi) \triangleq \frac{\gamma(m_+ + 1)(m_- + 1)L_1}{\beta} f_3(\xi)^\gamma - \frac{\gamma m_- + 1}{K} \xi^{\gamma m_+ + 1} f_3(\xi) + \frac{\gamma m_- (1 - v_1) I_1}{r} \xi^{\gamma m_+} - \frac{\gamma m_- (m_+ + 1) I_1}{r}.$$

Then, we see that

$$F(0+) = \frac{\gamma(m_+ + 1)(m_- + 1)L_1}{\beta} f_3(0+)^\gamma - \frac{\gamma m_- (m_+ + 1) I_1}{r} = -\infty$$

since $m_- < -1$, and

$$\begin{aligned} F(\widehat{\xi}-) &= \frac{\gamma m_- I_1}{r} \left[(1 - v_1) \widehat{\xi}^{\gamma m_+} - (m_+ + 1) \right] \\ &= \frac{\gamma m_- I_1}{r} \left[(1 - v_1) \widehat{\xi}^{\gamma m_+} - 1 \right] - \frac{\gamma m_+ m_- I_1}{r} \\ &> 0 \end{aligned}$$

since $m_+ > 0$, $m_- < -1$ and $(1 - v_1) \widehat{\xi}^{\gamma m_+} < 1$. Thus, $F(\xi) = 0$ has at least one solution for $\xi \in (0, \widehat{\xi})$.

Now, we suppose that there are two different solutions to the algebraic equation $F(\xi) = 0$ for $\xi \in (0, \widehat{\xi})$, that is, there are ξ_1 and ξ_2 ($0 < \xi_1 < \xi_2 < \widehat{\xi}$) such that $F(\xi_1) = 0$ and $F(\xi_2) = 0$. From (3.43), there exist \bar{c}_i ($i=1,2$) such that $\bar{c}_i = f_3(\xi_i)$ ($i = 1, 2$). Since $f_3(\xi)$ is a decreasing function, we have

$$\bar{c}_1 = f_3(\xi_1) > f_3(\xi_2) = \bar{c}_2.$$

However, Lemma 11 implies that $X_{pre}(c)$ for $c \in [\widehat{c}_1, \bar{c})$ in (3.13) and $X_{post}(c)$ for $c \in [\bar{c}_2, \infty)$ in (3.18) are both increasing functions, and \bar{x} is determined uniquely from (3.21). So $\bar{c} = \bar{c}_1 = \bar{c}_2$. This is contradiction to the assumption $\xi_1 < \xi_2$, so we have $\xi_1 = \xi_2$. \square

Theorem 15 (Verification Theorem) *The value function $V^*(x)$ of the optimization problem (3.2) is equal to the solution $V(x)$ in (3.7) to the Bellman equations (3.5) and (3.6). It therefore follows that the candidate optimal policies in Theorem 9 are also the optimal policies of the optimization problem (3.2).*

Proof The proof is similar to that of Theorem 3 in Lee et al. (2017), so we omit it. \square

4 Numerical illustrations

In this section, we present numerical illustrations. Firstly, we demonstrate how different borrowing constraints before/after retirement impact the determination of the threshold wealth level \bar{x} for (voluntary) retirement when the interest rate r is fixed. Then, we observe the threshold wealth level \bar{x} under different borrowing constraints when the interest rate r changes. Finally, we analyze the impact of the volatility of the risky asset and interest rate on the threshold wealth level.

Remark 16 It can be observed that a stronger borrowing constraint corresponds to a lower value of v_i ($i = 1, 2$), while a weaker borrowing constraint corresponds to a higher value of v_i ($i = 1, 2$). In addition, a lower threshold retirement wealth level \bar{x} leads to an earlier retirement time, whereas a higher threshold retirement wealth level \bar{x} results in a later retirement time.

