Computation of Grundy dominating sequences in (co-)bipartite graphs

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Abstract

A sequence S of vertices of a graph G is called a dominating sequence of G if (i) each vertex v of S dominates a vertex of G that was not dominated by any of the vertices preceding vertex v in S, and (ii) every vertex of G is dominated by at least one vertex of S. The GRUNDY DOMINATION problem is to find a longest dominating sequence for a given graph G. It has been known that the decision version of the GRUNDY DOMINATION problem is NP-complete even when restricted to chordal graphs. In this paper, we prove that the decision version of the GRUNDY DOMINATION problem is NP-complete for bipartite graphs and co-bipartite graphs. On the positive side, we present a linear-time algorithm that solves the GRUNDY DOMINATION problem for chain graphs, which form a subclass of bipartite graphs.

 ${\bf Keywords:}$ dominating sequence, bipartite graph, chain graphs, computational complexity, linear-time algorithm

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1 Introduction

Graph domination is an established area of graph theory with an extremely rich literature and a number of applications. Given a graph G = (V, E) a set $S \subseteq V$ is *dominating* if every vertex $x \in V \setminus S$ is adjacent to a vertex in S. A dominating set of G with minimum cardinality is called a *minimum dominating set* of G. The cardinality of a minimum dominating set of Gis the *domination number* of G and is denoted by $\gamma(G)$. The MINIMUM DOMINATION problem

is to find a dominating set of cardinality $\gamma(G)$. The MINIMUM DOMINATION problem and its variations have numerous applications in real world problems including social networks, facility location problems, routing problems. An extensive overview on graph domination can be found in three recent monographs (Haynes et al, 2020, 2021, 2022).

The domination game was introduced in (Brešar et al, 2010) using the approach of building a dominating set of a graph one vertex at a time. The game is played on a graph G by two players, named Dominator and Staller, who are building a dominating set of G. These two players have the opposite goals in the game: one wants the game to end in as few moves as possible while the other one wants to extend the length of the game. They are taking turns in which a player chooses a vertex that has to dominate a vertex that has not been dominated by any of the previously chosen vertices. The game ends when there is no more vertex to choose, that is, the set of vertices chosen during the game is a dominating set. Thus, during the game a sequence of vertices is selected. In several papers, authors considered a sequence obtained by the same basic rule, yet assuming that only the slow player plays the game. This leads to the following definitions.

Let $S = (v_1, v_2, \ldots, v_k)$ be a sequence of vertices of G, and let $\widehat{S} = \{v_1, \ldots, v_k\}$. A sequence $S = (v_1, v_2, \ldots, v_k)$ is a closed neighborhood sequence if $N[v_i] \setminus \bigcup_{j=1}^{i-1} N[v_j] \neq \emptyset$, holds for every $i \in \{2, 3, \ldots, k\}$. If, in addition, \widehat{S} is a dominating set of G, then S is a dominating sequence of G. Clearly, the length k of S is bounded from below by $\gamma(G)$. A dominating sequence of maximum length in G is a *Grundy dominating sequence* (or, *GD*-sequence for short) of G. The cardinality of such a sequence is called the *Grundy domination number* of G and is denoted by $\gamma_{gr}(G)$.

These concepts were introduced and studied in 2014 by Brešar, Gologranc, Milanič, Rall and Rizzi (Brešar et al, 2014), where motivation came from the domination game as described above. In addition, Grundy domination presents the worst-case scenario in the process of the online update of a dominating set in the expanding network. In 2021, domination games as well as Grundy domination were comprehensevely surveyed in the book (Brešar et al, 2021).

The optimization version of the Grundy domination problem is to find a GD-sequence (that is, a dominating sequence of maximum length) in a graph G. From the computational complexity point of view the decision version of the problem is defined as follows:

GRUNDY DOMINATION DECISION (GDD) Input: A graph G = (V, E) and $k \in \mathbb{Z}^+$. Question: Is there a dominating sequence of G of length at least k?

In the seminal paper (Brešar et al, 2014), the authors proved that the GDD problem is NPcomplete for chordal graphs. They also proved that a GD-sequence in trees, cographs and split graphs can be computed in polynomial time (Brešar et al, 2014). An additional study of Grundy domination in forests was used to give a partial confirmation of a conjecture concerning the Grundy domination number in strong products of graphs (Bell et al, 2021). Several combinatorial results have also been established for the parameter and its relatives in the literature (Brešar et al, 2017; Campêlo and Severín, 2021; Erey, 2020; Haynes and Hedetniemi, 2021; Lin, 2019). Concerning the computational complexity, it was shown that the GDD problem can be solved in polynomial time for interval graphs and Sierpiński graphs (Brešar et al, 2016a), as well as on some X-join products, lexicographic products and related classes of graphs (Nasini and Torres, 2020). A hierarchy presenting relationships between some classes of graphs that are relevant for this paper are shown in Fig. 1.



Fig. 1 Hierarchy of selected graph classes.

A close relation between dominating sequences in graphs and certain covering sequences in hypergraphs was found in the seminal paper (Brešar et al, 2014). Given a hypergraph $\mathcal{H} = (\mathcal{X}, \mathcal{E})$ with no isolated vertices, an *edge cover* of \mathcal{H} is a family of (hyper)edges from \mathcal{E} whose union is the vertex set \mathcal{X} . The smallest cardinality of an edge cover of \mathcal{H} is the *edge cover number* of \mathcal{H} and is denoted by $\rho(\mathcal{H})$.

Now, consider a sequence of edges $\mathcal{C} = (C_1, \ldots, C_r)$ of a hypergraph \mathcal{H} . If for each i, $i \in [r], C_i$ contains a vertex not contained in C_j , for all j < i, then \mathcal{C} is a *legal edge sequence* of \mathcal{H} . If $\mathcal{C} = (C_1, C_2, \ldots, C_r)$ is a legal edge sequence and the set $\widehat{\mathcal{C}} = \{C_1, C_2, \ldots, C_r\}$ is an edge cover of \mathcal{H} , then \mathcal{C} is an *edge covering sequence*. An edge covering sequence of maximum length in \mathcal{H} is a *Grundy covering sequence* of \mathcal{H} . The length of such a sequence is the *Grundy cover number*, $\rho_{gr}(\mathcal{H})$, of \mathcal{H} . Given a hypergraph $\mathcal{H} = (\mathcal{X}, \mathcal{E})$, the GRUNDY COVERING problem is to find a Grundy covering sequence, and thus establishing the Grundy cover number, and its decision version is the GRUNDY COVERING DECISION (GCD) problem. It was shown in (Brešar et al, 2014) that the GDD problem is NP-complete by reduction from the GCD problem, while for the NP-completeness of the GCD problem a reduction from the classical FEEDBACK ARC SET problem was used.

