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A functionally-fitted block hybrid Falkner method for Kepler equations and related problems

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Abstract

For the approximate solution of the Kepler equations and some related problems, a fourth-order convergent functionally-fitted block hybrid Falkner method which is based on the concepts of interpolation and collocation of the fitting function given as a linear combination of $\{1, \sin(\omega x), \cos(\omega x), \sinh(\omega x), \cosh(\omega x)\}$ is presented. The proposed method uses variable coefficients that are based on the product of the dominant frequency and the integration step length. This hybrid formula uses a block-wise implementation strategy to get over the difficulties of the predictor–corrector mode. In addition to being zero stable, the proposed method is applied to the Lambert–Watson linear stability test, which allows obtaining its stability region. Six numerical examples are provided to establish the performance of the proposed method.

Keywords Block hybrid method · Convergent · Falkner formulas · Kepler problem · Non-linear

Mathematics Subject Classification 65L05 · 65L06 · 65L12

1 Introduction

In science and engineering, problems involving non-linear equations to simulate real-life processes have a long history. Non-linear equations are used to describe the majority of phenomena in our environment. As a result, research on nonlinear oscillatory problems in physics, engineering and other physical sciences is of great importance. Non-linear oscillatory problems are important tools in physical sciences and other engineering disciplines, and non-linear differential equations with oscillatory solutions are linked to a variety of practical problems, including the Kepler's problem.

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In classical mechanics, the Kepler problem is a special case of the two-body problem, in which two bodies interact according to a certain law that provides closed orbits for every possible set of initial conditions. The magnitude, structure, and direction of the orbit can be deduced using the specified orbital elements. Depending on the kind of orbit, the elliptic, parabolic, hyperbolic, extended elliptic, extended hyperbolic, Gaussian, and universal Kepler problems are distinct. With the exception of the parabolic case, these equations have no analytical solutions and must be solved numerically (Fukushima 2003, 1996a, b). The structure of the Kepler problem and its significance can be found in the classical textbook by Butcher (2008). The Kepler problem, according to Stickler (2016), is used to introduce basic integrators. This problem is an excellent choice for evaluating numerical integration algorithms, and therefore the purpose of this research.

Our goal is to develop a functionally-fitted block-hybrid Falkner method and to evaluate its performance considering Kepler-type equations. Functionally-fitted methods have been widely employed to solve IVPs. In particular, when the solution presents an oscillatory behavior, the trigonometrically-fitted methods have shown their higher efficiency compared to non-fitted ones. Some methods of this type have appeared in Simos and Vigo-Aguiar (2001), Franco (2004), Wang (2006), and Fang and Wu (2008), or Abdulganiy et al. (2018) among others. The underlying idea is to consider approximate solutions expressed in terms of polynomials and trigonometric functions. The novelty of our approach is to also incorporate hyperbolic functions in the approximate solution, as described in the following sections.

In what follows, the numerical solution of second-order differential equations of the form

$$y''(x) = f(x, y(x)), \tag{1}$$

specifically when the system exhibits oscillatory or periodic behavior, and the frequency,ω,is approximately known in advance is investigated under the following initial conditions

$$y(x_0) = y_0, \quad y'(x_0) = y'_0, \quad x \in [x_0, x_N] \subset \mathbb{R}, \tag{2}$$

where the existence and uniqueness of the solution is guaranteed by assuming well-known appropriate conditions on f. Besides the Kepler problem, classical mechanics, circuit simulation, molecular dynamics, electronics, and astrophysics are only a few examples of Eq. (1) in applied sciences and engineering.

The effectiveness of numerical approaches for solving oscillatory problems can be improved, according to Kosti and Anastassi (2015), by exploiting crucial features that are specifically stated for oscillatory equations. Recent studies have proved the importance of two of these features, notably phase-lag and amplification error. If these quantities are nullified, for example by solving for part of the free coefficients, a technique with variable coefficients based on the product of the dominant frequency and the integration step-length is achieved (which is the approach adopted in this study).

In this paper, we propose a fourth-order convergent functionally-fitted block Falkner Method based on the concept of interpolation and collocation of the fitting function given by a linear combination of $\chi = \{1, \sin(\omega x), \cos(\omega x), \sinh(\omega x), \cosh(\omega x)\}$. Ramos et al. (2021) and Abdulganiy et al. (2021a, b) introduced this functional basis, which is a linear combination of monomial, trigonometric, and hyperbolic terms. It is motivated by its ease of analysis and the expectation that it will provide better approximations for second-order initial-value problems with periodic or oscillatory solutions.

According to Nguyen et al. (2006), functionally-fitted methods are generalizations of collocation techniques that integrate an IVP exactly if the solution is a linear combination of a set of specified basis functions. Classic algebraic collocation methods are recovered when these

basis functions are chosen as power functions. Non-polynomial functions or a combination of polynomial and non-polynomial functions are commonly employed as the fitted function in functionally-fitted methods.
The Falkner method, whose explicit and implicit forms are due to Falkner (1936) and Collatz (1966), is one of the classical numerical integrators considered to directly solve Eq. (1) subject to the conditions (2). Ramos et al. (2016, 2017), Ramos and Lorenzo (2010), Ramos and Rufai (2018) and Ramos (2019) presented some variations of the traditional Falkner methods whose basis functions are either polynomials or rational functions. In Ramos et al. (2017), a unified approach of k-step Falkner methods was given. In Ramos et al. (2016) it was presented a rational Falkner-type method for solving the kind of problems in (1) without assuming an oscillatory solution. A modification of k-step block Falkner methods including third derivatives was presented in Ramos and Rufai (2018). Finally, Ramos and Lorenzo (2010) and Ramos (2019) considered different explicit and implicit formulations of Falkner methods for solving the special second-order IVP. On the other hand, the adapted Falkner methods, which can be found in the works of Ehigie and Okunuga (2018), and Ramos et al. (2021), take advantage of the special periodic feature of the IVP solution. The use of adapted methods began with Gautschi (1961) elegant work. Few of the many extensions of such adapted methods that have been investigated are reported in Jator et al. (2013), Ramos and Vigo-Aguiar (2010) and Vigo-Aguiar and Ramos (2014). Only a few works in the literature, such as Ramos et al. (2021) and Abdulganiy et al. (2021a, b), have considered the use of a basis function other than a linear combination of polynomials and trigonometric terms, which is why the current study was motivated.

The need for more order of convergence in a numerical method while retaining excellent stability encouraged the use of the hybrid formulas. Hybrid formulas were first proposed to overcome the Dahlquist (1956) barrier in such a way that the traditional linear multistep formulas were improved by considering hybrid points between some grid points during the formulation process (Gear 1965). Despite the fact that these formulas retain both higher order and superb stability properties, hybrid formulas are marred by the need to develop predictors for the computation of the corrector at hybrid points, making the methodology more tiresome and inefficient (Lambert 1973). In this paper, a block-wise implementation approach is used instead of the traditional stepwise execution to avoid the predictor–corrector mode difficulty.

The following is how this paper is structured: Section 2 presents the mathematical formulation of the proposed BHFM. Section 3 investigates the BHFM's important properties. Section 4 includes some numerical experiments to show the good performance of the method, and Sect. 5 concludes with some closing remarks.

2 Mathematical formulation of the BHFM

For the mathematical formulation of the proposed method, we first consider $y(x)$ as a scalar function, although, as can be seen in the numerical section, the method can be applied in a component-wise formulation for solving differential systems. After that, a Continuous Block Hybrid Falkner Method (CBHFM) on the interval $\left[x_n, x_{n+1}\right]$ that produces two discrete formulas (one principal and one secondary) is constructed.One formula to approximate the first derivative is generated through the evaluation of CBHFM at x_{n+1}. The three formulas are then combined as a Block numerical integrator to form the proposed Block Hybrid Falkner Method (BHFM).

