Sharp bounds on the symmetric division deg index of graphs and line graphs

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Abstract For a graph G with vertex set V_G and edge set E_G , the symmetric division deg index is defined as $SDD(G) = \sum_{uv \in E_G} \left(\frac{d_u}{d_v} + \frac{d_v}{d_u}\right)$, where d_u denotes the degree of vertex u in G. In 2018, Furtula et al. confirmed the quality of SDD index exceeds that of some more popular VDB indices, in particular that of the GA index. They shown a close connection between the SDD index and the earlier well-established GA index. Thus it is meaningful and important to consider the chemical and mathematical properties of the SDD index. In this paper, we determine some sharp bounds on the symmetric division deg index of graphs and line graphs and characterize the corresponding extremal graphs.

Keywords line graph, symmetric division deg index, bound. **Mathematics Subject Classification:** 05C07, 05C09, 05C92

1 Introduction

We use |U| to denote the cardinality of set U. Let $G = (V_G, E_G)$ be a graph with vertex set V_G and edge set E_G . Let $n_G := |V_G|$ and $m_G := |E_G|$ be the order and size of G, respectively. Denote by $N_G(u)$ the neighbors of vertex u, $d_G(u) := |N_G(u)|$ the degree of vertex u in G. We use Δ_G and δ_G to denote the maximum degree and minimum degree in G, respectively. We call G is a δ -regular graph if $d_u = \delta$ for any $u \in V_G$. A (Δ, δ) -biregular graph is the bipartite graph with $d_u = \Delta$, $d_v = \delta$ for any $uv \in E_G$. For convenience, we sometimes write $d_G(u)$ as d_u without causing confusion. If $E_G \neq \emptyset$, we call G is a nontrivial graph, we only consider connected nontrivial graphs in this paper. Denote by C_n , K_n , S_n and P_n , the cycle, complete graph, star graph and path with order n, respectively. In this paper, all notations and terminologies used but not defined can refer to Bondy and Murty [2]. The line graph $\mathcal{L}(G)$ is the graph whose vertices set is the edge sets of G and two vertices in $\mathcal{L}(G)$ are adjacent if the corresponding two edges has one common vertex in G. We use Δ_G (resp., δ_G) to denote the maximum degree (resp., minimum degree) of graph G. We use $\Delta_{\mathcal{L}(G)}$ (resp., $\delta_{\mathcal{L}(G)}$) to denote the maximum degree (resp., minimum degree) of line graph $\mathcal{L}(G)$.

The first and second Zagreb indices [14] are defined as

$$M_1(G) = \sum_{uv \in E_G} (d_u + d_v) = \sum_{u \in V_G} d_u^2, \quad M_2(G) = \sum_{uv \in E_G} d_u d_v.$$

They are often used to study molecular complexity, chirality, and other chemical properties. Others see [7, 12, 13, 18] and the references within.

The first and second general Zagreb indices [3, 23] are defined as

$$M_1^{\alpha}(G) = \sum_{u \in V_G} d_u^{\alpha}, \quad M_2^{\alpha}(G) = \sum_{uv \in E_G} (d_u d_v)^{\alpha},$$

with $\alpha \in R$.

The general sum-connectivity index [33] is defined as

$$\chi_{\alpha}(G) = \sum_{uv \in E_G} (d_u + d_v)^{\alpha},$$

with $\alpha \in R$.

The geometric-arithmetic index (GA) [30] is defined as

$$GA(G) = \sum_{uv \in E_G} \frac{2\sqrt{d_u d_v}}{d_u + d_v}.$$

In 2010, Vukičević and Gašperov proposed the symmetric division deg index (SDD) [32], which is defined as

$$SDD(G) = \sum_{uv \in E_G} \left(\frac{d_u}{d_v} + \frac{d_v}{d_u}\right).$$

Since then, the SDD index has attracted much attention of researchers. Furtula et al. [11] showed that the SDD index gains the comparable correlation coefficient with a well-known geometric-arithmetic index, the applicative potential of SDD is comparable to already well-established VDB structure descriptors. Vasilyev [31] determined lower and upper bounds of symmetric division deg index in some classes of graphs and determine the corresponding extremal graphs. Das et al. [6] obtained some new bounds for SDD index and presented a relation between SDD index and other topological indices. Pan et

al. [26] determined the extremal SDD index among trees, unicyclic graphs and bicyclic graphs. They also determined the minimum SDD index of tricyclic graphs [21]. Ali et al. [1] characterized the graphs with fifth to ninth minimum SDD indices from the class of all molecular trees. One can refer to [9, 15–17, 25, 28, 29] for more details about SDD index.

The relations between GA index (resp. AG index, general sum-connectivity index, Harmonic index) and the line graphs had been considered in [4,5,24,27]. We take further the line by investigating the SDD index. In this paper, we first determine some sharp bounds on the SDD index of graphs, then determine some sharp bounds on the SDD index of line graphs. In this paper, we only consider connected nontrivial graphs. Let \mathcal{G}_n be the set of connected nontrivial graphs with order n, $\mathcal{G}_{n,m}$ the set of connected nontrivial graphs with order n and size m.

2 Preliminaries

Recall that we only consider connected nontrivial graphs in this paper. We write graphs to denote connected nontrivial graphs without causing confusion.