Table 1 Threshold wealth level \bar{x} when ν_1 and ν_2 are changed with $r = 0.02$ ($I_1 = 1$, $I_2 = L_2 = 0.2$, $\beta = 0.08$, $\mu = 0.08$, $\sigma = 0.2$, $\gamma = 2$)

$\nu_2 = 0$		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$\nu_1 = 0$	20.945	20.132	19.333	18.550	17.789	17.054	16.353	15.696	15.099	14.590
0.1	23.871	23.152	22.451	21.771	21.117	20.491	19.901	19.354	18.863	18.450
0.2	26.235	25.570	24.926	24.303	23.705	23.136	22.602	22.108	21.667	21.298
0.3	28.230	27.603	26.996	26.410	25.849	25.316	24.817	24.357	23.947	23.603
0.4	29.951	29.351	28.771	28.212	27.677	27.170	26.696	26.259	25.870	25.544
0.5	31.451	30.872	30.312	29.773	29.258	28.771	28.314	27.895	27.521	27.209
0.6	32.760	32.197	31.653	31.130	30.631	30.158	29.716	29.310	28.948	28.646
0.7	33.895	33.345	32.814	32.303	31.816	31.355	30.924	30.528	30.176	29.882
0.8	34.858	34.318	33.797	33.296	32.819	32.367	31.944	31.556	31.211	30.924
0.9	35.631	35.099	34.586	34.093	33.622	33.177	32.761	32.380	32.040	31.757

Table 2 Threshold wealth level \bar{x} when ν_1 and ν_2 are changed with $r = 0.05$ ($I_1 = 1$, $I_2 = L_2 = 0.2$, $\beta = 0.08$, $\mu = 0.08$, $\sigma = 0.2$, $\gamma = 2$)

$\nu_2 = 0$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$\nu_1 = 0$	15.847	15.835	15.826	15.819	15.814	15.810	15.807	15.806	15.804
0.1	16.203	16.192	16.184	16.177	16.172	16.168	16.166	16.164	16.163
0.2	16.480	16.470	16.462	16.455	16.450	16.447	16.444	16.443	16.442
0.3	16.690	16.680	16.672	16.666	16.661	16.658	16.656	16.654	16.653
0.4	16.844	16.835	16.827	16.821	16.816	16.813	16.810	16.809	16.808
0.5	16.953	16.944	16.936	16.930	16.925	16.922	16.920	16.918	16.917
0.6	17.026	17.016	17.009	17.003	16.998	16.995	16.993	16.991	16.990
0.7	17.070	17.061	17.053	17.047	17.043	17.039	17.037	17.036	17.035
0.8	17.093	17.084	17.076	17.070	17.066	17.063	17.060	17.059	17.058
0.9	17.102	17.093	17.085	17.079	17.075	17.071	17.069	17.068	17.067

Table 3 Threshold wealth level \bar{x} when r and v_2 are changed with $v_1 = 0.1$ ($I_1 = L_1 = 1$, $I_2 = L_2 = 0.2$, $\beta = 0.08$, $\mu = 0.08$, $\sigma = 0.2$, $\gamma = 2$)

	$v_2 = 0$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$r = 0.01$	29.139	26.826	24.495	22.144	19.773	17.380	14.966	12.535	10.099	7.701
0.02	23.871	23.152	22.451	21.771	21.117	20.491	19.901	19.354	18.863	18.450
0.03	20.939	20.700	20.476	20.269	20.079	19.908	19.758	19.629	19.526	19.452
0.04	18.503	18.438	18.381	18.330	18.288	18.252	18.224	18.202	18.187	18.179
0.05	16.203	16.192	16.184	16.177	16.172	16.168	16.166	16.164	16.163	16.163
0.06	13.978	13.977	13.977	13.977	13.977	13.977	13.976	13.976	13.976	13.976

Table 4 Threshold wealth level \bar{x} when r and v_1 are changed with $v_2 = 0.4$ ($I_1 = L_1 = 1$, $I_2 = L_2 = 0.2$, $\beta = 0.08$, $\mu = 0.08$, $\sigma = 0.2$, $\gamma = 2$)

	$v_1 = 0.1$	0.2	0.3	0.4	0.5	0.6
$r = 0.01$	19.773	26.070	31.011	35.177	38.810	42.036
0.02	21.117	23.705	25.849	27.677	29.258	30.631
0.03	20.079	21.329	22.372	23.249	23.986	24.599
0.04	18.288	18.899	19.397	19.797	20.115	20.359
0.05	16.172	16.450	16.661	16.816	16.925	16.998
0.06	13.977	14.079	14.145	14.183	14.205	14.215