The CBHFM has the general form

$$\phi(x,u) = y_n + \alpha h y'_n + h^2 \left(\tau_0(x,u) f_n + \tau_\varrho(x,u) f_{n+\varrho} + \tau_1(x,u) f_{n+1}\right), \tag{3}$$

where $u = \omega h$ is explicitly included into $\phi(x,u)$ to highlight the dependence on this parameter, $\varrho = \frac{1}{2}$ is the selected hybrid point, α, τ_j are coefficients to be determined uniquely, that depend on the parameter frequency ω, $x_{n+\varrho} = x_n + \frac{h}{2}$ is an intermediate point on the interval $\left[x_n, x_{n+1}\right]$ and h is the step size. As usual, $y_{n+j}, y'_{n+j}, f_{n+j}$ are respectively the numerical approximations of the exact values $y\left(x_{n+j}\right), y'(x_{n+j})$, and $f(x_{n+j}, y(x_{n+j}))$.
We consider that the true solution $y(x)$ is locally approximated on the block interval $\left[x_n, x_{n+1}\right]$ by a fitting function $\phi(x)$ of the form

$$\phi(x) = \eta_0 + \eta_1 \sin(\omega x) + \eta_2 \cos(\omega x) + \eta_3 \sinh(\omega x) + \eta_4 \cosh(\omega x), \tag{4}$$

where the coefficients η_i will be obtained demanding that the following system of equations is satisfied

$$\begin{cases} \phi(x_n) = y_n \\ \phi'(x_n) = y'_n \\ \phi''(x_{n+j}) = f_{n+j}, \quad j = 0, \varrho, 1. \end{cases} \tag{5}$$

The construction of the CBHFM follows from the following proposition.

Proposition 1 *Let $\phi(x)$ be the function given in Eq. (4) which satisfies the system in Eq. (5). The continuous approximation that will be used to obtain the BHFM is given by*

$$\phi(x) = \Upsilon^T \left(\sigma^{-1}\right)^T \left[\left(\Delta^{-1}\right)^T \chi(x)\right], \tag{6}$$

where σ and Δ are 5×5 invertible lower and upper triangular matrices given by

$$\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \sigma_{2,1} & 1 & 0 & 0 & 0 \\ \sigma_{3,1} & \sigma_{3,2} & 1 & 0 & 0 \\ \sigma_{4,1} & \sigma_{4,2} & \sigma_{4,3} & 1 & 0 \\ \sigma_{5,1} & \sigma_{5,2} & \sigma_{5,3} & \sigma_{5,4} & 1 \end{pmatrix}, \qquad \Delta = \begin{pmatrix} \Delta_{1,1} & \Delta_{1,2} & \Delta_{1,3} & \Delta_{1,4} & \Delta_{1,5} \\ 0 & \Delta_{2,2} & \Delta_{2,3} & \Delta_{2,4} & \Delta_{2,5} \\ 0 & 0 & \Delta_{3,3} & \Delta_{3,4} & \Delta_{3,5} \\ 0 & 0 & 0 & \Delta_{4,4} & \Delta_{4,5} \\ 0 & 0 & 0 & 0 & \Delta_{5,5} \end{pmatrix},$$

χ and Υ are vectors defined by $\chi(x) = (\chi_0(x), \chi_1(x), \chi_2(x), \chi_3(x), \chi_4(x))^T$, with $\left\{\chi_j(x)\right\}_{j=0}^{4} = \{1, \sin(\omega x), \cos(\omega x), \sinh(\omega x), \cosh(\omega x)\}$ and $\Upsilon = \left(y_n, y'_n, f_n, f_{n+\varrho}, f_{n+1}\right)$, respectively (the superscript T denotes the transpose).

Proof The proof can be easily obtained following Abdulganiy et al. (2021a) with little modifications to the symbols. □

Remark 1 • The specific form of matrices σ and Δ is provided in the "Appendix".

- We emphasize that Eq. (6) is of the form presented in Eq. (3) whose coefficients after substituting for $x_{n+1} = x_n + h$ and $x_{n+\varrho} = x_n + \varrho h$ and some simplifications are stated as follows

$$\alpha = \frac{\begin{pmatrix} -\sin(u/2)\sinh\left(\frac{u(h-x+x_n)}{h}\right) + \sin(u)\sinh\left(\frac{u(h-2x+2x_n)}{2h}\right) + \sin(u/2)\sinh\left(\frac{u(x-x_n)}{h}\right) \\ -\sinh(u)\sin\left(\frac{u(h-2x+2x_n)}{2h}\right) + \sinh(u/2)\sin\left(\frac{u(h-x+x_n)}{h}\right) - \sinh(u/2)\sin\left(\frac{u(x-x_n)}{h}\right) \\ +2\sin(u/2)\sinh(u) - 2\sinh(u/2)\sin(u) \end{pmatrix}}{u\begin{pmatrix} \sin(u/2)\cosh(u) + \cos(u/2)\sinh(u) - \sinh(u/2)\cos(u) \\ -\cosh(u/2)\sin(u) + \sin(u/2) - \sinh(u/2) \end{pmatrix}},$$

$$\tau_0(x,u) = \frac{\begin{pmatrix} -\cos(u)\sinh\left(\frac{u(h+2x_n-2x)}{2h}\right) - \cosh(u)\sin\left(\frac{u(h+2x_n-2x)}{2h}\right) + \cos(u/2)\sinh\left(\frac{u(x_n+h-x)}{h}\right) \\ +\cosh(u/2)\sin\left(\frac{u(x_n+h-x)}{h}\right) + \sin(u/2)\cosh\left(\frac{u(x-x_n)}{h}\right) + \sinh(u/2)\cos\left(\frac{u(x-x_n)}{h}\right) \\ -\cosh(u/2)\sin(u) + (\cos(u)-1)\sinh(u/2) - \cos(u/2)\sinh(u) + \sin(u/2)(\cosh(u)-1) \end{pmatrix}}{u\begin{pmatrix} \sin(u/2)\cosh(u) + \cos(u/2)\sinh(u) - \sinh(u/2)\cos(u) \\ -\cosh(u/2)\sin(u) + \sin(u/2) - \sinh(u/2) \end{pmatrix}},$$

$$\tau_\varrho(x,u) \quad \frac{\begin{pmatrix} \cos(u)\sinh\left(\frac{u(x-x_n)}{h}\right) + \sin(u)\cosh\left(\frac{u(x-x_n)}{h}\right) + \cosh(u)\sin\left(\frac{u(x-x_n)}{h}\right) + \sinh(u)\cos\left(\frac{u(x-x_n)}{h}\right) \\ -2\sin(u) - 2\sinh(u) + \sinh\left(\frac{u(x_n+h-x)}{h}\right) + \sin\left(\frac{u(x_n+h-x)}{h}\right) \end{pmatrix}}{u^2\begin{pmatrix} \sinh(u/2)\cos(u) + \cosh(u/2)\sin(u) - \sin(u/2)\cosh(u) \\ -\cos(u/2)\sinh(u) + \sinh(u/2) - \sin(u/2) \end{pmatrix}},$$

$$\tau_1(x,u) = \frac{\begin{pmatrix} -\cos(u/2)\sinh\left(\frac{u(x-x_n)}{h}\right) - \sin(u/2)\cosh\left(\frac{u(x-x_n)}{h}\right) - \cosh(u/2)\sin\left(\frac{u(x-x_n)}{h}\right) \\ -\sinh(u/2)\cos\left(\frac{u(x-x_n)}{h}\right) 2\sinh(u/2) + 2\sin(u/2) - \sinh\left(\frac{u(h+2x_n-2x)}{2h}\right) - \sin\left(\frac{u(h+2x_n-2x)}{2h}\right) \end{pmatrix}}{u^2\begin{pmatrix} \sinh(u/2)\cos(u) + \cosh(u/2)\sin(u) - \sin(u/2)\cosh(u) \\ -\cos(u/2)\sinh(u) + \sinh(u/2) - \sin(u/2) \end{pmatrix}},$$

