Developing solution algorithm for LR-type fully interval-valued intuitionistic fuzzy linear programming problems using lexicographic-ranking method

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ARTICLE INFO

Accuracy function

Keywords: Fuzzy mathematical programming Interval-valued intuitionistic fuzzy number LR-type fuzzy number Lexicographic ranking criterion Score function

ABSTRACT

In this article, a new concept of LR-type interval-valued intuitionistic fuzzy numbers (LR-type IVIFN) has been introduced. The theory has also been enriched by demonstrating diagrammatic representations of LR-type IVIFNs and establishing arithmetic operations among these fuzzy numbers. The total order properties of lexicographic criteria have been used for ranking LR-type IVIFNs. Further, a linear programming problem having both equality as well as inequality type constraints with all the parameters as LR-type IVIFNs and unrestricted decision variables has been formulated. An algorithm to find a unique optimal solution to the problem using the lexicographic ranking method has been developed. In the proposed methodology, the given linear programming problem is converted to an equivalent mixed 0–1 lexicographic non-linear programming problem. Various theorems have been proved to show the equivalence of the proposed problem and its different constructions. The model formulation, algorithm and discussed results have not only developed a new idea but also generalized various well-known related works existing in the literature. A numerical problem has also been exemplified to show the steps involved in the approach. Finally, a practical application in production planning is framed, solved and analyzed to establish the applicability of the study.

1. Introduction

Linear programming problems (LPPs) are the simplest kind of optimization problem that are widely used to solve many real-life problems. In conventional LPPs, all the parameters and decision variables are taken to be precise real numbers. However, in practical situations due to various uncontrollable factors, the data may not be available as crisp values. It may involve some vagueness/ambiguity in all or some of the parameters and/or decision variables of the problem. A general fuzzy LPP can be modeled as follows:

(P) max (or min)
$$\sum_{j=1}^{n} c_j x_j$$
subject to
$$\sum_{j=1}^{n} a_{ij} x_j \{ \leq, =, \geq \} b_i, \quad i = 1, 2, \dots, m,$$
$$x_j \geq 0, \quad j = 1, 2, \dots, n.$$

Zadeh [1] developed the concept of fuzzy sets which incorporated imprecision in the data successfully. Motivated by Zadeh's concept of fuzzy sets, Zimmermann [2] initiated and developed the theory to solve fuzzy linear programming problems (FLPPs). Tanaka and Asai [3] had first introduced the FLPPs in which both the parameters and decision variables were represented by fuzzy numbers. Initially, the researchers had extended the classical methods which were used to solve crisp LPPs to deal with FLPPs such as simplex algorithm, two-phase approach, etc. But, later on, these were proven to be incompatible with fuzzy theory. After that, linear ranking functions have been widely employed to convert FLPPs into crisp optimization problems. However, the ranking functions fail to order two such fuzzy numbers (FNs) which seems to be distinguished to a decision-maker. To overcome such limitations for the ordering of FNs, the idea of lexicographic ranking criteria came which uses multiple parameters at a time associated with an FN and

Page 1 of 43

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hence is a more effective and powerful ordering criterion. The fuzzy theory was later on extended to the intuitionistic fuzzy set theory, which is more general than the former. The detailed literature can be seen in Section 2 of this article. It was observed that while handling the uncertain and hesitant data, the intuitionistic fuzzy sets utilize exact or crisp real numbers to assign a membership and non-membership degrees for each element of the set. However, in practical situations, a decision-maker may fails to give these degrees with full confidence. Consequently, the notion of intuitionistic fuzzy sets are extended to the interval-valued intuitionistic fuzzy (IVIF) sets, which is the key motivation for our present study. The IVIF sets use an interval to define the acceptance and rejection degrees of each element in the set. Moreover, to represent a realistic situation mathematically, LR-type fuzzy numbers play a crucial role in optimization theory since any type of variation in the input data can be reflected in the mathematical model using different L and R functions. Thus, we have firstly defined the concept of LR-type IVIF numbers and then considered an LPP having all the parameters and decision variables as LR-type IVIF numbers. Further, an approach for solving such LPPs using a lexicographic ranking methodology has been proposed.

To the best of our knowledge, there is no study in the literature, describing the arithmetic operations on LR-type IVIF numbers (IVIFNs) and to find the unique optimal IVIF solution for an interval-valued intuitionistic fuzzy linear programming problem (IVIFLPP) having both linear equalities and inequalities with all the parameters represented by LR-type IVIFNs and decision variables as unrestricted LR-type IVIFNs. In many practical problems like selling / purchase of some units or profit / loss etc., unrestricted decision variables are required which can be handled through this formulation. To sum up, the key features of the present work are listed as follows:

- 1. On the basis of (α, β) -cut of *LR*-type IVIFNs, we define the score and accuracy indices of these numbers.
- 2. The basic arithmetic operations on unrestricted LR-type IVIFNs are developed using the (α, β) -cut.
- 3. Using the total order properties of the lexicographic criterion, a ranking of LR-type IVIFNs has been proposed.
- 4. Based on the introduced lexicographic ranking criterion, the LR-type IVIFLPP is converted to an equivalent mixed 0–1 lexicographic non-linear programming problem for finding a unique optimal IVIF solution of the LR-type fully IVIFLPP.
- 5. Various theorems are established to show the equivalence between the various problems obtained in the proposed algorithm.
- 6. A practical application in production planning is constructed, solved and examined using the proposed technique.

The rest of the paper is summarized as follows: In Section 2, a detailed literature review on fuzzy, intuitionistic fuzzy and IVIF theory is given. Section 3 includes some basic definitions and arithmetic operations on LR-type IVIFNs. A lexicographic ranking criterion is also proposed to rank two LR-type IVIFNs. The mathematical formulation of an IVIFLPP is described in Section 4 and a lexicographic method has been proposed to find the unique IVIF optimal solution of LR-type IVIFLPPs. In Section 5, the advantages of the proposed method are listed. Section 6 illustrates a numerical example to describe the proposed algorithm. A production planning problem along with its managerial insights is discussed in Section 7. The last section sums up the conclusions and some interesting future directions.

2. Literature review

2.1. Fuzzy LPP

In the model (P), if all the parameters are taken to be fuzzy numbers (FNs), then the problem is described as a fuzzy linear programming problem (FLPP). In the literature, several methods have been proposed to solve these models depending on which parameters and/or decision variables are taken to be fuzzy. A comprehensive survey on FLPPs can be found in Ebrahimnejad and Verdegay [4, ch. 2-4]. Hashemi et al. [5] considered a fully FLPP with inequality constraints having all the parameters and decision variables to be given by symmetric *LR*-type FNs and proposed a two-phase solution approach by using the lexicographic comparison of the mean and standard deviation of FNs. Allahviranloo et al. [6] used a ranking function to develop a solution algorithm to deal with the fully FLPPs having inequality constraints. Later on, Kumar et al. [7] proposed a methodology to solve the fully FLPPs with equality constraints where parameters were taken to be unrestricted and decision variables as non-negative triangular FNs. After that, Najafi and Edalatpanah [8] proposed some corrections to the methodology of Kumar et al. [7]. Khan et al. [9] studied a fully FLPP where parameters and decision variables were taken to be triangular FNs and proposed a method by making use of some ranking function. Ozkok et al. [10] extended the method of Kumar and Kaur [11] to solve fully FLPP with all types of constraints having parameters as unrestricted and decision variables as non-negative triangular FNs.

Page 2 of 43

Najafi et al. [12] examined a fully FLPP having equality constraints with parameters as well as decision variables to be expressed by unrestricted triangular FNs and developed a solution technique by converting the original model to a non-linear model. Later, Gong and Zhao [13] considered a fully FLPP with equality constraints and proposed a method in which the problem is first transformed into a crisp multi-objective LPP and then solved by using various approaches. Arana-Jiménez [14] presented a new method to find fuzzy optimal (nondominated) solutions of fully FLPPs having inequality constraints with triangular fuzzy numbers and not necessarily symmetric, via solving a multiobjective linear problem with crisp numbers. Kaur and Kumar [15] analyzed that by employing the existing approaches for solving fully FLPP, the obtained optimal solution is not necessarily unique. To overcome this limitation, they have defined a lexicographic criterion for ranking trapezoidal FNs and introduced an approach to find the unique optimal solution of fully FLPP having equality constraints with unrestricted parameters and non-negative decision variables. On similar lines, Ezzati et al. [16] introduced a lexicographic method to solve a fully FLPP with equality constraints having parameters as unrestricted triangular fuzzy and decision variables to be non-negative triangular fuzzy. Further, Mottaghi et al. [17] solved a fully FLPP with inequality constraints by introducing non-negative fuzzy slack and surplus variables for converting the inequalities into fuzzy equality constraints.

Based on a lexicographic criterion for ranking of LR-type FNs, Hosseinzadeh and Edalatpanah [18] devised a method to solve a fully FLPP having only equality constraints where the parameters and decision variables were taken to be non-positive or non-negative LR-type FNs. After that, Kaur and Kumar [19] introduced a lexicographic technique for obtaining the unique optimal solution of a fully FLPP with equality constraints having parameters and decision variables as unrestricted LR-type FNs. They pointed out that no method exists to obtain the unique optimal solution of a fully FLPP having inequalities in the set of constraints. But, some researchers [13, 17, 20] solved fully FLPPs with inequality constraints by transforming them into equality constraints using fuzzy slack and surplus variables. However, in the case of FNs, such transformations are not correct mathematically and may lead to infeasible solutions for the considered FLPP. Later, Das et al. [21] proposed a lexicographic method to solve a fully FLPP with all types of constraints keeping parameters as unrestricted and decision variables as non-negative trapezoidal FNs. But Ebrahimnejad and Verdegay [4, p. 298] demonstrated that this method is not suitable to deal with the fully FLPP having inequality constraints as the authors utilized different order relation for inequality constraints than that was used for the objective function, which is clearly false. Consequently, Ebrahimnejad and Verdegay [4, p. 299] suggested a correction by replacing the inequalities with a set of crisp linear inequalities. Pérez-Cañedo and Concepción-Morales [22] introduced a method to solve a fully FLPP having equality and inequality constraints with parameters and decision variables as unrestricted LR-type FNs, using the lexicographic ranking criterion for the objective function and the set of inequality constraints. Recently, Tadesse et al. [23] described a geometrical approach to handle the fully FLLP having non-negative decision variables. Further, some other significant applications of fuzzy theory can be found in studies of [24–26].

2.2. Intuitionistic fuzzy LPP

Atanassov [27] generalized Zadeh's concept of fuzzy sets by introducing intuitionistic fuzzy sets (IFSs) in order to include uncertainty as well as hesitation in the involved parameters. Angelov [28] was the first to apply the IFS theory to optimization problems. Mahapatra and Roy [29] developed the arithmetic operations on triangular intuitionistic fuzzy numbers (IFNs) and did reliability evaluation using these numbers. A linear programming problem (P) having equality and inequality constraints with all the parameters and decision variables expressed by IFNs is classified as a fully intuitionistic fuzzy linear programming problem (IFLPP). Nagoorgani and Ponnalagu [30] proposed a method to solve an IFLPP with inequality constraints only. Using a ranking function for IFNs, Suresh et al. [31] introduced a method to solve IFLPPs. Singh and Yadav [32] suggested the modelling and optimization of the multi-objective non-linear programming problem in an intuitionistic fuzzy environment.

Later, Arefi and Taheri [33] proposed the product of LR-type IFNs when both the numbers are either non-negative or non-positive or one is non-negative, and the other is non-positive. However, the remaining cases are not discussed. Then, Singh and Yadav [34] introduced the product of unrestricted LR-type IFNs using (α, β) -cut and proposed a method for solving fully IFLPPs using score and accuracy indices of LR-type IFNs. More review of IFS theory and its application to fully IFLPP can be seen in [35–40]. Later on, Pérez-Cañedo and Concepción-Morales [41] proposed a method using the total order properties of the lexicographic ranking criterion for finding the unique optimal intuitionistic fuzzy solution of a fully IFLPP having equality as well as inequality constraints with all the parameters and/or decision variables represented by unrestricted LR-type IFNs. Recently, Akram et al. [42] introduced a class of fully Pythagorean fuzzy linear programming problems with equality constraints and suggested a linear ranking function

: Page 3 of 43

based approach to handle such problems.

2.3. Interval-valued intuitionistic fuzzy LPP

In view of real-life situations, it is more flexible and viable to represent the membership and non-membership degrees of an element by intervals rather than crisp real numbers. Hence, Atanassov and Gargov [43] proposed the concept of interval-valued intuitionistic fuzzy (IVIF) sets. Optimizing a linear objective function over a set of linear constraints (P) where all the parameters and decision variables expressed by interval-valued intuitionistic fuzzy numbers (IVIFNs) is termed as a fully interval-valued intuitionistic fuzzy linear programming problem (IVIFLPP). Several researchers had used the idea of IVIF theory for dealing with realistic decision-making problems. Ishibuchi and Tanaka [44] were the first to solve a multi-objective programming problem in which coefficients of the objective function are intervals instead of crisp numbers. Şahin [45] suggested a ranking of IVIFNs. The basic theory and various rankings of interval-valued fuzzy numbers can be reviewed in works of [46–51]. Yang et al. [52] had studied the combination of interval-valued fuzzy sets and soft sets.

Zhang et al. [53] defined the *LR*-type interval-valued triangular FNs and proposed a method for solving multicriteria decision-making problems with *LR*-type interval –valued triangular fuzzy assessments and unknown weights. Garg et al. [54] gave an intuitionistic fuzzy optimization approach using an interval environment to solve multiobjective reliability optimization problems. Later on, Akbari and Hesamian [55] introduced signed-distance measures to rank *LR*-type interval-valued FNs and applied it to solve a multi-criteria group-decision making problem. Bharati and Singh [56] proposed a method to solve a multi-objective LPP in IVIF situations. Recently, Bharati and Singh [57] introduced an approach for solving an IVIFLPP having unrestricted parameters while decision variables are taken to be non-negative.

A brief description of the various approaches to deal with FLPPs, IFLPPs and our proposed methodology is presented in Table 1.

Table 1
Existing approaches to solve FLPPs, IFLPPs and contribution of our present study

Existing methods	Type of FN/IFN/ IVIFN	Unrestricted variables	Criterion	Type of constraints (equality and/or inequality)
Hashemi et al. [5]	LR-type FN	×	lexicographic	inequality only
Lotfi et al. [58]	Triangular FN	X	lexicographic	equality only
Kaur and Kumar [15]	Trapezoidal FN	Х	lexicographic	equality only
Kaur and Kumar [59]	LR-type FN	1	ranking function	equality and inequality both
Hosseinzadeh and Edalatpanah [18]	LR-type FN	X	lexicographic	equality only
Kaur and Kumar [19]	LR-type FN	✓	lexicographic	equality only
Pérez-Cañedo and Concepción-Morales [22]	LR-type FN	✓	lexicographic	equality and inequality both
Tadesse et al. [23]	Triangular FN	Х	geometric approach	inequality only
Nagoorgani and Ponnalagu [30]	Triangular IFN	X	score function	inequality only
Singh and Yadav [34]	LR-type IFN	1	weighted sum of score and accuracy indices	equality and inequality both
Pérez-Cañedo and Concepción-Morales [41]	LR-type IFN	✓	lexicographic	equality and inequality both
Akram et al. [42]	Pythagorean FN	/	ranking function	equality only
Bharati and Singh [57]	Triangular IVIFN	Х	expected value function	equality and inequality both
Present study	LR-type IVIFN	✓	lexicographic	equality and inequality both

3. Preliminaries

In this section, we have introduced the basic concepts related to LR-type IVIFNs followed by the arithmetic operations on them.

Definition 3.1 [43]. Let X be the universal set and Int[0,1] denote the set of all subintervals of the interval [0,1]. An interval-valued intuitionistic fuzzy set (IVIFS) is defined as a set $\tilde{A} = \{(x, \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x)) : x \in X\}$, where $\mu_{\tilde{A}}: X \to Int[0,1]$ and $\nu_{\tilde{A}}: X \to Int[0,1]$ represent the interval-valued membership and non-membership functions respectively, provided $0 \le \operatorname{Sup}(\mu_{\tilde{A}}(x)) + \operatorname{Sup}(\nu_{\tilde{A}}(x)) \le 1, \forall x \in X$.

Definition 3.2 A set $\tilde{A} = \{(x, \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x)) : x \in X\}$ where $\mu_{\tilde{A}} = [\mu_{\tilde{A}}^L, \mu_{\tilde{A}}^U]$ and $\nu_{\tilde{A}} = [\nu_{\tilde{A}}^L, \nu_{\tilde{A}}^U]$ is called a convex IVIFS if $\forall x_1, x_2 \in X$, $0 \le \lambda \le 1$, the following conditions are satisfied:

Page 4 of 43

- $\mu_{\tilde{a}}^{L}(\lambda x_1 + (1 \lambda)x_2) \ge \min\{\mu_{\tilde{a}}^{L}(x_1), \mu_{\tilde{a}}^{L}(x_2)\},$
- $\mu_{\bar{x}}^{U}(\lambda x_1 + (1 \lambda)x_2) \ge \min\{\mu_{\bar{x}}^{U}(x_1), \mu_{\bar{x}}^{U}(x_2)\},\$
- $\bullet \ v^L_{\tilde{\mathbf{A}}}(\lambda x_1 + (1-\lambda)x_2) \leq \max\{v^L_{\tilde{\mathbf{A}}}(x_1), v^L_{\tilde{\mathbf{A}}}(x_2)\} \text{ and }$
- $v_{\tilde{i}}^{U}(\lambda x_1 + (1 \lambda)x_2) \le \max\{v_{\tilde{i}}^{U}(x_1), v_{\tilde{i}}^{U}(x_2)\}.$

Definition 3.3 An IVIF set \tilde{A} in X is called normal IVIFS if there exist $x_1, x_2 \in X$ such that $\mu_{\tilde{A}}(x_1) = 1$ and $\nu_{\tilde{A}}(x_2) = 1$.

Definition 3.4 An IVIF set $\tilde{A} = \{(x, \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x)) : x \in \mathbb{R}\}$ is called an IVIFN if the following conditions hold:

- \tilde{A} is a convex IVIFS in \mathbb{R} .
- \bullet \tilde{A} is a normal IVIFS and
- $\mu_{\tilde{A}}^L$, $\mu_{\tilde{A}}^U$, $\nu_{\tilde{A}}^L$ and $\nu_{\tilde{A}}^U$ are piecewise continuous functions from \mathbb{R} to [0,1].

Mathematically, lower - upper membership and non-membership functions of an IVIFN \tilde{A} can be represented as:

$$\mu_{\tilde{A}}^{L}(x) = \begin{cases} 1, & \text{if } x = a, \\ g_{1}(x), & \text{if } a - l_{L}^{\mu} < x < a, \\ g_{2}(x), & \text{if } a < x < a + r_{L}^{\mu}, \\ 0, & \text{otherwise,} \end{cases} \qquad \mu_{\tilde{A}}^{U}(x) = \begin{cases} 1, & \text{if } x = a, \\ h_{1}(x), & \text{if } a - l_{U}^{\prime \mu} < x < a, \\ h_{2}(x), & \text{if } a < x < a + r_{U}^{\prime \mu}, \\ 0, & \text{otherwise,} \end{cases}$$

$$v_{\tilde{A}}^{L}(x) = \begin{cases} 0, & \text{if } x = a, \\ l_{1}(x), & \text{if } a - l_{L}^{v} < x < a, \\ l_{2}(x), & \text{if } a < x < a + r_{L}^{v}, \\ 1, & \text{otherwise} \end{cases} \quad \text{and} \quad v_{\tilde{A}}^{U}(x) = \begin{cases} 0, & \text{if } x = a, \\ m_{1}(x), & \text{if } a - l_{U}^{v} < x < a, \\ m_{2}(x), & \text{if } a < x < a + r_{U}^{v}, \\ 1, & \text{otherwise} \end{cases}$$

where

- (i). g_1, h_1, l_2 and m_2 are piecewise continuous and strictly increasing functions,
- (ii). g_2 , h_2 , l_1 and m_1 are piecewise continuous and strictly decreasing functions,
- $(iii). \ g_1(x) \leq h_1(x), \ g_2(x) \leq h_2(x), \ l_1(x) \leq m_1(x), \ l_2(x) \leq m_2(x), \ \forall \ x \in \mathbb{R},$
- (iv). a is called the mean value of \tilde{A} ,
- $\begin{array}{ll} (v). \ \ l_L^\mu, \ l_U^{\prime\mu}, \ l_L^\nu \ \ \text{and} \ \ l_U^{\prime\nu} \ \ \text{are respectively the left spreads of} \ \mu_{\tilde{A}}^L, \ \mu_{\tilde{A}}^U, \ v_{\tilde{A}}^L \ \ \text{and} \ v_{\tilde{A}}^U \ \ \text{and} \ \ v_{\tilde{A}}^U \ \ \ \text{and} \ \ v_{\tilde{A}}^U \ \ \ \text{and} \ \ v_{\tilde{A}}^U \ \ \text{and} \ \ \ v_{\tilde{A}}^U \ \ \text{and} \ \ \ v_{\tilde{A}}^U \ \ \ \text{and} \ \ v_{\tilde{A}}^U \ \ \ \text{and} \ \ v_{\tilde{A}}^U \ \ \ \text{and} \ \ \ v_{\tilde{A}}^U \ \ \ \text$

It can be represented as $\tilde{A} = (a; l_L^{\mu}, r_L^{\mu}, l_U^{\prime \mu}, r_U^{\prime \mu}; l_L^{\nu}, r_L^{\nu}, l_U^{\prime \nu}, r_U^{\prime \nu})$. The graphical representation of an IVIFN \tilde{A} is given in Fig. 1.

Definition 3.5 [60]. A triangular IVIFN (TIVIFN) is denoted by $\tilde{A} = \{(a_1^U, a_1^L, a_2, a_3^L, a_3^U), (b_1^L, b_1^U, a_2, b_2^U, b_3^L)\}$, and its membership and non-membership degrees are defined as follows:

Page 5 of 43

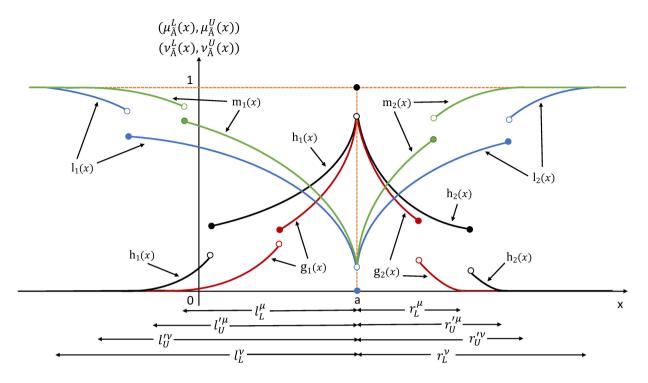


Figure 1: Graphical representation of an IVIFN

• Lower and upper membership functions are respectively given by:

$$\mu_{\tilde{A}}^{L}(x) = \begin{cases} 1, & \text{if } x = a_{2}, \\ \frac{x - a_{1}^{L}}{a_{2} - a_{1}^{L}}, & \text{if } a_{1}^{L} < x < a_{2}, \\ \frac{a_{3}^{L} - x}{a_{3}^{L} - a_{2}}, & \text{if } a_{2} < x < a_{3}^{L}, \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \mu_{\tilde{A}}^{U}(x) = \begin{cases} 1, & \text{if } x = a_{2}, \\ \frac{x - a_{1}^{U}}{a_{2} - a_{1}^{U}}, & \text{if } a_{1}^{U} < x < a_{2}, \\ \frac{a_{3}^{U} - x}{a_{3}^{U} - a_{2}}, & \text{if } a_{2} < x < a_{3}^{U}, \\ 0, & \text{otherwise} \end{cases}$$

• Lower and upper non-membership functions are respectively defined as:

$$v_{\tilde{A}}^{L}(x) = \begin{cases} 0, & \text{if } x = a_2, \\ \frac{a_2 - x}{a_2 - b_1^L}, & \text{if } b_1^L < x < a_2, \\ \frac{a_2 - x}{a_2 - b_3^L}, & \text{if } a_2 < x < b_3^L, \\ 1, & \text{otherwise} \end{cases} \quad \text{and} \quad v_{\tilde{A}}^{U}(x) = \begin{cases} 0, & \text{if } x = a_2, \\ \frac{x - a_2}{b_1^U - a_2}, & \text{if } b_1^U < x < a_2, \\ \frac{x - a_2}{b_1^U - a_2}, & \text{if } a_2 < x < b_3^U, \\ 1, & \text{otherwise} \end{cases}$$

where $b_1^L \le b_1^U \le a_1^U \le a_1^L \le a_2 \le a_3^L \le a_3^U \le b_3^U \le b_3^L$. The diagrammatic representation of a TIVIFN is shown in Fig. 2.

: Page 6 of 43

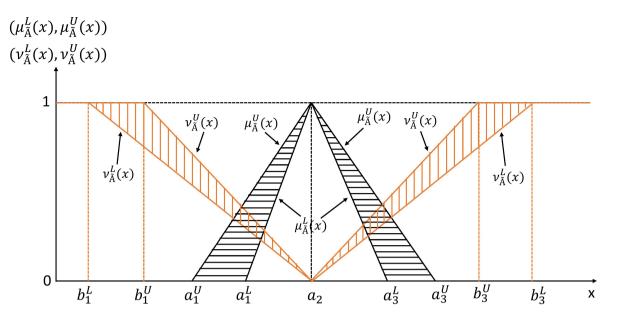


Figure 2: Triangular interval-valued intuitionistic fuzzy number

Definition 3.6 [34]. A function $f:[0,\infty) \to [0,1]$ is said to be shape function or reference function if it satisfies the following conditions:

- (i) f(0) = 1,
- (ii) f is invertible on $[0, \infty)$,
- (iii) f is continuous function on $[0, \infty)$,
- (iv) f is strictly decreasing on $[0, \infty)$ and
- $(v) \lim_{x \to \infty} f(x) = 0.$

Definition 3.7 An IVIFN \tilde{A} is said to be LR-type IVIFN if there exist shape functions L, R, L' and R', and positive real constants l_L^{μ} , r_L^{μ} , $l_L^{\prime \mu}$, $r_L^{\prime \nu}$, $l_L^{\prime \nu}$ and $r_L^{\prime \nu}$, such that its

• Lower and upper membership functions, respectively are defined as:

$$\mu_{\tilde{A}}^{L}(x) = \begin{cases} L\left(\frac{a-x}{l_{L}^{\mu}}\right), & a-l_{L}^{\mu} \leq x \leq a, \\ R\left(\frac{x-a}{r_{L}^{\mu}}\right), & a \leq x \leq a+r_{L}^{\mu}, \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \mu_{\tilde{A}}^{U}(x) = \begin{cases} L'\left(\frac{a-x}{l_{U}^{\prime\mu}}\right), & a-l_{U}^{\prime\mu} \leq x \leq a, \\ R'\left(\frac{x-a}{r_{U}^{\prime\mu}}\right), & a \leq x \leq a+r_{U}^{\prime\mu}, \\ 0, & \text{otherwise} \end{cases}$$

Page 7 of 43

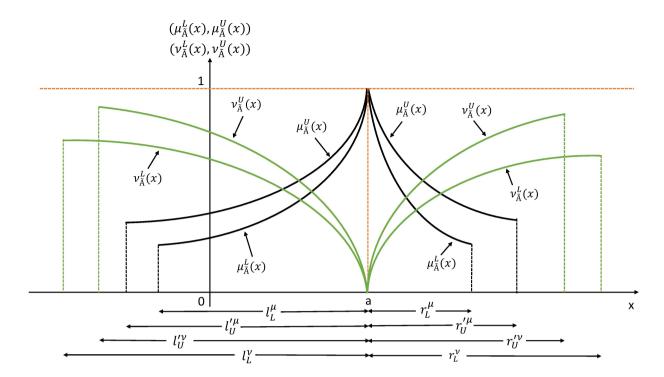


Figure 3: Graphical representation of an LR-type IVIFN

• Lower and upper non-membership functions, respectively are given by:

$$v_{\tilde{A}}^{L}(x) = \begin{cases} 1 - L\left(\frac{a-x}{l_{L}^{v}}\right), & a - l_{L}^{v} \leq x \leq a, \\ 1 - R\left(\frac{x-a}{r_{L}^{v}}\right), & a \leq x \leq a + r_{L}^{v}, \\ 1, & \text{otherwise} \end{cases} \quad \text{and} \quad v_{\tilde{A}}^{U}(x) = \begin{cases} 1 - L'\left(\frac{a-x}{l_{U}^{'v}}\right), & a - l_{U}^{'v} \leq x \leq a, \\ 1 - R'\left(\frac{x-a}{r_{U}^{'v}}\right), & a \leq x \leq a + r_{U}^{'v}, \\ 1, & \text{otherwise} \end{cases}$$

where $l_U'^{\mu} \geq l_L^{\mu}$, $r_U'^{\mu} \geq r_L^{\mu}$, $l_L^{\nu} \geq l_U'^{\nu}$, $r_L^{\nu} \geq r_U'^{\nu}$, $l_L^{\nu} \geq l_L^{\mu}$, $r_L^{\nu} \geq r_L^{\mu}$, $l_U'^{\nu} \geq l_U'^{\mu}$, $r_U'^{\nu} \geq r_U'^{\mu}$ and $0 \leq \sup\{\mu_{\tilde{A}}(x)\} + \sup\{\nu_{\tilde{A}}(x)\} \leq 1$, $\forall x \in \mathbb{R}$. a is called the mean value of \tilde{A} ; l_L^{μ} , $l_L'^{\mu}$, l_L^{ν} and $l_U'^{\nu}$ are respectively the left spreads of $\mu_{\tilde{A}}^L$, $\mu_{\tilde{A}}^U$, $\nu_{\tilde{A}}^L$ and $\nu_{\tilde{A}}^U$, and r_L^{μ} , $r_U'^{\mu}$, r_L^{ν} and $r_U'^{\nu}$ are the respective right spreads of $\mu_{\tilde{A}}^L$, $\mu_{\tilde{A}}^U$, $\nu_{\tilde{A}}^L$ and $\nu_{\tilde{A}}^U$. An LR-type IVIFN is denoted by $\tilde{A} = (a; l_L^{\mu}, r_L^{\mu}, l_U'^{\mu}, r_U'^{\mu}; l_L^{\nu}, r_L^{\nu}, l_U'^{\nu}, r_U'^{\nu})_{LR}$ and its possible general graphical representation is shown in Fig. 3. Let $IV(\mathbb{R})$ represents the set of all LR-type IVIFNs.

Remark 3.1 Taking $L(x) = R(x) = L'(x) = R'(x) = \max\{0, 1-x\}, \ \forall \ x \in \mathbb{R}$, the Definition 3.7 reduces to Definition 3.5.

Definition 3.8 An LR-type IVIFN $\tilde{A} = (a; l_L^{\mu}, r_L^{\mu}, l_U^{\prime \mu}, r_U^{\prime \mu}; l_L^{\nu}, r_L^{\nu}, l_U^{\prime \nu}, r_U^{\prime \nu})_{LR}$ is called an unrestricted LR-type IVIFN if a is any real number.

 $\textbf{Definition 3.9} \text{ An } LR \text{-type IVIFN } \tilde{A} = (a; l_L^\mu, r_L^\mu, l_U^{\prime\mu}, r_U^{\prime\mu}; l_L^\nu, r_L^\nu, l_U^{\prime\nu}, r_U^{\prime\nu})_{LR} \text{ is called non-negative (positive) if } \tilde{A} = (a; l_L^\mu, r_L^\mu, l_U^{\prime\mu}, r_U^{\prime\mu}; l_L^\nu, r_L^\nu, l_U^{\prime\nu}, r_U^{\prime\nu})_{LR} \text{ is called non-negative (positive) if } \tilde{A} = (a; l_L^\mu, r_L^\mu, l_L^\mu, r_L^\mu, l_L^\nu, r_L^\mu, l_L^\nu, r_L^\nu, l_L^\nu, l_L^\nu,$

: Page 8 of 43

 $a - l_L^{\nu} \ge (>) 0$ and non-positive (negative) if $a + r_L^{\nu} \le (<) 0$.