In Section 2, we establish basic notation and mention several preliminary results. In Section 3, we prove that the GDD problem is NP-complete even when restricted to bipartite or co-bipartite graphs. In contrast, we present a linear-time algorithm for determining the Grundy domination number of chain graphs in Section 4. In the final section, we add a few comments and open problems.

2 Preliminaries

Let $[n] = \{1, \ldots, n\}$ for any $n \in \mathbb{N}$. Given a graph G, the *neighborhood* of a vertex x is $N_G(x) = \{y \in V(G) : xy \in E(G)\}$, and the *closed neighborhood* of x is $N_G[x] = N_G(x) \cup \{x\}$. Vertices u and v in a graph G are *closed twins* (respectively, *open twins*) if $N_G[u] = N_G[v]$ (respectively, $N_G(u) = N_G(v)$). We may omit the indices in the above definitions if the graph G is understood from the context. The following statement has a straightforward proof using definitions.

Proposition 1. Let S be a (Grundy) dominating sequence in a graph G. If $u, v \in V(G)$ such that $N[u] \subseteq N[v]$ and $u, v \in \hat{S}$, then u appears before v in S.

If S is a closed neighborhood sequence, then we say that v_i footprints the vertices from $N[v_i] \setminus \bigcup_{j=1}^{i-1} N[v_j]$, and that v_i is the footprinter of every vertex $u \in N[v_i] \setminus \bigcup_{j=1}^{i-1} N[v_j]$. If $S_1 = (v_1, v_2, \ldots, v_n)$ and $S_2 = (u_1, u_2, \ldots, u_m)$, $n, m \ge 1$, are two sequences, then the concatenation of S_1 and S_2 is the sequence $S_1 \oplus S_2 = (v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_m)$.

Given a hypergraph $\mathcal{H} = (\mathcal{X}, \mathcal{E})$, a *legal transversal sequence* is a sequence $S = (v_1, \ldots, v_k)$ of vertices from \mathcal{X} such that for each *i* there exists an edge $\mathcal{E}_i \in \mathcal{E}$ such that $v_i \in \mathcal{E}_i$ and $v_j \notin \mathcal{E}_i$ for all *j*, where j < i. The maximum length of a legal transversal sequence in a hypergraph \mathcal{H} is denoted by $\tau_{\rm gr}(\mathcal{H})$. The following result was proved in (Brešar et al, 2016b, Proposition 8.3).

Proposition 2. For any hypergraph \mathcal{H} we have $\tau_{gr}(\mathcal{H}) = \rho_{gr}(\mathcal{H})$.

A set of vertices $A \subseteq V(G)$ is called an *independent set* of G if no two vertices of A are adjacent in G. A maximum independent set is an independent set of maximum cardinality. The size of a maximum independent set in G is the *independence number* of G and is denoted by $\alpha(G)$.

A bipartite graph is a graph whose vertex set can be partitioned into two independent sets. A co-bipartite graph is a graph, which is the complement of a bipartite graph. A bipartite graph G = (X, Y, E) is a chain graph if the neighborhoods of the vertices of Xform a chain, that is, the vertices of X can be linearly ordered, say $\{x_1, x_2, \ldots, x_{n_1}\}$ such that $N(x_1) \subseteq N(x_2) \subseteq \cdots \subseteq N(x_{n_1})$, where $n_1 = |X|$. If G = (X, Y, E) is a chain graph, then one can easily see that the neighborhoods of the vertices of Y also form a chain. If $n_2 = |Y|$, an ordering $\alpha = (x_1, x_2, \ldots, x_{n_1}, y_1, y_2, \ldots, y_{n_2})$ is called a *chain ordering* if $N(x_1) \subseteq N(x_2) \subseteq \cdots \subseteq N(x_{n_1})$ and $N(y_1) \supseteq N(y_2) \supseteq \cdots \supseteq N(y_{n_2})$.

If a vertex x_i appears before x_j in chain ordering, we write $x_i < x_j$. Given a chain graph G, a chain ordering of G can be computed in linear time (Heggernes and Kratsch, 2007). Now, suppose $A = \{x_{i_1}, x_{i_2}, \ldots, x_{i_r}\}, B = \{x_{j_1}, x_{j_2}, \ldots, x_{j_{r'}}\}$ be two disjoint subsets of X such that $x_{i_p} < x_{i_q}$ and $x_{j_p} < x_{j_q}$ for p < q. Then $(A) \oplus (B)$ denotes the sequence $(x_{i_1}, \ldots, x_{i_r}, x_{j_1}, \ldots, x_{j_{r'}})$. Similar notation can be used for vertices in Y.

Recall that two vertices u, v are called open twins if N(u) = N(v). We define a relation R on X such that vertices $x, x' \in X$ are related by R if and only if they are open twins. Clearly, R is an equivalence relation, and let X_1, X_2, \ldots, X_k be the parts in X that arise from R. Without loss of generality, for a chain graph G = (X, Y, E), we may also assume that $N(X_1) \subseteq N(X_2) \cdots \subseteq N(X_k)$. We denote the set of vertices in $N(X_1)$ by Y_1 . For $i \in \{2, 3, \ldots, k\}$, let $Y_i = N(X_i) \setminus \bigcup_{j=1}^{i-1} N(X_j)$. It is easy to see that every two vertices from Y_j are open twins for all $j \in [k]$. The following statement follows from the construction of the sets X_i and Y_j .

Observation 1. Let X_1, X_2, \ldots, X_k and Y_1, Y_2, \ldots, Y_k be the subsets of vertices of G as defined above. Then, $N(X_i) = \bigcup_{r=1}^i Y_r$ and $N(Y_j) = \bigcup_{r=j}^k X_r$ for each $i, j \in [k]$.

3 NP-completeness results

As mentioned in the introduction, the GDD problem is NP-complete for general graphs, and also when restricted to chordal graphs. In the following subsections, we prove that the problem remains NP-complete for bipartite and co-bipartite graphs.

3.1 Bipartite graphs

In this subsection, we prove the NP-completeness of the GDD problem for bipartite graphs. We reduce the GCD problem for hypergraphs to the GDD problem for bipartite graphs. Given a hypergraph $\mathcal{H} = (\mathcal{X}, \mathcal{E})$ with $|\mathcal{X}| = n$ and $|\mathcal{E}| = m$, $(n, m \ge 2)$, we construct an instance $G_{\mathcal{H}}$ of the GDD problem, where $G_{\mathcal{H}}$ is a bipartite graph, as follows.