2.1 Specification of the BHFM

We evaluate the continuous formula in Eq. (3) at $x = x_{n+1}$ and $x = x_{n+\varrho}$, while its first derivative is evaluated at $x = x_{n+1}$ respectively to obtain two primary formulas (one for method and one for derivative) and one secondary formula that forms the functionally fitted block method BHFM as follows

$$\begin{aligned} y_{n+\varrho} &= y_n + \alpha_1^1(u)hy_n' + h^2(\gamma_0^1(u)\,f_n + \gamma_\varrho^1(u)\,f_{n+\varrho} + \gamma_1^1(u)\,f_{n+1}) \\ y_{n+1} &= y_n + \alpha_1(u)hy_n' + h^2(\gamma_0(u)\,f_n + \gamma_\varrho(u)\,f_{n+\varrho} + \gamma_1(u)\,f_{n+1}) \\ hy_{n+1}' &= hy_n' + h^2(\bar{\gamma}_0(u)\,f_n + \bar{\gamma}_\varrho(u)\,f_{n+\varrho} + \bar{\gamma}_1(u)\,f_{n+1}) \end{aligned} \tag{7}$$

where the coefficients of Eq. (7) are respectively obtained as follows

$$\alpha_1^1(u) = 2\frac{\sin(u/2)\sinh(u/2)}{u(\sinh(u/2)\cos(u/2) + \sin(u/2)\cosh(u/2))}$$

$$\gamma_0^1(u) = \frac{\begin{pmatrix} (-\cos(u) - 2\cos(u/2) + 1)\sinh(u/2) + (-\cosh(u) - 2\cosh(u/2) + 1)\sin(u/2) \\ +\cosh(u/2)\sin(u) + \cos(u/2)\sinh(u) \end{pmatrix}}{u^2\begin{pmatrix} (\cos(u)+1)\sinh(u/2) + (-\cosh(u)-1)\sin(u/2) \\ +\cosh(u/2)\sin(u) - \cos(u/2)\sinh(u) \end{pmatrix}}$$

$$\gamma_\varrho^1(u) = \frac{\begin{pmatrix} (\cos(u)+1)\sinh(u/2) + \sin(u/2)(\cosh(u)+1) + \cosh(u/2)\sin(u) \\ +\cos(u/2)\sinh(u) - 2\sin(u) - 2\sinh(u) \end{pmatrix}}{u^2\begin{pmatrix} (\cos(u)+1)\sinh(u/2) + (-\cosh(u)-1)\sin(u/2) \\ +\cosh(u/2)\sin(u) - \cos(u/2)\sinh(u) \end{pmatrix}}$$

$$\gamma_1^1(u) = \frac{(-2\cos(u/2)+2)\sinh(u/2) - 2\sin(u/2)(\cosh(u/2)-1)}{u^2((\cos(u)+1)\sinh(u/2) + (-\cosh(u)-1)\sin(u/2) + \cosh(u/2)\sin(u) - \cos(u/2)\sinh(u))}$$

$$\alpha_1(u) = \frac{4\sin(u/2)\sinh(u/2)}{u(\sinh(u/2)\cos(u/2) + \sin(u/2)\cosh(u/2))}$$

$$\gamma_0(u) = \frac{(-3\cos(u)+1)\sinh(u/2) + (-3\cosh(u)+1)\sin(u/2) + \cosh(u/2)\sin(u) + \cos(u/2)\sinh(u)}{u^2((\cos(u)+1)\sinh(u/2) + (-\cosh(u)-1)\sin(u/2) + \cosh(u/2)\sin(u) - \cos(u/2)\sinh(u))}$$

$$\gamma_\varrho(u) = \frac{(\cos(u)-1)\sinh(u) + \sin(u)(\cosh(u)-1)}{u^2(-\cosh(u/2) + \cos(u/2))(\sinh(u/2)\cos(u/2) + \sin(u/2)\cosh(u/2))}$$

$$\gamma_1(u) = \frac{(-\cos(u)+3)\sinh(u/2) + (-\cosh(u)+3)\sin(u/2) - \cosh(u/2)\sin(u) - \cos(u/2)\sinh(u)}{u^2((\cos(u)+1)\sinh(u/2) + (-\cosh(u)-1)\sin(u/2) + \cosh(u/2)\sin(u) - \cos(u/2)\sinh(u))}$$

$$\bar{\gamma}_0(u) = \bar{\gamma}_1(u) = \frac{\begin{pmatrix} -\cosh(u/2)\cos(u) + \sinh(u/2)\sin(u) - \cos(u/2)\cosh(u) \\ -\sin(u/2)\sinh(u) + \cosh(u/2) + \cos(u/2) \end{pmatrix}}{u\begin{pmatrix} \cosh(u/2)\sin(u) + \sinh(u/2)\cos(u) - \cos(u/2)\sinh(u) \\ -\sin(u/2)\cosh(u) + \sinh(u/2) - \sin(u/2) \end{pmatrix}}$$

$$\bar{\gamma}_{\varrho}(u) = \frac{-2\sinh(u/2)\cos(u/2) + 2\sin(u/2)\cosh(u/2)}{u(\cosh(u/2) - \cos(u/2))}. \tag{8}$$

The coefficients of the BHFM may be subject to significant cancellations for small values of u. The expansion of the coefficients via Taylor series is preferred in that case. The series expansion of each coefficient to the twelfth order of approximation is as follows

$$\begin{aligned}
\alpha_1^1(u) &\simeq \frac{1}{2} + \frac{u^4}{1440} + \frac{u^8}{725760} + \frac{2879\,u^{12}}{1046139494400} \\
\gamma_0^1(u) &\simeq \frac{7}{96} + \frac{1003\,u^4}{7741440} + \frac{42139\,u^8}{163499212800} + \frac{105898279\,u^{12}}{205679393714995200} \\
\gamma_{\varrho}^1(u) &\simeq \frac{1}{16} + \frac{101\,u^4}{552960} + \frac{8023\,u^8}{22295347200} + \frac{74038871\,u^{12}}{102839696857497600} \\
\gamma_1^1(u) &\simeq -\frac{1}{96} - \frac{149\,u^4}{7741440} - \frac{2713\,u^8}{70071091200} - \frac{15900377\,u^{12}}{205679393714995200}
\end{aligned}$$

$$\begin{aligned}
\alpha_1(u) &\simeq 1 + \frac{u^4}{720} + \frac{u^8}{362880} + \frac{2879\,u^{12}}{523069747200} \\
\gamma_0(u) &\simeq \frac{1}{6} + \frac{31\,u^4}{120960} + \frac{659\,u^8}{1277337600} + \frac{103409\,u^{12}}{100429391462400} \\
\gamma_{\varrho}(u) &\simeq \frac{1}{3} + \frac{37\,u^4}{120960} + \frac{11191\,u^8}{15328051200} + \frac{2311013\,u^{12}}{1606870263398400} \\
\gamma_1(u) &\simeq -\frac{u^4}{24192} - \frac{59\,u^8}{766402560} - \frac{239\,u^{12}}{1545067560960} \\
\bar{\gamma}_0(u) = \bar{\gamma}_1(u) &\simeq \frac{1}{6} - \frac{u^4}{60480} + \frac{u^8}{383201280} - \frac{u^{12}}{2391175987200} \\
\bar{\gamma}_{\varrho}(u) &\simeq \frac{2}{3} - \frac{19\,u^4}{60480} + \frac{13\,u^8}{218972160} - \frac{2593\,u^{12}}{267811710566400}.
\end{aligned} \tag{9}$$

Remark 2 It is interesting to note that taking limit when $u \to 0$ in the power series expansion of the coefficients (or in the coefficients themselves), a traditional fourth-order block hybrid Falkner method is recovered (Nguyen et al. 2006).

3 Basic properties of the BHFM

The basic properties of the proposed BHFM are investigated in this section.

3.1 Local truncation error, order and consistency of the BHFM

The local truncation error for each of the formulas in Eq. (7) can be calculated in the traditional way by shifting all the terms to the left-hand side, substituting the approximate values for the true ones, and then expanding the resulting formula by Taylor series in powers of h. As a result, the local truncation errors listed below are obtained

$$\begin{aligned}
\mathcal{L}\left[y\left(x_{n+\varrho}\right);h\right] &= \tfrac{h^6}{1440}\left(y^{(6)}\left(x_n\right) + \omega^4 y''\left(x_n\right)\right) + O\left(h^7\right) \\
\mathcal{L}\left[y\left(x_{n+1}\right);h\right] &= \tfrac{h^6}{720}\left(y^{(6)}\left(x_n\right) + \omega^4 y''\left(x_n\right)\right) + O\left(h^7\right) \\
\mathcal{L}\left[hy'\left(x_{n+1}\right);h\right] &= -\tfrac{h^6}{2880}\left(y^{(6)}\left(x_n\right) + \omega^4 y''\left(x_n\right)\right) + O\left(h^7\right),
\end{aligned} \tag{10}$$

which indicates that the proposed BHFM is at least a fourth-order convergent method. The method is also consistent because the order of the BHFM is greater than one.