Lemma 2.1 Let $f(x, y) = \frac{x}{y} + \frac{y}{x}$, and real number a, b satisfied that $0 < a \le x \le y \le b$. Then $1 \le f(x, y) \le \frac{a}{b} + \frac{b}{a}$, with left equality if and only if x = y, right equality if and only if x = a, y = b.

Proof. The binary functions $f(x, y) = \frac{x}{y} + \frac{y}{x}$ and $0 < a \le x \le y \le b$. Let $g(t) = t + \frac{1}{t}$ $(t \ge 1)$. Since $g'(t) = 1 - \frac{1}{t^2} \ge 0$, then g(t) is monotonically increasing for $t \ge 1$.

Since $0 < a \le x \le y \le b$, then $1 \le \frac{y}{x} \le \frac{b}{a}$. Thus $2 = g(1) \le f(x, y) = g(t) \le g(\frac{b}{a}) = \frac{a}{b} + \frac{b}{a}$, with left equality if and only if x = y, right equality if and only if x = a, y = b.

Lemma 2.2 ([20]) Let G be a graph with maximum degree Δ and minimum degree δ , and $\alpha > 0$. Then

$$\frac{\delta^{\alpha}}{2}M_1^{\alpha+1}(G) \le M_2^{\alpha}(G) \le \frac{\Delta^{\alpha}}{2}M_1^{\alpha+1}(G),$$

with both equalities hold if and only if G is regular.

Lemma 2.3 ([27]) Let G be a graph and $G \ncong P_n$. Then $m_G \leq m_{\mathcal{L}(G)}$.

Lemma 2.4 ([4]) Let G be a graph. Then $m_{\mathcal{L}(G)} = \frac{1}{2}M_1(G) - m_G$.

We also need these simple Facts in the proof of our results.

Fact 2.5 ([24]) (i) If $\mathcal{L}(G) \cong S_2$, then $G \cong P_3$; (ii) If $\mathcal{L}(G) \cong S_3$, then $G \cong P_4$; (iii) If $\mathcal{L}(G) \cong S_n$ $(n \ge 4)$, then $G = \emptyset$; (iv) If $\mathcal{L}(G) \cong C_3$, then $G \cong C_3$ or S_4 ; (v) If $\mathcal{L}(G) \cong C_n$ $(n \ge 4)$, then $G = \emptyset$; (vi) If $\mathcal{L}(G) \cong P_n$, then $G \cong P_{n+1}$.

Fact 2.6 ([27]) Let G be a connected nontrivial graph. Then $\mathcal{L}(G)$ is regular if and only if G is regular or biregular.

Fact 2.7 ([24]) Let G be a connected nontrivial graph with maximum degree Δ , minimum degree δ . If $e = uv \in E_G$, then $e \in V_{\mathcal{L}(G)}$, $d_{\mathcal{L}(G)}(e) = d_G(u) + d_G(v) - 2$ and $\max\{2\delta - 2, 1\} \leq \delta_{\mathcal{L}(G)} \leq \Delta_{\mathcal{L}(G)} \leq 2\Delta - 2$, with left equality if and only if G is $\max\{2\delta - 2, 1\}$ -regular, with right equality if and only if G is $2\Delta - 2$ -regular.

3 Sharp bounds for the SDD index of graphs

Vasilyev [31] obtained some bounds for the SDD index of graphs, including the following lower bound of Theorem 3.1.

Theorem 3.1 Let $G \in \mathcal{G}_{n,m}$. Then $2m \leq SDD(G) \leq m(n-1+\frac{1}{n-1})$, with left equality if and only if G is regular, right equality if and only if $G \cong S_n$.

Proof. By Lemma 2.1, one has

$$SDD(G) = \sum_{uv \in E_G} \left(\frac{d_u}{d_v} + \frac{d_v}{d_u}\right) \ge 2m,$$

with equality if and only if G is regular.

$$SDD(G) = \sum_{uv \in E_G} \left(\frac{d_u}{d_v} + \frac{d_v}{d_u}\right) \le \sum_{uv \in E_G} \left(n - 1 + \frac{1}{n - 1}\right) = m(n - 1 + \frac{1}{n - 1}),$$

with equality if and only if $G \cong S_n$.

Since the numbers of cycle $\eta = m - n + 1 \ge 0$, thus $m \ge n - 1$. By Theorem 3.1, we have

Corollary 3.2 Let $G \in \mathcal{G}_n$. Then $SDD(G) \ge 2(n-1)$, with equality if and only if $G \cong K_2$.

 $ID(G) = \sum_{u \in V_G} \frac{1}{d_u}$ is called the inverse degree index [19].

Theorem 3.3 Let $G \in \mathcal{G}_n$ with maximum degree Δ and minimum degree δ . Then

$$\delta^2 \cdot ID(G) \le SDD(G) \le \Delta^2 \cdot ID(G),$$

with both equalities if and only if G is regular.

Proof. By the definition of SDD index

$$SDD(G) = \sum_{uv \in E_G} \left(\frac{d_u}{d_v} + \frac{d_v}{d_u}\right)$$
$$= \sum_{uv \in E_G} \left(\frac{1}{d_v^2} + \frac{1}{d_u^2}\right) d_u d_v$$
$$\geq \delta^2 \sum_{uv \in E_G} \left(\frac{1}{d_v^2} + \frac{1}{d_u^2}\right)$$
$$= \delta^2 \cdot ID(G),$$

with equality if and only if G is regular.