Let $V(G_{\mathcal{H}}) = A \cup X' \cup \mathcal{E}' \cup B$, where $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_m\}$. The sets X' and \mathcal{E}' contain n and m vertices respectively, where each vertex of X' corresponds to a vertex of \mathcal{X} in the hypergraph \mathcal{H} and each vertex of \mathcal{E}' correspond to an edge of \mathcal{H} . For an edge $\mathcal{E}_i \in \mathcal{E}$, we denote the corresponding vertex in \mathcal{E}' by e_i . All sets A, X', \mathcal{E}' , and B are independent sets in $G_{\mathcal{H}}$, and $A \cup X'$ induces the complete bipartite graph $K_{n,n}$, while $\mathcal{E}' \cup B$ induces the complete bipartite graph $K_{m,m}$. Now, a vertex x of X' is adjacent to a vertex of $e_i \in \mathcal{E}'$ in $G_{\mathcal{H}}$ if and only if $x \in \mathcal{E}_i$ in \mathcal{H} .

Clearly, $G_{\mathcal{H}}$ is a bipartite graph. See Fig. 2, which presents the construction of the graph $G_{\mathcal{H}}$ from a hypergraph \mathcal{H} , which is given by $(\mathcal{X} = \{x_1, x_2, x_3, x_4\}, \mathcal{E} = \{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4, \mathcal{E}_5\})$, where $\mathcal{E}_1 = \{x_1, x_2, x_4\}, \mathcal{E}_2 = \{x_2, x_3\}, \mathcal{E}_3 = \{x_1, x_2\}, \mathcal{E}_4 = \{x_2, x_3, x_4\}$ and $\mathcal{E}_5 = \{x_1, x_3, x_4\}$.



Fig. 2 Illustration of the construction of $G_{\mathcal{H}}$ from \mathcal{H} .

Theorem 1. If $\mathcal{H} = (\mathcal{X}, \mathcal{E})$ is a hypergraph with $|\mathcal{X}| = n$ and $|\mathcal{E}| = m$, where $n, m \ge 2$, then $\rho_{gr}(\mathcal{H}) \ge k$ if and only if $\gamma_{gr}(G_{\mathcal{H}}) \ge n + m + k$.

Proof. In the proof, let $G = G_{\mathcal{H}}$. First, let $(\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_{k'})$ be an edge covering sequence of length at least k in \mathcal{H} . Then the sequence $(b_1, b_2, \ldots, b_m, e_1, e_2, \ldots, e_{k'}, a_1, a_2, \ldots, a_n)$ is a dominating sequence of length at least n + m + k in G. Hence, $\gamma_{gr}(G) \ge n + m + k$.

For the converse, let us assume that $\gamma_{gr}(G) \geq n + m + k$ for some positive integer k. If $X' \cap \hat{S} \neq \emptyset$ for some dominating sequence S of G, then x_0 denotes the first vertex in S coming from X', and if $\mathcal{E}' \cap \hat{S} \neq \emptyset$ for some dominating sequence S of G, then e_0 denotes the first vertex in S coming from \mathcal{E}' . First, we prove two auxiliary claims.

Claim 1. There exists a dominating sequence S of length at least n + m + k in G such that all vertices of A appear in S and if $X' \cap \hat{S} \neq \emptyset$, then all vertices of A appear before x_0 .

Proof. Let S be a dominating sequence of length at least n + m + k in G (which exists, since $\gamma_{gr}(G) \ge n + m + k$). Suppose there exists a vertex $a_i \in A$, which is not appearing in S. Then, there exists a vertex from X' which is appearing in S to footprint a_i . Hence, $X' \cap \hat{S} \neq \emptyset$ and x_0 footprints a_i . Let P denotes the set of vertices appearing before x_0 and Q be the set of vertices appearing after x_0 in S. Now, two cases are possible.

Case 1: $Q \cap A = \emptyset$.

In this case, either $P \cap A = \emptyset$ or P contains some vertices of A. So, first assume that P contains no vertex of A. Then, we see that no vertex of A appears in the sequence S. If x_0 does not footprint any vertex in \mathcal{E}' , then we modify S by appending vertices of A in the order (a_1, a_2, \ldots, a_n) just before x_0 and removing all vertices from Q that footprinted a vertex of X' along with the vertex x_0 . Otherwise, if x_0 footprints some vertices of \mathcal{E}' , we perform the same modification without removing x_0 . Note that the number of vertices of Q that footprint a vertex of X' are at most n-1. In either case, we removed at most n vertices and we added n new vertices to S, by which the so modified sequence S is a dominating sequence of length at least n + m + k in G, which satisfies the statement of the claim.

Now, if P contains some vertices of A, then no vertex of Q footprints any vertex of X'. Again, if x_0 footprints only vertices of A, then we modify S by appending vertices of $A \setminus P$ in any order just before x_0 and removing the vertex x_0 . Otherwise, if x_0 footprints also some vertices of \mathcal{E}' , we perform the same modification, but without removing x_0 . In either case, we removed at most 1 vertex and we added at least 1 new vertex to the sequence S. With this, the so modified sequence S is a dominating sequence of length at least n + m + k in G, which satisfies the statement of the claim.

Case 2: $|Q \cap A| = 1$.

In this case, P contains no vertex of A. Let a_j be the vertex from $Q \cap A$ appearing in S. Note that the vertices in S, which footprint only vertices from X', do not appear after a_j in S. Note that there are at most n-2 vertices that appear in S between x_0 and a_j and footprint a vertex of X', and denote the set of these vertices by Q'. If x_0 does not footprint any vertex in \mathcal{E}' , then we modify S by appending vertices of A in the order (a_1, a_2, \ldots, a_n) just before x_0 and then removing all vertices of $Q' \cup \{x_0, a_j\}$. Otherwise, if x_0 footprints also some vertices of \mathcal{E}' , we perform the same modification without removing x_0 . In either case, we removed at most n-1 vertices and we added n-1 new vertices to S. Hence, the so modified sequence S is a dominating sequence of length at least n + m + k in G, which satisfies the statement of the claim. By the above claim, there exists a dominating sequence S of length n + m + k such that all vertices of A appear in S and they appear before x_0 if $X' \cap \hat{S} \neq \emptyset$. The proof of Claim 2 follows similar lines as the proof of the Claim 1.

Claim 2. There exists a dominating sequence S of length at least n + m + k in G such that all vertices of B appear in S and if $\mathcal{E}' \cap \hat{S} \neq \emptyset$, then all vertices of B appear before e_0 .