Proposition 2 *When the solution to the problem in Eq.* (1) *subject to condition* (2) *is a linear combination of the basis functions* $\{1, \sin(\omega x), cos(\omega x), \sinh(\omega x), \cosh(\omega x)\}$*, the local truncation error of the primary formula in the BHFM vanishes.*

Proof Solving the differential equation $y^{(5)}(x) + \omega^4 y'(x) = 0$ yields the following fundamental set of solutions $\{1, \sin(\omega x), cos(\omega x), \sinh(\omega x), \cosh(\omega x)\}$, which contains the basis functions of the BHFM, and the statement follows immediately. □

3.2 Analysis of convergence of the BHFM

The BHFM convergence analysis is carried out according to the following theorem (Abdulganiy et al. 2021a).

Theorem 1 *Let* $\overline{\Upsilon}$ *be a vector that approximates the true solution vector* Υ*, where* $\overline{\Upsilon}$ *is the solution of the system obtained from the BHFM given by the equations in* (7) *on the consecutive block intervals* $[x_0, x_1], [x_1, x_2], \ldots, [x_{N-1}, x_N]$*. If we denote the error vector by* $\Lambda = (\epsilon_1, \ldots, \epsilon_N, h\epsilon'_1, \ldots, h\epsilon'_N)^T$*, where* $\epsilon_j = y\left(x_j\right) - y_j$ *and* $h\epsilon'_j = hy'\left(x_j\right) - hy'_j$*,* $j = 1, 2, \ldots, N$*, assuming the solution in closed form is several times differentiable on the interval* $[x_0, x_N]$*, then, for sufficiently small* h *the BHFM is a fourth-order convergent method, that is,*

$$\|\Lambda\| = \left\|\overline{\Upsilon} - \Upsilon\right\| = \mathrm{O}(h^4).$$

Proof Assume the coefficients obtained from the BHFA are represented by the $(N \times N)$-matrices defined as follows:

$$\Pi_1 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0
\end{pmatrix},$$

$$\Pi_2 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & -\alpha_1^1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & -\alpha_1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -\alpha_1^1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -\alpha_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 1
\end{pmatrix},$$

$$\Theta = h^2 \begin{pmatrix} \gamma_0^1 & \gamma_\varrho^1 & \gamma_1^1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \gamma_0 & \gamma_\varrho & \gamma_1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \bar{\gamma}_0 & \bar{\gamma}_\varrho & \bar{\gamma}_1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \gamma_0^1 & \gamma_\varrho^1 & \gamma_1^1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \gamma_0 & \gamma_\varrho & \gamma_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \bar{\gamma}_0 & \bar{\gamma}_\varrho & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \gamma_0^1 & \gamma_\varrho^1 & \gamma_1^1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \gamma_0 & \gamma_\varrho & \gamma_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \bar{\gamma}_0 & \bar{\gamma}_\varrho & \bar{\gamma}_1 \end{pmatrix},$$

and the N-vector containing the known values given by

$$M = (-y_0 - \alpha_1^1 h y_0' - h^2 \gamma_0^1 f_0, -y_0 - \alpha_0 h y_0' - h^2 \gamma_0 f_0, -h y_0' - h^2 \bar{\gamma}_0 f_0, 0, \ldots, 0)^T.$$

We consider the vectors of exact values

$$\Upsilon = (y(x_1), \ldots, y(x_N), hy'(x_1), \ldots, hy'(x_N)),$$
$$\Phi = (f(x_1, y(x_1), y'(x_1)), \ldots, f(x_N, y(x_N), y'(x_N))),$$

the vectors of approximate values

$$\bar{\Upsilon} = (y_1, \ldots, y_N, hy_1', \ldots, hy_N'),$$
$$\bar{\Phi} = (f_1, \ldots, f_N),$$

and the vector of local truncation errors $L(h) = (L_1, \ldots, L_N)$.

Taking the $(N \times N)$-matrix $\Pi = (\Pi_1 | \Pi_2)$, the exact form of the system formed by the formulas in (7) along the one-step block intervals on $[x_0, x_N]$ is

$$\Pi \Upsilon - \Theta \Phi + M = L(h). \tag{11}$$

On the other hand, the system that gives the approximate values may be written as

$$\Pi \overline{\Upsilon} - \Theta \bar{\Phi} + C = 0. \tag{12}$$

Subtracting Eq. (12) from Eq. (11) to obtain

$$\Pi(\Upsilon - \overline{\Upsilon}) - \Theta(\Phi - \bar{\Phi}) = L(h), \tag{13}$$

and having in mind that $\Lambda = (\epsilon_1, \ldots, \epsilon_N, h\epsilon_1', \ldots, h\epsilon_N')^T$, the above equation becomes

$$\Pi \Lambda - \Theta \left(\Phi - \bar{\Phi} \right) = L(h). \tag{14}$$

Applying the Mean-Value Theorem, we obtain that $\Phi - \bar{\Phi} = J\Lambda$, where J is the $(N \times 2N)$-matrix given as

$$J = \begin{pmatrix} \frac{\partial f}{\partial y}(\zeta_1) & 0 & \cdots & 0 & 0 & \frac{1}{h}\frac{\partial f}{\partial y'}(\zeta_1) & 0 & \cdots & 0 \\ 0 & \frac{\partial f}{\partial y}(\zeta_2) & \cdots & 0 & 0 & 0 & \frac{1}{h}\frac{\partial f}{\partial y'}(\zeta_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\partial f}{\partial y}(\zeta_N) & 0 & 0 & 0 & \cdots & \frac{1}{h}\frac{\partial f}{\partial y'}(\zeta_N) \end{pmatrix},$$

and the partial derivatives are applied at intermediate points $\{\zeta_i\}_{i=1}^N$, which are on each corresponding line joining $\left(x_i, y(x_i), y'(x_i)\right)$ to $\left(x_i, y_i, y_i'\right)$. As a result, the equation in (14) can be written as

$$(\Pi - \Theta J)\Lambda = L(h).$$

Let φ denote the matrix $\varphi = -\Theta J$. Then, we have that

$$(\Pi + \varphi)\Lambda = L(h). \tag{15}$$

For sufficiently small h, the matrix $\Pi + \varphi$ is non-singular. Therefore, if we denote by

$$(\Pi + \varphi)^{-1} = \Omega, \tag{16}$$

and consider the maximum norm, we can obtain after expanding in Taylor series the terms in Ω that $\|\Omega\| = O(h^{-2})$. Finally, we have that

$$\begin{aligned} \|\Lambda\| &= \|\Omega L(h)\| = \|\Omega\| \|L(h)\| \\ &= |O(h^{-2})|\,|O(h^6)| = O(h^4). \end{aligned}$$

Therefore, the BHFM is a fourth-order convergent method. □

3.3 Stability of the BHFM

In numerical analysis, the concept of stability is crucial. In the context of ordinary differential equations, it refers to the extent to which a numerical scheme is appropriate for solving an initial value problem. A method is said to be stable if small changes in the data result in subtle changes in the solution it provides (Ramos and Lorenzo 2010).

A common method for studying stability is to write the proposed method as a one-step recurrence difference system, then apply the necessary definition on the resulting matrices as in the case of zero stability and linear stability.

Consequently, the BHFM specified by the formulas in Eq. (7) may be written as follows

$$(A_1 \otimes I)\Upsilon_{n+1} = (A_0 \otimes I)\Upsilon_n + h^2(B_0 \otimes I)\Xi_n + h^2(B_1 \otimes I)\Xi_{n+1}, \tag{17}$$

where $\Upsilon_{n+1} = (y_{n+\varrho}, y_{n+1}, hy'_{n+1})^T$, $\Upsilon_n = (y_{n-\varrho}, y_n, hy'_n)^T$, $\Xi_{n+1} = (f_{n+\varrho}, f_{n+1}, hf'_{n+1})^T$ and $\Xi_n = (f_{n-\varrho}, f_n, hf'_n)^T$, I is the identity matrix of dimension three, $\otimes$ denotes the Kronecker product of matrices. A_0, A_1, B_0 and B_1 are 3×3 matrices containing the coefficients of the formulas and are given as follows

$$A_0 = \begin{pmatrix} 0 & 1 & \alpha_1^1 \\ 0 & 1 & \alpha_1 \\ 0 & 0 & 1 \end{pmatrix}, A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, B_0 = \begin{pmatrix} 0 & \gamma_0^1 & 0 \\ 0 & \gamma_0 & 0 \\ 0 & \bar{\gamma}_0 & 0 \end{pmatrix}, B_1 = \begin{pmatrix} \gamma_\varrho^1 & \gamma_1^1 & 0 \\ \gamma_\varrho & \gamma_1 & 0 \\ \bar{\gamma}_\varrho & \bar{\gamma}_1 & 0 \end{pmatrix}. \tag{18}$$