The proof of the upper bound is similar, we omit it.

Theorem 3.4 Let G be a graph with $|E_G| = m$, maximum degree Δ and minimum degree δ . Then

$$SDD(G) \le m(\frac{\Delta}{\delta} + \frac{\delta}{\Delta}),$$

with equality if and only if G is regular or biregular.

Proof. Suppose that $1 \leq \delta \leq d_v \leq d_u \leq \Delta$, and by Lemma 2.1, we have

$$SDD(G) = \sum_{uv \in E_G} \left(\frac{d_u}{d_v} + \frac{d_v}{d_u}\right)$$
$$\leq \sum_{uv \in E_G} \left(\frac{\Delta}{\delta} + \frac{\delta}{\Delta}\right)$$
$$= m\left(\frac{\Delta}{\delta} + \frac{\delta}{\Delta}\right),$$

with equality if and only if $d_u = \Delta$ and $d_v = \delta$ for all $uv \in E_G$, i.e., G is regular or biregular.

Corollary 3.5 Let $G \in \mathcal{G}_{n,m}$ with maximum degree $\Delta \leq n-2$. Then

$$SDD(G) < m(n-2+\frac{1}{n-2}).$$

Proof. Suppose that $1 \le \delta \le \Delta \le n-2$, by Theorem 3.4, we have $SDD(G) \le m(n-2+\frac{1}{n-2})$ with equality if and only if G is (n-2, 1)-biregular, which is a contradiction with G is a connected graph. Thus $SDD(G) < m(n-2+\frac{1}{n-2})$.

Theorem 3.6 Let G be a graph with $|E_G| = m$, maximum degree Δ and minimum degree δ . Then

$$SDD(G) \ge \frac{2\delta^2 m^{\frac{\alpha+1}{\alpha}}}{(M_2^{\alpha}(G))^{\frac{1}{\alpha}}}, \quad SDD(G) \ge \frac{\delta^2(2m)^{\frac{\alpha+1}{\alpha}}}{\Delta(M_1^{\alpha}(G))^{\frac{1}{\alpha}}}$$

with both equalities if and only if G is regular.

Proof. By the definition of SDD index, we have

$$\begin{aligned} \frac{1}{m}SDD(G) &= \frac{1}{m}\sum_{uv\in E_G} \left(\frac{d_u^2 + d_v^2}{d_u d_v}\right) \\ &\geq \left(\prod_{uv\in E_G} \frac{d_u^2 + d_v^2}{d_u d_v}\right)^{\frac{1}{m}} \\ &\geq \left(2^m \delta^{2m} \prod_{uv\in E_G} \frac{1}{d_u d_v}\right)^{\frac{1}{m}} \end{aligned}$$

with first equality if and only if $\frac{d_u^2 + d_v^2}{d_u d_v}$ is a constant for any $uv \in E_G$, second equality if and only if $d_u = d_v = \delta$ for all $uv \in E_G$. Thus

$$(SDD(G))^{\alpha} \geq (2m)^{\alpha} \delta^{2\alpha} \left(\prod_{uv \in E_G} (\frac{1}{d_u d_v})^{\alpha} \right)^{\frac{1}{m}}$$
$$\geq (2m)^{\alpha} \delta^{2\alpha} \cdot \frac{m}{\sum_{uv \in E_G} (d_u d_v)^{\alpha}}$$
$$= \frac{2^{\alpha} m^{\alpha+1} \delta^{2\alpha}}{M_2^{\alpha}(G)},$$

with first equality if and only if G is regular, second equality if and only if $d_u d_v$ is a constant for any $uv \in E_G$. Thus $SDD(G) \geq \frac{2\delta^2 m^{\frac{\alpha+1}{\alpha}}}{(M_2^{\alpha}(G))^{\frac{1}{\alpha}}}$ with equality if and only if G is regular.

By Lemma 2.2, $M_2^{\alpha}(G) \leq \frac{\Delta^{\alpha}}{2} M_1^{\alpha+1}(G)$ with equality if and only if G is regular. Thus $SDD(G) \geq \frac{\delta^2(2m)^{\frac{\alpha+1}{\alpha}}}{\Delta(M_1^{\alpha}(G))^{\frac{1}{\alpha}}}$ with equality if and only if G is regular. $F(G) = \sum_{uv \in E_G} (d_u^2 + d_v^2)$ is called the forgotten index [10].

Theorem 3.7 Let G be a graph with $|E_G| = m$. Then

$$SDD(G) \ge \frac{2m^2}{M_2(G)}, \quad SDD(G) \ge \frac{4m^2}{F(G)}$$

with both equalities if and only if $G \cong K_2$.

Proof. Since

$$m = \sum_{uv \in E_G} \left(\frac{d_u d_v}{d_u^2 + d_v^2} \right)^{\frac{1}{2}} \left(\frac{d_u^2 + d_v^2}{d_u d_v} \right)^{\frac{1}{2}} \\ \leq \left(\sum_{uv \in E_G} \frac{d_u d_v}{d_u^2 + d_v^2} \right)^{\frac{1}{2}} \left(\sum_{uv \in E_G} \frac{d_u^2 + d_v^2}{d_u d_v} \right)^{\frac{1}{2}},$$

with equality if and only if $\frac{d_u^2 + d_v^2}{d_u d_v}$ is a constant for any $uv \in E_G$.