Combining the above two claims we infer that there exists a dominating sequence S of length at least n + m + k in G such that $|(X' \cup \mathcal{E}') \cap \hat{S}| \ge k$.

Claim 3. Either $X' \cap \hat{S} = \emptyset$ or $\mathcal{E}' \cap \hat{S} = \emptyset$.

Proof. If $X' \cap \hat{S} = \emptyset$, we are done. So, assume that $X' \cap \hat{S} \neq \emptyset$ and $\mathcal{E}' \cap \hat{S} \neq \emptyset$. Now, either e_0 appears before x_0 or e_0 appears after x_0 . In the former case, we see that before the vertex x_0 , all vertices of G are footprinted (and thus dominated) using Claims 1 and 2. So, x_0 does not footprint any vertex implying that this case is not possible. Similarly, the latter case is also not possible, which proves the claim.

Now, if $X' \cap \hat{S} = \emptyset$, then we have that $|\mathcal{E}' \cap \hat{S}| \ge k$. In addition, by Claim 2, since all vertices of B appear in S before any vertex of \mathcal{E}' appears in S, the subsequence of S of vertices in \mathcal{E}' corresponds to an edge covering sequence in the hypergraph \mathcal{H} , which is of length at least k. Thus, $\rho_{gr}(\mathcal{H}) \ge k$, as desired.

Otherwise, if $\mathcal{E}' \cap \hat{S} = \emptyset$, then we derive that $|X' \cap \hat{S}| \ge k$, where the subsequence formed by vertices of $X' \cap \hat{S}$ corresponds to a legal transversal sequence of the hypergraph \mathcal{H} of length at least k. By Proposition 2, $\tau_{\rm gr}(\mathcal{H}) = \rho_{\rm gr}(\mathcal{H})$, and so $\rho_{gr}(\mathcal{H}) \ge k$. The proof of the theorem is complete.

Based on Theorem 1 and earlier discussions we immediately derive the main result of this section.

Theorem 2. The GDD problem is NP-complete for bipartite graphs.

3.2 Co-bipartite graphs

In this subsection, we prove the NP-completeness of the GDD problem for co-bipartite graphs. Here, we reduce the GDD problem for general graphs to the GDD problem for co-bipartite graphs. Given a graph G = (V, E), where $V = \{v_1, v_2, \ldots, v_n\}$, we construct an instance $G' = (V_1 \cup V_2, E')$ of the GDD problem, where G' is a co-bipartite graph, as follows.

The vertex set of G' is $V_1 \cup V_2$, where $V_1 = \{v_i^1 : v_i \in V\}$ and $V_2 = \{v_i^2 : v_i \in V\}$. The set of edges of G' is given by $\{v_i^1 v_j^1 : 1 \le i < j \le n\} \cup \{v_i^2 v_j^2 : 1 \le i < j \le n\} \cup \{v_i^1 v_j^2 : v_j \in N_G[v_i], i, j \in [n]\}$. Note that G' is a co-bipartite graph. Fig. 3 provides an illustration of the construction of G' from G.

Claim 4. For a positive integer k, $\gamma_{gr}(G) \ge k$ if and only if $\gamma_{gr}(G') \ge k$.

Proof. First, let $S = (u_1, u_2, \ldots, u_t)$ be a dominating sequence of G of length t, where $t \ge k$. Then $S' = (u_1^1, u_2^1, \ldots, u_t^1)$ is a dominating sequence of length at least k in G'. Indeed, if v_i is a vertex footprinted by u_i with respect to S, then v_i^2 is footprinted by u_i^1 with respect to S'.

Conversely, let $S = (w_1, w_2, \ldots, w_t)$ be a dominating sequence of length t in G', where $t \geq k$ and $w_i \in V_1 \cup V_2$ for all $i \in [t]$. Without loss of generality, we may assume that $w_1 \in V_1$. Note that there can be at most one vertex from V_2 in S. If there is no such vertex,



Fig. 3 Construction of a co-bipartite graph G' from a graph G, where $G = P_3$.

then the sequence S corresponds to a sequence of vertices in G of length at least k, which is a dominating sequence of G. Now, suppose there exists a vertex from V_2 in S. Clearly, it has to be the last vertex of S, and let w_t be vertex $v_i^2 \in V_2$. Note that v_i^2 appears in S to footprint a vertex $v_j^2 \in V_2$ and $v_j^1 \notin \hat{S}$. We modify S by replacing the vertex v_i^2 with the vertex v_j^1 and get a new sequence S', which is again a dominating sequence of G' of length at least k. Since S' contains only vertices from V_1 , it corresponds to a sequence of vertices in G of length at least k, which is a dominating sequence of G. This completes the proof of the converse direction of the statement.

Now, we are ready to state the announced result.

Theorem 3. The GDD problem is NP-complete for co-bipartite graphs.

4 Efficient algorithm for chain graphs

In this section, we give a linear-time algorithm to compute a GD-sequence of a chain graph. Before discussing the main idea for chain graphs, we first give the Grundy domination number of a complete bipartite graph which is a subclass of chain graphs. The proof of this result is straightforward.

Proposition 3. If G = (X, Y, E) is a complete bipartite graph, then $\gamma_{gr}(G) = max\{|X|, |Y|\}$.

Recall that a chain graph G = (X, Y, E) has a chain ordering α , and based on the equivalence relation joining open twins, the sets X and Y partition into subsets X_1, X_2, \ldots, X_k and Y_1, Y_2, \ldots, Y_k , respectively. (Also recall that $N(X_i) = \bigcup_{r=1}^i Y_r$ and $N(Y_j) = \bigcup_{r=j}^k X_r$ for each $i, j \in [k]$.) Since the case k = 1 yields a complete bipartite graph, in the rest of this section we only consider the chain graphs with $k \geq 2$. We also assume that a chain graph G = (X, Y, E) is given along with the chain ordering α and the partitions of X and Y into k subsets.

The proof of the following observation is again easy, and hence is omitted.

Observation 2. Let $A \subseteq V(G)$ be a set of open twins in an arbitrary graph G. Then there exists a GD-sequence S of G such that all vertices of $A \cap \hat{S}$ appear together in S.

Observation 3. Let S be a GD-sequence of an arbitrary graph G and $A \subseteq V(G)$ be a set of open twins in G such that $A \cap \hat{S} \neq \emptyset$. If the first vertex of A in S footprints itself, then

there exists a GD-sequence of G in which all vertices of A appear and they appear together in that sequence.