Theorem 3.1. Let $\tilde{A} = (a; l_L^{\mu}, r_L^{\mu}, l_U^{\prime \mu}, r_U^{\prime \mu}; l_L^{\nu}, r_L^{\nu}, l_U^{\prime \nu}, r_U^{\prime \nu})_{LR}$ be an LR-type IVIFN. Then, $\forall \ \alpha, \beta \in (0, 1]$ and $\alpha + \beta \leq 1$,

(i). its lower α -cut for membership and lower β -cut for non-membership are respectively, given by

$$A_{\alpha}^{L} = [a - l_{L}^{\mu}L^{-1}(\alpha), a + r_{L}^{\mu}R^{-1}(\alpha)] \quad and \quad A_{\beta}^{L} = [a - l_{L}^{\nu}L^{-1}(1 - \beta), a + r_{L}^{\nu}R^{-1}(1 - \beta)].$$

(ii). its upper α -cut for membership and upper β -cut for non-membership, respectively are

$$A_{\alpha}^{U} = [a - l_{U}^{\prime \mu}(L^{\prime})^{-1}(\alpha), a + r_{U}^{\prime \mu}(R^{\prime})^{-1}(\alpha)] \quad and \quad A_{\beta}^{U} = [a - l_{U}^{\prime \nu}(L^{\prime})^{-1}(1 - \beta), a + r_{U}^{\prime \nu}(R^{\prime})^{-1}(1 - \beta)].$$

(iii). its lower and upper (α, β) -cut respectively, are

$$\begin{split} A^L_{\alpha,\beta} &= [a - l_L^\mu L^{-1}(\alpha), a + r_L^\mu R^{-1}(\alpha)] \ \cap \ [a - l_L^\nu L^{-1}(1-\beta), a + r_L^\nu R^{-1}(1-\beta)] \ and \\ A^U_{\alpha,\beta} &= [a - l_U^{\prime\mu}(L^\prime)^{-1}(\alpha), a + r_U^{\prime\mu}(R^\prime)^{-1}(\alpha)] \ \cap \ [a - l_U^{\prime\nu}(L^\prime)^{-1}(1-\beta), a + r_U^{\prime\nu}(R^\prime)^{-1}(1-\beta)]. \end{split}$$

Proof.

(i). For $\alpha \in (0, 1]$, $\mu_{\tilde{A}}^{L}(x) \ge \alpha$ implies

$$L\left(\frac{a-x}{l_L^{\mu}}\right) \ge \alpha \text{ and } R\left(\frac{x-a}{r_L^{\mu}}\right) \ge \alpha.$$

Since L and R are decreasing functions, therefore

$$\frac{a-x}{l_L^{\mu}} \le L^{-1}(\alpha), \ \frac{x-a}{r_L^{\mu}} \le R^{-1}(\alpha).$$

It further yields

$$a-l_L^\mu L^{-1}(\alpha) \leq x \leq a+r_L^\mu R^{-1}(\alpha).$$

Hence,

$$A_{\alpha}^{L} = [a - l_{I}^{\mu} L^{-1}(\alpha), a + r_{I}^{\mu} R^{-1}(\alpha)].$$

Now, for $\beta \in (0, 1]$ such that $\alpha + \beta \leq 1$,

 $v_{\tilde{A}}^{L}(x) \leq \beta$ gives

$$1 - L\left(\frac{a - x}{l_L^v}\right) \le \beta \text{ and } 1 - R\left(\frac{x - a}{r_L^v}\right) \le \beta$$

which implies

$$\frac{a-x}{l_L^{\nu}} \leq L^{-1}(1-\beta), \ \frac{x-a}{r_L^{\nu}} \leq R^{-1}(1-\beta).$$

Thus,

$$A_{\beta}^{L} = [a - l_{L}^{\nu} L^{-1} (1 - \beta), a + r_{L}^{\nu} R^{-1} (1 - \beta)].$$

This proves (i).

(ii). Applying α -cut on the upper membership function, that is, $\mu_{\tilde{A}}^U(x) \geq \alpha$, $\alpha \in (0,1]$, we get

$$L'\left(\frac{a-x}{l_U'^{\mu}}\right) \ge \alpha \text{ and } R'\left(\frac{x-a}{r_U'^{\mu}}\right) \ge \alpha.$$

Page 9 of 43

Using the fact that L' and R' are decreasing functions, it follows that

$$\frac{a-x}{l_{II}'^{\mu}} \le (L')^{-1}(\alpha) \text{ and } \frac{x-a}{r_{II}'^{\mu}} \le (R')^{-1}(\alpha).$$

This after simplification gives

$$a - l_U'^{\mu}(L')^{-1}(\alpha) \le x \le a + r_U'^{\mu}(R')^{-1}(\alpha).$$

Therefore,

$$A_{\alpha}^{U} = [a - l_{U}^{\prime \mu}(L^{\prime})^{-1}(\alpha), a + r_{U}^{\prime \mu}(R^{\prime})^{-1}(\alpha)].$$

Similarly, for $\beta \in (0, 1]$ such that $\alpha + \beta \le 1$, the expression $v_{\tilde{A}}^{U}(x) \le \beta$ yields

$$1 - L'\left(\frac{a - x}{l_U'^{\nu}}\right) \le \beta \text{ and } 1 - R'\left(\frac{x - a}{r_U'^{\nu}}\right) \le \beta.$$

This finally gives

$$A_{\beta}^{U} = [a - l_{U}^{\prime \nu}(L^{\prime})^{-1}(1 - \beta), a + r_{U}^{\prime \nu}(R^{\prime})^{-1}(1 - \beta)].$$

Hence proved part (ii).

(iii). From (i), the lower α -cut for membership and the lower β -cut for non-membership of \tilde{A} are respectively, given by

$$A_{\alpha}^{L} = [a - l_{I}^{\mu} L^{-1}(\alpha), a + r_{I}^{\mu} R^{-1}(\alpha)] \quad \text{and} \quad A_{\beta}^{L} = [a - l_{I}^{\nu} L^{-1}(1 - \beta), a + r_{I}^{\nu} R^{-1}(1 - \beta)].$$

It yields

$$\begin{split} A^L_{\alpha,\beta} &= A^L_{\alpha} \cap A^L_{\beta} \\ &= [a - l^{\mu}_L L^{-1}(\alpha), a + r^{\mu}_L R^{-1}(\alpha)] \, \cap \, [a - l^{\nu}_L L^{-1}(1 - \beta), a + r^{\nu}_L R^{-1}(1 - \beta)]. \end{split}$$

On the same lines, the proof of $A^U_{\alpha,\beta}$ can also be obtained. Hence, the result.

Definition 3.10 Let $\tilde{A} = (a; l_L^\mu, r_L^\mu, l_U'^\mu, r_U'^\mu; l_L^\nu, r_L^\nu, l_U'^\nu, r_U'^\nu)_{LR}$ be an LR-type IVIFN. Then, the score and accuracy indices of \tilde{A} are denoted by $S(\tilde{A})$ and $A(\tilde{A})$, respectively and are defined by:

$$S(\tilde{A}) := \frac{1}{4} \int_{0}^{1} \left(a - l_{L}^{\mu} L^{-1}(\alpha) + a + r_{L}^{\mu} R^{-1}(\alpha) + a - l_{U}^{\prime \mu}(L^{\prime})^{-1}(\alpha) + a + r_{U}^{\prime \mu}(R^{\prime})^{-1}(\alpha) \right) d\alpha - \frac{1}{4} \int_{0}^{1} \left(a - l_{L}^{\nu} L^{-1}(1 - \beta) + a + r_{U}^{\prime \nu}(R^{\prime})^{-1}(1 - \beta) \right) d\beta$$

$$\begin{split} A(\tilde{A}) := \frac{1}{4} \int_0^1 \left(a - l_L^\mu L^{-1}(\alpha) + a + r_L^\mu R^{-1}(\alpha) + a - l_U^{\prime\mu}(L^\prime)^{-1}(\alpha) + a + r_U^{\prime\mu}(R^\prime)^{-1}(\alpha) \right) d\alpha + \frac{1}{4} \int_0^1 \left(a - l_L^\nu L^{-1}(1 - \beta) + a + r_L^{\prime\nu}(R^\prime)^{-1}(1 - \beta) \right) d\beta \\ + a + r_L^\nu R^{-1}(1 - \beta) + a - l_U^{\prime\nu}(L^\prime)^{-1}(1 - \beta) + a + r_U^{\prime\nu}(R^\prime)^{-1}(1 - \beta) \right) d\beta \end{split}$$

Remark 3.2 If $l_L^{\mu} = l_U^{\prime \mu}$, $r_L^{\mu} = r_U^{\prime \mu}$, $l_L^{\nu} = l_U^{\prime \nu}$ and $r_L^{\nu} = r_U^{\prime \nu}$, then the Definition 3.10 reduces to the corresponding definition for LR-type IFNs given in Singh and Yadav [34].

Theorem 3.2. Let $\tilde{A} = (a; a - a_1^L, a_3^L - a, a - a_1^U, a_3^U - a; a - b_1^L, b_3^L - a, a - b_1^U, b_3^U - a)_{LR}$ be an LR-type TIVIFN. Then, the score and accuracy indices of LR-type TIVIFN \tilde{A} are respectively, given by:

$$S(\tilde{A}) = \frac{a_1^L + a_3^L + a_1^U + a_3^U - b_1^L - b_3^L - b_1^U - b_3^U}{8},$$

$$A(\tilde{A}) = \frac{a_1^L + a_3^L + a_1^U + a_3^U + 8a + b_1^L + b_3^L + b_1^U + b_3^U}{8}.$$

Page 10 of 43

Proof. From Definition 3.10, we have

$$S(\tilde{A}) := \frac{1}{4} \int_{0}^{1} \left(a - l_{L}^{\mu} L^{-1}(\alpha) + a + r_{L}^{\mu} R^{-1}(\alpha) + a - l_{U}^{\prime \mu} (L^{\prime})^{-1}(\alpha) + a + r_{U}^{\prime \mu} (R^{\prime})^{-1}(\alpha) \right) d\alpha - \frac{1}{4} \int_{0}^{1} \left(a - l_{L}^{\nu} L^{-1} (1 - \beta) + a + r_{U}^{\prime \nu} (R^{\prime})^{-1} (1 - \beta) + a + r_{U}^{\prime \nu} (R^{\prime})^{-1} (1 - \beta) \right) d\beta$$

$$(1)$$

$$A(\tilde{A}) := \frac{1}{4} \int_0^1 \left(a - l_L^{\mu} L^{-1}(\alpha) + a + r_L^{\mu} R^{-1}(\alpha) + a - l_U^{\prime \mu} (L^{\prime})^{-1}(\alpha) + a + r_U^{\prime \mu} (R^{\prime})^{-1}(\alpha) \right) d\alpha + \frac{1}{4} \int_0^1 \left(a - l_L^{\nu} L^{-1}(1 - \beta) + a - l_L^{\nu} (L^{\prime})^{-1}(1 - \beta) + a - l_L^{\nu} (R^{\prime})^{-1}(1 - \beta) \right) d\alpha$$

$$+a + r_L^{\nu} R^{-1} (1 - \beta) + a - l_U^{\prime \nu} (L^{\prime})^{-1} (1 - \beta) + a + r_U^{\prime \nu} (R^{\prime})^{-1} (1 - \beta) \Big) d\beta$$
 (2)

Now, since \tilde{A} is a TIVIFN, therefore

$$L(x) = R(x) = L'(x) = R'(x) = \max\{0, 1 - x\}, \ \forall \ x \in \mathbb{R}.$$

Hence, for $\alpha \in (0, 1]$, we have

$$L(\alpha) = L'(\alpha) = R(\alpha) = R'(\alpha) = 1 - \alpha. \tag{3}$$

This further implies

$$L^{-1}(\alpha) = R^{-1}(\alpha) = (L')^{-1}(\alpha) = (R')^{-1}(\alpha) = 1 - \alpha.$$
(4)

Substituting the expressions from the equations (3) and (4) in (1) and (2), we obtain

$$S(\tilde{A}) = \frac{a_1^L + a_3^L + a_1^U + a_3^U - b_1^L - b_3^L - b_1^U - b_3^U}{8} \quad \text{and} \quad A(\tilde{A}) = \frac{a_1^L + a_3^L + a_1^U + a_3^U + 8a + b_1^L + b_3^L + b_1^U + b_3^U}{8}.$$

Hence the result.

3.1. Arithmetic operations on LR-type IVIFNs

In this subsection, the basic arithmetic operations on LR-type IVIFNs are discussed. Here, we have introduced the addition operator (\bigoplus) , subtraction operator (\bigoplus) and product operator (\bigcirc) for LR-type IVIFNs. The following propositions discuss the detailed expressions for the addition, subtraction, scalar multiplication and product operations on these numbers.

Proposition 3.1.1. Let $\tilde{A}_1 = (a_1; l^{\mu}_{1L}, r^{\mu}_{1L}, l'^{\mu}_{1U}, r'^{\mu}_{1U}; l^{\nu}_{1L}, r^{\nu}_{1L}, l'^{\nu}_{1U}, r'^{\nu}_{1U})_{LR}$ and $\tilde{A}_2 = (a_2; l^{\mu}_{2L}, r^{\mu}_{2L}, l'^{\mu}_{2U}, r'^{\mu}_{2U}; l^{\nu}_{2L}, r^{\nu}_{2L}, l'^{\nu}_{2U}, r^{\nu}_{2L}, l'^{\nu}_{2U}, r^{\nu}_{2U})_{LR}$ be two LR-type IVIFNs. Then,

(i).
$$\tilde{A}_1 \oplus \tilde{A}_2 = (a_1 + a_2; l_{1L}^{\mu} + l_{2L}^{\mu}, r_{1L}^{\mu} + r_{2L}^{\mu}, l_{1U}^{\prime \mu} + l_{2U}^{\prime \mu}, r_{1U}^{\prime \mu} + r_{2U}^{\prime \mu}; l_{1L}^{\nu} + l_{2L}^{\nu}, r_{1L}^{\nu} + r_{2L}^{\nu}, l_{1U}^{\prime \nu} + l_{2U}^{\prime \nu}, r_{1U}^{\prime \nu} + r_{2U}^{\prime \nu})_{LR},$$
 where the conditions for LR-type representation of $\tilde{A}_1 \oplus \tilde{A}_2$ are satisfied.

(ii).
$$\tilde{A}_1 \ominus \tilde{A}_2 = (a_1 - a_2; l_{1L}^{\mu} + r_{2L}^{\mu}, r_{1L}^{\mu} + l_{2L}^{\mu}, l_{1U}^{\prime \mu} + r_{2U}^{\prime \mu}, r_{1U}^{\prime \mu} + l_{2U}^{\prime \mu}; l_{1L}^{\nu} + r_{2L}^{\nu}, r_{1L}^{\nu} + l_{2L}^{\nu}, l_{1U}^{\prime \nu} + r_{2U}^{\prime \nu}, r_{1U}^{\prime \nu} + l_{2U}^{\prime \nu})_{LR},$$
 where the conditions for LR-type representation of $\tilde{A}_1 \ominus \tilde{A}_2$ are fulfilled.

Proof. In view of Theorem 3.1, the α and β -cuts of \tilde{A}_1 and \tilde{A}_2 are respectively, given by

$$A_{1\alpha}^{L} = [a_{1} - l_{1L}^{\mu} L^{-1}(\alpha), a_{1} + r_{1L}^{\mu} R^{-1}(\alpha)],$$

$$A_{1\alpha}^{U} = [a_{1} - l_{1U}^{\prime \mu} (L')^{-1}(\alpha), a_{1} + r_{1U}^{\prime \mu} (R')^{-1}(\alpha)],$$

$$A_{1\beta}^{L} = [a_{1} - l_{1L}^{\nu} L^{-1}(1 - \beta), a_{1} + r_{1L}^{\nu} R^{-1}(1 - \beta)],$$

$$A_{1\beta}^{U} = [a_{1} - l_{1U}^{\prime \nu} (L')^{-1}(1 - \beta), a_{1} + r_{1U}^{\prime \nu} (R')^{-1}(1 - \beta)].$$

$$(5)$$

Page 11 of 43

$$A_{2\alpha}^{L} = [a_{2} - l_{2L}^{\mu} L^{-1}(\alpha), a_{2} + r_{2L}^{\mu} R^{-1}(\alpha)],$$

$$A_{2\alpha}^{U} = [a_{2} - l_{2U}^{\prime \mu} (L')^{-1}(\alpha), a_{2} + r_{2U}^{\prime \mu} (R')^{-1}(\alpha)],$$

$$A_{2\beta}^{L} = [a_{2} - l_{2L}^{\nu} L^{-1}(1 - \beta), a_{2} + r_{2L}^{\nu} R^{-1}(1 - \beta)],$$

$$A_{2\beta}^{U} = [a_{2} - l_{2U}^{\prime \nu} (L')^{-1}(1 - \beta), a_{2} + r_{2U}^{\prime \nu} (R')^{-1}(1 - \beta)].$$
(6)

(i). From the Eqs. (5) and (6), we get

$$\begin{split} \left(\tilde{A}_1 \oplus \tilde{A}_2\right)^L_{\alpha} &= A^L_{1\alpha} + A^L_{2\alpha} \\ &= [a_1 + a_2 - (l^{\mu}_{1L} + l^{\mu}_{2L})L^{-1}(\alpha), a_1 + a_2 + (r^{\mu}_{1L} + r^{\mu}_{2L})R^{-1}(\alpha)]. \end{split}$$

$$\begin{split} \left(\tilde{A}_1 \oplus \tilde{A}_2\right)^U_{\alpha} &= A^U_{1\alpha} + A^U_{2\alpha} \\ &= [a_1 + a_2 - (l'^{\mu}_{1U} + l'^{\mu}_{2U})(L')^{-1}(\alpha), a_1 + a_2 + (r'^{\mu}_{1U} + r'^{\mu}_{2U})(R')^{-1}(\alpha)]. \end{split}$$

$$\begin{split} \left(\tilde{A}_1 \oplus \tilde{A}_2\right)^L_{\beta} &= A^L_{1\beta} + A^L_{2\beta} \\ &= [a_1 + a_2 - (l^v_{1L} + l^v_{2L})L^{-1}(1-\beta), a_1 + a_2 + (r^v_{1L} + r^v_{2L})R^{-1}(1-\beta)]. \end{split}$$

$$\begin{split} \left(\tilde{A}_1 \oplus \tilde{A}_2\right)^U_{\beta} &= A^U_{1\beta} + A^U_{2\beta} \\ &= \left[a_1 + a_2 - (l'^{v}_{1U} + l'^{v}_{2U})(L')^{-1}(1-\beta), a_1 + a_2 + (r'^{v}_{1U} + r'^{v}_{2U})(R')^{-1}(1-\beta)\right]. \end{split}$$

Since L, R, L' and R' are decreasing functions on $[0, \infty)$ with L(0) = R(0) = L'(0) = R'(0) = 1, there exists $\alpha_0 \in (0, 1]$, such that $L^{-1}(\alpha_0) = R^{-1}(\alpha_0) = (L')^{-1}(\alpha_0) = (R')^{-1}(\alpha_0) = 1$. Hence,

$$\left(\tilde{A}_1 \oplus \tilde{A}_2\right)_{a_0}^L = \left[a_1 + a_2 - (l_{1L}^{\mu} + l_{2L}^{\mu}), a_1 + a_2 + (r_{1L}^{\mu} + r_{2L}^{\mu})\right]. \tag{7}$$

$$\left(\tilde{A}_1 \oplus \tilde{A}_2\right)_{a_0}^U = \left[a_1 + a_2 - (l_{1U}^{\prime \mu} + l_{2U}^{\prime \mu}), a_1 + a_2 + (r_{1U}^{\prime \mu} + r_{2U}^{\prime \mu})\right]. \tag{8}$$

Also, choosing $\beta_0 = 1 - \alpha_0 \in (0, 1]$, we get

$$\left(\tilde{A}_1 \oplus \tilde{A}_2\right)_{\theta_0}^L = \left[a_1 + a_2 - (l_{1L}^{\nu} + l_{2L}^{\nu}), a_1 + a_2 + (r_{1L}^{\nu} + r_{2L}^{\nu})\right]. \tag{9}$$

$$\left(\tilde{A}_1 \oplus \tilde{A}_2\right)_{\beta_0}^U = \left[a_1 + a_2 - (l_{1U}^{\prime \nu} + l_{2U}^{\prime \nu}), a_1 + a_2 + (r_{1U}^{\prime \nu} + r_{2U}^{\prime \nu})\right]. \tag{10}$$

Further,

$$(\tilde{A}_1 \oplus \tilde{A}_2)_{\alpha=1}^L = (\tilde{A}_1 \oplus \tilde{A}_2)_{\alpha=1}^U = (\tilde{A}_1 \oplus \tilde{A}_2)_{\beta=0}^L = (\tilde{A}_1 \oplus \tilde{A}_2)_{\beta=0}^U = [a_1 + a_2, a_1 + a_2].$$
 (11)

Now, since \tilde{A}_1 and \tilde{A}_2 are LR-type IVIFNs, therefore

$$\begin{split} &l_{1U}^{\prime\mu} \geq l_{1L}^{\mu} > 0, \quad r_{1U}^{\prime\mu} \geq r_{1L}^{\mu} > 0, \quad l_{1L}^{\nu} \geq l_{1U}^{\prime\nu} > 0, \quad r_{1L}^{\nu} \geq r_{1U}^{\prime\nu} > 0, \quad l_{1L}^{\nu} \geq l_{1L}^{\mu} > 0, \quad r_{1L}^{\nu} \geq r_{1L}^{\mu} > 0, \\ &l_{1U}^{\prime\nu} \geq l_{1U}^{\prime\mu} > 0, \quad r_{1U}^{\prime\nu} \geq r_{1U}^{\prime\mu} > 0, \quad \text{and} \\ &l_{2U}^{\prime\mu} \geq l_{2L}^{\mu} > 0, \quad r_{2U}^{\prime\mu} \geq r_{2L}^{\mu} > 0, \quad l_{2L}^{\nu} \geq l_{2U}^{\prime\nu} > 0, \quad r_{2L}^{\nu} \geq r_{2U}^{\prime\nu} > 0, \quad l_{2L}^{\nu} \geq l_{2L}^{\mu} > 0, \quad r_{2L}^{\nu} \geq r_{2L}^{\mu} > 0, \\ &l_{2U}^{\prime\nu} \geq l_{2U}^{\prime\mu} > 0, \quad r_{2U}^{\prime\nu} \geq r_{2U}^{\prime\mu} > 0. \end{split}$$

Thus,

$$\begin{aligned} l_{1U}^{\prime\mu} + l_{2U}^{\prime\mu} &\geq l_{1L}^{\mu} + l_{2L}^{\mu} > 0, \ r_{1U}^{\prime\mu} + r_{2U}^{\prime\mu} \geq r_{1L}^{\mu} + r_{2L}^{\mu} > 0, \ l_{1L}^{\nu} + l_{2L}^{\nu} \geq l_{1U}^{\prime\nu} + l_{2U}^{\prime\nu} > 0, \\ r_{1L}^{\nu} + r_{2L}^{\nu} &\geq r_{1U}^{\prime\nu} + r_{2U}^{\prime\nu} > 0, \ l_{1L}^{\nu} + l_{2L}^{\nu} \geq l_{1L}^{\mu} + l_{2L}^{\mu} > 0, \ r_{1L}^{\nu} + r_{2L}^{\nu} \geq r_{1L}^{\mu} + r_{2L}^{\mu} > 0, \\ l_{1U}^{\prime\nu} + l_{2U}^{\prime\nu} &\geq l_{1U}^{\prime\mu} + l_{2U}^{\prime\mu} > 0, \ r_{1U}^{\prime\nu} + r_{2U}^{\prime\nu} \geq r_{1U}^{\prime\mu} + r_{2U}^{\prime\mu} > 0. \end{aligned}$$

$$(12)$$

Page 12 of 43

Combining Eqs. (7) - (11), we have

$$\tilde{A}_{1} \oplus \tilde{A}_{2} = (a_{1} + a_{2}; l_{1L}^{\mu} + l_{2L}^{\mu}, r_{1L}^{\mu} + r_{2L}^{\mu}, l_{1U}^{\prime \mu} + l_{2U}^{\prime \mu}, r_{1U}^{\prime \mu} + r_{2U}^{\prime \mu}; l_{1L}^{\nu} + l_{2L}^{\nu}, r_{1L}^{\nu} + r_{2L}^{\nu}, l_{1U}^{\prime \nu} + l_{2U}^{\prime \nu}, r_{1U}^{\prime \nu} + r_{2U}^{\prime \nu})_{LR},$$
 where the conditions for LR -type form of $\tilde{A}_{1} \oplus \tilde{A}_{2}$ holds from (12). Hence, (i) is proved.

(ii). Using equations (5) and (6), we can write

$$\begin{split} \left(\tilde{A}_{1} \ominus \tilde{A}_{2}\right)_{\alpha}^{L} &= A_{1\alpha}^{L} - A_{2\alpha}^{L} = [a_{1} - a_{2} - l_{1L}^{\mu}L^{-1}(\alpha) - r_{2L}^{\mu}R^{-1}(\alpha), \ a_{1} - a_{2} + r_{1L}^{\mu}R^{-1}(\alpha) + l_{2L}^{\mu}L^{-1}(\alpha)]. \\ \left(\tilde{A}_{1} \ominus \tilde{A}_{2}\right)_{\alpha}^{U} &= A_{1\alpha}^{U} - A_{2\alpha}^{U} = [a_{1} - a_{2} - l_{1U}^{\prime\mu}(L^{\prime})^{-1}(\alpha) - r_{2U}^{\prime\mu}(R^{\prime})^{-1}(\alpha), \ a_{1} - a_{2} + r_{1U}^{\prime\mu}(R^{\prime})^{-1}(\alpha) + l_{2U}^{\prime\mu}(L^{\prime})^{-1}(\alpha)]. \\ \left(\tilde{A}_{1} \ominus \tilde{A}_{2}\right)_{\beta}^{L} &= A_{1\beta}^{L} - A_{2\beta}^{L} = [a_{1} - a_{2} - l_{1L}^{\prime\nu}L^{-1}(1 - \beta) - r_{2L}^{\prime\nu}R^{-1}(1 - \beta), a_{1} - a_{2} + r_{1L}^{\prime\nu}R^{-1}(1 - \beta) + l_{2L}^{\prime\nu}L^{-1}(1 - \beta)]. \\ \left(\tilde{A}_{1} \ominus \tilde{A}_{2}\right)_{\beta}^{U} &= A_{1\beta}^{U} - A_{2\beta}^{U} = [a_{1} - a_{2} - l_{1U}^{\prime\nu}(L^{\prime})^{-1}(1 - \beta) - r_{2U}^{\prime\nu}(R^{\prime})^{-1}(1 - \beta), a_{1} - a_{2} + r_{1U}^{\prime\nu}(R^{\prime})^{-1}(1 - \beta) + l_{2U}^{\prime\nu}(L^{\prime})^{-1}(1 - \beta)]. \end{split}$$

Now, taking $\alpha = \alpha_0$ and $\beta = \beta_0 \in (0, 1]$, we get

$$\begin{aligned}
&(\tilde{A}_{1} \ominus \tilde{A}_{2})_{a_{0}}^{L} = [a_{1} - a_{2} - l_{1L}^{\mu} - r_{2L}^{\mu}, a_{1} - a_{2} + r_{1L}^{\mu} + l_{2L}^{\mu}], \\
&(\tilde{A}_{1} \ominus \tilde{A}_{2})_{a_{0}}^{U} = [a_{1} - a_{2} - l_{1U}^{\prime \mu} - r_{2U}^{\prime \mu}, a_{1} - a_{2} + r_{1U}^{\prime \mu} + l_{2U}^{\prime \mu}], \\
&(\tilde{A}_{1} \ominus \tilde{A}_{2})_{\beta_{0}}^{L} = [a_{1} - a_{2} - l_{1L}^{\nu} - r_{2L}^{\nu}, a_{1} - a_{2} + r_{1L}^{\nu} + l_{2L}^{\nu}], \\
&(\tilde{A}_{1} \ominus \tilde{A}_{2})_{\beta_{0}}^{U} = [a_{1} - a_{2} - l_{1U}^{\prime \nu} - r_{2U}^{\prime \nu}, a_{1} - a_{2} + r_{1U}^{\prime \nu} + l_{2U}^{\prime \nu}].
\end{aligned} \tag{13}$$

Further, on substituting $\alpha = 1$ and $\beta = 0$, we obtain

$$(\tilde{A}_1 \ominus \tilde{A}_2)_{\alpha=1}^L = (\tilde{A}_1 \ominus \tilde{A}_2)_{\alpha=1}^U = (\tilde{A}_1 \ominus \tilde{A}_2)_{\beta=0}^L = (\tilde{A}_1 \ominus \tilde{A}_2)_{\beta=0}^L = [a_1 - a_2, a_1 - a_2].$$
 (14)

Also, using the fact \tilde{A}_1 and \tilde{A}_2 are LR-type IVIFNs, we have

$$\begin{aligned} &l_{1U}^{\prime\mu} + r_{2U}^{\prime\mu} \geq l_{1L}^{\mu} + r_{2L}^{\mu} > 0, & r_{1U}^{\prime\mu} + l_{2U}^{\prime\mu} \geq r_{1L}^{\mu} + l_{2L}^{\mu} > 0, & l_{1L}^{\nu} + r_{2U}^{\nu} \geq l_{1U}^{\prime\nu} + r_{2U}^{\prime\nu} > 0, & r_{1L}^{\nu} + l_{2L}^{\nu} \geq r_{1U}^{\prime\nu} + l_{2U}^{\prime\nu} > 0, \\ &l_{1L}^{\nu} + r_{2L}^{\nu} \geq l_{1L}^{\mu} + r_{2L}^{\mu} > 0, & r_{1L}^{\nu} + l_{2L}^{\nu} \geq r_{1L}^{\mu} + l_{2L}^{\mu} > 0, & l_{1U}^{\prime\nu} + r_{2U}^{\prime\nu} \geq l_{1U}^{\prime\mu} + r_{2U}^{\prime\mu} > 0, & r_{1U}^{\prime\nu} + l_{2U}^{\prime\nu} \geq r_{1U}^{\prime\mu} + l_{2U}^{\prime\nu} > 0. \end{aligned}$$

Finally, from the Eqs. (13) and (14), we have

$$\begin{split} \tilde{A}_1 & \ominus \tilde{A}_2 = (a_1 - a_2; l_{1L}^{\mu} + r_{2L}^{\mu}, r_{1L}^{\mu} + l_{2L}^{\mu}, l_{1U}^{\prime \mu} + r_{2U}^{\prime \mu}, r_{1U}^{\prime \mu} + l_{2U}^{\prime \mu}; l_{1L}^{\nu} + r_{2L}^{\nu}, r_{1L}^{\nu} + l_{2L}^{\nu}, l_{1U}^{\prime \nu} + r_{2U}^{\prime \nu}, r_{1U}^{\prime \nu} + l_{2U}^{\prime \nu})_{LR}, \\ \text{along-with } \tilde{A}_1 & \ominus \tilde{A}_2 \text{ retains the form of a LR-type IVIFN. This proves (i)}. \end{split}$$

Proposition 3.1.2. Let $\tilde{A} = (a; l_L^\mu, r_L^\mu, l_U^{\prime\mu}, r_U^{\prime\mu}; l_L^\nu, r_L^\nu, l_U^{\prime\nu}, r_U^{\prime\nu})_{LR}$ be an LR-type IVIFN and λ be any real number. Then

$$\lambda \tilde{A} = \begin{cases} (\lambda a; \lambda l_L^\mu, \lambda r_L^\mu, \lambda l_U^{\prime\mu}, \lambda r_U^{\prime\mu}; \lambda l_L^\nu, \lambda r_L^\nu, \lambda l_U^{\prime\nu}, \lambda r_U^{\prime\nu})_{LR} & \text{if } \lambda \geq 0, \\ (\lambda a; -\lambda r_L^\mu, -\lambda l_L^\mu, -\lambda r_U^{\prime\mu}, -\lambda l_U^{\prime\mu}; -\lambda r_L^\nu, -\lambda l_L^\nu, -\lambda r_U^{\prime\nu}, -\lambda l_U^{\prime\nu})_{LR} & \text{if } \lambda < 0. \end{cases}$$

Page 13 of 43

Proof. From Theorem 3.1, the α and β -cuts of \tilde{A} are given by

$$A_{\alpha}^{L} = [a - l_{L}^{\mu} L^{-1}(\alpha), a + r_{L}^{\mu} R^{-1}(\alpha)],$$

$$A_{\alpha}^{U} = [a - l_{U}^{\prime \mu} (L')^{-1}(\alpha), a + r_{U}^{\prime \mu} (R')^{-1}(\alpha)],$$

$$A_{\beta}^{L} = [a - l_{L}^{\nu} L^{-1}(1 - \beta), a + r_{L}^{\nu} R^{-1}(1 - \beta)],$$

$$A_{\beta}^{U} = [a - l_{U}^{\prime \nu} (L')^{-1}(1 - \beta), a + r_{U}^{\prime \nu} (R')^{-1}(1 - \beta)].$$

$$(15)$$

Using the expression (15), we have

$$\begin{split} (\lambda \tilde{A})_{\alpha}^{L} &= \lambda A_{\alpha}^{L} = [\lambda, \lambda] A_{\alpha}^{L} = [\lambda, \lambda] [a - l_{L}^{\mu} L^{-1}(\alpha), a + r_{L}^{\mu} R^{-1}(\alpha)], \\ (\lambda \tilde{A})_{\alpha}^{U} &= \lambda A_{\alpha}^{U} = [\lambda, \lambda] A_{\alpha}^{U} = [\lambda, \lambda] [a - l_{U}^{\prime \mu} (L^{\prime})^{-1}(\alpha), a + r_{U}^{\prime \mu} (R^{\prime})^{-1}(\alpha)], \\ (\lambda \tilde{A})_{\beta}^{L} &= \lambda A_{\beta}^{L} = [\lambda, \lambda] A_{\beta}^{L} = [\lambda, \lambda] [a - l_{L}^{\nu} L^{-1} (1 - \beta), a + r_{L}^{\nu} R^{-1} (1 - \beta)], \\ (\lambda \tilde{A})_{\beta}^{U} &= \lambda A_{\beta}^{U} = [\lambda, \lambda] A_{\beta}^{U} = [\lambda, \lambda] [a - l_{U}^{\prime \nu} (L^{\prime})^{-1} (1 - \beta), a + r_{U}^{\prime \nu} (R^{\prime})^{-1} (1 - \beta)]. \end{split}$$

Case 1. \tilde{A} is a non-negative LR-type IVIFN.