Remark 3 We note that the difference system in (17) can be written in the form

$$\Upsilon_{n+1} - \Upsilon_n = h\phi_\Xi(\Upsilon_n, \Upsilon_{n+1}; u, h),$$

where the subscript indicates that the dependence of ϕ on Υ_n, Υ_{n+1} is through the function Ξ. Thus, the numerical solution of the problem in Eq. (1) subject to Eq. (2) according to Abdulganiy et al. (2021b) is the one given by

$$\begin{cases} \Upsilon_{n+1} - \Upsilon_n = h\phi_\Xi(\Upsilon_n, \Upsilon_{n+1}; u, h), \\ \Upsilon_0 = \Upsilon(x_0), \quad n = 1, 2, \ldots, N-1. \end{cases} \tag{19}$$

3.3.1 Zero stability of BHFM

Zero stability is concerned with the stability of the difference system in Eq. (17) in the limit as $h \to 0$. Any numerical method for solving (1) subject to (2) will produce errors that can be interpreted as if we were solving a perturbed problem of the form

$$\begin{cases} \Psi_{n+1} - \Psi_n = h\left(\phi_{\Xi}(\Psi_n, \Psi_{n+1}; u, h) + \delta_n\right), \\ \Psi_0 = \Upsilon(x_0) + \varepsilon_0, \quad n = 0, 1, 2, \ldots, N-1. \end{cases}$$

The zero-stability may be defined according to Lambert (1991) as follows:

Definition 1 Let $\delta = \{\delta_i\}_{i=0}^{N-1}$, $\delta^* = \left\{\delta_i^*\right\}_{i=0}^{N-1}$ be any two perturbations of (19) and let $\{\Psi_i\}_{i=0}^{N}$, $\left\{\Psi_i^*\right\}_{i=0}^{N}$ be the corresponding solutions, respectively. Then if there exist constants K and h_0 such that for all $0 < h < h_0$ it is

$$\|\Psi_i - \Psi_i^*\| \leq K\epsilon, \quad i = 0, 1, \ldots, N,$$

whenever

$$\|\delta_i - \delta_i^*\| \leq \epsilon, \quad i = 0, 1, \ldots, N,$$

we say that the method (19) is zero-stable.

In practice, zero-stability is concerned with the roots of the difference system's first characteristic polynomial when $h \to 0$ (see Lambert 1991). Thus as $h \to 0$, the difference system in Eq. (17) becomes

$$A_1\Upsilon_{n+1} - A_0\Upsilon_n = 0, \tag{20}$$

where A_1 and A_0 are the matrices of dimension four in Eq. (18). The following definition given by Fatunla (1991) helps to establish the zero stability of the BHFM.

Definition 2 A block method is zero-stable if the roots of its first characteristic $\kappa(\xi) = \det(\xi A_1 - A_0)$, have modulus less than or equal to one, and the multiplicity of those with modulus one is less than or equal to the order of the differential equation (see Fatunla 1991).

Proposition 3 *The BHFM is zero-stable.*

Proof We can deduce from the normalized first characteristic polynomial of the BHFM that

$$\xi A_1 - A_0 = \begin{pmatrix} \xi & -1 & -\alpha_1^1 \\ 0 & \xi - 1 & -\alpha_1 \\ 0 & 0 & \xi - 1 \end{pmatrix},$$

so that the characteristic equation is $\kappa(\xi) = \det(\xi A_1 - A_0) = 0$, in this case, $\xi\,(\xi - 1)^2 = 0$, and thus, the BHFM is zero-stable according to Definition 2. □

3.3.2 Linear stability of the BHFA

It is necessary to have a notion of stability other than zero-stability to establish whether a numerical approach will yield satisfactory results with a given value of $h > 0$. The Lambert–Watson linear test equation given by

$$y'' = -\kappa^2 y, \tag{21}$$

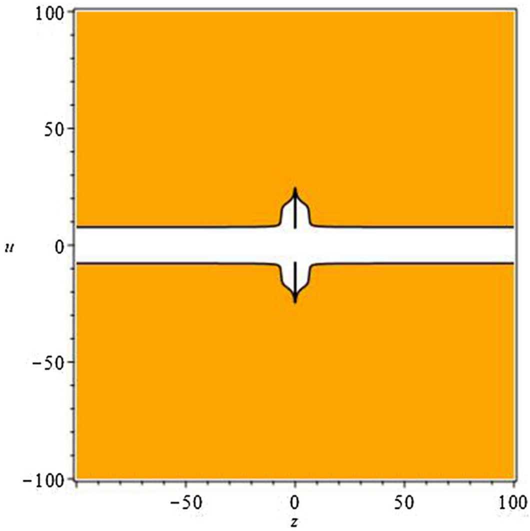

Fig. 1 Stability region for BHFM

is used to examine such stability property called linear stability.

The following recursion formula is obtained when the BHFM specified by the formulas in Eq. (7) is applied to Eq. (21)

$$\Upsilon_{n+1} = M(z)\Upsilon_n, \tag{22}$$

where $z = \kappa h$, and the matrix

$$M(z, u) = (A^{(1)} - z^2 B^{(1)})^{-1}(A^{(0)} + z^2 B^{(0)}), \tag{23}$$

is the stability matrix of the BHFM.
The stability region for the BHFM is plotted in Fig. 1, where the colored region (orange) is the stability region corresponding to the test problem $y'' = -\kappa^2 y$.

4 Implementation and numerical examples

The BHFM is simple to apply, the system in (7) must be solved sequencially until the end point of the integration interval is reached. Note that for a system of m equations the total number of equations in the algebraic system to be solved is $3m$. Obtaining the starting values is an easy task, since we may use the Taylor expansion to get

$$y_\rho^0 = y_0 + \rho h y_0' + \frac{(\rho h)^2}{2} f(x_0, y_0),$$
$$y_1^0 = y_0 + h y_0' + \frac{h^2}{2} f(x_0, y_0).$$

A suitable step size h can be selected based on the initial values, y_0, y_0', at the starting point, while keeping the accuracy needed in mind. For the first time, the equations begin with x_0, and the solutions of $y_{1/2}$, y_1, and y_1' for the points $x_{1/2}$ and x_1 are found by solving a set of implicit equations specified in the method. Since the equations are non-linear, any Newton-type technique can be used to solve them (in this instance, Maple 2016.1 has the command *fsolve* for solving non-linear equations). The procedure is then repeated until all of the desired points have values with x_1 taken as the new initial point while y_1 and y_1' are the new starting values for the next block.

For a fair comparison, a written algorithm in Maple 2016 is developed for every method chosen for comparison and the computations were carried out on the same work-station Laptop with the following specifications.

Processor: Intel(R) Core(TM) i7-3740QM CPU @ 2.70GHz 2.70 GHz
Installed RAM: 16.0 GB
System type: 64-bit operating system, x64-based processor
Edition Windows: 10 Pro
Version: 21H1
OS build: 19043.1706

For the numerical simulations, plots of the logarithm of the maximum errors $\left(\log_{10}(Err)\right)$ of the numerical results of the BHFM versus the logarithm of the number of function evaluations $\left(\log_{10}(NFE)\right)$ are used to measure the computational efficiency on the integration interval, whereas plots of absolute errors in the integration interval obtained using BHFM at a given point are used as measures of accuracy. The fitting frequencies considered in the numerical simulations are taken from the literature-referenced questions. However, Ramos and Vigo-Aguiar (2010) and Vigo-Aguiar and Ramos (2014) presented some frequency selection techniques that can be examined. To solve problems of type (1) subject to condition (2) and examine how the BHFM performs, a number of adapted block techniques whose performances have been published in the literature and have comparable properties to BHFM are used. For comparisons, the following adapted block methods are used

MBFM: the third-order block method developed in Ehigie and Okunuga (2018)
TBNM: the fourth-order block method developed in Jator et al. (2013)
BFFM: the fourth-order block method developed in Ramos et al. (2021)
FFBNM: the third-order block method developed in Abdulganiy et al. (2021a)
BHFM: the fourth-order block method developed in this paper