Proof. If $A \cap \hat{S} = A$, we have nothing to prove. So, suppose that $A \cap \hat{S} \neq A$. Then, there exists a vertex $a \in A$, which is not in S. Thus there exists a vertex $b \in \hat{S}$, which footprints a. Note that b appears after all vertices of $A \cap \hat{S}$. Now, we modify S by replacing b with the vertex a. By doing this repeatedly, we get a new GD-sequence S' of G which contains all vertices of A. We can rearrange all vertices of A so that all vertices of A appear together in S'.

In the remainder of the section we assume that G is a chain graph with the partition of its vertex set as described earlier.

Observation 4. There exists a GD-sequence S of G such that for every $i \in [k]$, we have $X_i \cap \hat{S} \neq \emptyset$ or $Y_i \cap \hat{S} \neq \emptyset$.

Proof. Let S be a GD-sequence such that $X_i \cap \hat{S} = \emptyset$ and $Y_i \cap \hat{S} = \emptyset$. This implies that there exist vertices $x \in \bigcup_{r=i+1}^k X_r$ and $y \in \bigcup_{r=1}^{i-1} Y_r$ in S to footprint the vertices of Y_i and X_i respectively. Note that $i \notin \{1, k\}$. Now, if x appears before y in S, then modifying the sequence S by replacing y by all vertices of X_i gives another GD-sequence of G, which includes some vertices from X_i . (Note that if $y \in Y_j$, then $X_j \cap \hat{S} \neq \emptyset$. To see this, assuming that $X_j \cap \hat{S} = \emptyset$ implies that y footprints vertices of both X_j and X_i in S. In this case, replacing the vertex y with all vertices of X_i and X_j results in a new dominating sequence of G of length bigger than S, a contradiction.) Similarly, if y appears before x in S, then modifying the sequence S by replacing x by all vertices of Y_i gives another GD-sequence of G, which includes some vertices from Y_i . Hence, there exists a GD-sequence S of G such that $X_i \cap \hat{S} \neq \emptyset$ or $Y_i \cap \hat{S} \neq \emptyset$.

Let A be a set of open twins in G. If $|A \cap \hat{S}| \geq 2$, then we see that the first vertex of A in S footprints itself. Thus we can assume that, $A \subseteq \hat{S}$, due to Observation 3. Hence, we have, $|A \cap \hat{S}| \leq 1$ or $A \subseteq \hat{S}$ for any set of open twins A in G. Note that the each of the sets $X_1, X_2, \ldots, X_k, Y_1, Y_2, \ldots, Y_k$ is a set of open twins in G.

Now, based on the Observations 2, 3 and 4, whenever we consider a GD-sequence S of G, we assume that S satisfies the following, for the rest of this section:

(1) For each $i \in [k], |X_i \cap \hat{S}| \leq 1$ or $X_i \subseteq \hat{S}$. If $X_i \subseteq \hat{S}$, then all vertices of X_i appear together in S.

(2) For each $i \in [k], |Y_i \cap \hat{S}| \leq 1$ or $Y_i \subseteq \hat{S}$. If $Y_i \subseteq \hat{S}$, then all vertices of Y_i appear together in S.

(3) For each $i \in [k]$, $X_i \cap \hat{S} \neq \emptyset$ or $Y_i \cap \hat{S} \neq \emptyset$.

Now, let S be a GD-sequence of G. Then S is one of the following type:

(a) $X \cap \hat{S} = \emptyset$, (b) $Y \cap \hat{S} = \emptyset$, (c) $X \cap \hat{S} \neq \emptyset$ and $Y \cap \hat{S} \neq \emptyset$.

We call the corresponding GD-sequences S to be of type (a), type (b), and type (c), respectively.

Lemma 1. Let $S^* = (v_1, v_2, ..., v_p)$ be a GD-sequence of G of type (c). The following statements hold:

(1) If $v_1 \in Y_1$, then there exists a type (a) GD-sequence of G.

(2) If $v_1 \notin Y_1$, then there exists a GD-sequence S of G such that $\bigcup_{r=1}^i X_r \subseteq \hat{S}$ for some $i \in [k]$.

Proof. First, we assume that $v_1 \in Y_1$. In this case, all vertices of X are footprinted by v_1 . So, all the vertices v_2, \ldots, v_p appear to footprint vertices of $Y \setminus \{v_1\}$ only. This implies that $\gamma_{gr}(G) \leq |Y|$. So, the sequence $S = (y_{n_2}, y_{n_2-1}, \ldots, y_1)$ is also a GD-sequence of G and it is of type (a). Next, we assume that $v_1 \notin Y_1$. Since S^* contains vertices from both X and Y, we have two cases to consider.

Case 1: $v_1 \in X$.

Let $v_1 \in \overline{X_i}$ $(i \in [k])$. So, we get that v_1 footprints all vertices of $N(X_i) \cup \{v_1\}$. Now, let u be a vertex of X such that $N(u) \subseteq N(v_1)$. If u is footprinted by some vertex from N(u), we modify the sequence S^* as follows. We remove the footprinter of u from S^* and include u just after v_1 and get a new sequence. But, if u is footprinted by itself, then we relocate u in S by putting it just after v_1 . We repeat the respective modifications for each vertex u such that $N(u) \subseteq N(v_1)$ and get a new sequence S which remains a GD-sequence of G. We again modify the ordering of vertices in S, so that it satisfies all the properties of Observations 2, 3 and 4, if required. Hence, we see that $\cup_{r=1}^i X_r \subseteq \hat{S}$ for some $i \in [k]$.

Case 2: $v_1 \in Y$.

Let $v_1 \in Y_i$ $(i \in [k])$. Note that i > 1. Here, we may assume that all vertices u from Y, where $N(u) \supseteq N(v_1)$, are in S^{*} and all vertices of $\bigcup_{r=i}^{k} Y_r \setminus \{v_1\}$ appear together just after v_1 . This is ensured because we can do modifications similar to the case 1, if it is not true. We rename vertices of S^* again by (v_1, v_2, \ldots, v_p) , if necessary. Now, let v_t be the vertex with the smallest index in the ordering (v_2, v_3, \ldots, v_p) such that $v_t \in X$. Suppose that $v_t \in X_{t'}$. Note that there exists an integer $r' \in \{2, 3, ..., k\}$ such that all vertices of $\bigcup_{r=r'}^k (X_r \cup Y_r)$ are footprinted before the vertex v_t appears in S, and all remaining vertices of G are not footprinted. If $v_t \in \bigcup_{r=1}^{r'-1} X_r$, we get that v_t footprints all vertices of $N(v_t) \cup \{v_t\}$. In this case, we put all vertices u of X such that $N(u) \subseteq N(v_t)$ just after v_t in any order, remove vertices from $N(u) \cap \widehat{S^*}$ which were appearing to footprint the vertex u and rearrange all vertices so that the new sequence S satisfies all the properties of Observation 2. Thus, we get that $\bigcup_{r=1}^{t'} X_r \subseteq \hat{S}$. But, if $v_t \in \bigcup_{r=r'}^k X_r$, then v_t footprints all vertices of $\bigcup_{r=1}^{r'-1} Y_r$. So, we may assume that $v_t \in X_{r'}$. Again, we put all vertices $u \in \bigcup_{r=1}^{r'-1} X_r$ just after v_t in any order, remove vertices from $N(u) \cap \widehat{S^*}$ which were appearing to footprint the vertex u and rearrange all vertices so that the new sequence S satisfies all the properties of Observation 2. Thus, we get that $\bigcup_{r=1}^{r'-1} X_r \subseteq \hat{S}$.