Sub-case 1. $\lambda \ge 0$.

Since, \tilde{A} is a non-negative LR-type IVIFN, i.e., $a-l_L^{\nu}\geq 0$. Thus, $a-l_L^{\mu}\geq 0$, $a-l_U^{\prime\mu}\geq 0$, $a-l_L^{\nu}\geq 0$, $a-l_U^{\prime\nu}\geq 0$. This further implies $a-l_L^{\mu}L^{-1}(\alpha)\geq 0$, $a-l_U^{\prime\mu}(L')^{-1}(\alpha)\geq 0$, $a-l_L^{\nu}L^{-1}(1-\beta)\geq 0$, $a-l_U^{\prime\nu}(L')^{-1}(1-\beta)\geq 0$, $\forall \alpha,\beta\in[0,1]$. Hence, we get

$$(\lambda \tilde{A})^{L}_{\alpha} = [\lambda, \lambda][a - l^{\mu}_{L}L^{-1}(\alpha), a + r^{\mu}_{L}R^{-1}(\alpha)] = [\lambda(a - l^{\mu}_{L}L^{-1}(\alpha)), \lambda(a + r^{\mu}_{L}R^{-1}(\alpha))],$$

$$(\lambda\tilde{A})^U_\alpha = [\lambda(a-l_U'^\mu(L')^{-1}(\alpha)), \lambda(a+r_U'^\mu(R')^{-1}(\alpha))],$$

$$(\lambda \tilde{A})_{\beta}^{L} = [\lambda (a - l_{I}^{\nu} L^{-1} (1 - \beta)), \lambda (a + r_{I}^{\nu} R^{-1} (1 - \beta))],$$

$$(\lambda \tilde{A})^U_{\beta} = [\lambda (a - l_U'^{\nu}(L')^{-1}(1-\beta)), \lambda (a + r_U'^{\nu}(R')^{-1}(1-\beta))].$$

Further, as L, R, L' and R' are decreasing functions on $[0, \infty)$ with L(0) = R(0) = L'(0) = R'(0) = 1, there exists $\alpha_0 \in (0, 1]$, such that $L^{-1}(\alpha_0) = R^{-1}(\alpha_0) = (L')^{-1}(\alpha_0) = (R')^{-1}(\alpha_0) = 1$. Therefore,

$$(\lambda \tilde{A})_{a_{0}}^{L} = [\lambda (a - l_{L}^{\mu}), \lambda (a + r_{L}^{\mu})] = [\lambda a - \lambda l_{L}^{\mu}, \lambda a + \lambda r_{L}^{\mu}],$$

$$(\lambda \tilde{A})_{a_{0}}^{U} = [\lambda (a - l_{U}^{\prime \mu}), \lambda (a + r_{U}^{\prime \mu})] = [\lambda a - \lambda l_{U}^{\prime \mu}, \lambda a + \lambda r_{U}^{\prime \mu}].$$
(16)

Choosing $\beta_0 = 1 - \alpha_0 \in (0, 1]$, we have

$$(\lambda \tilde{A})_{\beta_0}^L = [\lambda(a - l_L^{\nu}), \lambda(a + r_L^{\nu})] = [\lambda a - \lambda l_L^{\nu}, \lambda a + \lambda r_L^{\nu}],$$

$$(\lambda \tilde{A})_{\beta_0}^U = [\lambda(a - l_U^{\prime \nu}), \lambda(a + r_U^{\prime \nu})] = [\lambda a - \lambda l_U^{\prime \nu}, \lambda a + \lambda r_U^{\prime \nu}].$$
(17)

Putting $\alpha = 1$ and $\beta = 0$, we get

$$(\lambda \tilde{A})_{\alpha=1}^{L} = (\lambda \tilde{A})_{\alpha=1}^{U} = (\lambda \tilde{A})_{\beta=0}^{L} = (\lambda \tilde{A})_{\beta=0}^{U} = [\lambda a, \lambda a]. \tag{18}$$

Page 14 of 43

Since \tilde{A} is an *LR*-type IVIFN and $\lambda \geq 0$, we obtain

$$\lambda l_{U}^{\prime \mu} \geq \lambda l_{L}^{\mu} > 0, \quad \lambda r_{U}^{\prime \mu} \geq \lambda r_{L}^{\mu} > 0, \quad \lambda l_{L}^{\nu} \geq \lambda l_{U}^{\prime \nu} > 0, \quad \lambda r_{L}^{\nu} \geq \lambda r_{U}^{\prime \nu} > 0, \quad \lambda l_{L}^{\nu} \geq \lambda l_{L}^{\mu} > 0, \quad \lambda r_{L}^{\nu} \geq \lambda r_{L}^{\mu} > 0, \\ \lambda l_{U}^{\prime \nu} \geq \lambda l_{U}^{\prime \mu} > 0, \quad \lambda r_{U}^{\prime \nu} \geq \lambda r_{U}^{\prime \mu} > 0.$$

Hence, combining (16) - (18), we have

$$\lambda \tilde{A} = (\lambda a; \lambda l_L^\mu, \lambda r_L^\mu, \lambda l_U^{\prime \mu}, \lambda r_U^{\prime \mu}; \lambda l_L^\nu, \lambda r_L^\nu, \lambda l_U^{\prime \nu}, \lambda r_U^{\prime \nu})_{LR}.$$

Sub-case 2. $\lambda < 0$.

Since, \tilde{A} is a non-negative LR-type IVIFN, i.e., $a-l_L^{\nu} \geq 0$. Thus, $a-l_L^{\mu} \geq 0$, $a-l_U^{\prime \mu} \geq 0$, $a-l_L^{\nu} \geq 0$, $a-l_U^{\nu} \geq 0$ $\Rightarrow a-l_L^{\mu}L^{-1}(\alpha) \geq 0$, $a-l_U^{\prime \mu}(L')^{-1}(\alpha) \geq 0$, $a-l_L^{\nu}L^{-1}(1-\beta) \geq 0$, $a-l_U^{\prime \nu}(L')^{-1}(1-\beta) \geq 0$, $\forall \alpha, \beta \in [0,1]$. It follows that

$$\begin{split} &(\lambda \tilde{A})_{\alpha}^{L} = [\lambda,\lambda][a - l_{L}^{\mu}L^{-1}(\alpha),a + r_{L}^{\mu}R^{-1}(\alpha)] = [\lambda(a + r_{L}^{\mu}R^{-1}(\alpha)),\lambda(a - l_{L}^{\mu}L^{-1}(\alpha))],\\ &(\lambda \tilde{A})_{\alpha}^{U} = [\lambda(a + r_{U}^{\prime\mu}(R^{\prime})^{-1}(\alpha)),\lambda(a - l_{U}^{\prime\mu}(L^{\prime})^{-1}(\alpha))],\\ &(\lambda \tilde{A})_{\beta}^{L} = [\lambda(a + r_{L}^{\nu}R^{-1}(1 - \beta)),\lambda(a - l_{L}^{\nu}L^{-1}(1 - \beta))],\\ &(\lambda \tilde{A})_{\beta}^{U} = [\lambda(a + r_{U}^{\prime\nu}(R^{\prime})^{-1}(1 - \beta)),\lambda(a - l_{U}^{\prime\nu}(L^{\prime})^{-1}(1 - \beta))]. \end{split}$$

Taking $\alpha = \alpha_0$, $\beta = \beta_0 = 1 - \alpha_0$ and using the fact that $L^{-1}(\alpha_0) = R^{-1}(\alpha_0) = (L')^{-1}(\alpha_0) = (R')^{-1}(\alpha_0) = 1$, we have

$$(\lambda \tilde{A})_{a_{0}}^{L} = [\lambda(a + r_{L}^{\mu}), \lambda(a - l_{L}^{\mu})] = [\lambda a + \lambda r_{L}^{\mu}, \lambda a - \lambda l_{L}^{\mu}],$$

$$(\lambda \tilde{A})_{a_{0}}^{U} = [\lambda(a + r_{U}^{\prime \mu}), \lambda(a - l_{U}^{\prime \mu})] = [\lambda a + \lambda r_{U}^{\prime \mu}, \lambda a - \lambda l_{U}^{\prime \mu}],$$

$$(\lambda \tilde{A})_{\beta_{0}}^{L} = [\lambda(a + r_{L}^{\nu}), \lambda(a - l_{L}^{\nu})] = [\lambda a + \lambda r_{L}^{\nu}, \lambda a - \lambda l_{L}^{\nu}],$$

$$(\lambda \tilde{A})_{\beta_{0}}^{U} = [\lambda(a + r_{U}^{\prime \nu}), \lambda(a - l_{U}^{\prime \nu})] = [\lambda a + \lambda r_{U}^{\prime \nu}, \lambda - \lambda l_{U}^{\prime \nu}].$$
(19)

Putting $\alpha = 1$ and $\beta = 0$, we obtain

$$(\lambda \tilde{A})_{\alpha=1}^{L} = (\lambda \tilde{A})_{\alpha=1}^{U} = (\lambda \tilde{A})_{\beta=0}^{L} = (\lambda \tilde{A})_{\beta=0}^{U} = [\lambda a, \lambda a]. \tag{20}$$

Further, from the fact that \tilde{A} is an LR-type IVIFN and $\lambda < 0$, we have

$$\begin{split} -\lambda l_{U}'^{\mu} &\geq -\lambda l_{L}^{\mu} > 0, \quad -\lambda r_{U}'^{\mu} \geq -\lambda r_{L}^{\mu} > 0, \quad -\lambda l_{L}^{\nu} \geq -\lambda l_{U}'^{\nu} > 0, \quad -\lambda r_{L}^{\nu} \geq -\lambda r_{U}'^{\nu} > 0, \quad -\lambda l_{L}^{\nu} \geq -\lambda l_{L}^{\mu} > 0, \\ -\lambda r_{L}^{\nu} &\geq -\lambda r_{L}^{\mu} > 0, \quad -\lambda l_{U}'^{\nu} \geq -\lambda l_{U}'^{\mu} > 0, \quad -\lambda r_{U}'^{\nu} \geq -\lambda r_{U}'^{\mu} > 0. \end{split}$$

Hence, the expressions (19) and (20) finally yield

$$\lambda \tilde{A} = (\lambda a; -\lambda r_{I}^{\mu}, -\lambda l_{I}^{\mu}, -\lambda r_{II}^{\prime \mu}, -\lambda l_{II}^{\prime \mu}; -\lambda r_{I}^{\nu}, -\lambda l_{I}^{\nu}, -\lambda r_{II}^{\prime \nu}, -\lambda l_{II}^{\prime \nu})_{LR}.$$

Therefore, the result is proved for a non-negative LR-type IVIFN.

Case 2. \tilde{A} is an LR-type IVIFN such that $a - l_L^{\nu} < 0$ and $a - l_U^{\prime \nu} \ge 0$. Since, $a - l_L^{\mu} \ge 0$, $a - l_U^{\prime \mu} \ge 0$, $a - l_U^{\prime \nu} \ge 0$ $\implies a - l_L^{\mu} L^{-1}(\alpha) \ge 0$, $a - l_U^{\prime \mu}(L')^{-1}(\alpha) \ge 0$, $a - l_U^{\prime \nu}(L')^{-1}(1 - \beta) \ge 0$, $\forall \alpha, \beta \in [0, 1]$.

Page 15 of 43

So, expressions for $(\lambda \tilde{A})_{\alpha}^{L}$, $(\lambda \tilde{A})_{\alpha}^{U}$ and $(\lambda \tilde{A})_{\beta}^{U}$ are same as derived in Case 1, and

$$(\lambda \tilde{A})^L_{\beta} = \lambda A^L_{\beta} = [\lambda,\lambda] A^L_{\beta} = [\lambda,\lambda] [a-l^v_L L^{-1}(1-\beta),a+r^v_L R^{-1}(1-\beta)].$$

Now, if $a - l_L^{\nu} < 0$, then either

$$a - l_L^{\nu} L^{-1} (1 - \beta) < 0 \iff \frac{a}{l_L^{\nu}} < L^{-1} (1 - \beta) \iff \beta > 1 - L \left(\frac{a}{l_L^{\nu}}\right)$$

or

$$a - l_L^{\nu} L^{-1}(1-\beta) \geq 0 \iff \frac{a}{l_L^{\nu}} \geq L^{-1}(1-\beta) \iff \beta \leq 1 - L\left(\frac{a}{l_L^{\nu}}\right).$$

Hence,
$$a - l_L^{\nu} L^{-1}(1 - \beta) < 0$$
 for $\beta > 1 - L\left(\frac{a}{l_L^{\nu}}\right)$ and $a - l_L^{\nu} L^{-1}(1 - \beta) \ge 0$ for $\beta \le 1 - L\left(\frac{a}{l_L^{\nu}}\right)$.

(a). Let
$$1 - L\left(\frac{a}{l_L^{\nu}}\right) < \beta \le 1$$
 or $a - l_L^{\nu} L^{-1}(1 - \beta) < 0$.

Sub-case 1. $\lambda \ge 0$.

$$(\lambda \tilde{A})^L_{\beta} = [\lambda, \lambda][a - l^v_L L^{-1}(1-\beta), a + r^v_L R^{-1}(1-\beta)] = [\lambda (a - l^v_L L^{-1}(1-\beta)), \lambda (a + r^v_L R^{-1}(1-\beta))].$$

On the similar lines as in Case 1, we get

$$(\lambda \tilde{A})_{\beta_0}^L = [\lambda (a - l_L^{\nu}), \lambda (a + r_L^{\nu})] = [\lambda a - \lambda l_L^{\nu}, \lambda a + \lambda r_L^{\nu}],$$

This yields

$$\lambda \tilde{A} = (\lambda a; \lambda l_L^\mu, \lambda r_L^\mu, \lambda l_U^{\prime \mu}, \lambda r_U^{\prime \mu}; \lambda l_L^\nu, \lambda r_L^\nu, \lambda l_U^{\prime \nu}, \lambda r_U^{\prime \nu})_{LR}.$$

Sub-case 2. $\lambda < 0$.

$$(\lambda\tilde{A})^L_\beta = [\lambda(a+r^v_LR^{-1}(1-\beta)), \lambda(a-l^v_LL^{-1}(1-\beta))].$$

Further, we have

$$(\lambda \tilde{A})_{\beta_0}^L = [\lambda (a + r_L^{\nu}), \lambda (a - l_L^{\nu})] = [\lambda a + \lambda r_L^{\nu}, \lambda a - \lambda l_L^{\nu}]$$

which gives

$$\lambda \tilde{A} = (\lambda a; -\lambda r_L^\mu, -\lambda l_L^\mu, -\lambda r_U^{\prime\mu}, -\lambda l_U^{\prime\mu}; -\lambda r_L^\nu, -\lambda l_L^\nu, -\lambda r_U^{\prime\nu}, -\lambda l_U^{\prime\nu})_{LR}.$$

(b). Let
$$0 \le \beta \le 1 - L\left(\frac{a}{l_L^{\nu}}\right)$$
 or $a - l_L^{\nu} L^{-1}(1 - \beta) \ge 0$.

Sub-case 1. $\lambda \ge 0$.

$$(\lambda \tilde{A})^L_{\beta} = [\lambda, \lambda][a - l_L^{\nu} L^{-1}(1-\beta), a + r_L^{\nu} R^{-1}(1-\beta)] = [\lambda (a - l_L^{\nu} L^{-1}(1-\beta)), \lambda (a + r_L^{\nu} R^{-1}(1-\beta))].$$

Following the steps of Case 1, we obtain

$$(\lambda \tilde{A})_{\beta_0}^L = [\lambda (a - l_L^{\nu}), \lambda (a + r_L^{\nu})] = [\lambda a - \lambda l_L^{\nu}, \lambda a + \lambda r_L^{\nu}]$$

which yields

$$\lambda \tilde{A} = (\lambda a; \lambda l_L^{\mu}, \lambda r_L^{\mu}, \lambda l_U^{\prime \mu}, \lambda r_U^{\prime \mu}; \lambda l_L^{\nu}, \lambda r_L^{\nu}, \lambda l_U^{\prime \nu}, \lambda r_U^{\prime \nu})_{LR}.$$

Sub-case 2. $\lambda < 0$.

$$(\lambda \tilde{A})_{\beta}^{L} = [\lambda (a + r_{L}^{v} R^{-1} (1 - \beta)), \lambda (a - l_{L}^{v} L^{-1} (1 - \beta))].$$

Page 16 of 43

Proceeding on the lines of Case 1 and taking $\beta = \beta_0$, we get

$$(\lambda \tilde{A})_{\beta_0}^L = [\lambda(a + r_L^{\nu}), \lambda(a - l_L^{\nu})] = [\lambda a + \lambda r_L^{\nu}, \lambda a - \lambda l_L^{\nu}].$$

Therefore, we have

$$\lambda \tilde{A} = (\lambda a; -\lambda r_L^{\mu}, -\lambda l_L^{\mu}, -\lambda r_U^{\prime \mu}, -\lambda l_U^{\prime \mu}; -\lambda r_L^{\nu}, -\lambda l_L^{\nu}, -\lambda r_U^{\prime \nu}, -\lambda l_U^{\prime \nu})_{LR}.$$

Thus, the result follows. Rest all the Cases can also be proved on the similar lines. This completes the proof.

Corollary 3.1.1 Proposition 3.1.2 can also be restated as: If \tilde{A} be an LR-type IVIFN and λ be any arbitrary real number, then

$$\begin{split} \lambda \tilde{A} &= \left(\lambda a; \max\{\lambda l_L^\mu, -\lambda r_L^\mu\}, \max\{\lambda r_L^\mu, -\lambda l_L^\mu\}, \max\{\lambda l_U^{\prime\mu}, -\lambda r_U^{\prime\mu}\}, \max\{\lambda r_U^{\prime\mu}, -\lambda l_U^{\prime\mu}\}; \max\{\lambda l_L^\nu, -\lambda r_L^\nu\}, \\ \max\{\lambda r_L^\nu, -\lambda l_L^\nu\}, \max\{\lambda l_U^{\prime\nu}, -\lambda r_U^{\prime\nu}\}, \max\{\lambda r_U^{\prime\nu}, -\lambda l_U^{\prime\nu}\}\right)_{LR}. \end{split}$$

Proposition 3.1.3. Let $\tilde{A}_1 = (a_1; l_{1L}^{\mu}, r_{1L}^{\mu}, l_{1U}^{\prime \mu}, r_{1L}^{\prime \mu}, l_{1U}^{\prime \nu}, r_{1L}^{\prime \nu}, l_{1U}^{\prime \nu}, r_{1L}^{\prime \nu}, l_{1U}^{\prime \nu}, r_{1L}^{\prime \nu})_{LR}$ be an LR-type IVIFN such that $a_1 - l_{1L}^{\nu} < 0$, $a_1 - l_{1U}^{\prime \nu} \ge 0$ and $\tilde{A}_2 = (a_2; l_{2L}^{\mu}, r_{2L}^{\mu}, l_{2U}^{\prime \mu}, r_{2U}^{\prime \mu}, l_{2U}^{\prime \nu}, r_{2U}^{\prime \nu}, l_{2U}^{\prime \nu}, r_{2U}^{\prime \nu})_{LR}$ be any LR-type IVIFN. Then

$$\begin{split} \tilde{A}_1 \odot \tilde{A}_2 &= (a; l_L^\mu, r_L^\mu, l_U'^\mu, r_U'^\mu; l_L^\nu, r_L^\nu, l_U'^\nu, r_U'^\nu)_{LR} \quad where \\ a &= a_1 a_2, \\ l_L^\mu &= a_1 a_2 - \min\{(a_1 - l_{1L}^\mu)(a_2 - l_{2L}^\mu), \ (a_1 + r_{1L}^\mu)(a_2 - l_{2L}^\mu)\}, \\ r_L^\mu &= \max\{(a_1 - l_{1L}^\mu)(a_2 + r_{2L}^\mu), \ (a_1 + r_{1L}^\mu)(a_2 + r_{2L}^\mu)\} - a_1 a_2, \\ l_U''^\mu &= a_1 a_2 - \min\{(a_1 - l_{1U}'^\mu)(a_2 - l_{2U}'^\mu), \ (a_1 + r_{1U}'^\mu)(a_2 - l_{2U}'^\mu)\}, \\ r_U''^\mu &= \max\{(a_1 - l_{1U}'^\mu)(a_2 + r_{2U}'^\mu), \ (a_1 + r_{1U}'^\mu)(a_2 + r_{2U}'^\mu)\} - a_1 a_2, \\ l_L^\nu &= a_1 a_2 - \min\{(a_1 - l_{1L}^\nu)(a_2 + r_{2L}^\nu), \ (a_1 + r_{1L}^\mu)(a_2 + r_{2L}^\nu)\}, \\ r_L^\nu &= \max\{(a_1 - l_{1L}^\nu)(a_2 - l_{2L}^\nu), \ (a_1 + r_{1L}^\nu)(a_2 + r_{2L}^\nu)\} - a_1 a_2, \\ l_U''^\nu &= a_1 a_2 - \min\{(a_1 - l_{1U}'^\nu)(a_2 - l_{2U}'^\nu), \ (a_1 + r_{1U}'^\nu)(a_2 - l_{2U}'^\nu)\}, \\ r_U''^\nu &= \max\{(a_1 + r_{1U}'^\nu)(a_2 + r_{2U}'^\nu), \ (a_1 - l_{1U}'^\nu)(a_2 + r_{2U}'^\nu)\} - a_1 a_2, \\ where the conditions for LR-type representation of \tilde{A}_1 \odot \tilde{A}_2 \ are \ satisfied. \end{split}$$

Proof. Let $\tilde{A}_1 = (a_1; l_{1L}^{\mu}, r_{1L}^{\mu}, l_{1U}^{\prime \mu}, r_{1U}^{\prime \mu}; l_{1L}^{\nu}, r_{1L}^{\nu}, l_{1U}^{\prime \nu}, r_{1U}^{\prime \nu}, l_{1U}^{\prime \nu}, r_{1U}^{\prime \nu})_{LR}$ and $\tilde{A}_2 = (a_2; l_{2L}^{\mu}, r_{2L}^{\mu}, l_{2U}^{\prime \mu}, r_{2U}^{\prime \mu}; l_{2L}^{\nu}, r_{2L}^{\nu}, l_{2U}^{\prime \nu}, r_{2U}^{\prime \nu})_{LR}$ be two LR-type IVIFN with $a_1 - l_{1L}^{\nu} < 0$, $a_1 - l_{1U}^{\prime \nu} \ge 0$ and $a_2 - l_{2L}^{\nu}$, $a_2 - l_{2U}^{\prime \nu}$, $a_2 - l_{2U}^{\prime \mu}$, $a_2 - l_{2L}^{\mu}$, $a_2 + r_{2L}^{\mu}$, $a_2 + r_{2L}^{\prime \mu}$, $a_2 + r_{2L}^{\prime \nu}$, be any real numbers. Then, in view of Theorem 3.1, we can write

$$\begin{split} & \left(\tilde{A}_1 \odot \tilde{A}_2 \right)_{\alpha}^L = A_{1\alpha}^L \times A_{2\alpha}^L = \left[a_1 - l_{1L}^{\mu} L^{-1}(\alpha), a_1 + r_{1L}^{\mu} R^{-1}(\alpha) \right] \times \left[a_2 - l_{2L}^{\mu} L^{-1}(\alpha), a_2 + r_{2L}^{\mu} R^{-1}(\alpha) \right]. \\ & \left(\tilde{A}_1 \odot \tilde{A}_2 \right)_{\alpha}^U = A_{1\alpha}^U \times A_{2\alpha}^U = \left[a_1 - l_{1U}^{\prime \mu} (L')^{-1}(\alpha), a_1 + r_{1U}^{\prime \mu} (R')^{-1}(\alpha) \right] \times \left[a_2 - l_{2U}^{\prime \mu} (L')^{-1}(\alpha), a_2 + r_{2U}^{\prime \mu} (R')^{-1}(\alpha) \right]. \\ & \left(\tilde{A}_1 \odot \tilde{A}_2 \right)_{\beta}^L = A_{1\beta}^L \times A_{2\beta}^L = \left[a_1 - l_{1L}^{\nu} L^{-1} (1 - \beta), a_1 + r_{1L}^{\nu} R^{-1} (1 - \beta) \right] \times \left[a_2 - l_{2L}^{\nu} L^{-1} (1 - \beta), a_2 + r_{2L}^{\nu} R^{-1} (1 - \beta) \right]. \\ & \left(\tilde{A}_1 \odot \tilde{A}_2 \right)_{\beta}^U = A_{1\beta}^U \times A_{2\beta}^U = \left[a_1 - l_{1U}^{\prime \nu} (L')^{-1} (1 - \beta), a_1 + r_{1U}^{\prime \nu} (R')^{-1} (1 - \beta) \right] \times \left[a_2 - l_{2U}^{\prime \nu} (L')^{-1} (1 - \beta), a_2 + r_{2U}^{\prime \nu} (R')^{-1} (1 - \beta) \right]. \end{split}$$

Since $a_1 - l_{1L}^{\nu} < 0$ and $a_1 - l_{1U}^{\prime \nu} \ge 0$, therefore

$$a_1 - l_{1U}^{\prime \nu}(L^{\prime})^{-1}(1 - \beta) \ge 0 \text{ for } \beta \in [0, 1] \text{ and } a_1 - l_{1L}^{\nu}L^{-1}(1 - \beta) \le (\ge) 0 \text{ for } \beta \le (\ge) 1 - L\left(\frac{a_1}{l_{1L}^{\nu}}\right).$$

Now, to find the product $\tilde{A}_1 \odot \tilde{A}_2$, the following nine cases will arise and in each case, based on the sign of $a_1 - l_{1L}^{\nu} L^{-1} (1 - \beta)$, two sub-cases are there.

$$\begin{aligned} &\textbf{Case 1.} \ \ a_2 - l_{2L}^{\nu} \geq 0. \\ &\textbf{Then,} \ \ a_2 - l_{2L}^{\nu} L^{-1} (1 - \beta) \geq 0, \ \ a_2 - l_{2L}^{\prime \nu} (L^{\prime})^{-1} (1 - \beta) \geq 0, \ \ a_2 - l_{2L}^{\prime \mu} L^{-1} (\alpha) \geq 0, \ \ a_2 - l_{2L}^{\prime \mu} (L^{\prime})^{-1} (\alpha) \geq 0 \ \forall \ \alpha, \beta \in [0, 1]. \end{aligned}$$

Page 17 of 43

$$\begin{split} &\frac{\text{Sub-case }1.1.}{\text{Now,}} \quad a_1 - l_{1L}^{\nu} L^{-1}(1-\beta) \leq 0. \\ &\tilde{N} \\ &\tilde{O} \\ &\tilde{A}_2 \Big)_{\alpha}^{L} = [(a_1 - l_{1L}^{\mu} L^{-1}(\alpha))(a_2 - l_{2L}^{\mu} L^{-1}(\alpha)), \ (a_1 + r_{1L}^{\mu} R^{-1}(\alpha))(a_2 + r_{2L}^{\mu} R^{-1}(\alpha))] \\ &\tilde{A}_1 \odot \tilde{A}_2 \Big)_{\alpha}^{U} = [(a_1 - l_{1U}^{\prime \mu} (L^{\prime})^{-1}(\alpha))(a_2 - l_{2U}^{\prime \mu} (L^{\prime})^{-1}(\alpha)), \ (a_1 + r_{1U}^{\prime \mu} (R^{\prime})^{-1}(\alpha))(a_2 + r_{2U}^{\prime \mu} (R^{\prime})^{-1}(\alpha))] \\ &\tilde{A}_1 \odot \tilde{A}_2 \Big)_{\beta}^{L} = [(a_1 - l_{1L}^{\nu} L^{-1}(1-\beta))(a_2 + r_{2L}^{\nu} R^{-1}(1-\beta)), \ (a_1 + r_{1L}^{\nu} R^{-1}(1-\beta))(a_2 + r_{2L}^{\nu} R^{-1}(1-\beta))] \\ &\tilde{A}_1 \odot \tilde{A}_2 \Big)_{\beta}^{U} = [(a_1 - l_{1U}^{\prime \nu} (L^{\prime})^{-1}(1-\beta))(a_2 - l_{2U}^{\prime \nu} (L^{\prime})^{-1}(1-\beta)), \ (a_1 + r_{1U}^{\prime \nu} (R^{\prime})^{-1}(1-\beta))(a_2 + r_{2U}^{\prime \nu} (R^{\prime})^{-1}(1-\beta))]. \end{split}$$

Further, as L, R, L' and R' are decreasing functions on $[0, \infty)$ with L(0) = R(0) = L'(0) = R'(0) = 1, there exists $\alpha_0 \in (0, 1]$, such that $L^{-1}(\alpha_0) = R^{-1}(\alpha_0) = (L')^{-1}(\alpha_0) = (R')^{-1}(\alpha_0) = 1$. Hence,

$$\begin{split} & \left(\tilde{A}_{1} \odot \tilde{A}_{2} \right)_{\alpha_{0}}^{L} = \left[(a_{1} - l_{1L}^{\mu})(a_{2} - l_{2L}^{\mu}), \ (a_{1} + r_{1L}^{\mu})(a_{2} + r_{2L}^{\mu}) \right], \\ & \left(\tilde{A}_{1} \odot \tilde{A}_{2} \right)_{\alpha_{0}}^{U} = \left[(a_{1} - l_{1U}^{\prime \mu})(a_{2} - l_{2U}^{\prime \mu}), \ (a_{1} + r_{1U}^{\prime \mu})(a_{2} + r_{2U}^{\prime \mu}) \right]. \end{split}$$

Also, choosing $\beta_0 = 1 - \alpha_0 \in (0, 1]$, we have

$$\begin{pmatrix} \tilde{A}_{1} \odot \tilde{A}_{2} \end{pmatrix}_{\beta_{0}}^{L} = \left[(a_{1} - l_{1L}^{\nu})(a_{2} + r_{2L}^{\nu}), \ (a_{1} + r_{1L}^{\nu})(a_{2} + r_{2L}^{\nu}) \right], \\
\left(\tilde{A}_{1} \odot \tilde{A}_{2} \right)_{\beta_{0}}^{U} = \left[(a_{1} - l_{1U}^{\prime \nu})(a_{2} - l_{2U}^{\prime \nu}), \ (a_{1} + r_{1U}^{\prime \nu})(a_{2} + r_{2U}^{\prime \nu}) \right].$$
(22)

Taking $\alpha = 1$ and $\beta = 0$, we obtain

$$(\tilde{A}_1 \odot \tilde{A}_2)_{\alpha=1}^L = (\tilde{A}_1 \odot \tilde{A}_2)_{\alpha=1}^U = (\tilde{A}_1 \odot \tilde{A}_2)_{\beta=0}^L = (\tilde{A}_1 \odot \tilde{A}_2)_{\beta=0}^U = a_1 a_2.$$
 (23)

Now, using the property that \tilde{A}_1 and \tilde{A}_2 are LR-type IVIFNs, we obtain

$$\begin{split} a_1a_2 - (a_1 - l_{1U}'^\mu)(a_2 - l_{2U}'^\mu) &\geq a_1a_2 - (a_1 - l_{1L}^\mu)(a_2 - l_{2L}^\mu) \implies l_U'^\mu \geq l_L^\mu. \\ (a_1 + r_{1U}'^\mu)(a_2 + r_{2U}'^\mu) - a_1a_2 &\geq (a_1 + r_{1L}^\mu)(a_2 + r_{2L}^\mu) - a_1a_2 \implies r_U'^\mu \geq r_L^\mu. \\ (a_1 + r_{1L}^\nu)(a_2 + r_{2L}^\nu) - a_1a_2 &\geq (a_1 + r_{1U}'^\nu)(a_2 + r_{2U}'^\nu) - a_1a_2 \implies r_L^\nu \geq r_U'^\nu. \\ (a_1 + r_{1L}^\nu)(a_2 + r_{2L}^\nu) - a_1a_2 &\geq (a_1 + r_{1L}^\mu)(a_2 + r_{2L}^\mu) - a_1a_2 \implies r_L^\nu \geq r_L^\mu. \\ a_1a_2 - (a_1 - l_{1U}'^\nu)(a_2 - l_{2U}'^\nu) &\geq a_1a_2 - (a_1 - l_{1U}'^\mu)(a_2 - l_{2U}'^\mu) \implies l_U'^\nu \geq l_U'^\mu. \\ (a_1 + r_{1U}'^\nu)(a_2 + r_{2U}'^\nu) - a_1a_2 &\geq (a_1 + r_{1U}'^\mu)(a_2 + r_{2U}'^\mu) - a_1a_2 \implies r_U'^\nu \geq r_U'^\mu. \end{split}$$