Since $d_u \ge 1$ for any $u \in V_G$, then $\sum_{uv \in E_G} \frac{d_u d_v}{d_u^2 + d_v^2} \le \frac{1}{2} \sum_{uv \in E_G} d_u d_v = \frac{1}{2} M_2(G)$, with equality if and only if $d_u = d_v = 1$ for any $uv \in E_G$. Thus $SDD(G) \ge \frac{2m^2}{M_2(G)}$ with equality if and only if $G \cong K_2$.

Since $d_u \ge 1$ for any $u \in V_G$, then $\frac{d_u d_v}{d_u^2 + d_v^2} \le \frac{d_u^2 + d_v^2}{4}$, with equality if and only if $d_u = d_v = 1$ for any $uv \in E_G$. Then $\sum_{uv \in E_G} \frac{d_u d_v}{d_u^2 + d_v^2} \le \frac{1}{4} \sum_{uv \in E_G} d_u^2 + d_v^2 = \frac{1}{4} F(G)$. Thus $SDD(G) \ge \frac{4m^2}{F(G)}$ with equality if and only if $G \cong K_2$.

In the following, we consider the connected graphs with minimal SDD index.

Theorem 3.8 Let $G \in \mathcal{G}_{n,m}$. Then

- (i) $SDD(G) \ge 2$, with equality if and only if $G \cong K_2$;
- (ii) There is no such graphs with $2 < SDD(G) \le 4$;
- (*iii*) If $4 < SDD(G) \le 6$, then $G \in \{S_3, C_3\}$ with $SDD(S_3) = 5$ and $SDD(C_3) = 6$;
- (iv) If $6 < SDD(G) \le 8$, then $G \in \{P_4, C_4\}$ with $SDD(P_4) = 7$ and $SDD(C_4) = 8$.

Proof. (i) By Theorem 3.1, $SDD(G) \ge 2m \ge 2$, with equality if and only if $G \cong K_2$.

Suppose that $n \ge 3$, then we have (*ii*) If $2 < SDD(G) \le 4$, then $4 \le 2(n-1) \le 2m \le SDD(G) \le 4$, then n = 3 and m = 2. Thus $G \cong S_3$, while $SDD(S_3) = 5 > 4$, which is a contradiction.

(*iii*) If $4 < SDD(G) \le 6$, then $4 \le 2(n-1) \le 2m \le SDD(G) \le 6$, then n = 3or 4 and $m \le 3$. Thus $G \in \{S_3, S_4, P_4, C_3\}$, while $SDD(S_3) = 5$, $SDD(S_4) = 10 > 6$, $SDD(P_4) = 7 > 6$, $SDD(C_3) = 6$. Thus $G \in \{S_3, C_3\}$.

(*iv*) If $6 < SDD(G) \le 8$, then $4 \le 2(n-1) \le 2m \le SDD(G) \le 8$, then n = 3 or 4 or 5 and $m \le 4$. If n = 3 and $m \le 4$, then $G \in \{S_3, C_3\}$ which is a contradiction with

 $SDD(G) \leq 8$. If n = 4 and $m \leq 4$, then $G \in \{S_4, C_4, P_4, C_3^*\}$, where C_3^* is the graph obtained from C_3 by adding a pendent vertex to one vertex of C_3 . $SDD(S_4) = 10 > 8$, $SDD(P_4) = 7$, $SDD(C_4) = 8$, $SDD(C_3^*) = 9 + \frac{2}{3} > 8$. Thus $G \in \{P_4, C_4\}$ in this case.

If n = 5 and $m \le 4$, since $m \ge n - 1 = 4$, thus m = 4. then $G \in \{P_5, P_4^*, S_5\}$, where P_4^* is the graph obtained from P_4 by adding a pendent vertex to one vertex with degree two of P_4 . $SDD(P_5) = 9 > 8$, $SDD(P_4^*) = 11 + \frac{1}{3} > 8$, $SDD(S_5) = 17 > 8$. Thus $G = \emptyset$ in this case.

The inverse problem for the SDD index is also interesting, thus we propose the following problem.

Problem 3.1 Solve the inverse problem for the SDD index of graphs or chemical graphs.

We call $u_0v_0 \in E_G$ is a minimal edge in G if $d_{u_0} \leq d_u$ for all $u \in N_G(u_0) \setminus \{v_0\}$ and $d_{v_0} \leq d_u$ for all $u \in N_G(v_0) \setminus \{u_0\}$.

Theorem 3.9 Let G be a graph with a minimal edge u_0v_0 . Let $G^* = G - u_0v_0$. Then

$$SDD(G^*) > SDD(G) - \frac{(d_{u_0})^2 + (d_{v_0})^2}{d_{u_0}d_{v_0}}$$

Proof. Since $G^* = G - u_0 v_0$, then $V_G = V_{G^*}$. Let $d_u \in V_G$ and $d_u^* \in V_{G^*}$, then $d_{u_0}^* = d_{u_0} - 1$, $d_{v_0}^* = d_{v_0} - 1$ and $d_u^* = d_u$ for all $u \in V_G \setminus \{u_0, v_0\}$.