4.1 Kepler equations

4.1.1 Example 1

In the first example, we consider the following classical Kepler problem in Jator et al. (2013)

$$
\begin{aligned}
y_1''(x) &= -\frac{y_1(x)}{r^3}, \qquad y_1(0) = 1-\epsilon, \quad y_1'(0) = 0, \\
y_2''(x) &= -\frac{y_2(x)}{r^3}, \qquad y_2(0) = 0, \quad y_2'(0) = \sqrt{\frac{1+\epsilon}{1-\epsilon}}, \qquad 0 \le x \le 50\pi,
\end{aligned} \tag{24}
$$

where $r = \sqrt{y_1(x)^2 + y_2(x)^2}$, and ϵ $(0 \le \epsilon \le 1)$ is the eccentricity of the orbit. The exact solution of (24) is

$$
\begin{aligned}
y_1(x) &= \cos(\mu) - \epsilon, \\
y_2(x) &= \sqrt{1-\epsilon^2}\sin(\mu),
\end{aligned} \tag{25}
$$

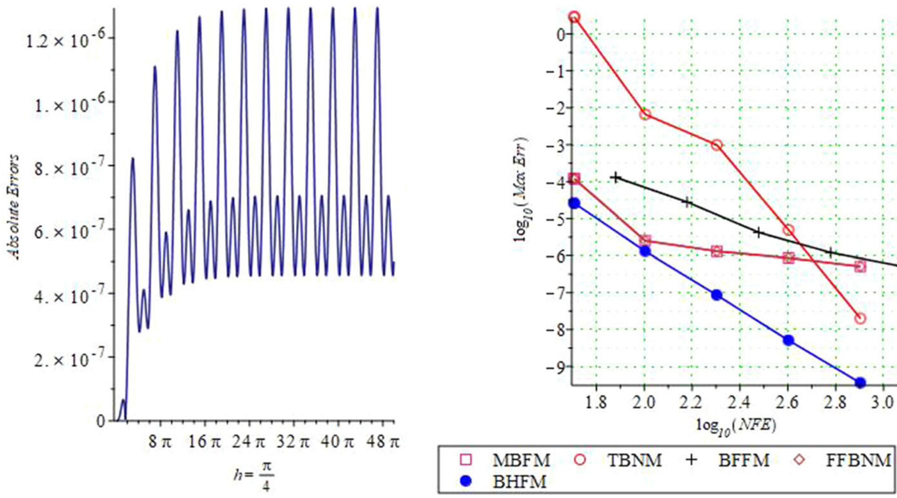

Fig. 2 The graphical representation of the solution to Example 1: accuracy (Left) and efficiency (Right)

where μ is the solution of the Kepler's equation $\mu = x + \epsilon \sin(\mu)$. We integrate the Kepler problem with the fitting parameter selected as $\omega = 1$, eccentricity $\epsilon = 0.05$ and the step sizes taken as $h = \pi/2^i$, $i = 1, 2, 3, 4, 5$. The results presented in Fig. 2 show the competitiveness of the BHFM. Whereas Fig. 2 (Left) presents the accuracy at $h = \pi/4$, Fig. 2 (Right) represents its efficiency.

4.1.2 Example 2

The following well-known mildly stiff Kepler problem in Ramos et al. (2021)

$$
\begin{aligned}
y_1''(x) &= -\frac{y_1}{r^3}, \qquad y_1(0) = 1, \quad y_1'(0) = 0, \\
y_2''(x) &= -\frac{y_2}{r^3}, \qquad y_2(0) = 0, \quad y_2'(0) = 0,
\end{aligned} \tag{26}
$$

where $r = \sqrt{y_1^2 + y_2^2}$, and whose analytic solution is given by $y_1(x) = \cos(x)$, $y_2(x) = \sin(x)$ is considered in the integration interval $0 \le x \le 30$. The fitting frequency ω is chosen as $\omega = 1$ and the step size h is chosen as $h = 1/2^i$, where $i = 1, 2, 3, 4, 5$. Figure 3 (Left) illustrates the numerical accuracy of BHFM, whereas Fig. 3 (Right) provides a visual explanation of its effectiveness, demonstrating its superiority.

4.1.3 Example 3

We consider the following perturbed Kepler problem studied by Wang Wang et al. (2015)

$$
\begin{aligned}
y_1'' &= -\frac{y_1}{r^3} - \frac{2\left(\epsilon + \epsilon^2\right) y_1}{r^5}, \qquad y_1(0) = 1, \quad y_1'(0) = 0, \\
y_2'' &= -\frac{y_2}{r^3} - \frac{2\left(\epsilon + \epsilon^2\right) y_2}{r^5}, \qquad y_2(0) = 0, \quad y_2'(0) = 1 + \epsilon,
\end{aligned} \tag{27}
$$

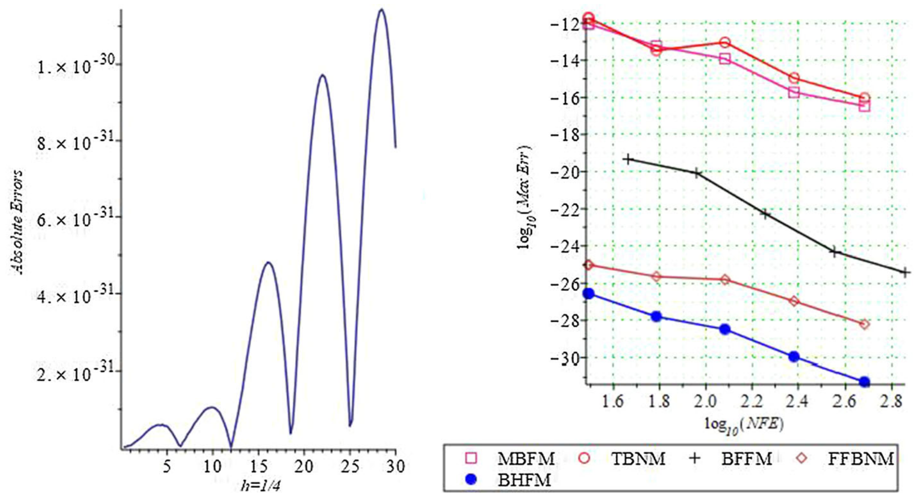

Fig. 3 The graphical representation of the solution to Example 2: accuracy (Left) and efficiency (Right)

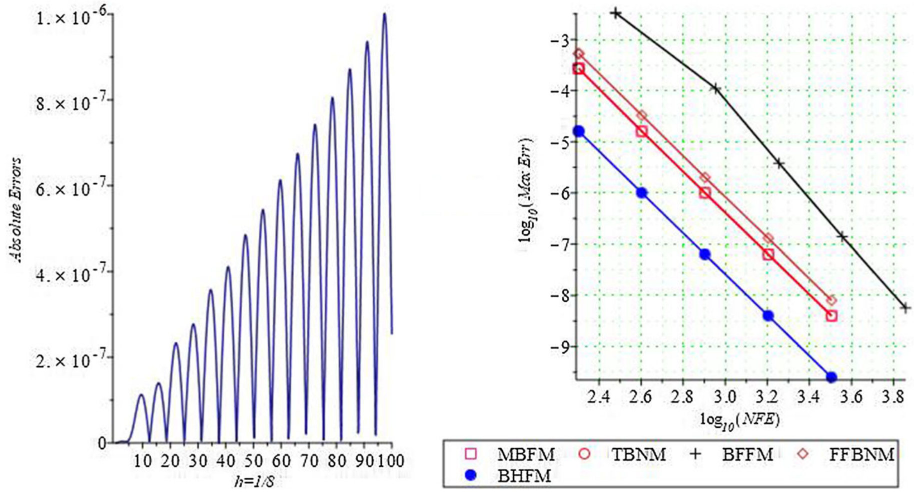

Fig. 4 The graphical representation of the solution to Example 3: accuracy (Left) and efficiency (Right)

with $r = \sqrt{y_1^2 + y_2^2}$, whose exact solution is given by $y_1 = \cos(x + \epsilon x)$, $y_2 = \sin(x + \epsilon x)$. The system in Eq. (27) is solved with parameter $\epsilon = 10^{-3}$ for different step-sizes $h = 1/2^i$, where $i = 2, 3, 4, 5, 6$. The numerical results of the BHFM with fitting frequency chosen as with $\omega = 1.01$ are represented graphically in Fig. 4 showing that BHFM is an accurate and efficiency method for the perturbed Kepler's equation. While Fig. 4 (Left) depicts the accuracy of BHFM at $h = 1/8$, the efficiency curves are represented in Fig. 4 (Right).