Further, since $a_2 - l_{2L}^{\nu} \le a_2 + r_{2L}^{\nu}$, $a_1 - l_{1L}^{\nu} \le a_1 - l_{1U}^{\prime \nu}$ and $a_2 - l_{2L}^{\nu} \le a_2 - l_{2U}^{\prime \nu}$, therefore

$$\begin{aligned} &a_1a_2-(a_1-l_{1U}^{\prime\nu})(a_2-l_{2U}^{\prime\nu})\leq a_1a_2-(a_1-l_{1L}^{\nu})(a_2+r_{2L}^{\nu}), \text{ that is, } \ l_U^{\prime\nu}\leq l_L^{\nu}\\ &\text{and from } a_1-l_{1L}^{\nu}<0, \ a_2-l_{2L}^{\nu}\leq a_2+r_{2L}^{\nu}, a_1-l_{1L}^{\nu}\leq a_1-l_{1L}^{\mu}, \ a_2-l_{2L}^{\nu}\leq a_2-l_{2L}^{\mu}, \text{ we have }\\ &a_1a_2-(a_1-l_{1L}^{\mu})(a_2-l_{2L}^{\mu})\leq a_1a_2-(a_2+r_{2L}^{\nu})(a_1-l_{1L}^{\nu}), \text{ that is, } l_L^{\mu}\leq l_L^{\nu}.\\ &\text{Moreover, } l_{2L}^{\mu}(a_1-l_{1L}^{\mu})+a_2l_{1L}^{\mu}\geq 0 \quad \text{and} \quad a_1r_{2L}^{\mu}+a_2r_{1L}^{\mu}+r_{1L}^{\mu}r_{2L}^{\mu}\geq 0 \quad \text{yield} \quad l_L^{\mu}\geq 0 \text{ and } r_L^{\mu}\geq 0, \text{ respectively.} \end{aligned}$$

Finally, in view of these inequalities and combining expressions (21), (22) and (23), we get

$$\begin{split} \tilde{A}_{1} \odot \tilde{A}_{2} &= \left(a_{1}a_{2}; a_{1}a_{2} - \left(a_{1} - l_{1L}^{\mu}\right)\left(a_{2} - l_{2L}^{\mu}\right), \left(a_{1} + r_{1L}^{\mu}\right)\left(a_{2} + r_{2L}^{\mu}\right) - a_{1}a_{2}, a_{1}a_{2} - \left(a_{1} - l_{1U}^{\prime\mu}\right)\left(a_{2} - l_{2U}^{\prime\mu}\right), \\ \left(a_{1} + r_{1U}^{\prime\mu}\right)\left(a_{2} + r_{2U}^{\prime\mu}\right) - a_{1}a_{2}; a_{1}a_{2} - \left(a_{1} - l_{1L}^{\nu}\right)\left(a_{2} + r_{2L}^{\nu}\right), \left(a_{1} + r_{1L}^{\nu}\right)\left(a_{2} + r_{2L}^{\nu}\right) - a_{1}a_{2}, \\ a_{1}a_{2} - \left(a_{1} - l_{1U}^{\prime\nu}\right)\left(a_{2} - l_{2U}^{\prime\nu}\right), \left(a_{1} + r_{1U}^{\prime\nu}\right)\left(a_{2} + r_{2U}^{\prime\nu}\right) - a_{1}a_{2}\right)_{LR}. \end{split}$$

Sub-case 1.2. $a_1 - l_{1L}^{\nu} L^{-1} (1 - \beta) \ge 0.$ Now,

Page 18 of 43

$$\left(\tilde{A}_1 \odot \tilde{A}_2 \right)_{\beta}^L = \left[\left(a_1 - l_{1L}^{\nu} L^{-1} (1 - \beta) \right) \left(a_2 - l_{2L}^{\nu} L^{-1} (1 - \beta) \right), \ \left(a_1 + r_{1L}^{\nu} R^{-1} (1 - \beta) \right) \left(a_2 + r_{2L}^{\nu} R^{-1} (1 - \beta) \right) \right].$$

Choosing $\beta = \beta_0$, we have

$$(\tilde{A}_1 \odot \tilde{A}_2)_{\beta_0}^L = [(a_1 - l_{1L}^{\nu}) (a_2 - l_{2L}^{\nu}), (a_1 + r_{1L}^{\nu}) (a_2 + r_{2L}^{\nu})].$$
 (24)

Hence, we have $l_L^{\nu} = a_1 a_2 - \left(a_1 - l_{1L}^{\nu}\right) \left(a_2 - l_{2L}^{\nu}\right)$ and remaining spreads of $\tilde{A}_1 \odot \tilde{A}_2$ will be same as derived in Sub-case 1.1. Further,

$$\begin{split} (a_1 - l_{1L}^{\nu})(a_2 - l_{2L}^{\nu}) & \leq (a_1 - l_{1U}^{\prime \nu})(a_2 - l_{2U}^{\prime \nu}) \implies l_L^{\nu} \geq l_U^{\prime \nu} \quad \text{and} \\ (a_1 - l_{1L}^{\nu})(a_2 - l_{2L}^{\nu}) & \leq (a_1 - l_{1L}^{\mu})(a_2 - l_{2L}^{\mu}) \implies l_L^{\nu} \geq l_L^{\mu}. \end{split}$$

Finally, combining Sub-cases 1.1 and 1.2, we get

$$\begin{split} \tilde{A}_1 & \odot \tilde{A}_2 = (a; l_L^\mu, r_L^\mu, l_U^{\prime\mu}, r_U^{\prime\mu}; l_L^\nu, r_L^\nu, l_U^{\prime\nu}, r_U^{\prime\nu})_{LR} \text{ where } \\ a &= a_1 a_2, \\ l_L^\mu &= a_1 a_2 - (a_1 - l_{1L}^\mu)(a_2 - l_{2L}^\mu), \\ r_L^\mu &= (a_1 + r_{1L}^\mu)(a_2 + r_{2L}^\mu) - a_1 a_2, \\ l_U^{\prime\mu} &= a_1 a_2 - (a_1 - l_{1U}^{\prime\mu})(a_2 - l_{2U}^{\prime\mu}), \\ r_U^{\prime\mu} &= (a_1 + r_{1U}^{\prime\mu})(a_2 + r_{2U}^{\prime\mu}) - a_1 a_2, \\ l_L^\nu &= a_1 a_2 - \min\{(a_1 - l_{1L}^\nu)(a_2 + r_{2L}^\nu), (a_1 - l_{1L}^\nu)(a_2 - l_{2L}^\nu)\}, \\ r_L^\nu &= (a_1 + r_{1L}^\nu)(a_2 + r_{2L}^\nu)\} - a_1 a_2, \\ l_U^{\prime\nu} &= a_1 a_2 - (a_1 - l_{1U}^{\prime\nu})(a_2 - l_{2U}^{\prime\nu}), \\ r_U^{\prime\nu} &= (a_1 + r_{1U}^\nu)(a_2 + r_{2U}^\prime) - a_1 a_2. \end{split}$$

Case 2. $a_2 - l_{2L}^{\nu} < 0$ and $a_2 - l_{2U}^{\prime \nu} \ge 0$.

Then, we have

$$a_2 - l_{2L}^{\nu} L^{-1} (1 - \beta) \le (\ge) 0 \text{ for } \beta \le (\ge) 1 - L \left(\frac{a_2}{l_{2L}^{\nu}} \right).$$

<u>Sub-case 2.1.</u> $a_1 - l_{1L}^{\nu} L^{-1} (1 - \beta) \le 0.$

Then, $(\tilde{A}_1 \odot \tilde{A}_2)^L_{\alpha}$, $(\tilde{A}_1 \odot \tilde{A}_2)^U_{\alpha}$, $(\tilde{A}_1 \odot \tilde{A}_2)^U_{\beta}$ remain same as in Sub-case 1.1, however, the expression for $(\tilde{A}_1 \odot \tilde{A}_2)^L_{\beta}$ will be changed and is given by

$$\begin{split} \left(\tilde{A}_{1} \odot \tilde{A}_{2}\right)_{\beta}^{L} &= A_{1\beta}^{L} \times A_{2\beta}^{L} \\ &= \left[\left(a_{1} - l_{1L}^{v} L^{-1}(1-\beta)\right), \left(a_{1} + r_{1L}^{v} R^{-1}(1-\beta)\right)\right] \times \\ \left[\left(a_{2} - l_{2L}^{v} L^{-1}(1-\beta)\right), \left(a_{2} + r_{2L}^{v} R^{-1}(1-\beta)\right)\right] \\ &= \left[\min\left\{\left(a_{1} - l_{1L}^{v} L^{-1}(1-\beta)\right)\left(a_{2} + r_{2L}^{v} R^{-1}(1-\beta)\right), \left(a_{2} - l_{2L}^{v} L^{-1}(1-\beta)\right)\left(a_{1} + r_{1L}^{v} R^{-1}(1-\beta)\right)\right\}, \\ \max\left\{\left(a_{1} - l_{1L}^{v} L^{-1}(1-\beta)\right)\left(a_{2} - l_{2L}^{v} L^{-1}(1-\beta)\right), \left(a_{1} + r_{1L}^{v} R^{-1}(1-\beta)\right)\left(a_{2} + r_{2L}^{v} R^{-1}(1-\beta)\right)\right\}\right]. \end{split}$$

Choosing $\beta = \beta_0$, we get

$$(\tilde{A}_{1} \odot \tilde{A}_{2})_{\beta_{0}}^{L} = \left[\min \left\{ \left(a_{1} - l_{1L}^{\nu} \right) \left(a_{2} + r_{2L}^{\nu} \right), \left(a_{2} - l_{2L}^{\nu} \right) \left(a_{1} + r_{1L}^{\nu} \right) \right\},$$

$$\max \left\{ \left(a_{1} - l_{1L}^{\nu} \right) \left(a_{2} - l_{2L}^{\nu} \right), \left(a_{1} + r_{1L}^{\nu} \right) \left(a_{2} + r_{2L}^{\nu} \right) \right\} \right].$$

$$(25)$$

Hence,

$$\begin{split} l_L^{\nu} &= a_1 a_2 - \min \left\{ \left(a_1 - l_{1L}^{\nu} \right) \left(a_2 + r_{2L}^{\nu} \right), \; \left(a_2 - l_{2L}^{\nu} \right) \left(a_1 + r_{1L}^{\nu} \right) \right\}, \\ r_L^{\nu} &= \max \left\{ \left(a_1 - l_{1L}^{\nu} \right) \left(a_2 - l_{2L}^{\nu} \right), \left(a_1 + r_{1L}^{\nu} \right) \left(a_2 + r_{2L}^{\nu} \right) \right\} - a_1 a_2. \end{split}$$

Page 19 of 43

Further, since $a_1 - l_{1L}^{\nu} \le a_1 - l_{1U}^{\prime \nu}$, $a_2 - l_{2U}^{\prime \nu} \le a_2 + r_{2U}^{\prime \nu}$, $a_2 + r_{2U}^{\prime \nu} \le a_2 + r_{2L}^{\nu}$ and $a_1 - l_{1L}^{\nu} < 0$, therefore

$$(a_1 - l_{1II}^{\prime \nu})(a_2 - l_{2II}^{\prime \nu}) \ge (a_1 - l_{1I}^{\nu})(a_2 + r_{2I}^{\nu}). \tag{26}$$

Also, $a_1 - l_{1II}^{\prime \nu} \le a_1 + r_{1II}^{\prime \nu} \le a_1 + r_{1II}^{\nu}$ and $a_2 - l_{2II}^{\nu} < 0$ implies

$$(a_1 - l_{1II}^{\prime \nu})(a_2 - l_{2II}^{\prime \nu}) \ge (a_1 + r_{1I}^{\nu})(a_2 - l_{2I}^{\nu}). \tag{27}$$

It follows from the inequalities (26) and (27) that

$$(a_1 - l_{1U}^{\prime \nu})(a_2 - l_{2U}^{\prime \nu}) \ge \min\{(a_1 - l_{1L}^{\nu})(a_2 + r_{2L}^{\nu}), (a_1 + r_{1L}^{\nu})(a_2 - l_{2L}^{\nu})\}, \text{ that is, } l_U^{\prime \nu} \le l_L^{\nu}.$$

Now, if $(a_1 - l_{11}^{\nu})(a_2 - l_{21}^{\nu}) \le (a_1 + r_{11}^{\nu})(a_2 + r_{21}^{\nu})$, then

$$(a_1 + r_{1U}^{\prime \nu})(a_2 + r_{2U}^{\prime \nu}) \le (a_1 + r_{1L}^{\nu})(a_2 + r_{2L}^{\nu}) \tag{28}$$

and if $(a_1 - l_{1I}^{\nu})(a_2 - l_{2I}^{\nu}) \ge (a_1 + r_{1I}^{\nu})(a_2 + r_{2I}^{\nu})$, then

$$(a_1 + r_{1U}^{\prime \nu})(a_2 + r_{2U}^{\prime \nu}) \le (a_1 + r_{1L}^{\nu})(a_2 + r_{2L}^{\nu}) \le (a_1 - l_{1L}^{\nu})(a_2 - l_{2L}^{\nu}). \tag{29}$$

Thus, the inequalities (28) and (29) yield

$$(a_1 + r_{1U}'^{\nu})(a_2 + r_{2U}'^{\nu}) \leq \max\{(a_1 + r_{1L}^{\nu})(a_2 + r_{2L}^{\nu}), \ (a_1 - l_{1L}^{\nu})(a_2 - l_{2L}^{\nu})\}, \ \text{that is, } r_U'^{\nu} \leq r_L^{\nu}.$$

Further, $a_1 - l_{1L}^{\nu} \le a_1 - l_{1L}^{\mu}$, $a_1 - l_{1L}^{\nu} < 0$ give

$$(a_1 - l_{1L}^{\mu})(a_2 - l_{2L}^{\mu}) \ge (a_1 - l_{1L}^{\nu})(a_2 + r_{2L}^{\nu}) \tag{30}$$

and

$$(a_1 - l_{1L}^{\mu})(a_2 - l_{2L}^{\mu}) \ge (a_1 - l_{1L}^{\mu})(a_2 - l_{2L}^{\nu}) \ge (a_1 + r_{1L}^{\nu})(a_2 - l_{2L}^{\nu}). \tag{31}$$

It follows from the expressions (30) and (31) that

$$(a_1 - l_{1I}^{\mu})(a_2 - l_{2I}^{\mu}) \ge \min\{(a_1 - l_{1I}^{\nu})(a_2 + r_{2I}^{\nu}), (a_1 + r_{1I}^{\nu})(a_2 - l_{2I}^{\nu})\}, \text{ that is, } l_I^{\mu} \le l_I^{\nu}.$$

Now, if $(a_1 - l_{11}^{\nu})(a_2 - l_{21}^{\nu}) \le (a_1 + r_{11}^{\nu})(a_2 + r_{21}^{\nu})$, then

$$(a_1 + r_{1L}^{\mu})(a_2 + r_{2L}^{\mu}) \le (a_1 + r_{1L}^{\nu})(a_2 + r_{2L}^{\nu}) \tag{32}$$

and if $(a_1 - l_{1I}^{\nu})(a_2 - l_{2I}^{\nu}) \ge (a_1 + r_{1I}^{\nu})(a_2 + r_{2I}^{\nu})$, then

$$(a_1 + r_{1L}^{\mu})(a_2 + r_{2L}^{\mu}) \le (a_1 + r_{1L}^{\nu})(a_2 + r_{2L}^{\nu}) \le (a_1 - l_{1L}^{\nu})(a_2 - l_{2L}^{\nu}). \tag{33}$$

Hence, the inequalities (32) and (33) give

$$(a_1 + r_{1L}^\mu)(a_2 + r_{2L}^\mu) \leq \max\{(a_1 + r_{1L}^\nu)(a_2 + r_{2L}^\nu), \; (a_1 - l_{1L}^\nu)(a_2 - l_{2L}^\nu)\}, \; \text{that is, } r_L^\mu \leq r_L^\nu.$$

<u>Sub-case 2.2.</u> $a_1 - l_{1L}^v L^{-1}(1 - \beta) \ge 0$. The proof of this part follows on the lines of Sub-case 1.1.

Hence, now combining Cases 1 and 2, we get $\tilde{A}_1 \odot \tilde{A}_2 = (a; l_I^{\mu}, r_I^{\mu}, l_U^{\prime \mu}, r_U^{\prime \mu}; l_I^{\nu}, r_I^{\nu}, l_U^{\prime \nu}, r_U^{\prime \nu})_{LR}$ where
$$\begin{split} l_L^{\mu} &= a_1 a_2 - (a_1 - l_{1L}^{\mu})(a_2 - l_{2L}^{\mu}), \\ r_L^{\mu} &= (a_1 + r_{1L}^{\mu})(a_2 + r_{2L}^{\mu}) - a_1 a_2, \\ l_U^{\prime \mu} &= a_1 a_2 - (a_1 - l_{1U}^{\prime \mu})(a_2 - l_{2U}^{\prime \mu}), \end{split}$$

$$l_U^{\prime\mu} = a_1 a_2 - (a_1 - l_{1U}^{\prime\mu})(a_2 - l_{2U}^{\prime\mu}),$$

Page 20 of 43

$$\begin{split} r_U'^{\mu} &= (a_1 + r_{1U}'^{\mu})(a_2 + r_{2U}'^{\mu}) - a_1 a_2, \\ l_L^{\nu} &= a_1 a_2 - \min\left\{\left(a_1 - l_{1L}^{\nu}\right)\left(a_2 + r_{2L}^{\nu}\right), \; \left(a_2 - l_{2L}^{\nu}\right)\left(a_1 + r_{1L}^{\nu}\right)\right\}, \\ r_L^{\nu} &= \max\left\{\left(a_1 - l_{1L}^{\nu}\right)\left(a_2 - l_{2L}^{\nu}\right), \left(a_1 + r_{1L}^{\nu}\right)\left(a_2 + r_{2L}^{\nu}\right)\right\} - a_1 a_2, \\ l_U^{\nu} &= a_1 a_2 - (a_1 - l_{1U}'^{\nu})(a_2 - l_{2U}'^{\nu}), \\ r_U^{\prime \nu} &= (a_1 + r_{1U}'^{\nu})(a_2 + r_{2U}'^{\nu}) - a_1 a_2. \end{split}$$

Case 3. $a_2 - l_{2U}^{\prime \nu} < 0$ and $a_2 - l_{2U}^{\prime \mu} \ge 0$.

Then,

$$a_2 - l_{2U}^{\prime \nu}(L^{\prime})^{-1}(1-\beta) \le (\ge) 0 \text{ for } \beta \le (\ge) 1 - L^{\prime} \left(\frac{a_2}{l_{2U}^{\prime \nu}}\right).$$

<u>Sub-case 3.1.</u> $a_1 - l_{1I}^{\nu} L^{-1} (1 - \beta) \le 0.$

Then, the expressions $(\tilde{A}_1 \odot \tilde{A}_2)_{\alpha}^L$, $(\tilde{A}_1 \odot \tilde{A}_2)_{\alpha}^U$, $(\tilde{A}_1 \odot \tilde{A}_2)_{\beta}^L$ are same as discussed in Case 2 but $(\tilde{A}_1 \odot \tilde{A}_2)_{\beta}^U$ is given by:

$$\begin{split} \left(\tilde{A}_{1} \odot \tilde{A}_{2}\right)_{\beta}^{U} &= A_{1\beta}^{U} \times A_{2\beta}^{U} \\ &= \left[\left(a_{1} + r_{1U}^{\prime v}(R^{\prime})^{-1}(1-\beta)\right)\left(a_{2} - l_{2U}^{\prime v}(L^{\prime})^{-1}(1-\beta)\right), \left(a_{1} + r_{1U}^{\prime v}(R^{\prime})^{-1}(1-\beta)\right)\left(a_{2} + r_{2U}^{\prime v}(R^{\prime})^{-1}(1-\beta)\right)\right]. \end{split}$$

Taking $\beta = \beta_0$, we get

$$\left(\tilde{A}_{1} \odot \tilde{A}_{2}\right)_{\beta_{0}}^{U} = \left[\left(a_{1} + r_{1U}^{\prime \nu}\right)\left(a_{2} - l_{2U}^{\prime \nu}\right), \left(a_{1} + r_{1U}^{\prime \nu}\right)\left(a_{2} + r_{2U}^{\prime \nu}\right)\right]$$

which gives

$$l_{II}^{\prime\nu} = a_1 a_2 - (a_1 + r_{1II}^{\prime\nu})(a_2 - l_{2II}^{\prime\nu}).$$

Also, we have

$$l_{I}^{\nu} = a_{1}a_{2} - \min\left\{\left(a_{1} - l_{1I}^{\nu}\right)\left(a_{2} + r_{2I}^{\nu}\right), \left(a_{2} - l_{2I}^{\nu}\right)\left(a_{1} + r_{1I}^{\nu}\right)\right\} \text{ and } l_{II}^{\prime\mu} = a_{1}a_{2} - (a_{1} - l_{1II}^{\prime\mu})(a_{2} - l_{2II}^{\prime\mu}).$$

Now, we claim that $l_L^{\nu} \ge l_U^{\prime \nu}$ and $l_U^{\prime \nu} \ge l_U^{\prime \mu}$.

Since $a_2 - l_{2L}^{\nu} < 0$, $a_2 - l_{2L}^{\nu} \le a_2 - l_{2U}^{\prime \nu}$ and $a_1 + r_{1U}^{\prime \nu} \le a_1 + r_{1L}^{\nu}$, therefore, we obtain

$$(a_1 + r_{1U}^{\prime \nu})(a_2 - l_{2U}^{\prime \nu}) \ge (a_1 + r_{1L}^{\nu})(a_2 - l_{2L}^{\nu}), \tag{34}$$

$$(a_1 + r_{1II}^{\prime \nu})(a_2 - l_{2II}^{\prime \nu}) \ge (a_1 - l_{1I}^{\nu})(a_2 + r_{2I}^{\nu}). \tag{35}$$

Thus, the inequalities (34) and (35) yield

$$(a_1 + r_{1U}^{\prime \nu})(a_2 - l_{2U}^{\prime \nu}) \ge \min\{(a_1 + r_{1L}^{\nu})(a_2 - l_{2L}^{\nu}), (a_1 - l_{1L}^{\nu})(a_2 + r_{2L}^{\nu})\} \text{ that is, } l_U^{\prime \nu} \le l_L^{\nu}.$$

Further, $a_2 - l_{2U}^{\prime \nu} < 0$, $a_1 - l_{1U}^{\prime \mu} \le a_1 + r_{1U}^{\prime \nu}$, $a_2 - l_{2U}^{\prime \nu} \le a_2 - l_{2U}^{\prime \mu}$ implies

$$(a_1 + r'^{\nu}_{1U})(a_2 - l'^{\nu}_{2U}) \le (a_1 - l'^{\mu}_{1U})(a_2 - l'^{\nu}_{2U}) \le (a_1 - l'^{\mu}_{1U})(a_2 - l'^{\mu}_{2U})$$
 that is, $l'^{\nu}_U \ge l'^{\mu}_{U}$.

<u>Sub-case 3.2.</u> $a_1 - l_{1L}^{\nu} L^{-1} (1 - \beta) \ge 0.$

The proof of this part follows on the lines of Sub-cases 1.1 and 2.2.

Hence, it is concluded from Cases 1 - 3 that

$$\begin{split} \tilde{A}_1 & \odot \tilde{A}_2 = (a; l_L^\mu, r_L^\mu, l_U^{\prime\mu}, r_U^{\prime\mu}; l_L^\nu, r_L^\nu, l_U^{\prime\nu}, r_U^{\prime\nu})_{LR} \text{ where } \\ a &= a_1 a_2, \\ l_L^\mu &= a_1 a_2 - (a_1 - l_{1L}^\mu)(a_2 - l_{2L}^\mu), \\ r_L^\mu &= (a_1 + r_{1L}^\mu)(a_2 + r_{2L}^\mu) - a_1 a_2, \\ l_U^{\prime\mu} &= a_1 a_2 - (a_1 - l_{1U}^{\prime\mu})(a_2 - l_{2U}^{\prime\mu}), \\ r_U^{\prime\mu} &= (a_1 + r_{1U}^\prime)(a_2 + r_{2U}^\prime) - a_1 a_2, \\ l_U^{\prime\mu} &= (a_1 + r_{1U}^{\prime\mu})(a_2 + r_{2U}^{\prime\mu}) - a_1 a_2, \\ l_L^\nu &= a_1 a_2 - \min \left\{ \left(a_1 - l_{1L}^\nu \right) \left(a_2 + r_{2L}^\nu \right), \, \left(a_2 - l_{2L}^\nu \right) \left(a_1 + r_{1L}^\nu \right) \right\}. \end{split}$$

 $r_{I}^{v} = \max\left\{ \left(a_{1} - l_{1I}^{v} \right) \left(a_{2}^{2} - l_{2I}^{v} \right), \left(a_{1}^{2} + r_{1I}^{v} \right) \left(a_{2}^{2} + r_{2I}^{v} \right) \right\} - a_{1}a_{2},$

: Page 21 of 43

$$\begin{split} l_U^{\prime \nu} &= a_1 a_2 - \min \left\{ (a_1 - l_{1U}^{\prime \nu}) (a_2 - l_{2U}^{\prime \nu}), \; (a_1 + r_{1U}^{\prime \nu}) (a_2 - l_{2U}^{\prime \nu}) \right\}, \\ r_U^{\prime \nu} &= (a_1 + r_{1U}^{\prime \nu}) (a_2 + r_{2U}^{\prime \nu}) - a_1 a_2. \end{split}$$

Case 4. $a_2 - l_{2U}^{\prime \mu} < 0$ and $a_2 - l_{2L}^{\mu} \ge 0$.

Then, $a_2 - l_{2U}^{\prime \mu}(L^{\prime})^{-1}(\alpha) \le (\ge) 0 \text{ for } \alpha \ge (\le) L^{\prime} \left(\frac{a_2}{l_{2U}^{\prime \mu}}\right).$

<u>Sub-case 4.1.</u> $a_1 - l_{1I}^{\nu} L^{-1} (1 - \beta) \le 0.$

Then, the expressions $(\tilde{A}_1 \odot \tilde{A}_2)^L_{\alpha}$, $(\tilde{A}_1 \odot \tilde{A}_2)^L_{\beta}$, $(\tilde{A}_1 \odot \tilde{A}_2)^U_{\beta}$ are same as in Cases 2 and 3 but $(\tilde{A}_1 \odot \tilde{A}_2)^U_{\alpha}$ is given by

$$\begin{split} \left(\tilde{A}_1 \odot \tilde{A}_2\right)^U_{\alpha} &= A^U_{1\alpha} \times A^U_{2\alpha} \\ &= \left[\left(a_1 + r'^{\mu}_{1U}(R')^{-1}(\alpha) \right) \left(a_2 - l'^{\mu}_{2U}(L')^{-1}(\alpha) \right), \ \left(a_1 + r'^{\mu}_{1U}(R')^{-1}(\alpha) \right) \left(a_2 + r'^{\mu}_{2U}(R')^{-1}(\alpha) \right) \right]. \end{split}$$

Choosing $\alpha = \alpha_0$, we get

$$(\tilde{A}_1 \odot \tilde{A}_2)_{a_0}^U = \left[(a_1 + r_{1U}^{\prime \mu}) (a_2 - l_{2U}^{\prime \mu}), (a_1 + r_{1U}^{\prime \mu}) (a_2 + r_{2U}^{\prime \mu}) \right].$$

This further yields

$$l_U^{\prime\mu} = a_1 a_2 - (a_1 + r_{1U}^{\prime\mu})(a_2 - l_{2U}^{\prime\mu}).$$

Moreover, from the preceding Case 3, we have

$$l_L^{\mu} = a_1 a_2 - (a_1 - l_{1L}^{\mu})(a_2 - l_{2L}^{\mu}), \ \ l_U^{\prime \nu} = a_1 a_2 - (a_1 + r_{1U}^{\prime \nu})(a_2 - l_{2U}^{\prime \nu}).$$

Next, it remain to prove that $l_{IJ}^{\prime\mu} \ge l_{IJ}^{\mu}$ and $l_{IJ}^{\prime\nu} \ge l_{IJ}^{\prime\mu}$.

Since $a_2 - l_{2U}^{\prime\mu} < 0$, $a_2 - l_{2U}^{\prime\mu} \le a_2 - l_{2L}^{\mu}$ and $a_1 - l_{1L}^{\mu} \le a_1 + r_{1L}^{\mu} \le a_1 + r_{1U}^{\prime\mu}$, therefore

$$a_1 a_2 - (a_2 - l_{2I}^{\mu})(a_1 - l_{1I}^{\mu}) \le a_1 a_2 - (a_2 - l_{2II}^{\prime \mu})(a_1 + r_{1II}^{\prime \mu}) \implies l_L^{\mu} \le l_{II}^{\prime \mu}.$$

Further, $a_2 - l_{2U}^{\prime \nu} \le a_2 - l_{2U}^{\prime \mu}$, $a_2 - l_{2U}^{\prime \mu} \le 0$ yield

$$a_1 a_2 - (a_2 - l_{2U}^{\prime \mu})(a_1 + r_{1U}^{\prime \mu}) \leq a_1 a_2 - (a_2 - l_{2U}^{\prime \nu})(a_1 + r_{1U}^{\prime \nu}) \implies l_U^{\prime \mu} \leq l_U^{\prime \nu}.$$

<u>Sub-case 4.2.</u> $a_1 - l_{1L}^{\nu} L^{-1} (1 - \beta) \ge 0.$ It follows on the lines of Sub-case 3.2.

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Hence, clubbing the Cases 1-4, we obtain $\tilde{A}_1 \odot \tilde{A}_2 = (a; l_L^\mu, r_L^\mu, l_U'^\mu, r_U'^\mu; l_L^\nu, r_L^\nu, l_U'^\nu, r_U'^\nu)_{LR} \text{ where } \\ a = a_1 a_2, \\ l_L^\mu = a_1 a_2 - (a_1 - l_{1L}^\mu)(a_2 - l_{2L}^\mu), \\ r_L^\mu = (a_1 + r_{1L}^\mu)(a_2 + r_{2L}^\mu) - a_1 a_2, \\ l_U'^\mu = a_1 a_2 - \min\left\{(a_1 - l_{1U}'^\mu)(a_2 - l_{2U}'^\mu), \ (a_1 + r_{1U}'^\mu)(a_2 - l_{2U}'^\mu)\right\}, \\ r_U'^\mu = (a_1 + r_{1U}'^\mu)(a_2 + r_{2U}'^\mu) - a_1 a_2, \\ l_L^\nu = a_1 a_2 - \min\left\{\left(a_1 - l_{1L}^\nu\right)\left(a_2 + r_{2L}^\nu\right), \ \left(a_2 - l_{2L}^\nu\right)\left(a_1 + r_{1L}^\nu\right)\right\}, \\ r_L^\nu = \max\left\{\left(a_1 - l_{1L}^\nu\right)\left(a_2 - l_{2L}^\nu\right), \ (a_1 + r_{1L}^\nu)\left(a_2 + r_{2L}^\nu\right)\right\} - a_1 a_2, \\ l_U'' = a_1 a_2 - \min\left\{(a_1 - l_{1U}')\left(a_2 - l_{2U}'\right), \ (a_1 + r_{1U}')\left(a_2 - l_{2U}'\right)\right\}, \\ r_U''^\nu = (a_1 + r_{1U}')(a_2 + r_{2U}') - a_1 a_2. \end{aligned}$

Now, for the succeeding cases, we have only derived the expressions for newer (or changed) spreads of the product $\tilde{A}_1 \odot \tilde{A}_2$ because the proofs for the bounds on the other spreads can be carried out in the similar pattern as is discussed in the preceding four cases.

: Page 22 of 43

Case 5. $a_2 - l_{2I}^{\mu} < 0$ and $a_2 \in \mathbb{R}$.

Then,
$$a_2 - l_{2L}^{\mu} L^{-1}(\alpha) \le (\ge) 0$$
 for $\alpha \ge (\le) L\left(\frac{a_2}{l_{2L}^{\mu}}\right)$.