Let $E_G \supseteq E_0 = \{ uv \in E_G | u \notin \{u_0, v_0\}, v \notin \{u_0, v_0\} \}$. Then

$$\begin{split} SDD(G) &- SDD(G^*) \\ = & \sum_{uv \in E_0} \left(\frac{d_u^2 + d_v^2}{d_u d_v} - \frac{(d_u^*)^2 + (d_v^*)^2}{d_u^* d_v^*} \right) + \sum_{u \in N_G^*(u_0)} \left(\frac{d_{u_0}^2 + d_u^2}{d_{u_0} d_u} - \frac{(d_{u_0}^*)^2 + (d_u^*)^2}{d_{u_0}^* d_u^*} \right) \\ &+ \sum_{u \in N_G^*(v_0)} \left(\frac{d_{v_0}^2 + d_u^2}{d_{v_0} d_u} - \frac{(d_{v_0}^*)^2 + (d_u^*)^2}{d_{v_0}^* d_u^*} \right) + \frac{d_{u_0}^2 + d_{v_0}^2}{d_{u_0} d_{v_0}} \\ &= \sum_{u \in N_G^*(u_0)} \left(\frac{d_{u_0}^2 + d_u^2}{d_{u_0} d_u} - \frac{(d_{u_0}^*)^2 + (d_u^*)^2}{d_{u_0}^* d_u^*} \right) + \sum_{u \in N_G^*(v_0)} \left(\frac{d_{v_0}^2 + d_u^2}{d_{v_0} d_u} - \frac{(d_{v_0}^*)^2 + (d_u^*)^2}{d_{v_0}^* d_u^*} \right) \\ &+ \frac{d_{u_0}^2 + d_{v_0}^2}{d_{u_0} d_{v_0}}. \end{split}$$

Since u_0v_0 is a minimal edge in G, then $1 \leq d_{u_0} \leq d_u$ for all $u \in N_G(u_0) \setminus \{v_0\}$. Since $d_u(d_{u_0}-1)(d_{u_0}^2+d_u^2) - d_u d_{u_0}((d_{u_0}-1)^2+d_v^2) = d_u(d_{u_0}^2-d_u^2-1) < 0$, then $\frac{d_{u_0}^2+d_u^2}{d_{u_0}d_u} - \frac{(d_{u_0}^*)^2+(d_u^*)^2}{d_{v_0}d_u^*} < 0$. Similarly, we also have $\frac{d_{v_0}^2+d_u^2}{d_{v_0}d_u} - \frac{(d_{v_0}^*)^2+(d_u^*)^2}{d_{v_0}^*d_u^*} < 0$. Thus we have $SDD(G^*) > SDD(G) - \frac{(d_{u_0})^2+(d_{v_0})^2}{d_{u_0}d_{v_0}}$.

4 Sharp bounds for the SDD index of line graphs

It is obvious that $SDD(\mathcal{L}(G)) = 0$ if and only if G is a trivial graph, i.e., $G \cong K_2$. Thus in the following, we suppose $G \ncong K_2$.

Theorem 4.1 Let G be a graph with $|E_G| = m$. Then

(i) If $G \cong P_{m+1}$, then $SDD(\mathcal{L}(G)) \ge 2m$, with equality if and only if $G \in \{S_4, C_m\}$;

(ii) If $G \ncong K_2$, then $SDD(\mathcal{L}(G)) \le (\frac{1}{2}M_1(G) - m_G)(m_G - 1 + \frac{1}{m_G - 1})$, with equality if and only if $G \in \{P_3, P_4\}$.

Proof. By Lemma 2.3, $m_{\mathcal{L}(G)} \ge m_G = n_{\mathcal{L}(G)}$ with equality if and only if $\mathcal{L}(G)$ is a unicyclic graph. By Theorem 3.1, $SDD(\mathcal{L}(G)) \ge 2m_{\mathcal{L}(G)} \ge 2m$, with equality if and only if $\mathcal{L}(G)$ is regular unicyclic graph, i.e., $\mathcal{L}(G) \cong C_m$, then $G \in \{S_4, C_m\}$.

By Fact 2.5, Theorem 3.1 and Lemma 2.4, we have that if $G \ncong K_2$, then $SDD(\mathcal{L}(G)) \leq (\frac{1}{2}M_1(G) - m_G)(m_G - 1 + \frac{1}{m_G - 1})$, with equality if and only if $G \in \{P_3, P_4\}$. Combine Theorem 3.3 and Fact 2.5, we have

Theorem 4.2 Let G be a graph with maximum degree Δ and minimum degree δ . If $G \ncong K_2$, then

$$\max\{4(\delta-1)^2, 1\} \cdot ID(\mathcal{L}(G)) \le SDD(\mathcal{L}(G)) \le 4(\Delta-1)^2 \cdot ID(\mathcal{L}(G)),$$

with left equality if and only if G is regular or $G \cong S_3$, right equality if and only if G is regular.