First, we examine how the threshold wealth level \bar{x} changes with different values of v_i , $i = 1, 2$. Tables 1 and 2 display the results for low and high interest rates, $r = 0.02$ and $r = 0.05$, respectively. Both Tables 1 and 2 show that, as v_1 increases while v_2 remains fixed, \bar{x} also increases. This effect is particularly pronounced when the interest rate r is 0.02. On the other hand, when v_2 increases while v_1 remains fixed, \bar{x} decreases, although this effect is relatively minor. This implies that if the borrowing constraint becomes stronger before retirement (i.e., v_1 decreases), the agent is likely to retire early to avoid the strong borrowing constraint. Conversely, if the borrowing constraint becomes stronger after retirement (i.e., v_2 decreases), the agent is likely to delay retirement to avoid the strong borrowing constraint after retirement. However, the impact of post-retirement borrowing constraint on retirement decisions is less significant compared to the pre-retirement borrowing constraint. This is because the pre-retirement borrowing constraint has a much stronger impact on retirement decisions than the post-retirement borrowing constraint.

The results of Table 3 show that as v_2 increases with fixed v_1 , \bar{x} decreases under low interest rate. However, when the interest rate r is 0.06, there is little change in \bar{x} ($\bar{x} \approx 13.98$) as v_2 increases. Hence, it can be observed that if the interest rate is relatively high, then v_2 has almost no impact on retirement decisions. Conversely, if the interest rate is relatively low, v_2 affects retirement decisions to some degree. As the post-retirement borrowing constraint weakens, the agent retires earlier under a low interest rate to avoid the pre-retirement borrowing constraint.

Furthermore, it can be observed that as the interest rate increases, the retirement threshold level \bar{x} decreases under low v_2 . However, under high v_2 , \bar{x} increases and then decreases as the interest rate increases, which is one of the novel features of our numerical results. It can be seen that as the interest rate increases, both the pre-retirement discounted income I_1/r and the post-retirement discounted income I_2/r decrease. Consequently, as the interest rate increases, the retirement threshold level becomes lower under low v_2 . However, in the case of high v_2 , the decreasing effect of post-retirement discounted income is stronger than that of pre-retirement discounted income. Thus, even though the interest rate increases, the retirement threshold level \bar{x} may increase and then decrease.

In addition, Table 4 presents similar results to those in Table 3 for a fixed v_2 . It is evident that for a fixed v_2 , \bar{x} increases as v_1 increases. However, the rate of change of \bar{x} differs depending on the interest rate r . When the interest rate is low, \bar{x} increases significantly as v_1 increases. On the other hand, if the interest rate is relatively high, such as $r = 0.06$, \bar{x} hardly changes ($\bar{x} \approx 13.98 \sim 14.22$) as v_1 increases. Similar to the above results of Table 3, v_1 has almost no impact on retirement decisions if the interest rate is relatively high. Conversely, when the interest rate is relatively low, v_1 affects retirement decisions to a certain extent; in