<u>Sub-case 5.1.</u> $a_1 - l_{1L}^{\nu} L^{-1} (1 - \beta) \le 0.$

Then,

$$\left(\tilde{A}_1 \odot \tilde{A}_2 \right)_{\alpha}^L = A_{1\alpha}^L \times A_{2\alpha}^L = [(a_1 + r_{1L}^{\mu} R^{-1}(\alpha))(a_2 - l_{2L}^{\mu} L^{-1}(\alpha)), \ (a_1 + r_{1L}^{\mu} R^{-1}(\alpha))(a_2 + r_{2L}^{\mu} R^{-1}(\alpha))].$$

Taking $\alpha = \alpha_0$, we get

$$\begin{split} \left(\tilde{A}_{1} \odot \tilde{A}_{2}\right)_{a_{0}}^{L} &= [(a_{1} + r_{1L}^{\mu})(a_{2} - l_{2L}^{\mu}), (a_{1} + r_{1L}^{\mu})(a_{2} + r_{2L}^{\mu})]. \\ l_{L}^{\mu} &= a_{1}a_{2} - (a_{1} + r_{1L}^{\mu})(a_{2} - l_{2L}^{\mu}). \end{split}$$

It further yields

Moreover, from above Cases, we have

$$\begin{split} l_U^{\prime\mu} &= a_1 a_2 - \min \left\{ (a_1 - l_{1U}^{\prime\mu}) (a_2 - l_{2U}^{\prime\mu}), (a_1 + r_{1U}^{\prime\mu}) (a_2 - l_{2U}^{\prime\mu}) \right\}, \\ l_U^{\nu} &= a_1 a_2 - \min \left\{ \left(a_1 - l_{1U}^{\nu} \right) \left(a_2 + r_{2U}^{\nu} \right), \left(a_2 - l_{2U}^{\nu} \right) \left(a_1 + r_{1U}^{\nu} \right) \right\}. \end{split}$$

Next, the claim that $l_{II}^{\prime\mu} \ge l_{II}^{\mu}$ and $l_{II}^{\nu} \ge l_{II}^{\mu}$ can be proved following the lines of Sub-case 3.1.

<u>Sub-case 5.2.</u> $a_1 - l_{1L}^{\nu} L^{-1} (1 - \beta) \ge 0$. The proof can be carried out following the lines of Sub-case 1.1.

Case 6. $a_2 + r_{2L}^{\mu} < 0$ and $a_2 + r_{2L}^{\prime \mu} \ge 0$.

Then,
$$a_2 + r_{2L}^{\mu} R^{-1}(\alpha) \le (\ge) 0$$
 for $\alpha \le (\ge) R\left(-\frac{a_2}{r_{2L}^{\mu}}\right)$.

<u>Sub-case 6.1.</u> $a_1 - l_{11}^{\nu} L^{-1} (1 - \beta) \le 0.$

Then,

$$\left(\tilde{A}_{1} \odot \tilde{A}_{2}\right)_{\alpha}^{L} = A_{1\alpha}^{L} \times A_{2\alpha}^{L} = \left[(a_{1} + r_{1L}^{\mu}R^{-1}(\alpha))(a_{2} - l_{2L}^{\mu}L^{-1}(\alpha)), \ (a_{1} - l_{1L}^{\mu}L^{-1}(\alpha))(a_{2} + r_{2L}^{\mu}R^{-1}(\alpha))\right].$$

Choosing $\alpha = \alpha_0$, we obtain

$$\begin{split} \left(\tilde{A}_{1} \odot \tilde{A}_{2}\right)_{\alpha_{0}}^{L} &= [(a_{1} + r_{1L}^{\mu})(a_{2} - l_{2L}^{\mu}), (a_{1} - l_{1L}^{\mu})(a_{2} + r_{2L}^{\mu})] \\ r_{L}^{\mu} &= (a_{1} - l_{1L}^{\mu})(a_{2} + r_{2L}^{\mu}) - a_{1}a_{2}. \end{split}$$

which gives

Further, we have

$$r_{II}^{\prime\mu} = (a_1 + r_{II}^{\prime\mu})(a_2 + r_{2II}^{\prime\mu}) - a_1 a_2 \text{ and } r_L^{\nu} = \max\{\left(a_1 - l_{1L}^{\nu}\right)\left(a_2 - l_{2L}^{\nu}\right), \left(a_1 + r_{1L}^{\nu}\right)\left(a_2 + r_{2L}^{\nu}\right)\} - a_1 a_2.$$

The proof that $r_{IJ}^{\prime\mu} \ge r_{IJ}^{\mu}$ and $r_{IJ}^{\nu} \ge r_{IJ}^{\mu}$ can be obtained on the lines of Sub-cases 2.1 and 3.1.

Sub-case 6.2. $a_1 - l_{1L}^{\nu} L^{-1} (1 - \beta) \ge 0$. This part can be proved on the lines of Sub-case 1.1.

Case 7. $a_2 + r'^{\mu}_{2II} < 0$ and $a_2 + r'^{\nu}_{2II} \ge 0$.

Then,
$$a_2 + r'^{\mu}_{2U}(R')^{-1}(\alpha) \le (\ge) 0 \text{ for } \alpha \le (\ge) R' \left(-\frac{a_2}{r'^{\mu}_{2U}} \right).$$

Page 23 of 43

Sub-case 7.1. $a_1 - l_{11}^{\nu} L^{-1}(1 - \beta) \le 0$.

Then,

$$(\tilde{A}_1 \odot \tilde{A}_2)_{\alpha}^U = A_{1\alpha}^U \times A_{2\alpha}^U = [(a_1 + r_{1U}'^{\mu}(R')^{-1}(\alpha))(a_2 - l_{2U}'^{\mu}(L')^{-1}(\alpha)), \ (a_1 - l_{1U}'^{\mu}(L')^{-1}(\alpha))(a_2 + r_{2U}'^{\mu}(R')^{-1}(\alpha))].$$

Taking $\alpha = \alpha_0$, we obtain

$$\left(\tilde{A}_1 \odot \tilde{A}_2\right)_{\alpha_0}^U = [(a_1 + r_{1U}'^{\mu})(a_2 - l_{2U}'^{\mu}), (a_1 - l_{1U}'^{\mu})(a_2 + r_{2U}'^{\mu})].$$

It follows that

$$r_{II}^{\prime\mu} = (a_1 - l_{1II}^{\prime\mu})(a_2 + r_{2II}^{\prime\mu}) - a_1 a_2.$$

Also, we have

$$r_L^\mu = \max\{(a_1 - l_{1L}^\mu)(a_2 + r_{2L}^\mu), \ (a_1 + r_{1L}^\mu)(a_2 + r_{2L}^\mu)\} - a_1 a_2 \ \text{ and } \ r_U^{\prime \nu} = (a_1 + r_{1U}^{\prime \nu})(a_2 + r_{2U}^{\prime \nu}) - a_1 a_2.$$

Further, the inequalities $r_U'^{\mu} \ge r_L^{\mu}$ and $r_U'^{\nu} \ge r_U'^{\mu}$ can be established on the lines of Sub-cases 2.1 and 3.1.

<u>Sub-case 7.2.</u> $a_1 - l_{1L}^{\nu} L^{-1} (1 - \beta) \ge 0$. This part can be proved on the similar lines as Sub-case 1.1.

Case 8. $a_2 + r_{2U}^{\prime v} < 0$ and $a_2 + r_{2L}^{v} \ge 0$.

Then,
$$a_2 + r_{2U}^{\prime \nu}(R^{\prime})^{-1}(1-\beta) \le (\ge) 0 \text{ for } \beta \ge (\le) 1 - R^{\prime} \left(-\frac{a_2}{r_{2U}^{\prime \nu}} \right).$$

<u>Sub-case 8.1.</u> $a_1 - l_{1L}^{\nu} L^{-1} (1 - \beta) \le 0.$

$$\begin{split} &(\tilde{A}_1 \odot \tilde{A}_2)^U_{\beta} = A^U_{1\beta} \times A^U_{2\beta} \\ &= [(a_1 + r'^v_{1U}(R')^{-1}(1-\beta))(a_2 - l'^v_{2U}(L')^{-1}(1-\beta)), \ (a_1 - l'^v_{1U}(L')^{-1}(1-\beta))(a_2 + r'^v_{2U}(R')^{-1}(1-\beta))]. \end{split}$$

Choosing $\beta = \beta_0$, we get

$$(\tilde{A}_1 \odot \tilde{A}_2)_{\beta_0}^U = [(a_1 + r_{1U}^{\prime v})(a_2 - l_{2U}^{\prime v}), (a_1 - l_{1U}^{\prime v})(a_2 + r_{2U}^{\prime v})].$$

This further yields

$$r_U'^{\nu} = (a_1 - l_{1U}'^{\nu})(a_2 + r_{2U}'^{\nu}) - a_1 a_2.$$

Moreover, we have

$$\begin{split} r_L^{\nu} &= \max \left\{ \left(a_1 - l_{1L}^{\nu} \right) \left(a_2 - l_{2L}^{\nu} \right), \left(a_1 + r_{1L}^{\nu} \right) \left(a_2 + r_{2L}^{\nu} \right) \right\} - a_1 a_2 \quad \text{and} \quad \\ r_U^{\prime \mu} &= \max \left\{ (a_1 - l_{1U}^{\prime \mu}) (a_2 + r_{2U}^{\prime \mu}), \ (a_1 + r_{1U}^{\prime \mu}) (a_2 + r_{2U}^{\prime \mu}) \right\} - a_1 a_2. \end{split}$$

Next, $r_L^{\nu} \ge r_U^{\prime \nu}$ and $r_U^{\prime \nu} \ge r_U^{\prime \mu}$ can be proved following the lines of Sub-cases 2.1 and 3.1.

<u>Sub-case 8.2.</u> $a_1 - l_{1L}^v L^{-1} (1 - \beta) \ge 0$. The proof of this part follows on the lines of Sub-case 1.1.

Case 9. $a_2 + r_{2L}^{\nu} < 0$.

Then,
$$a_2 + r_{2L}^{\nu} R^{-1} (1 - \beta) \le (\ge) 0 \text{ for } \beta \ge (\le) 1 - R \left(-\frac{a_2}{r_{2L}^{\nu}} \right).$$

Page 24 of 43

<u>Sub-case 9.1.</u> $a_1 - l_{1L}^{\nu} L^{-1} (1 - \beta) \le 0.$

Then.

$$\begin{split} &(\tilde{A}_1 \odot \tilde{A}_2)_{\beta}^L = A_{1\beta}^L \times A_{2\beta}^L \\ &= [(a_1 + r_{1L}^v R^{-1}(1-\beta))(a_2 - l_{2L}^v L^{-1}(1-\beta)), \ (a_1 - l_{1L}^v L^{-1}(1-\beta))(a_2 - l_{2L}^v L^{-1}(1-\beta))]. \end{split}$$

Taking $\beta = \beta_0$, we obtain

$$\begin{split} \left(\tilde{A}_{1} \odot \tilde{A}_{2}\right)_{\beta_{0}}^{L} &= [(a_{1} + r_{1L}^{\nu})(a_{2} - l_{2L}^{\nu}), (a_{1} - l_{1L}^{\nu})(a_{2} - l_{2L}^{\nu})]. \\ r_{L}^{\nu} &= (a_{1} - l_{1L}^{\nu})(a_{2} - l_{2L}^{\nu}) - a_{1}a_{2}. \end{split}$$

It follows that

Further,

$$\begin{split} r_U^{\prime \nu} &= \max \left\{ (a_1 + r_{1U}^{\prime \nu})(a_2 + r_{2U}^{\prime \nu}), \; (a_1 - l_{1U}^{\prime \nu})(a_2 + r_{2U}^{\prime \nu}) \right\} - a_1 a_2 \quad \text{and} \\ r_L^{\mu} &= \max \left\{ (a_1 - l_{1L}^{\mu})(a_2 + r_{2L}^{\mu}), \; (a_1 + r_{1L}^{\mu})(a_2 + r_{2L}^{\mu}) \right\} - a_1 a_2. \end{split}$$

Finally, the proof of the inequalities $r_I^{\nu} \geq r_I^{\prime \nu}$ and $r_I^{\nu} \geq r_I^{\mu}$ can be obtained on the lines of Sub-cases 2.1 and 3.1.

<u>Sub-case 9.2.</u> $a_1 - l_{1L}^{\nu} L^{-1} (1 - \beta) \ge 0.$

The proof of this part can be established on the lines of Sub-case 1.1.

Finally, combining all the Cases (1)
$$-$$
 (9), we have
$$\tilde{A}_1 \odot \tilde{A}_2 = (a; l_L^\mu, r_L^\mu, l_U^{\prime\mu}, r_U^{\prime\mu}; l_L^\nu, r_L^\nu, l_U^{\prime\nu}, r_U^{\prime\nu})_{LR} \text{ where } \\ a = a_1 a_2, \\ l_L^\mu = a_1 a_2 - \min\{(a_1 - l_{1L}^\mu)(a_2 - l_{2L}^\mu), \ (a_1 + r_{1L}^\mu)(a_2 - l_{2L}^\mu)\}, \\ r_L^\mu = \max\{(a_1 - l_{1L}^\mu)(a_2 + r_{2L}^\mu), \ (a_1 + r_{1L}^\mu)(a_2 + r_{2L}^\mu)\} - a_1 a_2, \\ l_U^{\prime\mu} = a_1 a_2 - \min\{(a_1 - l_{1U}^{\prime\mu})(a_2 - l_{2U}^{\prime\mu}), \ (a_1 + r_{1U}^{\prime\mu})(a_2 - l_{2U}^{\prime\mu})\}, \\ r_U^{\prime\mu} = \max\{(a_1 - l_{1U}^{\prime\mu})(a_2 + r_{2U}^{\prime\mu}), \ (a_1 + r_{1U}^{\prime\mu})(a_2 + r_{2U}^{\prime\mu})\} - a_1 a_2, \\ l_L^\nu = a_1 a_2 - \min\{(a_1 - l_{1L}^\nu)(a_2 + r_{2L}^\nu), \ (a_1 + r_{1L}^\prime)(a_2 + r_{2L}^\prime)\}, \\ r_L^\nu = \max\{(a_1 - l_{1L}^\prime)(a_2 - l_{2L}^\prime), \ (a_1 + r_{1L}^\prime)(a_2 + r_{2L}^\prime)\} - a_1 a_2, \\ l_U^{\prime\nu} = a_1 a_2 - \min\{(a_1 - l_{1U}^\prime)(a_2 - l_{2U}^\prime), \ (a_1 + r_{1L}^{\prime\nu})(a_2 - l_{2U}^\prime)\}, \\ r_U^{\prime\nu} = \max\{(a_1 + r_{1U}^{\prime\prime})(a_2 + r_{2U}^\prime), \ (a_1 + r_{1U}^{\prime\prime})(a_2 + r_{2U}^\prime)\} - a_1 a_2, \\ l_U^{\prime\nu} = \max\{(a_1 + r_{1U}^{\prime\prime})(a_2 + r_{2U}^{\prime\prime}), \ (a_1 - l_{1U}^{\prime\prime})(a_2 + r_{2U}^{\prime\prime})\}, \\ r_U^{\prime\nu} = \max\{(a_1 + r_{1U}^{\prime\prime})(a_2 + r_{2U}^{\prime\prime}), \ (a_1 - l_{1U}^{\prime\prime})(a_2 + r_{2U}^{\prime\prime})\} - a_1 a_2, \\ l_U^{\prime\nu} = \max\{(a_1 + r_{1U}^{\prime\prime})(a_2 + r_{2U}^{\prime\prime}), \ (a_1 + r_{1U}^{\prime\prime})(a_2 + r_{2U}^{\prime\prime})\}, \\ r_U^{\prime\nu} = \max\{(a_1 + r_{1U}^{\prime\prime})(a_2 + r_{2U}^{\prime\prime}), \ (a_1 + r_{1U}^{\prime\prime})(a_2 + r_{2U}^{\prime\prime})\} - a_1 a_2, \\ l_U^{\prime\nu} = \max\{(a_1 + r_{1U}^{\prime\prime})(a_2 + r_{2U}^{\prime\prime}), \ (a_1 + r_{1U}^{\prime\prime})(a_2 + r_{2U}^{\prime\prime})\}, \\ l_U^{\prime\nu} = \max\{(a_1 + r_{1U}^{\prime\prime})(a_2 + r_{2U}^{\prime\prime}), \ (a_1 + r_{1U}^{\prime\prime})(a_2 + r_{2U}^{\prime\prime})\} - a_1 a_2, \\ l_U^{\prime\nu} = \max\{(a_1 + r_{1U}^{\prime\prime})(a_2 + r_{2U}^{\prime\prime}), \ (a_1 + r_{1U}^{\prime\prime})(a_2 + r_{2U}^{\prime\prime})\}, \\ l_U^{\prime\nu} = \max\{(a_1 + r_{1U}^{\prime\prime})(a_2 + r_{2U}^{\prime\prime}), \ (a_1 + r_{1U}^{\prime\prime})(a_2 + r_{2U}^{\prime\prime})\} - a_1 a_2, \\ l_U^{\prime\nu} = \max\{(a_1 + r_{1U}^{\prime\prime})(a_2 + r_{2U}^{\prime\prime}), \ (a_1 + r_{1U}^{\prime\prime})(a_2 + r_{2U}^{\prime\prime})\}, \\ l_U^{\prime\nu} = \max\{(a_1 + r_{1U}^{\prime\prime})(a_2 + r_{2U}^{\prime\prime}), \ (a_1 + r_{1U}^{\prime\prime})(a_2 + r_{2U}^{\prime\prime})\} - a_1 a_2, \\ l_U^{\prime\nu} = \max\{(a_1 + r_{1U}^{\prime\prime})(a_2 + r_{2U}^{\prime$$

where
$$l_U^{\prime\mu} \geq l_L^{\mu} > 0$$
, $r_U^{\prime\mu} \geq r_L^{\mu} > 0$, $l_L^{\nu} \geq l_U^{\prime\nu} > 0$, $r_L^{\nu} \geq r_U^{\prime\nu} > 0$, $l_L^{\nu} \geq l_L^{\mu} > 0$, $r_L^{\nu} \geq r_L^{\mu} > 0$, $l_U^{\prime\nu} \geq l_U^{\prime\mu} > 0$, $r_L^{\nu} \geq r_L^{\nu} > 0$.

Hence, the result.

 $\begin{aligned} & \textbf{Proposition 3.1.4.} \ Let \ \tilde{A}_1 = (a_1; l_{1L}^{\mu}, r_{1L}^{\mu}, l_{1U}^{\prime \mu}, r_{1U}^{\prime \mu}, l_{1U}^{\prime \nu}, r_{1U}^{\prime \nu}, l_{1U}^{\prime \nu}, r_{1U}^{\prime \nu}, l_{1U}^{\prime \nu}, r_{1U}^{\prime \nu})_{LR} \ be \ an \ LR \ type \ IVIFN \ such \ that \ a_1 - l_{1U}^{\prime \nu} < 0, \\ & a_1 - l_{1U}^{\prime \mu} \geq 0 \ and \ \tilde{A}_2 = (a_2; l_{2L}^{\mu}, r_{2L}^{\mu}, l_{2U}^{\prime \nu}, r_{2U}^{\prime \nu}, l_{2U}^{\prime \nu}, r_{2U}^{\prime \nu})_{LR} \ be \ any \ LR \ type \ IVIFN. \ Then \\ & \tilde{A}_1 \odot \tilde{A}_2 = (a; l_L^{\mu}, r_L^{\mu}, l_U^{\prime \mu}, r_U^{\prime \mu}, l_U^{\prime \nu}, r_U^{\nu}, l_U^{\prime \nu}, r_U^{\prime \nu})_{LR} \ where \\ & a = a_1 a_2, \\ & l_L^{\mu} = a_1 a_2 - min\{(a_1 + r_{1L}^{\mu})(a_2 - l_{2L}^{\mu}), \ (a_1 - l_{1L}^{\mu})(a_2 + r_{2L}^{\mu})\}, \ (a_1 - l_{1L}^{\mu})(a_2 + r_{2L}^{\mu})\} - a_1 a_2, \\ & l_U^{\mu} = a_1 a_2 - min\{(a_1 + r_{1U}^{\prime \mu})(a_2 - l_{2U}^{\prime \mu}), \ (a_1 - l_{1U}^{\prime \mu})(a_2 - l_{2U}^{\prime \mu})\}, \\ & r_U^{\prime \mu} = max\{(a_1 + r_{1U}^{\prime \mu})(a_2 + r_{2U}^{\prime \mu}), \ (a_1 - l_{1U}^{\prime \mu})(a_2 + r_{2U}^{\prime \mu})\} - a_1 a_2, \\ & l_U^{\nu} = a_1 a_2 - min\{(a_1 - l_{1L}^{\prime \nu})(a_2 + r_{2L}^{\prime \nu}), \ (a_1 + r_{1L}^{\prime \nu})(a_2 - l_{2L}^{\prime \nu})\}, \ (a_1 + r_{1U}^{\prime \nu})(a_2 - l_{2L}^{\prime \nu})\}, \\ & r_U^{\nu} = max\{(a_1 + r_{1L}^{\prime \mu})(a_2 + r_{2L}^{\prime \nu}), \ (a_1 - l_{1L}^{\prime \mu})(a_2 - l_{2L}^{\prime \nu})\}, \ (a_1 + r_{1U}^{\prime \nu})(a_2 - l_{2U}^{\prime \nu})\}, \\ & r_U^{\prime \nu} = a_1 a_2 - min\{(a_1 - l_{1U}^{\prime \nu})(a_2 + r_{2U}^{\prime \nu}), \ (a_1 + r_{1U}^{\prime \nu})(a_2 - l_{2U}^{\prime \nu})\}, \ (a_1 + r_{1U}^{\prime \nu})(a_2 + r_{2U}^{\prime \nu}), \ (a_1 + r_{1U}^{\prime \nu})(a_2 - l_{2U}^{\prime \nu})\}, \\ & r_U^{\prime \nu} = max\{(a_1 + r_{1U}^{\prime \prime})(a_2 + r_{2U}^{\prime \prime}), \ (a_1 - l_{1U}^{\prime \prime})(a_2 - l_{2U}^{\prime \prime})\}, \ (a_1 - l_{1U}^{\prime \prime})(a_2 - l_{2U}^{\prime \prime})\}, \ (a_1 - l_{1U}^{\prime \prime})(a_2 - l_{2U}^{\prime \prime})\}, \\ & r_U^{\prime \nu} = max\{(a_1 + r_{1U}^{\prime \prime})(a_2 + r_{2U}^{\prime \prime}), \ (a_1 - l_{1U}^{\prime \prime})(a_2 - l_{2U}^{\prime \prime})\}, \ (a_1 - l_{1U}^{\prime \prime})(a_2 - l_{2U}^{\prime \prime})\}, \ (a_1 - l_{1U}^{\prime \prime})(a_2 - l_{2U}^{\prime \prime})\}, \ (a_1 - l_{1U}^{\prime \prime})(a_2 - l_$

Page 25 of 43

where the conditions for LR-type representation of $\tilde{A}_1 \odot \tilde{A}_2$ are satisfied.

Proof. Similar to the Proposition 3.1.3.

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 \begin{aligned} & \textbf{Proposition 3.1.5. } \ Let \ \tilde{A}_1 = (a_1; l_{1L}^{\mu}, r_{1L}^{\mu}, l_{1U}^{\prime}, r_{1U}^{\prime \mu}; l_{1L}^{\nu}, r_{1L}^{\prime \nu}, l_{1U}^{\prime \nu}, r_{1U}^{\prime \nu}, l_{2U}^{\prime \nu}, r_{2U}^{\prime \nu}, l_{2U}^{\prime \nu},
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Proof. Similar to the Proposition 3.1.3.

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 \begin{aligned} & \textbf{Proposition 3.1.6. } \ Let \ \tilde{A}_1 = (a_1; l_{1L}^{\mu}, r_{1L}^{\mu}, l_{1U}^{\prime}, r_{1U}^{\prime \mu}; l_{1L}^{\nu}, r_{1L}^{\prime \nu}, l_{1U}^{\prime \nu}, r_{1U}^{\prime \nu}, l_{1U}^{\prime \nu}, r_{1U}^{\prime \nu})_{LR} \ be \ an \ LR-type \ IVIFN \ such \ that \ a_1 - l_{1L}^{\mu} < 0, \\ & a_1 \geq 0 \ and \ \tilde{A}_2 = (a_2; l_{2L}^{\mu}, r_{2L}^{\mu}, l_{2U}^{\prime \mu}, r_{2U}^{\prime \nu}; l_{2L}^{\nu}, r_{2L}^{\nu}, l_{2U}^{\prime \nu}, r_{2U}^{\prime \nu})_{LR} \ be \ any \ LR-type \ IVIFN. \ Then \\ & \tilde{A}_1 \odot \tilde{A}_2 = (a; l_L^{\mu}, r_L^{\mu}, l_U^{\prime \mu}, r_U^{\prime \mu}; l_L^{\nu}, r_L^{\nu}, l_U^{\prime \nu}, r_U^{\prime \nu})_{LR} \ where \\ & a = a_1 a_2, \\ & l_L^{\mu} = a_1 a_2 - min \{ (a_1 - l_{1L}^{\mu}) (a_2 + r_{2L}^{\mu}), \ (a_1 + r_{1L}^{\mu}) (a_2 - l_{2L}^{\mu}) \}, \\ & r_L^{\mu} = max \{ (a_1 + r_{1L}^{\prime \mu}) (a_2 + r_{2L}^{\prime \mu}), \ (a_1 - l_{1L}^{\prime \mu}) (a_2 - l_{2L}^{\prime \mu}) \} - a_1 a_2, \\ & l_U^{\prime \mu} = a_1 a_2 - min \{ (a_1 - l_{1U}^{\prime \mu}) (a_2 + r_{2U}^{\prime \mu}), \ (a_1 + r_{1U}^{\prime \mu}) (a_2 - l_{2U}^{\prime \mu}) \}, \\ & r_L^{\prime \mu} = max \{ (a_1 + r_{1L}^{\prime \mu}) (a_2 + r_{2U}^{\prime \mu}), \ (a_1 - l_{1U}^{\prime \mu}) (a_2 - l_{2U}^{\prime \mu}) \} - a_1 a_2, \\ & l_L^{\nu} = a_1 a_2 - min \{ (a_1 - l_{1L}^{\prime \prime}) (a_2 + r_{2U}^{\prime \prime}), \ (a_1 - l_{1U}^{\prime \prime}) (a_2 - l_{2L}^{\prime \prime}) \}, \\ & r_L^{\nu} = max \{ (a_1 + r_{1L}^{\prime \mu}) (a_2 + r_{2L}^{\prime \nu}), \ (a_1 - l_{1L}^{\prime \prime}) (a_2 - l_{2L}^{\prime \prime}) \} - a_1 a_2, \\ & l_U^{\nu} = a_1 a_2 - min \{ (a_1 - l_{1U}^{\prime \prime}) (a_2 + r_{2U}^{\prime \prime}), \ (a_1 - l_{1L}^{\prime \prime}) (a_2 - l_{2L}^{\prime \prime}) \} - a_1 a_2, \\ & l_U^{\prime \prime} = max \{ (a_1 + r_{1L}^{\prime \prime}) (a_2 + r_{2U}^{\prime \prime}), \ (a_1 - l_{1L}^{\prime \prime}) (a_2 - l_{2U}^{\prime \prime}) \} - a_1 a_2, \\ & l_U^{\prime \prime} = max \{ (a_1 + r_{1U}^{\prime \prime}) (a_2 + r_{2U}^{\prime \prime}), \ (a_1 - l_{1U}^{\prime \prime}) (a_2 - l_{2U}^{\prime \prime}) \} - a_1 a_2, \\ & l_U^{\prime \prime} = max \{ (a_1 + r_{1U}^{\prime \prime}) (a_2 + r_{2U}^{\prime \prime}), \ (a_1 + r_{1U}^{\prime \prime}) (a_2 - l_{2U}^{\prime \prime}) \} - a_1 a_2, \\ & l_U^{\prime \prime} = a_1 a_2 - min \{ (a_1 - l_{1U}^{\prime \prime}) (a_2 + r_{2U}^{\prime \prime}), \ (a_1 + l_{1U}^{\prime \prime}) (a_2 - l_{2U}^{\prime \prime}) \} - a_1 a_2, \\ & l_U^{\prime \prime} = max \{ (a_1 + l_{1U}^{\prime \prime}) (a_2 + l_{2U}^{\prime \prime}), \ (a_1 - l_{1U}^{\prime \prime}) (a_2 - l_{2U}^{\prime \prime}) \} - a_1 a_2, \\ & l_U
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Proof. Similar to the Proposition 3.1.3.

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 \begin{aligned} & \textbf{Proposition 3.1.7.} \ Let \ \tilde{A}_1 = (a_1; l_{1L}^{\mu}, r_{1L}^{\mu}, l_{1U}^{\prime}, r_{1U}^{\prime\prime}, l_{1L}^{\prime\prime}, r_{1U}^{\prime\prime}, l_{1U}^{\prime\prime}, r_{1U}^{\prime\prime}, l_{1U}^{\prime\prime}, r_{1U}^{\prime\prime})_{LR} \ be \ an \ LR-type \ IVIFN \ such \ that \ a_1 < 0, \\ & a_1 + r_{1L}^{\mu} \geq 0 \ and \ \tilde{A}_2 = (a_2; l_{2L}^{\mu}, r_{2L}^{\mu}, l_{2U}^{\prime\prime}, r_{2U}^{\prime\prime}; l_{2L}^{\nu}, r_{2U}^{\prime\prime}, r_{2U}^{\prime\prime\prime}, r_{2U}^{\prime\prime})_{LR} \ be \ any \ LR-type \ IVIFN. \ Then \\ & \tilde{A}_1 \odot \tilde{A}_2 = (a; l_L^{\mu}, r_L^{\mu}, l_U^{\prime\prime}, r_U^{\prime\prime}; l_L^{\nu}, r_L^{\nu}, l_U^{\prime\prime}, r_U^{\prime\prime})_{LR} \ where \\ & a = a_1 a_2, \\ & l_L^{\mu} = a_1 a_2 - min \{(a_1 - l_{1L}^{\mu})(a_2 + r_{2L}^{\mu}), \ (a_1 + r_{1L}^{\mu})(a_2 - l_{2L}^{\mu})\}, \\ & r_L^{\mu} = max \{(a_1 + r_{1L}^{\prime\prime})(a_2 + r_{2L}^{\prime\prime}), \ (a_1 - l_{1L}^{\prime\prime})(a_2 - l_{2L}^{\prime\prime})\} - a_1 a_2, \\ & l_U^{\prime\prime} = a_1 a_2 - min \{(a_1 - l_{1U}^{\prime\prime})(a_2 + r_{2U}^{\prime\prime}), \ (a_1 + r_{1U}^{\prime\prime})(a_2 - l_{2U}^{\prime\prime})\} - a_1 a_2, \\ & l_L^{\nu} = max \{(a_1 + r_{1L}^{\prime\prime})(a_2 + r_{2U}^{\prime\prime}), \ (a_1 - l_{1U}^{\prime\prime})(a_2 - l_{2U}^{\prime\prime})\} - a_1 a_2, \\ & l_L^{\nu} = a_1 a_2 - min \{(a_1 - l_{1L}^{\prime\prime})(a_2 + r_{2U}^{\prime\prime}), \ (a_1 - l_{1U}^{\prime\prime})(a_2 - l_{2L}^{\prime\prime})\} - a_1 a_2, \\ & l_L^{\nu} = a_1 a_2 - min \{(a_1 - l_{1L}^{\prime\prime})(a_2 + r_{2L}^{\prime\prime}), \ (a_1 - l_{1L}^{\prime\prime})(a_2 - l_{2L}^{\prime\prime})\} - a_1 a_2, \\ & l_U^{\prime\prime} = a_1 a_2 - min \{(a_1 - l_{1U}^{\prime\prime})(a_2 + r_{2U}^{\prime\prime}), \ (a_1 + r_{1U}^{\prime\prime})(a_2 - l_{2U}^{\prime\prime})\} - a_1 a_2, \\ & l_U^{\prime\prime} = a_1 a_2 - min \{(a_1 - l_{1U}^{\prime\prime})(a_2 + r_{2U}^{\prime\prime}), \ (a_1 + l_{1U}^{\prime\prime})(a_2 - l_{2U}^{\prime\prime})\} - a_1 a_2, \\ & l_U^{\prime\prime} = a_1 a_2 - min \{(a_1 - l_{1U}^{\prime\prime})(a_2 + r_{2U}^{\prime\prime}), \ (a_1 + l_{1U}^{\prime\prime})(a_2 - l_{2U}^{\prime\prime})\} - a_1 a_2, \\ & l_U^{\prime\prime} = a_1 a_2 - min \{(a_1 - l_{1U}^{\prime\prime})(a_2 + r_{2U}^{\prime\prime}), \ (a_1 + l_{1U}^{\prime\prime})(a_2 - l_{2U}^{\prime\prime})\} - a_1 a_2, \\ & l_U^{\prime\prime} = a_1 a_2 - min \{(a_1 - l_{1U}^{\prime\prime})(a_2 + r_{2U}^{\prime\prime}), \ (a_1 - l_{1U}^{\prime\prime})(a_2 - l_{2U}^{\prime\prime})\} - a_1 a_2, \\ & l_U^{\prime\prime} = a_1 a_2 - min \{(a_1 - l_{1U}^{\prime\prime})(a_2 + r_{2U}^{\prime\prime}), \ (a_1 - l_{1U}^{\prime\prime})(a_2 - l_{2U}^{\prime\prime})\} - a_1 a_2, \\ & l_U^{\prime\prime} = a_1 a_2 - min \{(a_1 - l_{1U}^{\prime\prime}
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Page 26 of 43

Proof. Similar to the Proposition 3.1.3.