Theorem 4.3 Let G be a graph with $|E_G| = m$, maximum degree Δ and minimum degree δ . If $G \ncong K_2$, then

$$M_1(G) - 2m \le SDD(\mathcal{L}(G)) \le \frac{1}{2}(M_1(G) - 2m)\left(\frac{2\Delta - 2}{\max\{2\delta - 2, 1\}} + \frac{\max\{2\delta - 2, 1\}}{2\Delta - 2}\right),$$

with left equality if and only if G is regular or biregular, right equality if and only if $G \cong P_4$ or G is regular.

Proof. By Lemma 2.4 and Theorem 3.1, we have $SDD(\mathcal{L}(G)) \geq 2m_{\mathcal{L}(G)} = M_1(G) - 2m$, with equality if and only if $\mathcal{L}(G)$ is regular, i.e., G is regular or biregular.

By Theorem 3.4, Lemma 2.1, Lemma 2.4 and Fact 2.7, we have

$$SDD(\mathcal{L}(G)) \leq m_{\mathcal{L}(G)} \left(\frac{\Delta_{\mathcal{L}(G)}}{\delta_{\mathcal{L}(G)}} + \frac{\delta_{\mathcal{L}(G)}}{\Delta_{\mathcal{L}(G)}} \right)$$

$$= \frac{1}{2} (M_1(G) - 2m) \left(\frac{\Delta_{\mathcal{L}(G)}}{\delta_{\mathcal{L}(G)}} + \frac{\delta_{\mathcal{L}(G)}}{\Delta_{\mathcal{L}(G)}} \right)$$

$$\leq \frac{1}{2} (M_1(G) - 2m) \left(\frac{2\Delta - 2}{\max\{2\delta - 2, 1\}} + \frac{\max\{2\delta - 2, 1\}}{2\Delta - 2} \right),$$

with first equality if and only if $\mathcal{L}(G)$ is regular or biregular, second equality if and only if $\delta_{\mathcal{L}(G)} = \max\{2\delta - 2, 1\}$ and $\Delta_{\mathcal{L}(G)} = 2\Delta - 2$.

If $G \cong P_4$ or G is regular, we have the equality holds. $SDD(\mathcal{L}(P_4)) = SDD(S_3) = 5 = \frac{1}{2}(10 - 2 \times 3)(\frac{2 \times 2 - 2}{1} + \frac{1}{2 \times 2 - 2})$, and $SDD(\mathcal{L}(G)) = 2m_{\mathcal{L}(G)} = M_1(G) - 2m = \frac{1}{2}(M_1(G) - 2m)\left(\frac{2\Delta - 2}{\max\{2\delta - 2, 1\}} + \frac{\max\{2\delta - 2, 1\}}{2\Delta - 2}\right)$.

In the following, we suppose that $\mathcal{L}(G)$ is regular or biregular, and $\delta_{\mathcal{L}(G)} = \max\{2\delta - 2, 1\}$ and $\Delta_{\mathcal{L}(G)} = 2\Delta - 2$.

Case 1. $\delta = 1$.

Then $\delta_{\mathcal{L}(G)} = \max\{2\delta - 2, 1\} = 1$. Since $G \ncong K_2$, then $\Delta \ge 2$ and $\Delta_{\mathcal{L}(G)} = 2\Delta - 2 \ge 2$. Then $\mathcal{L}(G)$ is a $(\Delta_{\mathcal{L}(G)}, 1)$ -biregular graph. Thus $\mathcal{L}(G) \cong S_{\Delta_{\mathcal{L}(G)}+1}$ with $n_{\mathcal{L}(G)} = \Delta_{\mathcal{L}(G)} + 1 \ge 3$. By Fact 2.5, we have $G \cong P_4$.

Case 2. $\delta \geq 2$.

In this case, $\mathcal{L}(G)$ is regular or biregular, and $2\delta - 2 = \delta_{\mathcal{L}(G)} \leq \Delta_{\mathcal{L}(G)} = 2\Delta - 2$. If $\mathcal{L}(G)$ is biregular, we have $\delta_{\mathcal{L}(G)} < \Delta_{\mathcal{L}(G)}$, then $\delta < \Delta$, which is a contradiction with the definition of biregular graphs and $2\delta - 2 = \delta_{\mathcal{L}(G)} \leq \Delta_{\mathcal{L}(G)} = 2\Delta - 2$. Then $\mathcal{L}(G)$ is regular, thus $2\delta - 2 = \delta_{\mathcal{L}(G)} = 2\Delta - 2$. Thus G is regular in this case.

In the following, we consider the Nordhaus-Gaddum-type results for the SDD index of a graph G and its line graph $\mathcal{L}(G)$.

Corollary 4.4 Let G be a graph with maximum degree Δ and minimum degree δ . If $G \ncong K_2$, then

$$M_1(G) \le SDD(G) + SDD(\mathcal{L}(G)) \le \frac{1}{2}M_1(G) \left(\frac{2\Delta - 2}{\max\{2\delta - 2, 1\}} + \frac{\max\{2\delta - 2, 1\}}{2\Delta - 2}\right),$$

with both equalities if and only if G is regular.

Proof. Combine Theorem 3.1 and Theorem 4.3, we have $SDD(G) + SDD(\mathcal{L}(G)) \ge M_1(G)$, with equality if and only if G is regular.