Therefore, there exists a GD-sequence S of G such that $\bigcup_{r=1}^{i} X_r \subseteq \hat{S}$ for some $i \in [k]$. \square

Analogous to Lemma 1, we give a symmetric lemma for the set Y of G, whose proof follows similar lines and is omitted.

Lemma 2. Let $S^* = (v_1, v_2, ..., v_p)$ be a GD-sequence of G of type (c). The following statements hold:

- (1) If $v_1 \in X_k$, then there exists a type (b) GD-sequence of G.
- (2) If $v_1 \notin X_k$, then there exists a GD-sequence S of G such that $\bigcup_{r=j}^k Y_r \subseteq \hat{S}$ for some $j \in [k]$.

Lemma 3. Let S be a GD-sequence of G of type (c) and let $i \in [k]$ be the largest index such that $\bigcup_{r=1}^{i} X_r \subseteq \hat{S}$. If i < k, then the following is true:

- (1) If $i \leq k-2$, then there exists a GD-sequence S' of G such that either $(\bigcup_{r=i+1}^{k} X_r) \cap \hat{S}' = \emptyset$ or $(\bigcup_{r=i+2}^{k} X_r) \cap \hat{S}' = \emptyset$ and $|X_{i+1} \cap \hat{S}'| = 1$.
- (2) If i = k 1, then either $X_k \cap \hat{S} = \emptyset$ or $|X_k \cap \hat{S}| = 1$.

Proof. Let S be a GD-sequence of G in which we have a largest index i such that $\bigcup_{r=1}^{i} X_r \subseteq \hat{S}$. Now, assume that i < k and so, $X_{i+1} \not\subseteq \hat{S}$. If i = k - 1 and $X_k \cap \hat{S} \neq \emptyset$ then either $|X_k \cap \hat{S}| = 1$ or $|X_k \cap \hat{S}| \ge 2$. In the latter case, we get that first vertex of X_k in S footprints itself. So, Observation 3 ensures that $|X_k \cap \hat{S}| = 1$.

Next, we show that if $i \leq k-2$, then $(\bigcup_{r=i+2}^{k} X_r) \cap \hat{S} = \emptyset$. So, let $t \in \{i+2,\ldots,k\}$ be the minimum index such that $X_t \cap \hat{S} \neq \emptyset$. This means that there are some vertices of $\bigcup_{r=i+1}^{t-1} X_r$ which are not appearing in the sequence S. Let A denotes the set of these vertices. Note that A is not the empty set. As $X_t \cap \hat{S} \neq \emptyset$, vertices of $X_t \cap \hat{S}$ appear to footprint some vertices of $N[X_t \cap \hat{S}] = (X_t \cap \hat{S}) \cup (\bigcup_{r=1}^{t} Y_r)$. Let x be the vertex of X_t which appears first in S. We discuss two cases here.

Case 1: x footprints itself.

In this case, we have that no neighbor of X_t appears in S before x. So, all vertices in S, which footprint vertices of A, appear after x. Now, we modify S by removing all such vertices and including all vertices of A in the sequence just after all vertices of $X_t \cap \hat{S}$. We call the modified sequence again by S as it remains a GD-sequence of G. Thus, we get a contradiction on i being the largest index satisfying $\bigcup_{r=1}^i X_r \subseteq \hat{S}$. So, this case is not possible.

Case 2: x does not footprint itself.

In this case, we get that all vertices of X_t are footprinted by some vertex of Y, which appears before x in S. So, x footprints some vertices from the set $\bigcup_{r=1}^{t} Y_r$. Thus, $|X_t \cap \hat{S}| = 1$ and $X_t \cap \hat{S} = \{x\}$. Now, let y be the vertex which footprints vertices of $A \cap X_{i+1}$ in S. Then, there can be two subcases:

Subcase 2.1: y appears after x.

In this subcase, we modify S by removing y and including all vertices of $A \cap X_{i+1}$ in the sequence just after x. We call the modified sequence again by S as it remains a GD-sequence of G. Thus, we get a contradiction on i being the largest index satisfying $\bigcup_{r=1}^{i} X_r \subseteq \hat{S}$. So, this subcase is not possible.

Subcase 2.2: y appears before x.

Here, all vertices of A are footprinted before the appearance of x. Recall that $x \in (\bigcup_{r=i+2}^{k} X_r) \cap \hat{S}$. We get that all vertices of $\bigcup_{r=i+1}^{k} X_r$ are footprinted before the appearance of x. Note that the vertex x itself is footprinted before the appearance of x. So, we have, x appears to footprint some vertices of $\bigcup_{r=1}^{t} Y_r$. This can be further divided in two cases: (i) x does not footprint the vertices of Y_t . (ii) x footprints the vertices of Y_t .

In the first case, x footprints some vertices of the set $\cup_{r=1}^{t-1} Y_r$. Note that $A \cap X_{t-1} \neq \emptyset$ and vertices of $A \cap X_{t-1}$ do not appear in S. Now, we modify S by replacing the vertex xby a vertex of $A \cap X_{t-1}$ and a get a new GD-sequence in which no vertex of X_t appears and one vertex of X_{t-1} appears. If t = i+2, then after applying the modification once, we get a GD-sequence S' such that $X_{i+2} \cap \hat{S}' = \emptyset$ and $|X_{i+1} \cap \hat{S}'| = 1$. Otherwise, if t > i+2, then after applying the modification once, we get a GD-sequence S' such that $X_t \cap \hat{S}' = \emptyset$ and $|X_{t-1} \cap \hat{S}'| = 1$. Thus, we have another index $t' = t-1 \in \{i+2, \ldots, k\}$ such that $X_{t'} \cap \hat{S}' \neq \emptyset$.