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 \begin{aligned} & \textbf{Proposition 3.1.8. } \ Let \ \tilde{A}_1 = (a_1; l_{1L}^{\mu}, r_{1L}^{\mu}, l_{1U}^{\prime \mu}, r_{1U}^{\prime \mu}; l_{1L}^{\nu}, r_{1U}^{\nu}, l_{1U}^{\prime \nu}, r_{1U}^{\prime \nu}, l_{1U}^{\prime \nu}, r_{1U}^{\prime \nu})_{LR} \ be \ an \ LR-type \ IVIFN \ such \ that \ a_1 + r_{1L}^{\mu} < 0, \\ & a_1 + r_{1U}^{\prime \mu} \geq 0 \ and \ \tilde{A}_2 = (a_2; l_{2L}^{\mu}, r_{2L}^{\mu}, l_{2U}^{\prime \mu}, r_{2U}^{\prime \nu}, l_{2U}^{\nu}, r_{2U}^{\nu}, l_{2U}^{\prime \nu}, r_{2U}^{\prime \nu})_{LR} \ be \ any \ LR-type \ IVIFN. \ Then \\ & \tilde{A}_1 \odot \tilde{A}_2 = (a_3; l_{L}^{\mu}, r_{L}^{\mu}, l_{U}^{\prime \mu}, r_{U}^{\prime \nu}; l_{L}^{\nu}, r_{U}^{\nu}, l_{U}^{\prime \nu}, r_{U}^{\prime \nu})_{LR} \ where \\ & a = a_1 a_2, \\ & l_{L}^{\mu} = a_1 a_2 - min\{(a_1 + r_{1L}^{\mu})(a_2 + r_{2L}^{\mu}), \ (a_1 - l_{1L}^{\mu})(a_2 + r_{2L}^{\mu})\}, \\ & r_{L}^{\mu} = max\{(a_1 - l_{1L}^{\mu})(a_2 - l_{2L}^{\mu}), \ (a_1 + r_{1L}^{\mu})(a_2 - l_{2L}^{\mu})\} - a_1 a_2, \\ & l_{U}^{\prime \mu} = a_1 a_2 - min\{(a_1 - l_{1U}^{\prime \mu})(a_2 + r_{2U}^{\prime \mu}), \ (a_1 + r_{1U}^{\prime \mu})(a_2 - l_{2U}^{\prime \mu})\}, \\ & r_{U}^{\prime \mu} = max\{(a_1 + r_{1U}^{\prime \mu})(a_2 + r_{2U}^{\prime \mu}), \ (a_1 - l_{1U}^{\prime \mu})(a_2 - l_{2U}^{\prime \mu})\} - a_1 a_2, \\ & l_{U}^{\prime \nu} = a_1 a_2 - min\{(a_1 - l_{1U}^{\prime \mu})(a_2 + r_{2L}^{\prime \nu}), \ (a_1 + r_{1U}^{\prime \mu})(a_2 - l_{2U}^{\prime \nu})\}, \\ & r_{L}^{\prime \nu} = max\{(a_1 + r_{1L}^{\prime \mu})(a_2 + r_{2L}^{\prime \nu}), \ (a_1 - l_{1L}^{\prime \mu})(a_2 - l_{2L}^{\prime \nu})\}, \\ & r_{L}^{\prime \nu} = max\{(a_1 + r_{1L}^{\prime \nu})(a_2 + r_{2L}^{\prime \nu}), \ (a_1 - l_{1L}^{\prime \nu})(a_2 - l_{2U}^{\prime \nu})\}, \\ & r_{L}^{\prime \nu} = max\{(a_1 + r_{1L}^{\prime \nu})(a_2 + r_{2U}^{\prime \nu}), \ (a_1 - l_{1L}^{\prime \nu})(a_2 - l_{2U}^{\prime \nu})\}, \\ & r_{U}^{\prime \nu} = max\{(a_1 + r_{1U}^{\prime \nu})(a_2 + r_{2U}^{\prime \nu}), \ (a_1 - l_{1U}^{\prime \nu})(a_2 - l_{2U}^{\prime \nu})\}, \\ & r_{U}^{\prime \nu} = max\{(a_1 + r_{1U}^{\prime \nu})(a_2 + r_{2U}^{\prime \nu}), \ (a_1 - l_{1U}^{\prime \nu})(a_2 - l_{2U}^{\prime \nu})\}, \\ & r_{U}^{\prime \nu} = max\{(a_1 + r_{1U}^{\prime \nu})(a_2 + r_{2U}^{\prime \nu}), \ (a_1 - l_{1U}^{\prime \nu})(a_2 - l_{2U}^{\prime \nu})\}, \\ & r_{U}^{\prime \nu} = max\{(a_1 + r_{1U}^{\prime \nu})(a_2 + r_{2U}^{\prime \nu}), \ (a_1 - l_{1U}^{\prime \nu})(a_2 - l_{2U}^{\prime \nu})\}, \\ & r_{U}^{\prime \nu} = max\{(a_1 + r_{1U}^{\prime \nu})(a_2 + r_{2U}^{\prime \nu}), \ (a_1 - l_{1U}^{\prime \nu})(a_2 - l_{2U}^{\prime
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Proof. Similar to the Proposition 3.1.3.

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 \begin{aligned} & \textbf{Proposition 3.1.9. } \ Let \ \tilde{A}_1 = (a_1; l_{1L}^{\mu}, r_{1L}^{\mu}, l_{1U}^{\prime \mu}, r_{1L}^{\prime \mu}; l_{1L}^{\prime \nu}, r_{1L}^{\prime \nu}, l_{1U}^{\prime \nu}, r_{1U}^{\prime \nu})_{LR} \ be \ an \ LR-type \ IVIFN \ such \ that \ a_1 + r_{1U}^{\prime \mu} < 0, \\ & a_1 + r_{1U}^{\prime \nu} \geq 0 \ and \ \tilde{A}_2 = (a_2; l_{2L}^{\mu}, r_{2L}^{\mu}, l_{2U}^{\prime \mu}, r_{2U}^{\prime \mu}; l_{2L}^{\nu}, r_{2U}^{\nu}, l_{2U}^{\prime \nu}, r_{2U}^{\prime \nu})_{LR} \ be \ any \ LR-type \ IVIFN. \ Then \\ & \tilde{A}_1 \odot \tilde{A}_2 = (a; l_{L}^{\mu}, r_{L}^{\mu}, l_{U}^{\prime \mu}, r_{U}^{\prime \mu}; l_{L}^{\nu}, r_{L}^{\nu}, l_{U}^{\prime \nu}, r_{U}^{\prime \nu})_{LR} \ where \\ & a = a_1 a_2, \\ & l_{L}^{\mu} = a_1 a_2 - min\{(a_1 + r_{1L}^{\mu})(a_2 + r_{2L}^{\mu}), \ (a_1 - l_{1L}^{\mu})(a_2 + r_{2L}^{\mu})\}, \ max\{(a_1 - l_{1L}^{\mu})(a_2 - l_{2L}^{\prime \mu}), \ (a_1 + r_{1L}^{\prime \mu})(a_2 + r_{2U}^{\prime \mu})\}, \ a_1 - l_{1U}^{\prime \mu})(a_2 + r_{2U}^{\prime \mu})\}, \\ & r_{L}^{\prime \mu} = max\{(a_1 - l_{1L}^{\prime \mu})(a_2 + r_{2U}^{\prime \mu}), \ (a_1 - l_{1L}^{\prime \mu})(a_2 + r_{2U}^{\prime \mu})\} - a_1 a_2, \\ & l_{U}^{\prime \nu} = a_1 a_2 - min\{(a_1 + r_{1U}^{\prime \mu})(a_2 + r_{2U}^{\prime \mu}), \ (a_1 + r_{1U}^{\prime \mu})(a_2 - l_{2U}^{\prime \nu})\} - a_1 a_2, \\ & l_{L}^{\nu} = a_1 a_2 - min\{(a_1 - l_{1L}^{\prime \nu})(a_2 + r_{2U}^{\prime \nu}), \ (a_1 + r_{1L}^{\prime \nu})(a_2 - l_{2L}^{\prime \nu})\}, \\ & r_{L}^{\prime \nu} = max\{(a_1 + r_{1L}^{\prime \nu})(a_2 + r_{2L}^{\prime \nu}), \ (a_1 - l_{1L}^{\prime \nu})(a_2 - l_{2L}^{\prime \nu})\}, \ a_1 + r_{1L}^{\prime \nu})(a_2 - l_{2L}^{\prime \nu})\}, \\ & r_{L}^{\prime \nu} = max\{(a_1 + r_{1L}^{\prime \nu})(a_2 + r_{2L}^{\prime \nu}), \ (a_1 + r_{1U}^{\prime \nu})(a_2 - l_{2L}^{\prime \nu})\}, \ a_1 + r_{1U}^{\prime \nu})(a_2 - l_{2U}^{\prime \nu})\}, \\ & r_{L}^{\prime \nu} = max\{(a_1 + r_{1U}^{\prime \nu})(a_2 + r_{2U}^{\prime \nu}), \ (a_1 + r_{1U}^{\prime \nu})(a_2 - l_{2U}^{\prime \nu})\}, \ a_1 + r_{1U}^{\prime \nu})(a_2 - l_{2U}^{\prime \nu})\}, \\ & r_{U}^{\prime \nu} = max\{(a_1 + r_{1U}^{\prime \prime})(a_2 + r_{2U}^{\prime \nu}), \ (a_1 + r_{1U}^{\prime \prime})(a_2 - l_{2U}^{\prime \nu})\}, \ a_2 \ a_2 \ a_2 \ a_2 \ a_2 \ a_3 \ a_2 \ a_3 \ a_3
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Proof. Similar to the Proposition 3.1.3.

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 \begin{aligned} & \textbf{Proposition 3.1.10.} \ Let \ \tilde{A}_1 = (a_1; l_{1L}^{\mu}, r_{1L}^{\mu}, l_{1U}^{\prime \mu}, r_{1U}^{\prime \mu}; l_{1L}^{\nu}, r_{1U}^{\nu}, l_{1U}^{\prime \nu}, r_{1U}^{\prime \nu}, l_{1U}^{\prime \nu}, r_{1U}^{\prime \nu})_{LR} \ be \ an \ LR-type \ IVIFN \ such \ that \ a_1 + r_{1U}^{\prime \nu} < 0, \\ & a_1 + r_{1L}^{\nu} \geq 0 \ and \ \tilde{A}_2 = (a_2; l_{2L}^{\mu}, r_{2L}^{\mu}, l_{2U}^{\prime \mu}, r_{2U}^{\prime \nu}, l_{2U}^{\prime \nu}, r_{2U}^{\prime \nu}, l_{2U}^{\prime \nu}, r_{2U}^{\prime \nu})_{LR} \ be \ any \ LR-type \ IVIFN. \ Then \\ & \tilde{A}_1 \odot \tilde{A}_2 = (a; l_{L}^{\mu}, r_{L}^{\mu}, l_{U}^{\prime \mu}, r_{U}^{\prime \mu}; l_{L}^{\nu}, r_{L}^{\nu}, l_{U}^{\prime \nu}, r_{U}^{\prime \nu})_{LR} \ where \\ & a = a_1 a_2, \\ & l_{L}^{\mu} = a_1 a_2 - min\{(a_1 + r_{L}^{\mu})(a_2 + r_{2L}^{\mu}), \ (a_1 - l_{1L}^{\mu})(a_2 + r_{2L}^{\mu})\}, \\ & r_{L}^{\mu} = max\{(a_1 - l_{1L}^{\mu})(a_2 - l_{2L}^{\mu}), \ (a_1 + r_{1L}^{\prime \mu})(a_2 - l_{2L}^{\prime \mu})\} - a_1 a_2, \\ & l_{U}^{\prime \mu} = a_1 a_2 - min\{(a_1 + r_{1U}^{\prime \mu})(a_2 + r_{2U}^{\prime \mu}), \ (a_1 - l_{1U}^{\prime \mu})(a_2 + r_{2U}^{\prime \mu})\}, \\ & r_{U}^{\prime \mu} = max\{(a_1 - l_{1U}^{\prime \mu})(a_2 - l_{2U}^{\prime \nu}), \ (a_1 + r_{1U}^{\prime \mu})(a_2 + r_{2U}^{\prime \mu})\} - a_1 a_2, \\ & l_{U}^{\prime \nu} = a_1 a_2 - min\{(a_1 - l_{1L}^{\prime \prime})(a_2 + r_{2L}^{\prime \prime}), \ (a_1 + r_{1U}^{\prime \prime})(a_2 - l_{2L}^{\prime \prime})\}, \\ & r_{L}^{\prime \nu} = max\{(a_1 - l_{1U}^{\prime \prime})(a_2 + r_{2L}^{\prime \prime}), \ (a_1 - l_{1U}^{\prime \prime})(a_2 - l_{2L}^{\prime \prime})\}, \\ & r_{L}^{\prime \nu} = max\{(a_1 + r_{L}^{\prime \prime})(a_2 + r_{2L}^{\prime \prime}), \ (a_1 - l_{1L}^{\prime \prime})(a_2 - l_{2L}^{\prime \prime})\}, \\ & r_{L}^{\prime \nu} = max\{(a_1 + r_{L}^{\prime \prime})(a_2 + r_{2L}^{\prime \prime}), \ (a_1 - l_{1L}^{\prime \prime})(a_2 + r_{2U}^{\prime \prime})\}, \\ & r_{L}^{\prime \prime} = max\{(a_1 - l_{1U}^{\prime \prime})(a_2 + r_{2U}^{\prime \prime}), \ (a_1 - l_{1U}^{\prime \prime})(a_2 + r_{2U}^{\prime \prime})\}, \\ & r_{L}^{\prime \prime} = max\{(a_1 - l_{1U}^{\prime \prime})(a_2 + r_{2U}^{\prime \prime}), \ (a_1 - l_{1U}^{\prime \prime})(a_2 - l_{2U}^{\prime \prime})\}, \\ & r_{L}^{\prime \prime} = max\{(a_1 - l_{1U}^{\prime \prime})(a_2 - l_{2U}^{\prime \prime}), \ (a_1 - l_{1U}^{\prime \prime})(a_2 - l_{2U}^{\prime \prime})\}, \\ & r_{L}^{\prime \prime} = max\{(a_1 - l_{1U}^{\prime \prime})(a_2 - l_{2U}^{\prime \prime}), \ (a_1 - l_{1U}^{\prime \prime})(a_2 - l_{2U}^{\prime \prime})\}, \\ & r_{L}^{\prime \prime} = max\{(a_1 - l_{1U}^{\prime \prime})(a_2 - l_{2U}^{\prime \prime}), \ (a_1 - l_{1U}^{\prime \prime})(a_2 - l_{2U}
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Page 27 of 43

Proof. Similar to the Proposition 3.1.3.

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Proposition 3.1.11. Let \tilde{A}_1 = (a_1; l_{1L}^{\mu}, r_{1L}^{\mu}, l_{1U}^{\prime \mu}, r_{1U}^{\prime \mu}; l_{1L}^{\nu}, r_{1L}^{\prime \nu}, l_{1U}^{\prime \nu}, r_{1U}^{\prime \nu}, r_{1U}^{\prime \nu})_{LR} be an LR-type IVIFN such that a_1 + r_{1L}^{\nu} < 0 and \tilde{A}_2 = (a_2; l_{2L}^{\mu}, r_{2L}^{\mu}, l_{2U}^{\prime \mu}, r_{2U}^{\prime \nu}; l_{2L}^{\nu}, r_{2L}^{\nu}, l_{2U}^{\prime \nu}, r_{2U}^{\prime \nu})_{LR} be any LR-type IVIFN. Then \tilde{A}_1 \odot \tilde{A}_2 = (a; l_{L}^{\mu}, r_{L}^{\mu}, l_{U}^{\prime \mu}, r_{U}^{\prime \nu}; l_{L}^{\nu}, r_{L}^{\nu}, l_{U}^{\prime \nu}, r_{U}^{\prime \nu})_{LR} where a = a_1 a_2, l_{L}^{\mu} = a_1 a_2 - \min\{(a_1 + r_{1L}^{\mu})(a_2 + r_{2L}^{\mu}), (a_1 - l_{1L}^{\mu})(a_2 + r_{2L}^{\mu})\}, r_{L}^{\mu} = \max\{(a_1 - l_{1L}^{\mu})(a_2 - l_{2L}^{\mu}), (a_1 + r_{1L}^{\prime \mu})(a_2 + r_{2L}^{\prime \mu})\}, r_{U}^{\prime \mu} = a_1 a_2 - \min\{(a_1 + r_{1U}^{\prime \mu})(a_2 + r_{2U}^{\prime \mu}), (a_1 - l_{1U}^{\prime \mu})(a_2 + r_{2U}^{\prime \mu})\}, r_{U}^{\prime \mu} = \max\{(a_1 - l_{1U}^{\prime \mu})(a_2 - l_{2U}^{\prime \mu}), (a_1 + l_{1U}^{\prime \mu})(a_2 + r_{2U}^{\prime \mu})\}, r_{U}^{\prime \mu} = \max\{(a_1 - l_{1U}^{\prime \mu})(a_2 - l_{2U}^{\prime \mu}), (a_1 - l_{1U}^{\prime \mu})(a_2 + r_{2U}^{\prime \mu})\}, r_{U}^{\prime \mu} = \max\{(a_1 - l_{1U}^{\prime \mu})(a_2 - l_{2U}^{\prime \mu}), (a_1 - l_{1U}^{\prime \mu})(a_2 + r_{2L}^{\prime \nu})\}, r_{U}^{\prime \mu} = \max\{(a_1 - l_{1U}^{\prime \mu})(a_2 - l_{2U}^{\prime \mu}), (a_1 - l_{1U}^{\prime \mu})(a_2 + r_{2L}^{\prime \nu})\}, r_{U}^{\prime \mu} = \max\{(a_1 - l_{1L}^{\prime \mu})(a_2 + r_{2U}^{\prime \mu}), (a_1 - l_{1U}^{\prime \mu})(a_2 + r_{2L}^{\prime \nu})\}, r_{U}^{\prime \mu} = \max\{(a_1 - l_{1L}^{\prime \mu})(a_2 + r_{2U}^{\prime \mu}), (a_1 - l_{1U}^{\prime \mu})(a_2 + r_{2U}^{\prime \nu})\}, r_{U}^{\prime \mu} = \max\{(a_1 - l_{1U}^{\prime \mu})(a_2 + r_{2U}^{\prime \mu}), (a_1 + l_{1U}^{\prime \mu})(a_2 + r_{2U}^{\prime \mu})\}, r_{U}^{\prime \mu} = \max\{(a_1 - l_{1U}^{\prime \mu})(a_2 + r_{2U}^{\prime \mu}), (a_1 + l_{1U}^{\prime \mu})(a_2 + r_{2U}^{\prime \mu})\}, r_{U}^{\prime \mu} = \max\{(a_1 - l_{1U}^{\prime \mu})(a_2 - l_{2U}^{\prime \mu}), (a_1 + l_{1U}^{\prime \mu})(a_2 + r_{2U}^{\prime \mu})\}, r_{U}^{\prime \mu} = \max\{(a_1 - l_{1U}^{\prime \mu})(a_2 - l_{2U}^{\prime \mu}), (a_1 + l_{1U}^{\prime \mu})(a_2 + r_{2U}^{\prime \mu})\}, r_{U}^{\prime \mu} = \max\{(a_1 - l_{1U}^{\prime \mu})(a_2 - l_{2U}^{\prime \mu}), (a_1 + l_{1U}^{\prime \mu})(a_2 + l_{2U}^{\prime \mu})\}, r_{U}^{\prime \mu} = \max\{(a_1 - l_{1U}^{\prime \mu})(a_2 - l_{2U}^{\prime \mu}), (a_1 - l_{1U}^{\prime
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Proof. Similar to the Proposition 3.1.3.

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 \begin{aligned} & \textbf{Proposition 3.1.12.} \ Let \ \tilde{A}_1 = (a_1; l_{1L}^{\mu}, r_{1L}^{\mu}, l_{1L}^{\prime \mu}, r_{1U}^{\prime \mu}; l_{1L}^{\nu}, r_{1L}^{\prime \nu}, l_{1U}^{\prime \nu}, r_{1U}^{\prime \nu}, l_{1U}^{\prime \nu}, r_{1U}^{\prime \nu})_{LR} \ be \ an \ LR-type \ IVIFN \ such \ that \ a_1 - l_{1L}^{\nu} \geq 0 \ and \ \tilde{A}_2 = (a_2; l_{2L}^{\mu}, r_{2L}^{\mu}, l_{2U}^{\prime \mu}, r_{2U}^{\prime \nu}; l_{2L}^{\nu}, r_{2U}^{\nu}, l_{2U}^{\prime \nu}, r_{2U}^{\prime \nu})_{LR} \ be \ any \ LR-type \ IVIFN. \ Then \\ \tilde{A}_1 \odot \tilde{A}_2 = (a; l_{L}^{\mu}, r_{L}^{\mu}, l_{U}^{\prime \mu}, r_{U}^{\prime \mu}; l_{L}^{\nu}, r_{L}^{\nu}, l_{U}^{\prime \nu}, r_{U}^{\prime \nu}, l_{U}^{\prime \nu}, r_{U}^{\prime \nu})_{LR} \ where \\ a = a_1 a_2, \\ l_{L}^{\mu} = a_1 a_2 - min\{(a_1 - l_{1L}^{\mu})(a_2 - l_{2L}^{\mu}), \ (a_1 + r_{1L}^{\mu})(a_2 - l_{2L}^{\mu})\}, \\ r_{L}^{\mu} = max\{(a_1 - l_{1L}^{\mu})(a_2 + r_{2L}^{\mu}), \ (a_1 + r_{1L}^{\prime \mu})(a_2 + r_{2L}^{\prime \mu})\} - a_1 a_2, \\ l_{U}^{\prime \mu} = a_1 a_2 - min\{(a_1 - l_{1U}^{\prime \mu})(a_2 - l_{2U}^{\prime \mu}), \ (a_1 + r_{1U}^{\prime \mu})(a_2 - l_{2U}^{\prime \mu})\}, \\ r_{U}^{\prime \mu} = max\{(a_1 - l_{1U}^{\prime \mu})(a_2 + r_{2U}^{\prime \mu}), \ (a_1 + r_{1U}^{\prime \mu})(a_2 + r_{2U}^{\prime \mu})\} - a_1 a_2, \\ l_{U}^{\nu} = a_1 a_2 - min\{(a_1 - l_{1U}^{\prime \mu})(a_2 - l_{2L}^{\prime \nu}), \ (a_1 + r_{1U}^{\prime \mu})(a_2 - l_{2L}^{\prime \nu})\}, \\ r_{U}^{\nu} = max\{(a_1 - l_{1L}^{\prime \prime})(a_2 + r_{2L}^{\prime \prime}), \ (a_1 + r_{1L}^{\prime \prime})(a_2 - l_{2L}^{\prime \prime})\}, \\ r_{U}^{\prime \nu} = max\{(a_1 - l_{1L}^{\prime \prime})(a_2 + r_{2L}^{\prime \prime}), \ (a_1 + r_{1L}^{\prime \prime})(a_2 - l_{2L}^{\prime \prime})\}, \\ r_{U}^{\prime \nu} = max\{(a_1 - l_{1L}^{\prime \prime})(a_2 + r_{2L}^{\prime \prime}), \ (a_1 + r_{1L}^{\prime \prime})(a_2 - l_{2U}^{\prime \prime})\}, \\ r_{U}^{\prime \nu} = max\{(a_1 + l_{1U}^{\prime \prime})(a_2 + r_{2U}^{\prime \prime}), \ (a_1 - l_{1U}^{\prime \prime})(a_2 + r_{2U}^{\prime \prime})\} - a_1 a_2, \\ l_{U}^{\prime \nu} = max\{(a_1 + l_{1U}^{\prime \prime})(a_2 + r_{2U}^{\prime \prime}), \ (a_1 - l_{1U}^{\prime \prime})(a_2 + r_{2U}^{\prime \prime})\} - a_1 a_2, \\ l_{U}^{\prime \nu} = max\{(a_1 + l_{1U}^{\prime \prime})(a_2 + r_{2U}^{\prime \prime}), \ (a_1 - l_{1U}^{\prime \prime})(a_2 + r_{2U}^{\prime \prime})\} - a_1 a_2, \\ l_{U}^{\prime \nu} = max\{(a_1 + l_{1U}^{\prime \prime})(a_2 + r_{2U}^{\prime \prime}), \ (a_1 - l_{1U}^{\prime \prime})(a_2 + r_{2U}^{\prime \prime})\} - a_1 a_2, \\ l_{U}^{\prime \nu} = max\{(a_1 + l_{1U}^{\prime \prime})(a_2 + r_{2U}^{\prime \prime}), \ (a_1 - l_{1U}^{\prime \prime})(a_2 + r
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Proof. Similar to the Proposition 3.1.3.

3.2. Proposed lexicographic criteria for the ranking of LR-type IVIFNs

In the literature, there exist several ranking criteria to define the ordering of FNs, IFNs and IVIFNs. Most of the researchers have used linear ranking function which maps the set of these numbers to real-line and then a comparison is carried out on the basis of the usual ordering of real numbers. However, as pointed out in Pérez-Cañedo and Concepción-Morales [41] that it may possible that two of these numbers may look different for the decision-maker but can be mapped to the same real number. To overcome this limitation, lexicographic ranking criteria with total order properties seems to be more logical and relevant [19, 22, 61]. In this section, we discuss the lexicographic criteria for the ordering of *LR*-type IVIFNs.

Definition 3.2.1 [22]. For $x, y \in \mathbb{R}^n$, the strict lexicographic inequality $x \prec_{lex} y$ holds, iff there is $1 \le i \le n$ so that $x_i = y_i$ holds for j < i and $x_i < y_i$. The weak lexicographic inequality $x \leq_{lex} y$ holds, iff $x \prec_{lex} y$ or x = y.

 $\begin{aligned} \mathbf{Definition\,3.2.2} \, \text{Let} \, \tilde{A}_1 &= (a_1; l_{1L}^{\mu}, r_{1L}^{\prime \mu}, l_{1U}^{\prime \mu}, r_{1U}^{\prime \mu}; l_{1L}^{\nu}, r_{1U}^{\prime \nu}, l_{1U}^{\prime \nu}, r_{1U}^{\prime \nu})_{LR} \, \text{and} \, \tilde{A}_2 &= (a_2; l_{2L}^{\mu}, r_{2L}^{\mu}, l_{2U}^{\prime \mu}, r_{2U}^{\prime \mu}; l_{2L}^{\nu}, r_{2L}^{\nu}, l_{2U}^{\prime \nu}, r_{2U}^{\prime \nu}, l_{2U}^{\prime \nu},$

$$a_1 = a_2, l_{1L}^{\mu} = l_{2L}^{\mu}, r_{1L}^{\mu} = r_{2L}^{\mu}, l_{1U}^{\prime \mu} = l_{2U}^{\prime \mu}, r_{1U}^{\prime \mu} = r_{2U}^{\prime \mu}, l_{1L}^{\nu} = l_{2L}^{\nu}, r_{1L}^{\nu} = r_{2L}^{\nu}, l_{1U}^{\prime \nu} = l_{2U}^{\prime \nu}, r_{1U}^{\prime \nu} = r_{2U}^{\prime \nu}, r_{2U}^{\prime \nu} = r_{2$$

: Page 28 of 43

Definition 3.2.3 Let \leq_{lex} be the lexicographic order relation on \mathbb{R}^n . For $\tilde{A}=(a;l_L^\mu,r_L^\mu,l_U^\prime,r_U^\prime;l_L^\nu,r_L^\nu,l_U^\prime,r_U^\prime)_{LR}\in IV(\mathbb{R})$, let $S(\tilde{A})$ and $A(\tilde{A})$ be the score and accuracy indices of \tilde{A} , respectively; let $M(\tilde{A}):=a$ (the mean value of \tilde{A}), $C(\tilde{A}):=a-l_L^\mu$, $D(\tilde{A}):=a-l_U^\prime$, $G(\tilde{A}):=a-l_U^\prime$ and $H(\tilde{A}):=a-l_L^\nu$. Then, for any $\tilde{A}_1, \tilde{A}_2 \in IV(\mathbb{R})$, the strict inequality $\tilde{A}_1 < \tilde{A}_2$ holds iff

$$\left(S(\tilde{A}_1),A(\tilde{A}_1),M(\tilde{A}_1),C(\tilde{A}_1),D(\tilde{A}_1),G(\tilde{A}_1),H(\tilde{A}_1)\right) \prec_{lex} \left(S(\tilde{A}_2),A(\tilde{A}_2),M(\tilde{A}_2),C(\tilde{A}_2),D(\tilde{A}_2),G(\tilde{A}_2),H(\tilde{A}_2)\right).$$

The weak inequality $\tilde{A}_1 \leq \tilde{A}_2$ holds iff either

$$(S(\tilde{A}_1), A(\tilde{A}_1), M(\tilde{A}_1), C(\tilde{A}_1), D(\tilde{A}_1), G(\tilde{A}_1), H(\tilde{A}_1)) \prec_{lex} (S(\tilde{A}_2), A(\tilde{A}_2), M(\tilde{A}_2), C(\tilde{A}_2), D(\tilde{A}_2), G(\tilde{A}_2), H(\tilde{A}_2))$$

or
$$(S(\tilde{A}_1), A(\tilde{A}_1), M(\tilde{A}_1), C(\tilde{A}_1), D(\tilde{A}_1), G(\tilde{A}_1), H(\tilde{A}_1)) = (S(\tilde{A}_2), A(\tilde{A}_2), M(\tilde{A}_2), C(\tilde{A}_2), D(\tilde{A}_2), G(\tilde{A}_2), H(\tilde{A}_2)).$$

Remark 3.2.1 The specific order in which the functions S, A, M, C, D, G and H appear in the ranking of LR-type IVIFNs gives a description of the relative importance of these functions in deciding the ordering of the IVIFNs. However, one may consider a different permutation of these parameters depending on the priority of these functions.

Remark 3.2.2 Since the expressions of all the functions S, A, M, C, D, G, H are linear, therefore, each $\Phi \in \{S, A, M, C, D, G, H\}$ follows the linearity property, that is,

$$\Phi(\lambda_1 \tilde{A}_1 \oplus \lambda_2 \tilde{A}_2) = \lambda_1 \Phi(\tilde{A}_1) + \lambda_2 \Phi(\tilde{A}_2)$$
 for all $\tilde{A}_1, \tilde{A}_2 \in IV(\mathbb{R})$ and $\lambda_1, \lambda_2 \in \mathbb{R}$.

Theorem 3.2.1. The order relation on $IV(\mathbb{R})$ given in Definition 3.2.3, has the total order properties and yields a complete ranking on the set of all LR-type IVIFNs of the same type.

Proof. The order relation given in Definition 3.2.3 is a total order due to its following properties:

- 1. (reflexivity) $\tilde{A}_1 \leq \tilde{A}_1 \ \ \forall \ \tilde{A}_1 \in IV(\mathbb{R}),$
- 2. (anti-symmetry) $\tilde{A}_1 \leq \tilde{A}_2$ and $\tilde{A}_2 \leq \tilde{A}_1$ $\implies \tilde{A}_1 = \tilde{A}_2 \ \forall \ \tilde{A}_1, \tilde{A}_2 \in IV(\mathbb{R}),$
- 3. (transitivity) $\tilde{A}_1 \leq \tilde{A}_2$ and $\tilde{A}_2 \leq \tilde{A}_3$
- $\Longrightarrow \tilde{A}_1 \leq \tilde{A}_3 \ \forall \tilde{A}_1, \tilde{\tilde{A}}_2, \tilde{A}_3 \in IV(\mathbb{R}),$

4. (comparability) $\tilde{A}_1 \leq \tilde{A}_2$ or $\tilde{A}_2 \leq \tilde{A}_1 \ \forall \ \tilde{A}_1, \tilde{A}_2 \in IV(\mathbb{R})$.