For the upper bound of $SDD(G) + SDD(\mathcal{L}(G))$, we consider the following two cases. Case 1. $\delta = 1$. Then G is not a regular graph $(G \not\cong K_2)$, thus $\Delta \geq 2$. By Theorem 3.4 and Theorem 4.3, we have

$$SDD(G) + SDD(\mathcal{L}(G)) < m\left(\frac{\Delta^2 + 1}{\Delta}\right) + \frac{1}{2}(M_1(G) - 2m)\left(\frac{4(\Delta - 1)^2 + 1}{2(\Delta - 1)}\right) = \frac{1}{4}M_1(G)\left(\frac{4(\Delta - 1)^2 + 1}{\Delta - 1}\right) + m\left(\frac{\Delta^2 + 1}{\Delta} - \frac{4(\Delta - 1)^2 + 1}{2(\Delta - 1)}\right) \leq \frac{1}{4}M_1(G)\left(\frac{4(\Delta - 1)^2 + 1}{\Delta - 1}\right).$$

Case 2. $\delta \geq 2$.

It is easy to proof that $\frac{\Delta^2 + \delta^2}{\Delta \delta} \leq \frac{(\Delta - 1)^2 + (\delta - 1)^2}{(\Delta - 1)(\delta - 1)}$. Then

$$SDD(G) + SDD(\mathcal{L}(G))$$

$$\leq m\left(\frac{\Delta^{2} + \delta^{2}}{\Delta\delta}\right) + \frac{1}{2}(M_{1}(G) - 2m)\left(\frac{(\Delta - 1)^{2} + (\delta - 1)^{2}}{(\Delta - 1)(\delta - 1)}\right)$$

$$\leq m\left(\frac{(\Delta - 1)^{2} + (\delta - 1)^{2}}{(\Delta - 1)(\delta - 1)}\right) + \frac{1}{2}(M_{1}(G) - 2m)\left(\frac{(\Delta - 1)^{2} + (\delta - 1)^{2}}{(\Delta - 1)(\delta - 1)}\right)$$

$$= \frac{1}{2}M_{1}(G)\left(\frac{(\Delta - 1)^{2} + (\delta - 1)^{2}}{(\Delta - 1)(\delta - 1)}\right),$$

with equality if and only if G is regular.

Thus we have $SDD(G) + SDD(\mathcal{L}(G)) \leq \frac{1}{2}M_1(G)\left(\frac{2\Delta-2}{\max\{2\delta-2,1\}} + \frac{\max\{2\delta-2,1\}}{2\Delta-2}\right)$, with equality if and only if G is regular.

Theorem 4.5 Let G be a graph with $|E_G| = m$, maximum degree Δ and minimum degree δ . Then

$$SDD(\mathcal{L}(G)) \ge \frac{\Delta \delta^2 \chi_{\alpha+1}(G)}{(\Delta-1)^2 (\chi_{\alpha}(G))^{\frac{1}{\alpha}}}$$

with equality if and only if G is regular.

Proof. It is obvious that the conclusion holds for $\delta = 1$. In the following, we consider $\delta \geq 2$.

By Fact 2.7, Lemma 2.4 and Theorem 3.6, we have

$$SDD(\mathcal{L}(G)) \geq \frac{(\delta_{\mathcal{L}(G)})^2 (2m_{\mathcal{L}(G)})^{\frac{\alpha+1}{\alpha}}}{\Delta_{\mathcal{L}(G)} (M_1^{\alpha}(\mathcal{L}(G)))^{\frac{1}{\alpha}}} \geq \frac{2(\delta-1)^2 (M_1(G)-2m)^{\frac{\alpha+1}{\alpha}}}{(\Delta-1)(M_1^{\alpha}(\mathcal{L}(G)))^{\frac{1}{\alpha}}},$$

with equality if and only if $\mathcal{L}(G)$ is $(2\Delta - 2)$ -regular.

Since $M_1^{\alpha}(\mathcal{L}(G)) \leq (\frac{\Delta-1}{\Delta})^{\alpha} \chi_{\alpha}(G)$ for $\alpha > 0$, with equality if and only if G is Δ -regular [24]. Thus $SDD(\mathcal{L}(G)) \geq \frac{\Delta \delta^2 \chi_{\alpha+1}(G)}{(\Delta-1)^2 (\chi_{\alpha}(G))^{\frac{1}{\alpha}}}$, with equality if and only if G is regular.

Theorem 4.6 Let G be a graph with $|E_G| = m$ and maximum degree Δ . If $G \ncong K_2$, then

$$SDD(\mathcal{L}(G)) > \frac{\Delta^3 (M_1(G) - 2m)^2}{(\Delta - 1)^3 \chi_3(G)}$$

Proof. Since $\frac{d_u+d_v-2}{d_u+d_v} \leq \frac{\Delta-1}{\Delta}$ for any vertices $u, v \in V_G$ with maximum Δ , the equality holds if and only if $d_u = d_v = \Delta$. Then $d_{\mathcal{L}(G)}(uv) = d_u + d_v - 2 \leq (d_u + d_v) \frac{\Delta-1}{\Delta}$.

Since $F(\mathcal{L}(G)) = \sum_{uv \in V_{\mathcal{L}(G)}} (d_{\mathcal{L}(G)}(uv))^3 \leq (\frac{\Delta-1}{\Delta})^3 \sum_{uv \in E_G} (d_u + d_v)^3 = \frac{\Delta^3 (M_1(G) - 2m)^2}{(\Delta-1)^3 \chi_3(G)}$, with equality if and only if G is Δ -regular.