In the second case, x footprints vertices of Y_t and we modify S by replacing x with all vertices of Y_t . Note that no vertex of Y_t was appearing in the sequence prior to this modification. If t = i + 2, then after applying the modification once, we get a GD-sequence S' such that $X_{i+2} \cap \hat{S}' = \emptyset$. Otherwise, if t > i + 2, then after applying the modification once, we get a GD-sequence S' such that $X_t \cap \hat{S}' = \emptyset$ and $X_{t-1} \cap \hat{S}' = \emptyset$. Now, we have, either there is no index t' in the set $\{i + 2, \ldots, k\}$ such that $X_{t'} \cap \hat{S}' \neq \emptyset$ or there is some $t' \in \{i + 2, \ldots, k\}$ (t' > t) such that $X_{t'} \cap \hat{S}' \neq \emptyset$.

In both the cases, we end up with a new GD-sequence S' of G. In the former case, we get that $X_t \cap \hat{S}' = \emptyset$ and $|X_{t-1} \cap \hat{S}'| = 1$. The latter case ensures that $X_t \cap \hat{S}' = \emptyset$ and $X_{t-1} \cap \hat{S}' = \emptyset$ (if t > i+2). By repeating the above arguments we get that, there is a GD-sequence S of G such that either $(\bigcup_{r=i+1}^{k} X_r) \cap \hat{S} = \emptyset$ or $(\bigcup_{r=i+2}^{k} X_r) \cap \hat{S} = \emptyset$ and $|X_{i+1} \cap \hat{S}| = 1$ for some $i \in [k]$.

Analogous to Lemma 3, we give a symmetric lemma for the set Y of G, whose proof follows similar lines, and is omitted.

Lemma 4. Let S be a GD-sequence of G of type (c) and let $j \in [k]$ be the smallest index such that $\cup_{r=j}^{k} Y_r \subseteq \hat{S}$. If j > 1, then the following is true:

(1) If $j \ge 3$, then there exists a GD-sequence S' of G such that either $(\bigcup_{r=1}^{j-1} Y_r) \cap \hat{S}' = \emptyset$ or $(\bigcup_{r=1}^{j-2} Y_r) \cap \hat{S}' = \emptyset \text{ and } |Y_{j-1} \cap \hat{S}'| = 1.$

(2) If j = 2, then either $Y_1 \cap \hat{S} = \emptyset$ or $|Y_1 \cap \hat{S}| = 1$.

Lemma 5. Let S be a GD-sequence of G satisfying all properties of Lemmas 1, 2, 3 and 4, let $i \in [k]$ have the role as in Lemma 3, and let $j \in [k]$ have the role as in Lemma 4. The following statements are true:

(1) $j \in \{i, i+1\};$

(2) if j = i then $|X_i| = 1$ or $|Y_i| = 1$; (3) if j = i + 1, then either $|X_{i+1} \cap \hat{S}| = 1$ or $|Y_i \cap \hat{S}| = 1$.

Proof. Since S satisfies all properties of Lemmas 1, 2, 3 and 4, there are integers $i \in$ $[k-1], j \in \{2, \ldots, k\}$ such that $\cup_{r=1}^{i} X_r \subseteq \hat{S}$ and $\cup_{r=j}^{k} Y_r \subseteq \hat{S}$. It also holds that either $X_{i+1} \cap \hat{S} = \emptyset$ or $|X_{i+1} \cap \hat{S}| = 1$. Similarly, either $Y_{j-1} \cap \hat{S} = \emptyset$ or $|Y_{j-1} \cap \hat{S}| = 1$. Using Lemmas 3 and 4, we can also say that if $i \leq k-2$, then $(\bigcup_{r=i+2}^{k} X_r) \cap \hat{S} = \emptyset$ and, if $j \geq 3$, then $(\bigcup_{r=1}^{j-2} Y_r) \cap \hat{S} = \emptyset$.

First, we show that $j \ge i$. To the contrary, assume that j < i. This implies that $X_{i-1} \cup$ $Y_{i-1} \cup X_i \cup Y_i \subseteq \hat{S}$. If X_i appears before Y_i , then Y_{i-1} appears after Y_i . In this case, we see that X_{i-1} can not appear anywhere in the sequence. So, this case is not possible. If Y_i appears before X_i , then X_{i-1} appears after X_i . Here, Y_{i-1} can not appear anywhere in the sequence. So, this case is also not possible. So, we get that $j \ge i$. Now, either j = i or $j \ge i+1.$

First, we assume j = i. If X_i appears before Y_i , then an eventual second vertex from Y_i does not footprint any vertex, a contradiction. So, $|Y_i| = 1$. In the similar way, we get that $|X_i| = 1$, when Y_i appears before X_i . Thus, property (2) holds.

Next, assume that $j \ge i + 1$. We need to show that j = i + 1. First, we show that $j \leq i+3$. If $j \geq i+4$, then $X_{i+2} \cap \hat{S} = \emptyset, Y_{i+2} \cap \hat{S} = \emptyset$. This contradicts Observation 4. So, $j \in \{i+1, i+2, i+3\}$. If j = i+3, then we see that $\gamma_{gr}(G) \leq \alpha + \beta + 2$, where $\alpha = \sum_{r=1}^{i} |X_r| \text{ and } \beta = \sum_{r=i+3}^{k} |Y_r|. \text{ But, the sequence } (X_1) \oplus (X_2) \oplus \cdots \oplus (X_i) \oplus (X_{i+1}) \oplus (Y_k) \oplus (Y_{k-1}) \oplus \cdots \oplus (Y_{i+3}) \oplus (x) \oplus (X_{i+2}), \text{ where } x \in X_{i+3} \text{ is a dominating sequence of } G$ of length at least $\alpha + \beta + 3$. So, $j \neq i + 3$. If j = i + 2, then we see that $\gamma_{gr}(G) \leq \alpha + \beta' + 2$, where $\beta' = \sum_{r=i+2}^{k} |Y_r|$. Now, consider the sequence $S_0 = (X_1) \oplus (X_2) \oplus \cdots \oplus (X_i) \oplus (Y_k) \oplus (Y_{k-1}) \oplus \cdots \oplus (Y_{i+2}) \oplus (y) \oplus (Y_{i+1})$, where $y \in Y_i$. If $|Y_{i+1}| > 1$, S_0 is a dominating sequence of G having length at least $\alpha + \beta' + 3$. This implies that $|Y_{i+1}| = 1$ and S_0 is also a GDsequence of G. So, we consider S_0 as a GD-sequence of G as it also satisfies Lemmas 1, 2, 3 and 4 and thus, j = i + 1. Thus, property (1) holds.