Further, it is to be noted that the system of linear equations $S(\tilde{A}_1) = S(\tilde{A}_2)$, $A(\tilde{A}_1) = A(\tilde{A}_2)$, $M(\tilde{A}_1) = M(\tilde{A}_2)$, $C(\tilde{A}_1) = C(\tilde{A}_2)$, $D(\tilde{A}_1) = D(\tilde{A}_2)$, and $D(\tilde{A}_1) = D(\tilde{A}_2)$, $D(\tilde{A}_1) = D(\tilde{A}_2)$, $D(\tilde{A}_1) = D(\tilde{A}_2)$, and $D(\tilde{A}_1) = D(\tilde{A}_2)$, $D(\tilde{A}_1) = D(\tilde{A}_2)$, $D(\tilde{A}_1) = D(\tilde{A}_2)$, $D(\tilde{A}_1) = D(\tilde{A}_2)$, $D(\tilde{A}_1) = D(\tilde{A}_2)$, and $D(\tilde{A}_1) = D(\tilde{A}_2)$, $D(\tilde{A}_1) = D(\tilde{A}_$

4. Model formulation and solution algorithm

Consider a linear programming problem in which each parameter \tilde{c}_j , \tilde{a}_{ij} , \tilde{b}_i and decision variables \tilde{x}_j are taken to be in the form of LR-type IVIFNs. The model of such a fully LR-type interval-valued intuitionistic fuzzy linear programming problem along-with unrestricted decision variables can be mathematically formulated as:

$$\begin{aligned} \text{(P1)} \ \max \ & \tilde{Z} = \sum_{j=1}^n \tilde{c}_j \odot \tilde{x}_j \\ \text{s.t.} \ & \sum_{j=1}^n \tilde{a}_{ij} \odot \tilde{x}_j = \tilde{b}_i, \quad \text{for } i \in I_1 := \{1, 2, \dots, m_1\}, \\ & \sum_{j=1}^n \tilde{a}_{ij} \odot \tilde{x}_j \leq \tilde{b}_i, \quad \text{for } i \in I_2 := \{m_1 + 1, m_1 + 2, \dots, m_2\}, \\ & \sum_{j=1}^n \tilde{a}_{ij} \odot \tilde{x}_j \geq \tilde{b}_i, \quad \text{for } i \in I_3 := \{m_2 + 1, m_2 + 2, \dots, m\}, \end{aligned}$$

Page 29 of 43

$$\tilde{x}_j$$
 are unrestricted in sign, for $j \in J := \{1, 2, ..., n\}$

where the inequalities " \leq " and " \geq " in the problem (P1) is in accordance with the lexicographic ranking given in Definition 3.2.3.

Definition 4.1 Let X denote the set of all feasible solutions of (P1). A vector $\hat{x} = (\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n) \in X$ is said to be an optimal solution of (P1) if

$$\sum_{j=1}^n \tilde{c}_j \odot \tilde{x}_j \leq \sum_{j=1}^n \tilde{c}_j \odot \hat{\bar{x}}_j, \quad \text{for all } \tilde{x} = (\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n) \in X.$$

Next, we propose a method to find the unique optimal solution of (P1). We use lexicographic ranking criteria for ordering of LR-type IVIFNs. The steps of the proposed approach are as follows:

Step 1. Let
$$\tilde{a}_{ij} = (a_{ij}; \alpha^{\mu}_{ijL}, \beta^{\mu}_{ijL}, \alpha^{\prime\mu}_{ijU}, \beta^{\prime\mu}_{ijU}; \alpha^{\nu}_{ijL}, \beta^{\nu}_{ijL}, \alpha^{\prime\nu}_{ijU}, \beta^{\prime\nu}_{ijU})_{LR}, \ \tilde{c}_{j} = (c_{j}; \sigma^{\mu}_{jL}, \rho^{\mu}_{jL}, \sigma^{\prime\mu}_{jU}, \rho^{\prime\mu}_{jU}; \sigma^{\nu}_{jL}, \rho^{\nu}_{jU}, \rho^{\prime\nu}_{jU})_{LR},$$

$$\tilde{b}_{i} = (b_{i}; \gamma^{\mu}_{iL}, \delta^{\mu}_{iL}, \gamma^{\prime\mu}_{iU}, \delta^{\prime\mu}_{iU}; \gamma^{\nu}_{iL}, \delta^{\nu}_{iL}, \gamma^{\prime\nu}_{iU}, \delta^{\prime\nu}_{iU}, \gamma^{\prime\nu}_{iU}, \delta^{\prime\nu}_{iU})_{LR}, \text{ and } \tilde{x}_{j} = (x_{j}; \xi^{\mu}_{iL}, \eta^{\mu}_{iL}, \xi^{\prime\mu}_{iU}, \eta^{\prime\mu}_{iU}; \xi^{\nu}_{iL}, \eta^{\nu}_{iL}, \xi^{\prime\nu}_{iU}, \eta^{\prime\nu}_{iU})_{LR}.$$

Then, (P1) can be recast as:

$$\begin{aligned} & \max \quad \sum_{j=1}^{n} (c_{j}; \sigma_{jL}^{\mu}, \rho_{jL}^{\mu}, \sigma_{jU}^{\prime \mu}, \rho_{jU}^{\prime \mu}; \sigma_{jL}^{\nu}, \rho_{jU}^{\prime \nu}, \sigma_{jU}^{\prime \nu}, \rho_{jU}^{\prime \nu})_{LR} \odot (x_{j}; \xi_{jL}^{\mu}, \eta_{jL}^{\mu}, \xi_{jU}^{\prime \mu}, \eta_{jU}^{\prime \mu}; \xi_{jL}^{\nu}, \eta_{jU}^{\nu}, \xi_{jU}^{\prime \nu}, \eta_{jU}^{\prime \nu})_{LR} \\ & \text{s.t.} \quad \sum_{j=1}^{n} (a_{ij}; \alpha_{ijL}^{\mu}, \beta_{ijL}^{\mu}, \alpha_{ijU}^{\prime \mu}, \beta_{ijL}^{\prime \mu}; \alpha_{ijL}^{\nu}, \beta_{ijL}^{\prime \nu}, \alpha_{ijU}^{\prime \nu}, \beta_{ijU}^{\prime \nu}, \beta_{ijU}^{\prime \nu})_{LR} \odot (x_{j}; \xi_{jL}^{\mu}, \eta_{jL}^{\mu}, \xi_{jU}^{\prime \mu}, \eta_{jU}^{\prime \mu}; \xi_{jL}^{\nu}, \eta_{jU}^{\prime \nu}, \xi_{jU}^{\prime \nu}, \eta_{jU}^{\prime \nu})_{LR} \\ & = (b_{i}; \gamma_{iL}^{\mu}, \delta_{iL}^{\mu}, \gamma_{iU}^{\prime \mu}, \delta_{iU}^{\prime \mu}; \gamma_{iL}^{\nu}, \delta_{iL}^{\nu}, \gamma_{iU}^{\prime \nu}, \delta_{iU}^{\prime \nu}, \gamma_{iU}^{\prime \nu}, \delta_{iU}^{\prime \nu})_{LR}, \quad \text{for } i \in I_{1}, \\ & \sum_{j=1}^{n} (a_{ij}; \alpha_{iL}^{\mu}, \beta_{iL}^{\mu}, \alpha_{ijU}^{\prime \mu}, \beta_{ijU}^{\prime \mu}; \alpha_{ijL}^{\nu}, \beta_{iU}^{\nu}, \alpha_{iU}^{\prime \nu}, \beta_{iU}^{\prime \nu}, \beta_{iJU}^{\prime \nu})_{LR} \odot (x_{j}; \xi_{jL}^{\mu}, \eta_{jL}^{\mu}, \xi_{jU}^{\prime \mu}, \eta_{jU}^{\prime \mu}; \xi_{jL}^{\nu}, \eta_{jU}^{\prime \nu})_{LR} \\ & \leq (b_{i}; \gamma_{iL}^{\mu}, \delta_{iL}^{\mu}, \gamma_{iU}^{\prime \mu}, \delta_{iU}^{\prime \mu}; \gamma_{iL}^{\nu}, \delta_{iL}^{\nu}, \gamma_{iU}^{\prime \nu}, \delta_{iU}^{\prime \nu})_{LR}, \quad \text{for } i \in I_{2}, \\ & \sum_{j=1}^{n} (a_{ij}; \alpha_{iL}^{\mu}, \beta_{iL}^{\mu}, \alpha_{iU}^{\prime \mu}, \beta_{iU}^{\prime \mu}; \alpha_{iJ}^{\nu}, \beta_{iU}^{\prime \nu}, \beta_{iU}^{\prime \nu}, \beta_{iU}^{\prime \nu}, \beta_{iU}^{\prime \nu})_{LR} \odot (x_{j}; \xi_{jL}^{\mu}, \eta_{jL}^{\mu}, \xi_{jU}^{\prime \mu}, \eta_{jU}^{\prime \mu}; \xi_{jL}^{\nu}, \eta_{jU}^{\prime \nu})_{LR} \\ & \geq (b_{i}; \gamma_{iL}^{\mu}, \delta_{iL}^{\mu}, \gamma_{iU}^{\prime \mu}, \delta_{iU}^{\prime \mu}; \gamma_{iL}^{\prime \nu}, \delta_{iU}^{\prime \nu}, \gamma_{iU}^{\prime \nu}, \delta_{iU}^{\prime \nu}, \gamma_{iU}^{\prime \nu}, \delta_{iU}^{\prime \nu})_{LR}, \quad \text{for } i \in I_{3}, \\ & \geq (b_{i}; \gamma_{iL}^{\mu}, \delta_{iL}^{\mu}, \gamma_{iU}^{\prime \mu}, \delta_{iU}^{\prime \mu}; \xi_{iL}^{\nu}, \eta_{iU}^{\prime \nu}, \xi_{iU}^{\prime \nu}, \delta_{iU}^{\prime \nu}, \gamma_{iU}^{\prime \nu}, \delta_{iU}^{\prime \nu})_{LR}, \quad \text{are unrestricted in sign, for } j \in J. \end{cases}$$

Step 2. After applying the multiplication operation (Section 3.1), let

$$\begin{split} \tilde{c}_j \odot \tilde{x}_j &= (p_j; \tau^\mu_{jL}, \omega^\mu_{jL}, \tau'^\mu_{jU}, \omega'^\mu_{jU}; \tau^\nu_{jL}, \omega^\nu_{jL}, \tau'^\nu_{jU}, \omega'^\nu_{jU})_{LR} \quad \text{and} \\ \tilde{a}_{ij} \odot \tilde{x}_j &= (m_{ij}; s^\mu_{ijL}, \lambda^\mu_{ijL}, s'^\mu_{ijU}, \lambda'^\mu_{ijU}; s^\nu_{ijL}, \lambda^\nu_{ijL}, s'^\nu_{ijU}, \lambda'^\nu_{ijU})_{LR}. \end{split}$$

Then, the problem (P1) in step 1 becomes:

$$\begin{aligned} \text{(P2)} \ \max \ \ & \sum_{j=1}^{n} (p_{j}; \tau^{\mu}_{jL}, \omega^{\mu}_{jL}, \tau^{\prime \mu}_{jU}, \omega^{\prime \mu}_{jU}; \tau^{\nu}_{jL}, \omega^{\nu}_{jL}, \tau^{\prime \nu}_{jU}, \omega^{\prime \nu}_{jU})_{LR} \\ \text{s.t.} \ & \sum_{j=1}^{n} (m_{ij}; s^{\mu}_{ijL}, \lambda^{\mu}_{ijL}, s^{\prime \mu}_{ijU}, \lambda^{\prime \mu}_{ijU}; s^{\nu}_{ijL}, \lambda^{\nu}_{ijL}, s^{\prime \nu}_{ijU}, \lambda^{\prime \nu}_{ijU})_{LR} \\ & = (b_{i}; \gamma^{\mu}_{iL}, \delta^{\mu}_{iI}, \gamma^{\prime \mu}_{iU}, \delta^{\prime \mu}_{iIU}; \gamma^{\nu}_{iL}, \delta^{\nu}_{iL}, \gamma^{\prime \nu}_{iU}, \delta^{\prime \nu}_{iU})_{LR}, \ \ \text{for } i \in I_{1} \end{aligned}$$

Page 30 of 43

$$\begin{split} &\sum_{j=1}^{n}(m_{ij};s_{ijL}^{\mu},\lambda_{ijL}^{\mu},s_{ijU}^{\prime\mu},\lambda_{ijL}^{\prime\mu};s_{ijL}^{\nu},\lambda_{ijL}^{\nu},s_{ijU}^{\prime\nu},\lambda_{ijU}^{\prime\nu})_{LR} \\ &\leq (b_{i};\gamma_{iL}^{\mu},\delta_{iL}^{\mu},\gamma_{iU}^{\prime\mu},\delta_{iU}^{\prime\mu};\gamma_{iL}^{\nu},\delta_{iL}^{\nu},\gamma_{iU}^{\prime\nu},\delta_{iU}^{\prime\nu})_{LR}, \quad \text{for } i \in I_{2}, \\ &\sum_{j=1}^{n}(m_{ij};s_{ijL}^{\mu},\lambda_{ijL}^{\mu},s_{ijU}^{\prime\mu},\lambda_{ijU}^{\prime\mu};s_{ijL}^{\nu},\lambda_{ijL}^{\nu},s_{ijU}^{\prime\nu},\lambda_{ijU}^{\prime\nu})_{LR} \\ &\geq (b_{i};\gamma_{iL}^{\mu},\delta_{iL}^{\mu},\gamma_{iU}^{\prime\mu},\delta_{iU}^{\prime\mu};\gamma_{iL}^{\nu},\delta_{iL}^{\nu},\gamma_{iU}^{\prime\nu},\delta_{iU}^{\prime\nu})_{LR}, \quad \text{for } i \in I_{3}, \\ &(x_{j};\xi_{iL}^{\mu},\eta_{iL}^{\mu},\xi_{iU}^{\prime\mu},\eta_{iU}^{\prime\mu};\xi_{iL}^{\nu},\eta_{iU}^{\nu},\xi_{iU}^{\prime\nu},\eta_{iU}^{\prime\nu})_{LR} \quad \text{are unrestricted in sign, for } j \in J. \end{split}$$

Step 3. Define
$$\tilde{l}_i := \sum_{j=1}^n (m_{ij}; s^{\mu}_{ijL}, \lambda^{\mu}_{ijL}, s'^{\mu}_{ijU}, \lambda'^{\mu}_{ijU}; s^{\nu}_{ijL}, \lambda^{\nu}_{ijL}, s'^{\nu}_{ijU}, \lambda'^{\nu}_{ijU})_{LR}$$
 and $\tilde{r}_i := (b_i; \gamma^{\mu}_{iL}, \delta^{\mu}_{iL}, \gamma'^{\mu}_{iU}, \delta'^{\mu}_{iU}; \gamma^{\nu}_{iL}, \delta^{\nu}_{iL}, \gamma'^{\nu}_{iU}, \delta'^{\nu}_{iU})_{LR}.$

Further, by taking the order relation \leq_{lex} as in Definition 3.2.3, according to the addition operation from Section 3.1 and the equality between LR-type IVIFNs given in Definition 3.2.2, we can write the constraint set of (P2) as:

$$\sum_{j=1}^{n} m_{ij} = b_{i}, \quad \sum_{j=1}^{n} s_{ijL}^{\mu} = \gamma_{iL}^{\mu}, \quad \sum_{j=1}^{n} \lambda_{ijL}^{\mu} = \delta_{iL}^{\mu}, \quad \sum_{j=1}^{n} s_{ijU}^{\prime \mu} = \gamma_{iU}^{\prime \mu}, \quad \sum_{j=1}^{n} \lambda_{ijU}^{\prime \mu} = \delta_{iU}^{\prime \mu}, \quad \sum_{j=1}^{n} s_{ijL}^{\nu} = \gamma_{iL}^{\nu},$$

$$\sum_{j=1}^{n} \lambda_{ijL}^{\nu} = \delta_{iL}^{\nu}, \quad \sum_{j=1}^{n} s_{ijU}^{\prime \nu} = \gamma_{iU}^{\prime \nu}, \quad \sum_{j=1}^{n} \lambda_{ijU}^{\prime \nu} = \delta_{iU}^{\prime \nu}, \quad \text{for } i \in I_{1},$$

$$(3a)$$

$$\left(S(\tilde{l}_i),A(\tilde{l}_i),M(\tilde{l}_i),C(\tilde{l}_i),D(\tilde{l}_i),G(\tilde{l}_i),H(\tilde{l}_i)\right) \leq_{lex} \left(S(\tilde{r}_i),A(\tilde{r}_i),M(\tilde{r}_i),C(\tilde{r}_i),D(\tilde{r}_i),G(\tilde{r}_i),H(\tilde{r}_i)\right), \text{ for } i \in I_2, \tag{3b}$$

$$\left(S(\tilde{l}_i),A(\tilde{l}_i),M(\tilde{l}_i),C(\tilde{l}_i),D(\tilde{l}_i),G(\tilde{l}_i),H(\tilde{l}_i)\right)\succeq_{lex}\left(S(\tilde{r}_i),A(\tilde{r}_i),M(\tilde{r}_i),C(\tilde{r}_i),D(\tilde{r}_i),G(\tilde{r}_i),H(\tilde{r}_i)\right),\ \ \text{for}\ i\in I_3,$$

$$\begin{aligned} \xi_{jU}^{\prime\mu} &\geq \xi_{jL}^{\mu}, \ \eta_{jU}^{\prime\mu} \geq \eta_{jL}^{\mu}, \ \xi_{jL}^{\nu} \geq \xi_{jU}^{\prime\nu}, \ \eta_{jL}^{\nu} \geq \eta_{jU}^{\prime\nu}, \ \xi_{jL}^{\nu} \geq \xi_{jL}^{\mu}, \ \eta_{jL}^{\nu} \geq \eta_{jL}^{\mu}, \ \xi_{jU}^{\prime\nu} \geq \xi_{jU}^{\prime\mu}, \ \eta_{jU}^{\prime\nu} \geq \eta_{jU}^{\prime\mu}, \\ \xi_{jL}^{\mu} &\geq 0, \ \eta_{jL}^{\mu} \geq 0, \ \xi_{jU}^{\prime\mu} \geq 0, \ \xi_{jU}^{\prime\nu} \geq 0, \ \eta_{jL}^{\nu} \geq 0, \ \xi_{jU}^{\prime\nu} \geq 0, \ \eta_{jU}^{\prime\nu} \geq 0, \ \text{for} \ j \in J. \end{aligned}$$

$$(3d)$$

The objective function of the problem (P2) can be recast as:

lex max
$$\left(S(\tilde{Z}), A(\tilde{Z}), M(\tilde{Z}), C(\tilde{Z}), D(\tilde{Z}), G(\tilde{Z}), H(\tilde{Z})\right)$$

where
$$\tilde{Z} = \sum_{j=1}^{n} \tilde{c}_{j} \odot \tilde{x}_{j} = \sum_{j=1}^{n} (p_{j}; \tau_{jL}^{\mu}, \omega_{jL}^{\mu}, \tau_{jU}^{\prime \mu}, \omega_{jU}^{\prime \mu}; \tau_{jL}^{\nu}, \omega_{jL}^{\nu}, \tau_{jU}^{\prime \nu}, \omega_{jU}^{\prime \nu})_{LR}.$$

Hence, the problem (P1) is finally transformed into the following optimization problem (P3):

(P3) lex max
$$\left(S(\tilde{Z}), A(\tilde{Z}), M(\tilde{Z}), C(\tilde{Z}), D(\tilde{Z}), G(\tilde{Z}), H(\tilde{Z})\right)$$
 subject to constraints $(3a)$ – $(3d)$.

Step 4. By introducing new binary variables z_i^S , z_i^A , z_i^M , z_i^C , z_i^D , z_i^G and z_i^H for $i \in I_2 \cup I_3$ following Pérez-Cañedo and Concepción-Morales [41] along-with a new constraint (4p), for positive real values of k and K sufficiently small and large respectively, the lexicographic constraints (3b) and (3c) can be converted into the following set of constraints (4a) – (4p):

$$kz_i^S \le S(\tilde{r}_i) - S(\tilde{l}_i) \le Kz_i^S$$
, for $i \in I_2$, (4a)

$$-Kz_i^S + kz_i^A \le A(\tilde{r}_i) - A(\tilde{l}_i) \le Kz_i^A, \text{ for } i \in I_2, \tag{4b}$$

: Page 31 of 43

$$-K(z_i^S + z_i^A) + kz_i^M \le M(\tilde{r}_i) - M(\tilde{l}_i) \le Kz_i^M, \text{ for } i \in I_2,$$

$$(4c)$$

$$-K(z_i^S + z_i^A + z_i^M) + kz_i^C \le C(\tilde{r}_i) - C(\tilde{l}_i) \le Kz_i^C, \text{ for } i \in I_2,$$

$$(4d)$$

$$-K(z_{i}^{S} + z_{i}^{A} + z_{i}^{M} + z_{i}^{C}) + kz_{i}^{D} \le D(\tilde{r}_{i}) - D(\tilde{l}_{i}) \le Kz_{i}^{D}, \text{ for } i \in I_{2},$$

$$(4e)$$

$$-K(z_{i}^{S}+z_{i}^{A}+z_{i}^{M}+z_{i}^{C}+z_{i}^{D})+kz_{i}^{G} \leq G(\tilde{r}_{i})-G(\tilde{l}_{i}) \leq Kz_{i}^{G}, \text{ for } i \in I_{2}, \tag{4}f)$$

$$-K(z_{i}^{S} + z_{i}^{A} + z_{i}^{M} + z_{i}^{C} + z_{i}^{D} + z_{i}^{G}) + kz_{i}^{H} \le H(\tilde{r}_{i}) - H(\tilde{l}_{i}) \le Kz_{i}^{H}, \text{ for } i \in I_{2},$$

$$(4g)$$

$$kz_i^S \le S(\tilde{l}_i) - S(\tilde{r}_i) \le Kz_i^S$$
, for $i \in I_3$, (4h)

$$-Kz_i^S + kz_i^A \le A(\tilde{l}_i) - A(\tilde{r}_i) \le Kz_i^A, \text{ for } i \in I_3,$$

$$\tag{4}i$$

$$-K(z_i^S + z_i^A) + kz_i^M \le M(\tilde{l}_i) - M(\tilde{r}_i) \le Kz_i^M, \text{ for } i \in I_3,$$

$$(4j)$$

$$-K(z_i^S + z_i^A + z_i^M) + kz_i^C \le C(\tilde{l}_i) - C(\tilde{r}_i) \le Kz_i^C, \text{ for } i \in I_3, \tag{4k}$$

$$-K(z_{i}^{S} + z_{i}^{A} + z_{i}^{M} + z_{i}^{C}) + kz_{i}^{D} \le D(\tilde{I}_{i}) - D(\tilde{r}_{i}) \le Kz_{i}^{D}, \text{ for } i \in I_{3},$$

$$(4l)$$

$$-K(z_i^S+z_i^A+z_i^M+z_i^C+z_i^D)+kz_i^G\leq G(\tilde{I}_i)-G(\tilde{r}_i)\leq Kz_i^G, \ \text{for}\ i\in I_3, \eqno(4m)$$

$$-K(z_{i}^{S}+z_{i}^{A}+z_{i}^{M}+z_{i}^{C}+z_{i}^{D}+z_{i}^{G})+kz_{i}^{H}\leq H(\tilde{l}_{i})-H(\tilde{r}_{i})\leq Kz_{i}^{H}, \ \text{ for } i\in I_{3}, \tag{4n}$$

$$z_i^S, z_i^A, z_i^M, z_i^C, z_i^D, z_i^G, z_i^H \in \{0, 1\}, \quad \text{for } i \in I_2 \cup I_3,$$
 (40)

$$z_i^S \le z_i^A \le z_i^M \le z_i^C \le z_i^D \le z_i^G \le z_i^H, \text{ for } i \in I_2 \cup I_3. \tag{4p}$$

Step 5. Convert problem (P3) into the following mixed 0 - 1 lexicographic non-linear programming problem (P4):

(P4) lex max
$$\left(S(\tilde{Z}), A(\tilde{Z}), M(\tilde{Z}), C(\tilde{Z}), D(\tilde{Z}), G(\tilde{Z}), H(\tilde{Z})\right)$$
 subject to constraints $(3a), (3d), (4a) - (4p)$.

- Step 6. Solve the problem (P4) by optimizing orderly one objective at a time subject to constraints (3a), (3d), (4a) (4p); including all previously optimized objectives in the constraint set. Thus, by using the lexicographic method of multi-objective optimization [62] and a suitable optimization solver, we can find the optimal solution $x_j, \xi_{jL}^{\mu}, \eta_{jL}^{\mu}, \xi_{jU}^{\prime\mu}, \eta_{jU}^{\prime\mu}, \xi_{jL}^{\prime\nu}, \eta_{jU}^{\prime\nu}, \xi_{jU}^{\prime\nu}, \eta_{jU}^{\prime\nu}$ for $j \in J$. Hence, $\tilde{x}_j = (x_j; \xi_{jL}^{\mu}, \eta_{jL}^{\mu}, \xi_{jU}^{\prime\mu}, \eta_{jU}^{\prime\nu}, \xi_{jU}^{\prime\nu}, \eta_{jU}^{\prime\nu})_{LR}$ for $j \in J$ can be obtained.
- **Step 7.** Finally, by substituting the values of \tilde{x}_j 's into $\sum_{j=1}^n \tilde{c}_j \odot \tilde{x}_j$, we obtain the optimal interval-valued intuitionistic fuzzy value of the problem (P1).

Now, we present Theorems 4.1 and 4.2 to prove the equivalence of the models (P1), (P3) and (P3), (P4), respectively. The similar results for fuzzy LPP were discussed by Pérez-Cañedo and Concepción-Morales [22]. However, we have shown the results for LR-type IVIFLPPs.

Theorem 4.1. If $\hat{x} = (\hat{x}_1, \hat{x}_2, \hat{x}_3, \dots, \hat{x}_n)$ is an optimal solution of problem (P3), then it is also an optimal solution of (P1).

Proof. We will prove the result by the method of contradiction. Let $\hat{x} = (\hat{x}_1, \hat{x}_2, \hat{x}_3, \dots, \hat{x}_n)$ be an optimal solution of problem (P3) but not an optimal solution of (P1). Then, there exists a feasible solution $\tilde{x}^* = (\tilde{x}_1^*, \tilde{x}_2^*, \dots, \tilde{x}_n^*)$ of the

Page 32 of 43

problem (P1), such that

$$\sum_{j=1}^{n} \tilde{c}_{j} \odot \hat{x}_{j} < \sum_{j=1}^{n} \tilde{c}_{j} \odot \tilde{x}_{j}^{*}.$$

In view of Definition 3.2.3, \tilde{x}^* is a feasible solution of the problem (P3) for which

$$\left(S\left(\sum_{j=1}^{n} \tilde{c}_{j} \odot \hat{x}_{j}^{*} \right), A\left(\sum_{j=1}^{n} \tilde{c}_{j} \odot \hat{x}_{j}^{*} \right), M\left(\sum_{j=1}^{n} \tilde{c}_{j} \odot \hat{x}_{j}^{*} \right), C\left(\sum_{j=1}^{n} \tilde{c}_{j} \odot \hat{x}_{j}^{*} \right), D\left(\sum_{j=1}^{n} \tilde{c}_{j} \odot \hat{x}_{j}^{*} \right), G\left(\sum_{j=1}^{n} \tilde{c}_{j} \odot \hat{x}_{j}^{*} \right), A\left(\sum_{j=1}^{n} \tilde{c}_{j} \odot \tilde{x}_{j}^{*} \right), M\left(\sum_{j=1}^{n} \tilde{c}_{j} \odot \tilde{x}_{j}^{*} \right), C\left(\sum_{j=1}^{n} \tilde{c}_{j} \odot \tilde{x}_{j}^{*} \right), D\left(\sum_{j=1}^{n} \tilde{c}_{j} \odot \tilde{x}_{j}^{*} \right), A\left(\sum_{j=1}^{n} \tilde{c}_{j} \odot \tilde{x}_{j}^{*} \right), M\left(\sum_{j=1}^{n} \tilde{c}_{j} \odot \tilde{x}_{j}^{*} \right), C\left(\sum_{j=1}^{n} \tilde{c}_{j} \odot \tilde{x}_{j}^{*} \right), D\left(\sum_{j=1}^{n} \tilde{c}_{j} \odot \tilde{x}_{j}^{*} \right), A\left(\sum_{j=1}^{n} \tilde{c}_{j} \odot \tilde{c}_{j}^{*} \right), A\left(\sum_{j=1}^{n} \tilde{c}$$

that is, there exists a feasible solution of (P3) with the higher objective function value. This contradicts the fact that \hat{x} is an optimal solution of (P3). Hence, the result.

Theorem 4.2. The optimal solution of the problem (P4) is also optimal for (P3). The converse of the statement is also true.

Proof. Let
$$\tilde{l}_i := \sum_{j=1}^n (m_{ij}; s_{ijL}^{\mu}, \lambda_{ijL}^{\mu}, s_{ijU}^{\prime \mu}, \lambda_{ijL}^{\prime \mu}; s_{ijL}^{\nu}, \lambda_{ijU}^{\nu}, s_{ijU}^{\prime \nu}, \lambda_{ijU}^{\prime \nu}, \lambda_{ijU}^{\prime \nu})_{LR}, \ \tilde{r}_i := (b_i; \gamma_{iL}^{\mu}, \delta_{iL}^{\mu}, \gamma_{iU}^{\prime \mu}, \delta_{iU}^{\prime \mu}, \gamma_{iU}^{\prime \nu}, \delta_{iU}^{\prime \nu}, \gamma_{iU}^{\prime \nu}, \delta_{iU}^{\prime \nu})_{LR}$$
 and $z_i := (z_i^S, z_i^A, z_i^M, z_i^C, z_i^D, z_i^G, z_i^H).$

In order to prove this theorem, we need to establish the equivalence between constraint sets (3)[(3b),(3c)] and (4)[(4a)-(4p)] for positive real values of k and K sufficiently small and large, respectively. Here, we will prove the result for the I_2 set of constraints only. For the remaining set of constraints, the result can be proved on the same lines. To show that any solution satisfying constraint set (4) also satisfies set (3), we consider the following cases:

- 1. If $z_i = (1, \star, \star, \star, \star, \star, \star)$, where $\star = 0$ or 1, then by substituting into constraint set (4), we get $k \leq S(\tilde{r}_i) S(\tilde{l}_i) \leq K$, which implies $S(\tilde{l}_i) < S(\tilde{r}_i)$; hence, according to Definition 3.2.3, set of constraints (3) are satisfied.
- 2. If $z_i = (0, 1, \star, \star, \star, \star, \star)$, where $\star = 0$ or 1, then by the constraint set (4) we get $0 \le S(\tilde{r}_i) S(\tilde{l}_i) \le 0$, and $k \le A(\tilde{r}_i) A(\tilde{l}_i) \le K$, which implies $S(\tilde{l}_i) = S(\tilde{r}_i)$ and $A(\tilde{l}_i) < A(\tilde{r}_i)$; hence, from Definition 3.2.3, the constraint set (3) are satisfied.

Now, it is to be shown that any solution satisfying constraint set (3) also satisfies constraints (4). For this, let us consider the following cases:

- 1. If $S(\tilde{r}_i) = S(\tilde{l}_i)$, $A(\tilde{r}_i) = A(\tilde{l}_i)$, $M(\tilde{r}_i) = M(\tilde{l}_i)$, $C(\tilde{r}_i) = C(\tilde{l}_i)$, $D(\tilde{r}_i) = D(\tilde{l}_i)$, $G(\tilde{r}_i) = G(\tilde{l}_i)$ and $H(\tilde{r}_i) = H(\tilde{l}_i)$, then by constraint set (4), we get $z_i = (0,0,0,0,0,0)$.
- 2. If $S(\tilde{r}_i) > S(\tilde{l}_i)$, i.e., $S(\tilde{r}_i) S(\tilde{l}_i) = s_i > 0$, then by constraint (4a), we get $kz_i^S \le s_i \le Kz_i^S$, this inequality is satisfied for $z_i^S = 1$ and for positive real values of k and K, sufficiently small and large. Further, (4o) and (4p) give $(z_i^A, z_i^M, z_i^C, z_i^D, z_i^G, z_i^H) = (1, 1, 1, 1, 1, 1)$, due to which rest of the constraints (4b) (4g) are automatically satisfied. Also, $S(\tilde{r}_i) > S(\tilde{l}_i)$ implies that the constraint (3b) is satisfied.
- 3. If $S(\tilde{r}_i) = S(\tilde{l}_i)$ and $A(\tilde{r}_i) > A(\tilde{l}_i)$ i.e., $A(\tilde{r}_i) A(\tilde{l}_i) = a_i > 0$, then by constraints (4a) and (4b), we get $s_i = 0$ and $kz_i^A \le a_i \le Kz_i^A$, this inequality is satisfied for $z_i^S = 0$, $z_i^A = 1$ and for positive real values of k and K, sufficiently small and large. Further, (4o) and (4p) yield $(z_i^M, z_i^C, z_i^D, z_i^G, z_i^H) = (1, 1, 1, 1, 1)$, due to which rest of the constraints (4c) (4g) are obviously satisfied. Further, $S(\tilde{r}_i) = S(\tilde{l}_i)$ and $A(\tilde{r}_i) > A(\tilde{l}_i)$ implies (3b) holds.

The remaining cases can be similarly obtained. This establishes that the problems (P3) and (P4) are equivalent. Hence proved.