Then By Lemma 2.4 and Theorem 3.7, we have

$$SDD(\mathcal{L}(G)) \ge \frac{4(m_{\mathcal{L}(G)})^2}{F(\mathcal{L}(G))} = \frac{(M_1(G) - 2m)^2}{F(\mathcal{L}(G))} \ge \frac{\Delta^3(M_1(G) - 2m)^2}{(\Delta - 1)^3\chi_3(G)},$$

with first equality if and only if $\mathcal{L}(G) \cong P_2$, i.e., $G \cong P_3$, second equality if and only if G is regular. This is a contradiction. $SDD(\mathcal{L}(G)) > \frac{\Delta^3(M_1(G)-2m)^2}{(\Delta-1)^3\chi_3(G)}$.

In the following, we consider the line graphs with minimal SDD index. Combine the Fact 2.5 and Theorem 3.8, we have the following result. The proof of 4.7 is similar to Theorem 3.8, we omit it.

Theorem 4.7 Let G be a graph with $G \ncong K_2$. Then

(i) $SDD(\mathcal{L}(G)) \geq 2$, with equality if and only if $G \cong P_3$;

(ii) There is no such graphs with $2 < SDD(\mathcal{L}(G)) \leq 4$;

(*iii*) If $4 < SDD(\mathcal{L}(G)) \leq 6$, then $G \in \{P_4, C_3, S_4\}$ with $SDD(\mathcal{L}(P_4)) = 5$ and $SDD(\mathcal{L}(C_3)) = SDD(\mathcal{L}(S_4)) = 6$;

(*iv*) If $6 < SDD(\mathcal{L}(G)) \le 8$, then $G \in \{P_5, C_4\}$ with $SDD(\mathcal{L}(P_5)) = 7$ and $SDD(\mathcal{L}(C_4)) = 8$.

The inverse problem for the SDD index of line graphs $\mathcal{L}(G)$ is also interesting, thus we propose the following problem.

Problem 4.1 Solve the inverse problem for the SDD index of line graphs $\mathcal{L}(G)$.

Theorem 4.8 Let $G \in \mathcal{G}_n$ with maximum degree Δ and minimum degree δ , and $G \ncong K_2$. Then

$$\frac{SDD(\mathcal{L}(G))}{SDD(G)} \le \frac{\Delta^2 - \delta}{4\delta} \left(\frac{4(\Delta - 1)^2 + (\max\{2\delta - 2, 1\})^2}{(\Delta - 1) \cdot \max\{2\delta - 2, 1\}} \right)$$

with equality if and only if G is regular.

Proof. We know that $G \ncong K_2, \Delta_{\mathcal{L}(G)} \leq 2\Delta - 2, \delta_{\mathcal{L}(G)} \geq \max\{2\delta - 2, 1\}.$

Since $M_1(G) \leq \frac{2m\Delta^2}{\delta}$, with equality if and only if G is regular [24]. By Theorem 3.4 and Lemma 2.1, we have

$$\begin{aligned} SDD(\mathcal{L}(G)) &\leq m_{\mathcal{L}(G)} \left(\frac{\Delta_{\mathcal{L}(G)}}{\delta_{\mathcal{L}(G)}} + \frac{\delta_{\mathcal{L}(G)}}{\Delta_{\mathcal{L}(G)}} \right) \\ &= \frac{1}{2} (M_1(G) - 2m) \left(\frac{\Delta_{\mathcal{L}(G)}}{\delta_{\mathcal{L}(G)}} + \frac{\delta_{\mathcal{L}(G)}}{\Delta_{\mathcal{L}(G)}} \right) \\ &\leq \frac{1}{2} (M_1(G) - 2m) \left(\frac{2\Delta - 2}{\max\{2\delta - 2, 1\}} + \frac{\max\{2\delta - 2, 1\}}{2\Delta - 2} \right) \\ &\leq \left(\frac{m\Delta^2}{\delta} - m \right) \left(\frac{2\Delta - 2}{\max\{2\delta - 2, 1\}} + \frac{\max\{2\delta - 2, 1\}}{2\Delta - 2} \right) \\ &= 2m \left(\frac{\Delta^2 - \delta}{2\delta} \right) \left(\frac{2\Delta - 2}{\max\{2\delta - 2, 1\}} + \frac{\max\{2\delta - 2, 1\}}{2\Delta - 2} \right) \\ &\leq SDD(G) \cdot \left(\frac{\Delta^2 - \delta}{2\delta} \right) \left(\frac{2\Delta - 2}{\max\{2\delta - 2, 1\}} + \frac{\max\{2\delta - 2, 1\}}{2\Delta - 2} \right), \end{aligned}$$

with equality if and only if G is regular.

5 Conclusions

In a recent paper [11], Furtula et al. determined the quality of SDD index exceeds that of some more popular VDB indices, in particular that of the GA index. They shown a close connection between the SDD index and the earlier well-established GA index. Thus it is meaningful and important to consider the chemical and mathematical properties of the SDD index.

Liu et al. [21] determined the minimum and second minimum SDD index of tricyclic graphs. By the way, using a similar way of [8, 22], we can also determine minimum and second minimum SDD index of tetracyclic (chemical) graphs.

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