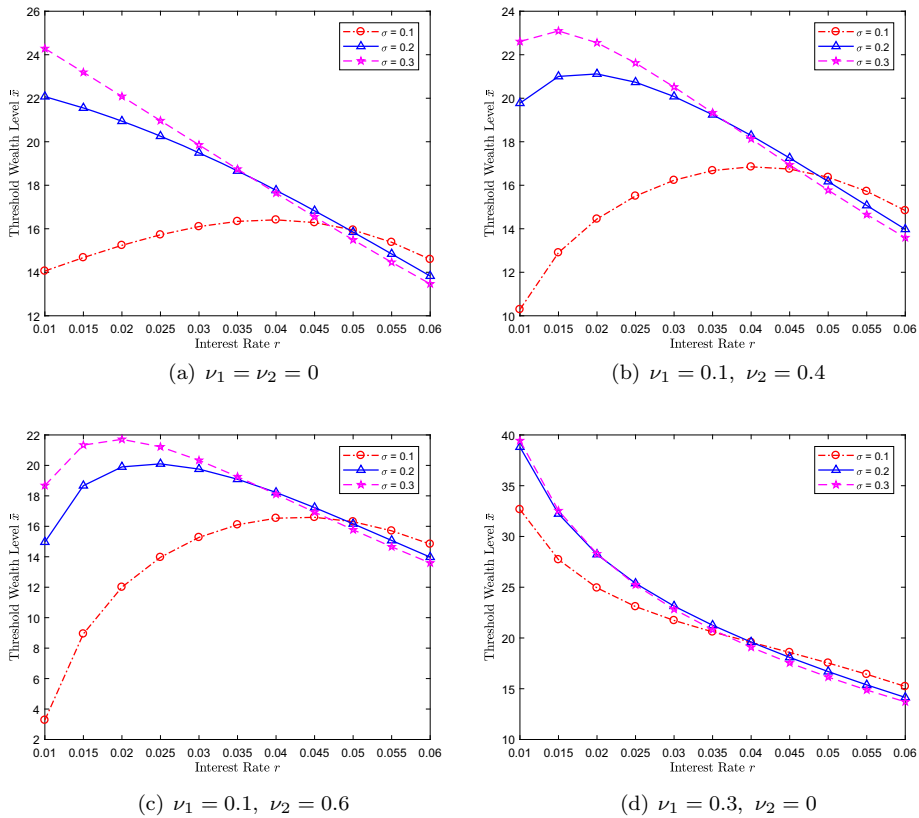


Fig. 1 $I_1 = L_1 = 1$, $I_2 = L_2 = 0.2$, $\beta = 0.08$, $\mu = 0.08$, $\gamma = 2$

particular, as the pre-retirement borrowing constraint becomes weak, the agent retires late under a low interest rate to avoid the post-retirement borrowing constraint.

Furthermore, it can be observed that if ν_1 is relatively high, the retirement threshold level \bar{x} decreases as the interest rate increases. However, if ν_1 is relatively low, \bar{x} initially increases and then decreases as the interest rate rises. This indicates that when ν_1 is considerably low, the impact of the decreasing pre-retirement discounted income is stronger than that of post-retirement discounted income. Consequently, even though the interest rate increases, the retirement threshold level \bar{x} may first increase and then decrease.

In the final analysis, we examine the effects of the volatility of the risky asset σ and the interest rate r on the retirement wealth level \bar{x} . The impact of the interest rate on the threshold wealth level remains significant even when different levels of volatility are implemented. It is noteworthy that as interest rates increase, the effect of σ on \bar{x} diminishes. For instance, in Fig. 1(a), when the interest rate is 0.01, the values of \bar{x} vary significantly across different σ values, approximately 14, 22, and 24. However, when the interest rate is raised to 0.06, the values of \bar{x} become similar for each σ , briefly 14. This implies that, at low interest rates, the threshold wealth level increases as volatility increases. However, at high interest rates, the threshold wealth level increases despite decreasing volatility. Furthermore, the impact of volatility on the threshold wealth level is relatively small. This proposes that at high interest rates, the impact of interest rates on retirement wealth levels is much stronger than that of

volatility. In other words, as interest rates increase, the effect of volatility decreases, resulting in threshold wealth levels that tend to have similar values regardless of the volatilities. This indicates that high interest rates minimize the impact of market liquidity on individual retirement.

5 Concluding remarks

In this paper, we have investigated an optimization problem in which an infinitely-lived working agent with an option to retire maximizes her expected lifetime utility. The agent receives labor (or annuity) income before/after retirement and experiences dis-utility due to labor. Additionally, there are pre-/post-retirement borrowing constraints that allow the agent to partially borrow against her future labor income before/after retirement. In light of these conditions, we derived closed-form solutions for the utility maximization problem.

Our numerical illustrations reveal three primary findings. Firstly, when the interest rate is held constant, different pre-/post-retirement borrowing constraints have varying effects on the determination of the retirement threshold level. Specifically, when pre-retirement (resp. post-retirement) borrowing constraints become strong, the agent retires early (resp. late) to avoid pre-retirement (resp. post-retirement) borrowing constraints.

Secondly, we observed how the retirement threshold level responds to different interest rates, which is one of the main features of our numerical results. Generally, as the interest rate increases, the retirement threshold level decreases. However, under certain conditions, the retirement threshold level may increase and then decrease. This occurs because the decreasing effect of post-retirement (resp. pre-retirement) discounted income is stronger than that of pre-retirement (resp. post-retirement) discounted income. As a result, even if the interest rate increases, the retirement threshold level may increase and then decrease.

Finally, we analyzed the impact of the volatility of the risky asset and the interest rate on the retirement wealth level. Notably, we found that at low levels of interest rates, the retirement wealth level increases as the volatility increases, while at high levels of interest rates, the retirement wealth level increases with relatively little change despite the volatility decreasing. This insists that as interest rates increase, the effect of volatility on the threshold wealth level diminishes.

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Declarations

Conflict of interest The authors declare no financial or non-financial interests that are directly or indirectly related to the work submitted for publication.

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