: Page 33 of 43

5. Advantages of the proposed method

Some of the main advantages of the proposed algorithm over the existing approaches are listed below:

- 1. The existing method [57] can only be used to solve IVIFLPPs in which all the decision variables are taken to be non-negative crisp parameters. However, our proposed method can be employed successfully to handle IVIFLPPs having all the decision variables represented by unrestricted IVIFNs.
- The existing study [34] defined a new product operator and the basic arithmetic operations on unrestricted LR-type IFNs. But, there is no study on LR-type IVIFNs. Consequently, we have introduced the definition of LR-type IVIFNs and developed the arithmetic operations on unrestricted LR-type IVIFNs.
- 3. The existing models [19, 22, 34, 41] can be used only to deal with *LR*-type FLPPs and IFLPPs. But, in this article, we have solved the IVIFLPP in which all the parameters and decision variables are represented by *LR*-type IVIFNs, which is more general. Hence, the proposed method can be successfully reduced to solve IFLPPs as well as FLPPs.
- 4. In the present study, we have considered the model parameters and decision variables to be *LR*-type IVIFNs, as a result, the proposed method can also be utilized to solve LPPs where parameters and variables are represented by Triangular/Trapezoidal IVIFNs.
- 5. The proposed algorithm can be used to solve the models having some/all decision variables as unrestricted LR-type IVIFNs or non-negative LR-type IVIFNs.

6. Numerical illustration

In this section, we present a numerical example to demonstrate the steps involved in the proposed algorithm. Consider the following LPP having all the parameters as LR-type IVIFN and unrestricted crisp variables:

(S1) max
$$\tilde{Z} = \tilde{5} \odot x_1 \oplus \tilde{8} \odot x_2$$

s.t. $\tilde{12} \odot x_1 \oplus \tilde{4} \odot x_2 = \tilde{100}$, $\tilde{6} \odot x_1 \oplus \tilde{10} \odot x_2 \leq \tilde{150}$, x_1 and x_2 are unrestricted in sign

where

$$\begin{split} \tilde{\mathbf{5}} &= (5; 2, 2, 3, 3; 5, 5, 5, 4)_{LR}, \quad \tilde{\mathbf{8}} &= (8; 1, 1, 2, 2; 4, 4, 2, 3)_{LR}, \quad \tilde{\mathbf{12}} &= (12; 2, 3, 4, 4; 6, 8, 4, 4)_{LR}, \\ \tilde{\mathbf{4}} &= (4; 1, 1, 2, 2; 4, 4, 2, 2)_{LR}, \quad \tilde{\mathbf{6}} &= (6; 3, 4, 4, 4; 6, 6, 4, 4)_{LR}, \quad \tilde{\mathbf{10}} &= (10; 3, 4, 4, 5; 6, 8, 5, 5)_{LR}, \\ \tilde{\mathbf{100}} &= (100; 25, 35, 50, 50; 80, 100, 50, 50)_{LR}, \quad \tilde{\mathbf{150}} &= (150; 50, 60, 50, 70; 120, 100, 80, 70)_{LR}, \\ L(x) &= R(x) = L'(x) = R'(x) = \max\{0, 1 - x\} \ \forall \ x \in \mathbb{R}. \end{split}$$

Solution:

Step 1. Substituting the expressions of various parameters, the problem (S1) can be re-written as follows:

$$\max \ \tilde{Z} = (5; 2, 2, 3, 3; 5, 5, 5, 4)_{LR} \odot x_1 \oplus (8; 1, 1, 2, 2; 4, 4, 2, 3)_{LR} \odot x_2$$
 s.t.
$$(12; 2, 3, 4, 4; 6, 8, 4, 4)_{LR} \odot x_1 \oplus (4; 1, 1, 2, 2; 4, 4, 2, 2)_{LR} \odot x_2 = (100; 25, 35, 50, 50; 80, 100, 50, 50)_{LR},$$

$$(6; 3, 4, 4, 4; 6, 6, 4, 4)_{LR} \odot x_1 \oplus (10; 3, 4, 4, 5; 6, 8, 5, 5)_{LR} \odot x_2 \leq (150; 50, 60, 50, 70; 120, 100, 80, 70)_{LR},$$

$$x_1 \text{ and } x_2 \text{ are unrestricted in sign.}$$

Step 2. Using the multiplication operation (Corollary 3.1.1) on LR-type IVIFNs along-with the fact that

$$\max\{a, b\} = \frac{1}{2}(a + b + |a - b|),$$

the problem (S1) is converted to the following equivalent problem:

$$(\mathbf{S2}) \max \ \tilde{Z} = \left(5x_1; 2|x_1|, 2|x_1|, 3|x_1|, 3|x_1|; 5|x_1|, 5|x_1|, \frac{1}{2}(x_1 + 9|x_1|), \frac{1}{2}(-x_1 + 9|x_1|)\right)_{LR} \oplus \left(8x_2; |x_2|, |x_2|, 2|x_2|; 4|x_2|, 4|x_2|, \frac{1}{2}(-x_2 + 5|x_2|), \frac{1}{2}(x_2 + 5|x_2|)\right)_{LR}$$

Page 34 of 43

$$\begin{aligned} \text{s.t. } & \left(12x_1; \frac{1}{2}(-x_1+5|x_1|), \frac{1}{2}(x_1+5|x_1|), 4|x_1|, 4|x_1|; -x_1+7|x_1|, x_1+7|x_1|, 4|x_1|, 4|x_1|\right)_{LR} \\ & \oplus \left(4x_2; |x_2|, |x_2|, 2|x_2|, 2|x_2|; 4|x_2|, 4|x_2|, 2|x_2|, 2|x_2|\right)_{LR} = (100; 25, 35, 50, 50; 80, 100, 50, 50)_{LR}, \\ & \left(6x_1; \frac{1}{2}(-x_1+7|x_1|), \frac{1}{2}(x_1+7|x_1|), 4|x_1|, 4|x_1|; 6|x_1|, 6|x_1|, 4|x_1|, 4|x_1|\right)_{LR} \oplus \left(10x_2; \frac{1}{2}(-x_2+7|x_2|), \frac{1}{2}(x_2+7|x_2|), \frac{1}{2}(-x_2+9|x_2|); -x_2+7|x_2|, x_2+7|x_2|, 5|x_2|, 5|x_2|\right)_{LR} \\ & \leq (150; 50, 60, 50, 70; 120, 100, 80, 70)_{LR}, \\ & x_1 \text{ and } x_2 \text{ are unrestricted in sign.} \end{aligned}$$

Step 3. By employing the lexicographic ordering as in Definition 3.2.3, using the addition operation, equality between LR-type IVIFNs and the corresponding definitions of S, A, M, C, D, G and H, we get the following non-linear programming problem:

(S3) lex max
$$\left(\frac{1}{8}(x_1 - x_2), \frac{1}{8}(79x_1 + 129x_2), 5x_1 + 8x_2, 5x_1 + 8x_2 - 2|x_1| - |x_2|, 5x_1 + 8x_2 - 3|x_1| - 2|x_2|, \frac{9}{2}x_1 + \frac{17}{2}x_2 - \frac{9}{2}|x_1| - \frac{5}{2}|x_2|, 5x_1 + 8x_2 - 5|x_1| - 4|x_2|\right)$$
s.t. $3x_1 + x_2 = 25, \quad -x_1 + 5|x_1| + 2|x_2| = 50, \quad x_1 + 5|x_1| + 2|x_2| = 70,$
 $2|x_1| + |x_2| = 25, \quad -x_1 + 7|x_1| + 4|x_2| = 80, \quad x_1 + 7|x_1| + 4|x_2| = 100,$
 $\left(\frac{x_1}{8}, \frac{1}{8}(97x_1 + 164x_2), 6x_1 + 10x_2, \frac{13}{2}x_1 + \frac{21}{2}x_2 - \frac{7}{2}|x_1| - \frac{7}{2}|x_2|, 6x_1 + \frac{21}{2}x_2 - 4|x_1| - \frac{9}{2}|x_2|, 6x_1 + 10x_2 - 4|x_1| - 5|x_2|, 6x_1 + 11x_2 - 6|x_1| - 7|x_2|\right)$
 $\leq_{lex} (7.5, 300, 150, 100, 100, 70, 30),$
 x_1 and x_2 are unrestricted in sign.

Step 4. To convert the lexicographic constraint \leq_{lex} into its equivalent form, we introduce the binary variables z_1^S , $z_1^A, z_1^M, z_1^C, z_1^D, z_1^G$ and z_1^H and for $k = 10^{-4}$ and K = 1000, we have the following set of constraints:

$$kz_1^S \le 7.5 - \frac{1}{8}(x_1) \le Kz_1^S,$$
 (5a)

$$-Kz_1^S + kz_1^A \le 300 - \frac{1}{8} (97x_1 + 164x_2) \le Kz_1^A, \tag{5b}$$

$$-K(z_1^S + z_1^A) + kz_1^M \le 150 - 6x_1 - 10x_2 \le Kz_1^M, \tag{5c}$$

$$-K(z_1^S + z_1^A + z_1^M) + kz_1^C \le 100 - \frac{13}{2}x_1 - \frac{21}{2}x_2 + \frac{7}{2}|x_1| + \frac{7}{2}|x_2| \le Kz_1^C, \tag{5d}$$

$$-K(z_1^S + z_1^A + z_1^M + z_1^C) + kz_1^D \le 100 - 6x_1 - \frac{21}{2}x_2 + 4|x_1| + \frac{9}{2}|x_2| \le Kz_1^D, \tag{5e}$$

$$-K(z_1^S + z_1^A + z_1^M + z_1^C + z_1^D) + kz_1^G \le 70 - 6x_1 - 10x_2 + 4|x_1| + 5|x_2| \le Kz_1^G,$$

$$-K(z_1^S + z_1^A + z_1^M + z_1^C + z_1^D + z_1^G) + kz_1^H \le 30 - 6x_1 - 11x_2 + 6|x_1| + 7|x_2| \le Kz_1^H,$$

$$(5f)$$

$$-K(z_1^S + z_1^A + z_1^M + z_1^C + z_1^D + z_1^G) + kz_1^H \le 30 - 6x_1 - 11x_2 + 6|x_1| + 7|x_2| \le Kz_1^H, \tag{5g}$$

$$z_1^S, z_1^A, z_1^M, z_1^C, z_1^D, z_1^G, z_1^H \in \{0, 1\},$$

$$(5h)$$

$$z_{1}^{S}, z_{1}^{A}, z_{1}^{M}, z_{1}^{C}, z_{1}^{D}, z_{1}^{G}, z_{1}^{H} \in \{0, 1\},$$

$$z_{1}^{S} \leq z_{1}^{A} \leq z_{1}^{M} \leq z_{1}^{C} \leq z_{1}^{D} \leq z_{1}^{G} \leq z_{1}^{H}.$$

$$(5h)$$

Step 5. Finally, the equivalent mixed 0-1 lexicographic non-linear optimization problem obtained is as follows:

(S4) lex max
$$\left(\frac{1}{8}(x_1 - x_2), \frac{1}{8}(79x_1 + 129x_2), 5x_1 + 8x_2, 5x_1 + 8x_2 - 2|x_1| - |x_2|, 5x_1 + 8x_2 - 3|x_1| - 2|x_2|, \frac{9}{2}x_1 + \frac{17}{2}x_2 - \frac{9}{2}|x_1| - \frac{5}{2}|x_2|, 5x_1 + 8x_2 - 5|x_1| - 4|x_2|\right)$$

s.t. $3x_1 + x_2 = 25, \quad -x_1 + 5|x_1| + 2|x_2| = 50, \quad x_1 + 5|x_1| + 2|x_2| = 70,$ $2|x_1| + |x_2| = 25, \quad -x_1 + 7|x_1| + 4|x_2| = 80, \quad x_1 + 7|x_1| + 4|x_2| = 100,$ constraints $(5a) - (5i),$ x_1 and x_2 are unrestricted in sign.

Step 6. To solve the problem (S4), we start by optimizing the first component of the objective function along-with imposing all the constraints of problem (S4). As a result, we solve the single-objective non-linear programming

Page 35 of 43

problem (S4-1) given by:

(S4-1) max
$$S = \frac{1}{8}(x_1 - x_2)$$

s.t. all the constraints of (S4).

Here, we have solved all the crisp optimization models by using a software "LINGO-17.0" on a MacBook Air system with 1.8 GHz Dual-Core Intel Core i5 processor and 8 GB RAM. The optimal solution of (S4-1) gives the optimal objective value as S = 1.875. Next, for optimizing the second component of objective function of (S4), we solve the problem (S4-2).

(S4-2) max
$$A = \frac{1}{8}(79x_1 + 129x_2)$$

s.t. all the constraints of (S4), $\frac{1}{8}(x_1 - x_2) \ge 1.875$.

The optimal objective value of (S4-2) is A = 18.125 and for optimizing the third objective, the problem (S4-3) is solved.

(S4-3) max
$$M = 5x_1 + 8x_2$$

s.t. all the constraints of (S4),
$$\frac{1}{8}(x_1 - x_2) \ge 1.875,$$
$$\frac{1}{8}(79x_1 + 129x_2) \ge 18.125.$$

The optimal solution of (S4-3) gives M = 10. Further, the fourth objective is optimized by solving the problem (S4-4).

(S4-4) max
$$C = 5x_1 + 8x_2 - 2|x_1| - |x_2|$$

s.t. all the constraints of (S4),

$$\frac{1}{8}(x_1 - x_2) \ge 1.875,$$

$$\frac{1}{8}(79x_1 + 129x_2) \ge 18.125,$$

$$5x_1 + 8x_2 \ge 10.$$

The optimal objective value of (S4-4) is C = -15 and for optimizing the fifth objective, we solve the problem (S4-5).

(S4-5) max
$$D = 5x_1 + 8x_2 - 3|x_1| - 2|x_2|$$

s.t. all the constraints of (S4),

$$\frac{1}{8}(x_1 - x_2) \ge 1.875,$$

$$\frac{1}{8}(79x_1 + 129x_2) \ge 18.125,$$

$$5x_1 + 8x_2 \ge 10,$$

$$5x_1 + 8x_2 - 2|x_1| - |x_2| \ge -15.$$

The optimal solution of (S4-5) gives the optimal objective value as D = -30 and to optimize the sixth objective, the problem (S4-6) is solved.

(S4-6) max
$$G = \frac{9}{2}x_1 + \frac{17}{2}x_2 - \frac{9}{2}|x_1| - \frac{5}{2}|x_2|$$

s.t. all the constraints of (S4),

$$\frac{1}{8}(x_1 - x_2) \ge 1.875,$$

$$\frac{1}{8}(79x_1 + 129x_2) \ge 18.125,$$

$$5x_1 + 8x_2 \ge 10,$$

$$5x_1 + 8x_2 - 2|x_1| - |x_2| \ge -15,$$

$$5x_1 + 8x_2 - 3|x_1| - 2|x_2| \ge -30.$$

Page 36 of 43

The optimal solution of (S4-6) gives G = -55. Finally, the seventh objective is optimized by solving the problem (S4-7).

(S4-7) max
$$H = 5x_1 + 8x_2 - 5|x_1| - 4|x_2|$$

s.t. all the constraints of (S4),

$$\frac{1}{8}(x_1 - x_2) \ge 1.875,$$

$$\frac{1}{8}(79x_1 + 129x_2) \ge 18.125,$$

$$5x_1 + 8x_2 \ge 10,$$

$$5x_1 + 8x_2 - 2|x_1| - |x_2| \ge -15,$$

$$5x_1 + 8x_2 - 3|x_1| - 2|x_2| \ge -30,$$

$$\frac{9}{2}x_1 + \frac{17}{2}x_2 - \frac{9}{2}|x_1| - \frac{5}{2}|x_2| \ge -55.$$

The optimal objective value and optimal solution of problem (S4-7) are respectively equal to H = -60, and $x_1 = 10$, $x_2 = -5$.

Step 7. By putting the optimal solution values as obtained in step 6, we get the unique optimal IVIF value

$$\tilde{Z} = (10; 25, 25, 40, 40; 70, 70, 65, 50)_{LR}$$

7. An application in production planning

A bicycle manufacturing company produces two models of bicycles, namely, road bikes and mountain bikes. Steel alloy and rubber constitute the primary raw materials required in the production process of the main body of the bicycles. It is observed from the past experiences that no complete unit of a road bicycle can be manufactured if 1.5 hours of skilled labourers are employed per unit. On the other hand, the company can't bear to spend more than 2.5 hours of labour time per unit of the road bike as it will reduce the efficiency of the process. From the past data, it is estimated that complete manufacturing of each unit of a road bike requires about 1.75 to 2.25 hours of skilled labourers. It was also judged that the production curve of road bikes peaks near to 2 hours of labour time per unit. Further, it is known from the judgement of the decision-maker that values of all the parameters follow linear variations and involve uncertainty as well as some inherent hesitation. In this context, all the information related to the skilled labour hours is summarized in Table 2 while the data referring to the requirement of raw material is tabulated in Table 3.

The manager of the company has approximated that about 100 hours of labour time is available for the production process. Further, due to the various uncontrollable factors, there occurs a fluctuation in the supply and consumption of Steel alloy and rubber in the manufacturing firm. Thus, the manager has estimated that around 300 units of Steel alloy and nearly 120 units of rubber will be available for this production run. However, the manufacturing firm can purchase the additional required units of Steel alloy at a price of about \$10 or can sell leftover units at the same price. The company estimated the selling price per unit of road bike at a price nearly \$80 and that of mountain bike at about \$120. Finally, the firm manager wants to find the number of optimal units of Steel alloy need to be purchased or sold and units of the road bikes and mountain bikes be produced consequently so as to maximize the total profit.

Table 2
Data related to the requirement of skilled labour hours per bicycle

Type of bicycle	Time in which no unit prepared completely	Estimated time range to prepare each unit	Peak Production Time	Time unacceptable by the company for each unit ¹
Road bike	1.5	1.75 to 2.25	2	2.5
Mountain bike	3	3.75 to 4.5	4	4.7

 $^{^{1}}$ This represents the labour hours which, if employed for a unit of the bicycle then it will reduce the efficiency of the production run.

Page 37 of 43

Table 3
Raw material requirement (in Kilograms) per unit of bicycle

Type of bicycle	Type of material	Amount of material that can't produce one complete frame of bicycle	Estimated range of material per unit of bike	Amount of material giving maximum production rate	Units of material can't be utilized per bicycle ²
Road bike	Steel alloy Rubber	5 1.5	6 to 8 1.75 to 2.3	7 2	8.5 2.7
Mountain bike	Steel alloy Rubber	6.5	7.2 to 8.9 3.5 to 4.7	8 4	10 5

² It describes the units of the raw material which if used per frame of the bicycle then it will reduce the profit of the company.

Problem formulation: Since all the parameters of the problem are known to be in the estimated/uncertain form, therefore, it is more relevant to represent these estimated numbers by LR-type IVIFNs. Following, the data given in Tables 2 and 3, the given estimated parameters can be presented as follows:

- 1. Labour time (hrs.):
 - (a). For Road bike: $\tilde{2} = (2; 0.1, 0.1, 0.25, 0.25; 0.5, 0.5, 0.3, 0.3)_{LR}$.
 - (b). For Mountain bike: $\tilde{4} = (4, 0.2, 0.2, 0.25, 0.5, 1, 0.7, 0.5, 0.6)_{LR}$.
- 2. Raw material (Kgs.):
 - (a). For Road bike:
 - (i). Steel alloy: $\tilde{7} = (7; 0.5, 0.5, 1, 1; 2, 1.5, 1.5, 1)_{LR}$.
 - (ii). Rubber: $\tilde{2} = (2; 0.1, 0.2, 0.25, 0.3; 0.5, 0.7, 0.3, 0.5)_{LR}$.
 - (b). For Mountain bike:
 - (i). Steel alloy: $\tilde{8} = (8; 0.5, 0.5, 0.8, 0.9; 1.5, 2, 1.2, 1.5)_{LR}$.
 - (ii). Rubber: $\tilde{4} = (4, 0.3, 0.3, 0.5, 0.7, 1, 1, 0.7, 0.8)_{LR}$.
- 3. Availability:
 - (a). Labour time (hrs.): $\tilde{100} = (100; 8, 8, 10, 10; 20, 22, 15, 15)_{LR}$.
 - (b). Steel alloy (Kgs.): $3\tilde{0}0 = (300; 10, 12, 15, 15; 30, 30, 20, 25)_{LR}$.
 - (c). Rubber (Kgs.): $1\tilde{2}0 = (120; 10, 8, 15, 15; 30, 30, 18, 20)_{LR}$.
- 4. Estimated Cost (\$):
 - (a). For Steel alloy: $\tilde{10} = (10; 1, 1.5, 2, 2; 4, 5, 3, 3.5)_{LR}$.
 - (b). For Road bike: $\tilde{80} = (80; 5, 5, 7, 7; 10, 10, 8, 9)_{LR}$.
 - (c). For Mountain bike: $1\tilde{2}0 = (120; 8, 7, 10, 10; 15, 15, 12, 11)_{LR}$.

Based on this data and as per the statement of the problem, it can be formulated as follows:

(M1) max
$$\tilde{Z} = \tilde{80} \odot \tilde{x}_1 \oplus 1\tilde{20} \odot \tilde{x}_2 \oplus 1\tilde{0} \odot \tilde{y}_1$$

s.t. $\tilde{7} \odot \tilde{x}_1 \oplus \tilde{8} \odot \tilde{x}_2 \oplus \tilde{y}_1 = 3\tilde{00}$,
 $\tilde{2} \odot \tilde{x}_1 \oplus \tilde{4} \odot \tilde{x}_2 \leq 1\tilde{20}$,
 $\tilde{2} \odot \tilde{x}_1 \oplus \tilde{4} \odot \tilde{x}_2 \leq 1\tilde{00}$,
 $\tilde{x}_1, \tilde{x}_2 \geq 0$, \tilde{y}_1 unrestricted

where $\tilde{x}_1 = (x_1; \xi_{1I}^{\mu}, \eta_{1I}^{\mu}, \xi_{1II}^{\prime \mu}, \eta_{1II}^{\prime \mu}; \xi_{1I}^{\nu}, \eta_{1II}^{\prime \nu}, \xi_{1II}^{\prime \nu}, \eta_{1II}^{\prime \nu})_{LR} = \text{Number of units of road bike to be produced,}$

 $\tilde{x}_2 = (x_2; \xi_{2L}^{\mu}, \eta_{2L}^{\mu}, \xi_{2U}^{\prime \mu}, \eta_{2U}^{\prime \mu}; \xi_{2L}^{\nu}, \eta_{2L}^{\prime \nu}, \xi_{2U}^{\prime \nu}, \eta_{2U}^{\prime \nu})_{LR} = \text{Number of units of mountain bike to be produced,}$

 $\tilde{y}_1 = (y_1; \pi_{1L}^{\mu}, \theta_{1L}^{\mu}, \pi_{1U}^{\prime \mu}, \theta_{1U}^{\prime \mu}; \pi_{1L}^{\nu}, \theta_{1L}^{\nu}, \pi_{1U}^{\prime \nu}, \theta_{1U}^{\prime \nu})_{LR} = \text{Number of additional units of Steel alloy to be purchased or sold.}$

Solution: Substituting the values of the parameters, the problem (M1) can be further expressed as:

$$(\textbf{M2}) \ \max \ \tilde{Z} = (80; 5, 5, 7, 7; 10, 10, 8, 9)_{LR} \odot \tilde{x}_1 \oplus (120; 8, 7, 10, 10; 15, 15, 12, 11)_{LR} \odot \tilde{x}_2 \oplus \\ (10; 1, 1.5, 2, 2; 4, 5, 3, 3.5)_{LR} \odot \tilde{y}_1$$

: Page 38 of 43

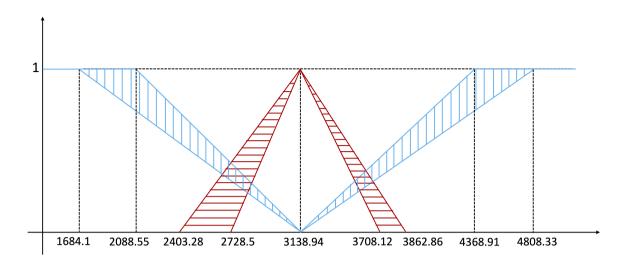


Figure 4: Representation of the optimal profit function as an LR-type TIVIFN

```
 \begin{array}{l} \text{s.t.} \ (7;0.5,0.5,1,1;2,1.5,1.5,1)_{LR} \odot \ \tilde{x}_1 \oplus (8;0.5,0.5,0.8,0.9;1.5,2,1.2,1.5)_{LR} \odot \ \tilde{x}_2 \oplus \ \tilde{y}_1 \\ = (300;10,12,15,15;30,30,20,25)_{LR}, \\ (2;0.1,0.2,0.25,0.3;0.5,0.7,0.3,0.5)_{LR} \odot \ \tilde{x}_1 \oplus (4;0.3,0.3,0.5,0.7;1,1,0.7,0.8)_{LR} \odot \ \tilde{x}_2 \\ \leq (120;10,8,15,15;30,30,18,20)_{LR}, \\ (2;0.1,0.1,0.25,0.25;0.5,0.5,0.3,0.3)_{LR} \odot \ \tilde{x}_1 \oplus (4;0.2,0.2,0.25,0.5;1,0.7,0.5,0.6)_{LR} \odot \ \tilde{x}_2 \\ \leq (100;8,8,10,10;20,22,15,15)_{LR}, \\ \tilde{x}_1,\tilde{x}_2 \geq 0, \ \ \tilde{y}_1 \ \text{unrestricted}. \end{array}
```

Now, since the input data follows the linear trend, therefore, $L(x) = R(x) = L'(x) = R'(x) = \max\{0, 1-x\} \ \forall \ x \in \mathbb{R}$. Further, using "LINGO-17.0" software for solving the equivalent crisp model obtained after applying Steps 1 – 7 of the proposed algorithm, we get the following optimal solution of problem (M1):

```
\tilde{x}_1 = (1.71; 0, 0, 0, 0; 1.7, 0, 0.64, 0)_{LR}, \quad \tilde{x}_2 = (3.05; 1.02, 1.13, 1.5, 1.18; 2.08, 2.13, 1.5, 1.97)_{LR} and \tilde{y}_1 = (263.69; 0, 0, 0, 0; 0, 0, 0, 0)_{LR} with the optimal IVIF value of the objective function equals
```

the definition of the objective function equals

$$\tilde{Z} = (3138.94; 410.44, 569.18, 735.66, 723.92; 1454.84, 1669.39, 1050.39, 1229.97)_{LR}$$

7.1. Results and discussion

The optimal solution of problem (M1) calls for selling nearly 263 Kgs. of leftover Steel alloy, manufacturing about 2 units of the road bike and 3 units of mountain bike to gain the maximum profit in the given scenario. The graphical representation of the objective function value \tilde{Z} as an LR-type IVIFN is shown in Fig. 4. The interpretation of the profit function can be viewed as follows:

The company's acceptance increases if the profit value increases from nearly \$2403.28 to \$3138.94 while the degree of attainability of profit decreases if profit further increases from \$3138.94 to \$3862.86. The manager is fully satisfied at a profit of \$3138.94. However, when the profit increases from nearly \$1684.1 to \$3138.94, the degree of non –attainability (or rejection) decreases continuously while the non –attainability degree increases if the profit grows from \$3138.94 to \$4808.33. Further, the company's profit can't go below \$1684.1 and a profit above \$4808.33 is also not achievable.

7.2. Managerial insights

Our modelling of linear optimization problems in LR-type IVIF environment integrates two significant variations in the data to solve the realistic problems. Firstly, this modelling allows different types of variation in the input data using

: Page 39 of 43

the suitable choice of L and R functions. Secondly, the model parameters are taken as IVIFNs with interval degrees which handle the uncertain data in a most appropriate manner. Hence, it is crucial for a policy-maker to understand and judge how the optimal strategy varies using different L, R functions and to evaluate the optimal solution which suits best to the concerned organization.

The formulation, solution and analysis of the production planning problem (M1) successfully incorporate the uncertain and vague data in the model and further provides the flexible and optimal production strategy to the company manager.

7.3. Comparison with other cases

The problem (M1) is also solved by transforming the inequality constraints of the model into equality by introducing the non-negative LR-type IVIF slack variables. Additionally, the model is alternatively solved using the usual order relation \leq in place of \leq_{lex} . The values obtained are mentioned in Table 4.

Further, by comparing the optimal values of \tilde{Z} , it is observed that the solution obtained by the proposed methodology yields better results. However, it may also be noticed that the optimal solutions are close to each other. Moreover, using the Definition 3.2.3 for comparing the objective function values, we get

$$S(\tilde{Z}_{slack}) = -75.34 < S(\tilde{Z}_{order\ relation} \leq) = -58.96 < S(\tilde{Z}_{proposed}) = -30.89$$

implying that $\tilde{Z}_{slack} \prec \tilde{Z}_{order\ relation} \leq \tilde{Z}_{proposed}$. Therefore, it can be inferred that the proposed algorithm yields better results than the two alternative approaches.

Table 4
Optimal solution obtained using other cases and proposed algorithm

Solution	Proposed method	Adding non-negative LR -type IVIF slack variables	Using order relation \leq instead of \leq_{lex}
\tilde{x}_1	(1.71; 0, 0, 0, 0;	(0; 0, 0, 0, 0;	(0; 0, 0, 0, 0.1;
	$1.7, 0, 0.64, 0)_{LR}$	$(0,0.3,0,0.875)_{LR}$	$(0, 0.794, 0, 0.687)_{LR}$
\tilde{x}_2	(3.05; 1.02, 1.13, 1.5, 1.18;	(3.682; 0, 0, 0, 0;	(5.75; 0, 0, 0, 0;
	$2.08, 2.13, 1.5, 1.97)_{LR}$	$(0,0,0,0)_{LR}$	$1.5, 0, 0.294, 0)_{LR}$
\tilde{y}_1	(263.69; 0, 0, 0, 0;	(246.67; 7, 8.67, 9.67, 9;	(240; 6.25, 8.25, 9, 8.25;
	$(0,0,0,0)_{LR}$	$16.67, 20, 12, 15)_{LR}$	$9, 18.25, 9, 15.25)_{LR}$
$ ilde{Z}$	(3138.94; 410.44, 569.18, 735.66, 723.92;	(2908.54; 339.13, 495.48, 607.52, 638.16;	(3090; 342.25, 495.12, 609.5, 645.2;
	$1454.84, 1669.39, 1050.39, 1229.97)_{LR}$	$1141.93, 1615.58, 868.19, 1184.22)_{LR}$	$1257.75, 1631.46, 883.75, 1170.27)_{LR}$

8. Conclusions and future research scope

The study proposes the definition of LR-type interval-valued intuitionistic fuzzy numbers and defines the basic operation on unrestricted LR-type IVIFNs with the help of α -cut and β -cut. We have also derived the expressions of the various ranking indices for these numbers. Further, a methodology has been proposed to solve a class of unrestricted fully LR-type IVIF linear programming problems. The current study also generalizes the results and theory of fuzzy, intuitionistic fuzzy and LR-type fuzzy / intuitionistic fuzzy numbers. Finally, it is to be pointed out that all the existing models of LPPs [19, 22, 30, 34, 41] can also be solved using the proposed algorithm. However, many real-world problems which fail to have crisp parameters can only be solved efficiently using the proposed technique by representing the uncertain data using intervals. Moreover, for ranking of the LR-type IVIFNs, a lexicographic criteria (S, A, M, C, D, G, H) has been used. However, the choice of this ranking criterion is not fixed. There are a total of 7! permutations possible. It totally depends on the decision-maker's attitude towards the preferences of various parameters involved in the lexicographic ranking. Furthermore, the practical applicability of the proposed model and solution algorithm is demonstrated by solving the production planning problem of a manufacturing company. Additionally, the optimal solution values are also compared with the two alternative approaches that can be used to solve the problem. The comparative results conclude that the proposed technique perform better than the alternate methods.

Some of the future directions of the present work are as follows:

: Page 40 of 43

- 1. It can be observed that although the method is able to handle a very generalized class of linear optimization problems under uncertain conditions but the algorithm involves a large number of computation steps. Since, to obtain the final optimal value, one needs to solve seven mixed 0 1 integer non-linear programming problems. Therefore, it will be interesting to devise a more computationally efficient approach to handle such problems.
- 2. In the future, different optimization problems such as supply chain problems, portfolio optimization problems, transportation problems, etc. can be solved under the *LR*-type IVIF environment.
- 3. An important futuristic research aspect would be to formulate and to devise a solution algorithm for multiobjective LPPs under IVIF scenario.
- 4. In future research, the methodology can be extended to solve the quadratic programming problems, non-linear / fractional problems having parameters / variables as *LR*-type IVIFNs.

CRediT authorship contribution statement

Manisha Malik: Conceptualization of this study, Methodology, Software, Validation, Formal analysis, Writing - original draft. S. K. Gupta: Visualization, Methodology, Investigation, Supervision, Writing - review & editing. Manuel Arana-Jiménez: Methodology, Investigation, Supervision.

Declaration of competing interest

We wish to confirm that there are no known conflicts of interest associated with this article and there has been no significant financial support for this work that could have influenced its outcome.

Funding

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

Data availability

No data was used for the research described in the article.

Acknowledgement

The first author is thankful to the Ministry of Human Resource Development, India, for financial support, to carry out this research work.

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Page 41 of 43

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: Page 42 of 43

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Page 43 of 43