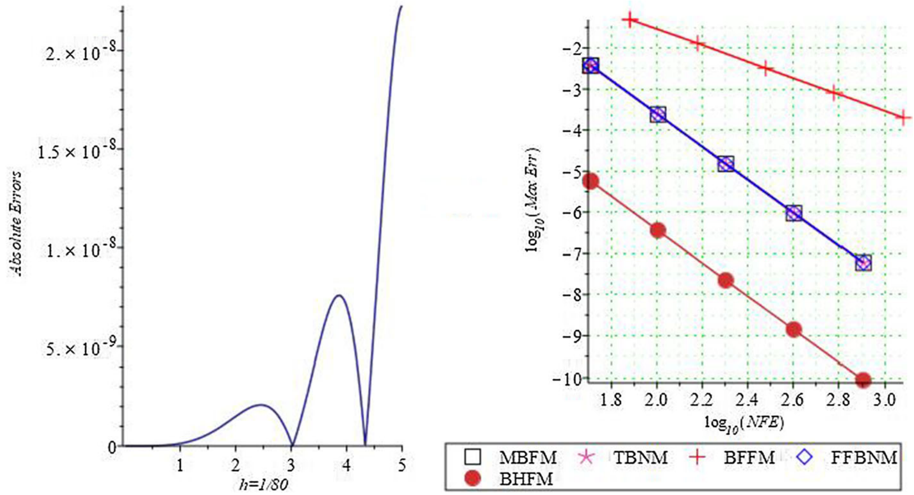

Fig. 5 The graphical representation of the solution to Example 4: accuracy (Left) and efficiency (Right)

4.2 Related problems

The proposed Functionally-Fitted method in this study, in addition to the Kepler equations, can be applied to solve different forms of oscillatory problems, as stated in Sect. 1. To establish further the efficiency of the BHFM, we solve a number of such problems.

4.2.1 Example 4

We consider the following non-linear oscillatory problem in the interval $0 \leq x \leq 5$

$$\begin{aligned} y_1''(x) &= -4x^2 y_1(x) - 2\,\frac{y_2(x)}{r}, \qquad && y_1(0) = 1, \quad y_1'(0) = 0, \\ y_2''(x) &= -4x^2 y_2(x) + 2\,\frac{y_1(x)}{r}, \qquad && y_2(0) = 0, \quad y_2'(0) = 0, \end{aligned} \tag{28}$$

where $r = \sqrt{y_1^2 + y_2^2}$, and whose exact solution is given by $y_1(x) = \cos\left(x^2\right)$, $\quad y_2(x) = \sin\left(x^2\right)$. The accuracy of the BHFM with fitting frequency $\omega = 1$ for different points on the interval of integration in comparison with the exact solution is shown in Fig. 5 (Left), whereas the efficiency curves plotted in Fig. 5 (Right) for different step sizes clearly show that the BHFM outperforms other adapted block methods appeared in the recent literature. It is worth noting that the methods MBFM, TBNM, and FFBNM all behave similarly for this problem, as shown by the efficiency curves in Fig. 5 (Right).

We observe also that for Eq. (28), the absolute errors between the BHFM results and the exact solutions in the integration interval for the fitting frequency $\omega = 5$ behave similarly to that of the fitting frequency $\omega = 1$ with errors less than 10^{-8}. Figure 6 shows an example with $h = 1/80$.

4.2.2 Example 5

We consider the following oscillatory non-linear system

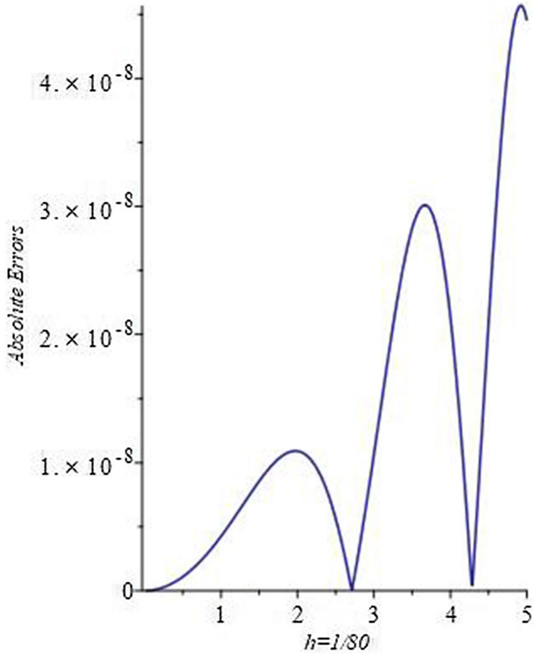

Fig. 6 The accuracy of Example 4 for $\omega = 5$

$$\begin{pmatrix} y_1''(x) \\ y_2''(x) \end{pmatrix} + \begin{pmatrix} 13 & -12 \\ -12 & 13 \end{pmatrix} \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} = \begin{pmatrix} \dfrac{\partial V}{\partial y} \end{pmatrix}, \quad y(0) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \; y'(0) = \begin{pmatrix} -5 \\ 5 \end{pmatrix}, \tag{29}$$

with $V(y) = y_1(x)\, y_2(x)\, (y_1(x) + y_2(x))^3$, whose solution in closed form is given as

$$\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} = \begin{pmatrix} -\sin(5x) - \cos(5x) \\ \sin(5x) + \cos(5x) \end{pmatrix}. \tag{30}$$

Figure 7 (Left) reports the comparison between the theoretical solution and the numerical approximations provided by the proposed BHFA with step size $h = 1/16$. The efficiency of the of the BHFM and other methods it compares is reported in Fig. 8 (Right) showing the good performance of the BHFM.

4.2.3 Example 6

As our last experiment, the following non-linear perturbed system

$$\begin{aligned} y_1''(x) &= \epsilon\varphi_1(x) - 25y_1(x) - \epsilon\left((y_1(x))^2 + (y_2(x))^2\right) & y_1(0) = 1, \;\; y_1'(0) = 0, \\ y_2''(x) &= \epsilon\varphi_2(x) - 25y_2(x) - \epsilon\left((y_1(x))^2 + (y_2(x))^2\right) & y_2(0) = \epsilon, \;\; y_2'(0) = 5, \end{aligned} \tag{31}$$

where

$$\begin{aligned} \varphi_1(x) &= 1 + \epsilon^2 + 2\epsilon \sin\left(5x + x^2\right) + 2\cos\left(x^2\right) + \left(25 - 4x^2\right)\sin\left(x^2\right), \\ \varphi_2(x) &= 1 + \epsilon^2 + 2\epsilon \sin\left(5x + x^2\right) - 2\sin\left(x^2\right) + \left(25 - 4x^2\right)\cos\left(x^2\right), \end{aligned}$$

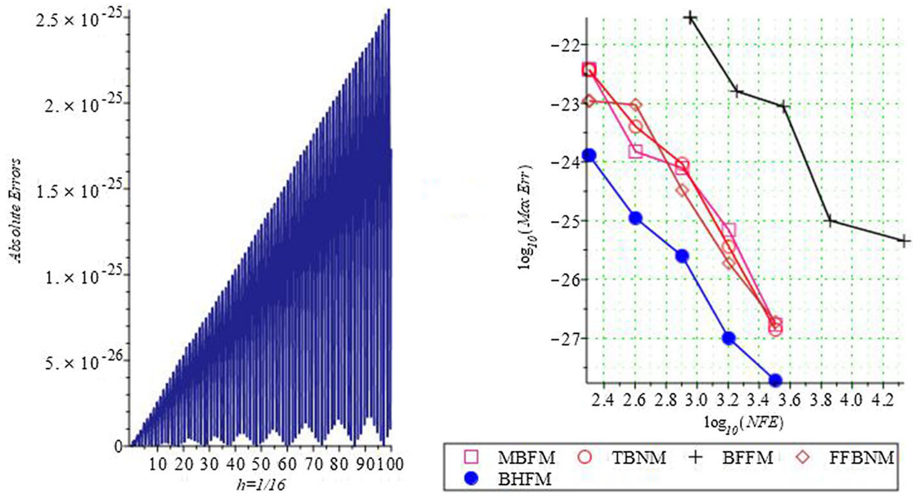

Fig. 7 The graphical representation of the solution to Example 5: accuracy (Left) and efficiency (Right)

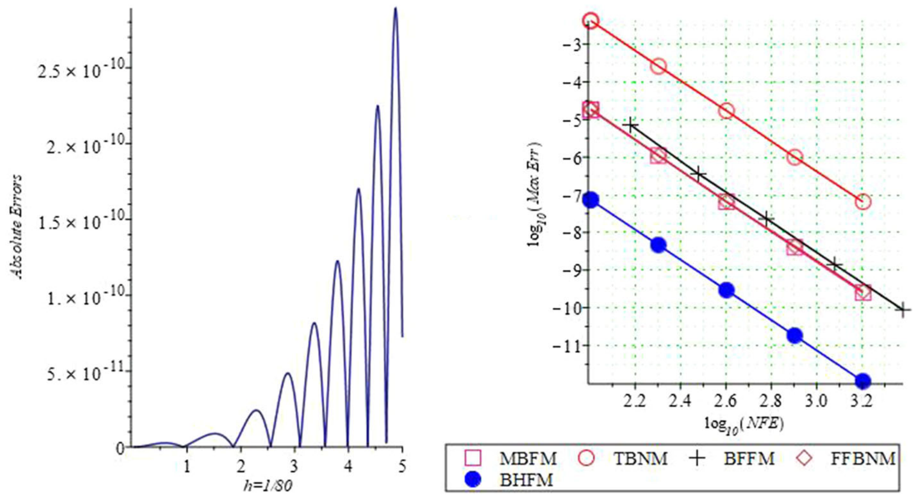

Fig. 8 The graphical representation of the solution to Example 6: accuracy (Left) and efficiency (Right)

on the integration interval [0, 10] with parameter $\epsilon = 10^{-3}$ is considered. The analytical solution of Eq. (31) which represents a periodic motion of constant frequency with a small perturbation of variable frequency is given by $y_1(x) = \cos(5x) + \epsilon \sin\left(x^2\right)$, $y_2(x) = \sin(5x) + \epsilon \cos\left(x^2\right)$. Figure 8 (Left) compares the theoretical solution to the numerical approximations provided by the proposed BHFM with step size $h = 1/80$. Figure 8 (Right) illustrates the efficiency of the BHFM and its advantageous performance.

Table 1 Rate of convergence of BHFM

Example 1		Example 2		Example 3	
Err	ROC	Err	ROC	Err	ROC
2.6E−5	–	2.8E−27	–	1.6E−5	–
1.3E−6	4.32	1.6E−28	4.12	1.0E−6	4.00
8.6E−8	3.92	3.3E−29	2.27	6.3E−8	3.98
5.1E−9	4.07	1.1E−30	4.91	4.0E−9	3.97
3.6E−10	3.82	5.6E−32	4.29	2.5E−10	4.00
Example 4		Example 5		Example 6	
Err	ROC	Err	ROC	Err	ROC
5.7E−6	–	1.3E−24	–	7.3E−8	–
3.6E−7	3.98	1.1E−25	3.56	4.6E−9	3.99
2.2E−8	4.03	2.5E−26	2.14	2.9E−10	3.99
1.4E−9	3.97	1.0E−27	4.64	1.8E−11	4.00
8.7E−11	4.00	1.9E−28	2.40	1.1E−12	4.03

4.3 Rate of convergence of BHFM

The order of convergence determines the rate of convergence (ROC). The following formula given in Jator and Oladejo (2017) is used to calculate the convergence rate for the proposed method

$$ROC = \log_2\left(\frac{E^{h_1}}{E^{h_2}}\right), \tag{32}$$

where E^h is the error obtained using the step size h. The numerical approximations of the BHFM convergence rate are included in Table 1 to showcase its theoretical results as well as its performance. Table 1 shows that BHFM is consistent with its theoretical order of convergence.

5 Conclusion

A fourth-order convergent functionally fitted block hybrid Falkner method is proposed for the numerical integration of Kepler equations and some other related problems. In addition to being zero-stable, the proposed method is consistent and converges rapidly to the analytic solution. For fair comparison and superiority's sake, six numerical experiments were considered to illustrate the performance of the proposed method. Whereas the simulations in Figs. 2, 3, 4, 5, 6, 7 and 8 (Left) showcase the agreement between the exact and the approximate solutions of the BHFM at some specified step sizes which confirms its accuracy with less errors between 10^{-30} and 10^{-6}, the simulations in Figs. 2, 3, 4, 5, 6, 7 and 8 (Right) provide the superiority in terms of efficiency of the BHFM over some adapted block methods in the recent literature.

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Code Availability Not applicable.

Declarations

Conflict of interest The authors declare no conflict of interest

Appendix

Specification of entries of The Lower Triangular Matrix σ, The Upper Triangular Matrix Δ

We have simplified $x_{n+1} = x_n + h$ and $x_{n+\varrho} = x_n + \varrho h$ accordingly.

$$\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -\frac{\omega\,\sin(\omega\,x_n)}{\cos(\omega\,x_n)} & 1 & 0 & 0 \\ 0 & -\frac{\omega\,\sin(h\varrho\,\omega+\omega\,x_n)}{\cos(\omega\,x_n)} & \cos\,(h\varrho\,\omega) & 1 & 0 \\ 0 & -\frac{\omega\,\sin(h\omega+\omega\,x_n)}{\cos(\omega\,x_n)} & \cos\,(h\omega) & \frac{\sinh(\omega\,x_n)\cos(h\omega)-\cosh(\omega\,x_n)\sin(h\omega)-\sinh(h\omega+\omega\,x_n)}{\sinh(\omega\,x_n)\cos(h\varrho\,\omega)-\cosh(\omega\,x_n)\sin(h\varrho\,\omega)-\sinh(h\varrho\,\omega+\omega\,x_n)} & 1 \end{pmatrix},$$

$$\Delta = \begin{pmatrix} 1 & \sin\,(\omega\,x_n) & \cos\,(\omega\,x_n) & \sinh\,(\omega\,x_n) & \cosh\,(\omega\,x_n) \\ 0 & \omega\,\cos\,(\omega\,x_n) & -\omega\,\sin\,(\omega\,x_n) & \omega\,\cosh\,(\omega\,x_n) & \sinh\,(\omega\,x_n)\,\omega \\ 0 & 0 & -\frac{\omega^2}{\cos(\omega\,x_n)} & \frac{\omega^2\sin(\omega\,x_n)\cosh(\omega\,x_n)+\omega^2\sinh(\omega\,x_n)\cos(\omega\,x_n)}{\cos(\omega\,x_n)} & \frac{\omega^2\sin(\omega\,x_n)\sinh(\omega\,x_n)+\cosh(\omega\,x_n)\omega^2\cos(\omega\,x_n)}{\cos(\omega\,x_n)} \\ 0 & 0 & 0 & \begin{pmatrix} -\sinh\,(\omega\,x_n)\cos\,(h\varrho\,\omega)\,\omega^2+\cosh\,(\omega\,x_n)\sin\,(h\varrho\,\omega)\,\omega^2 \\ +\sinh\,(h\varrho\,\omega+\omega\,x_n)\,\omega^2 \end{pmatrix} & \begin{pmatrix} \sinh\,(\omega\,x_n)\sin\,(h\varrho\,\omega)\,\omega^2-\cosh\,(\omega\,x_n)\cos\,(h\varrho\,\omega)\,\omega^2 \\ +\cosh\,(h\varrho\,\omega+\omega\,x_n)\,\omega^2 \end{pmatrix} \\ 0 & 0 & 0 & 0 & \frac{\begin{pmatrix} -\sinh\,(h\omega)\cos\,(h\varrho\,\omega)\,\omega^2-\cosh\,(h\omega)\sin\,(h\varrho\,\omega)\,\omega^2 \\ +\cos\,(h\omega)\sinh\,(h\varrho\,\omega)\,\omega^2+\sin\,(h\omega)\cosh\,(h\varrho\,\omega)\,\omega^2 \\ +\sin\,(h\varrho\,\omega-h\omega)\,\omega^2-\sinh\,(h\varrho\,\omega-h\omega)\,\omega^2 \end{pmatrix}}{\begin{pmatrix} \sinh\,(\omega\,x_n)\cos\,(h\varrho\,\omega)-\cosh\,(\omega\,x_n)\sin\,(h\varrho\,\omega) \\ -\sinh\,(h\varrho\,\omega+\omega\,x_n) \end{pmatrix}} \end{pmatrix}$$

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