For the property (3), we show that either $|X_{i+1} \cap \hat{S}| = 1$ or $|Y_i \cap \hat{S}| = 1$. So, first we assume that neither is true, that is, $|X_{i+1} \cap \hat{S}| = 0$ and $|Y_i \cap \hat{S}| = 0$. Now, if Y_{i+1} appears before X_i , then the length of S can be increased by including a vertex of X_{i+1} just before X_i , but S is a dominating sequence of G of maximum length. So, Y_{i+1} appears after X_i , thus length of S can be increased by including a vertex of Y_i just before Y_{i+1} , but S is a dominating sequence of \hat{G} of maximum length. Hence, $|X_{i+1} \cap \hat{S}| = 1$ or $|Y_i \cap \hat{S}| = 1$. If $|X_{i+1} \cap \hat{S}| = 1$ and $|Y_i \cap \hat{S}| = 1$, then suppose that $X_{i+1} \cap \hat{S} = \{a\}$ and $Y_i \cap \hat{S} = \{b\}$. Clearly, vertices of the four sets $\{a\}, \{b\}, X_i$ and Y_{i+1} appear in S. Let $\mathcal{K} = \{\{a\}, \{b\}, X_i, Y_{i+1}\}$. Recall that all vertices of X_i appear together in S. Similarly, all vertices of Y_{i+1} appear together in S. Let $A \in \mathcal{K}$ be the set whose vertices appear after the other three sets of \mathcal{K} in the sequence S. Then, all vertices of N[A] are footprinted before the appearance of vertices of A. Therefore, either $|X_{i+1} \cap \hat{S}| = 1$ or $|Y_i \cap \hat{S}| = 1$. Thus property (3) holds.

Lemma 6. Let S be a GD-sequence of G of type (c) and satisfies all properties of Lemmas 1, 2, 3, 4 and 5. Then the sequence S satisfies the following properties

(1) If j = i then, $X_{i+1} \cap \hat{S} = \emptyset$ and $Y_{i-1} \cap \hat{S} = \emptyset$; (2) If j = i+1, then either $|X_{i+1} \cap \hat{S}| = 1$ and $Y_i \cap \hat{S} = \emptyset$ or $|Y_i \cap \hat{S}| = 1$ and $X_{i+1} \cap \hat{S} = \emptyset$.

Proof. First, let j = i and $X_{i+1} \cap \hat{S} \neq \emptyset$. This implies that $|X_{i+1} \cap \hat{S}| = 1$. There are two cases.

Case 1: A vertex of X_{i+1} footprints itself.

In this case, both Y_i and Y_{i+1} appear after X_{i+1} . Note that Y_{i+1} appears before Y_i . Here, we see that X_i can not appear after Y_i as all vertices in the closed neighborhood of X_i are footprinted before its appearance. So, X_i appears before Y_i , but, then all vertices in the closed neighborhood of Y_i are footprinted before its appearance.

Case 2: Vertices of X_{i+1} are footprinted before their appearance in S.

Here, at least one of Y_i and Y_{i+1} appear before X_{i+1} . So, first we assume that Y_i appears before X_{i+1} . Let A denotes the set of vertices which appear before Y_i , B denotes the set of vertices which appear before X_{i+1} and after Y_i and C denotes the set of vertices which appear after X_{i+1} in S. In this case, $X_i \not\subseteq C$, so $X_i \subseteq A \cup B$. But, then one of Y_{i+1} and X_{i+1} does not footprint any vertex, a contradiction. Similar arguments can be given when Y_{i+1} appears before X_{i+1} . Hence, $X_{i+1} \cap \hat{S} = \emptyset$. In the similar manner, we can prove that $Y_{i-1} \cap \hat{S} = \emptyset.$

Next, we assume that j = i + 1 and $|X_{i+1} \cap \hat{S}| = 1$. We need to show that $Y_i \cap \hat{S} = \emptyset$. On the contrary, suppose that $Y_i \cap \hat{S} \neq \emptyset$. Here, we see that S contain vertices from all of the sets X_i, X_{i+1}, Y_i and Y_{i+1} . Then there exists a vertex $a \in X_i \cup X_{i+1} \cup Y_i \cup Y_{i+1}$ whose closed neighborhood is footprinted before its appearance, a contradiction. In the similar way, we can prove that if $|Y_i \cap \hat{S}| = 1$, then $X_{i+1} \cap \hat{S} = \emptyset$.

Lemma 7. Let $S = (v_1, v_2, \dots, v_p)$ be a GD-sequence of G, which satisfies $X \subseteq \hat{S}$. Then one of the following statements is true.

1. $\gamma_{gr}(G) = |X|$ 2. $\gamma_{gr}(G) = |Y|$ 3. $\gamma_{qr}(G) = |X| + |Y_k|$

Proof. If S is not a type (c) GD-sequence of G, then $Y \cap \hat{S} = \emptyset$ and so, $\gamma_{qr}(G) = |X|$. So, assume that S is a type (c) GD-sequence of G.

Case 1: $v_1 \in Y_1$.

In this case, all vertices of X are footprinted by v_1 . So, all the vertices v_2, \ldots, v_p appear to footprint vertices of $Y \setminus \{v_1\}$ only. This implies that $\gamma_{gr}(G) \leq |Y|$ and so, $\gamma_{gr}(G) = |Y|$. Case 2: $v_1 \notin Y_1$.

Here, we see that S satisfies all conditons of part (2) of Lemma 1 with i = k. Using Lemma 4, we get that there exists an integer $j \in [k]$ such that $\bigcup_{r=j}^{k} Y_r \subseteq \hat{S}$ and Lemma 5 ensures that j = k. Hence, we have that $Y_k \subseteq \hat{S}$ and $(\bigcup_{r=1}^{k-2} Y_r) \cap \hat{S} = \emptyset$, if $k \ge 3$. Thus, we have that $X \cup Y_k \subseteq \hat{S}$ and so, $\gamma_{gr}(G) \ge |X| + |Y_k|$. Now, if there is no vertex of Y before the vertices of X_k in S, then $\gamma_{gr}(G) \leq |X|$. So, all vertices of Y_k appear before vertices of X_k .

Next, we have that $Y_{k-1} \cap \widehat{S} = \emptyset$ using Lemma 6. Therefore, $\widehat{S} = X \cup Y_k$ which implies that $\gamma_{gr}(G) = |X| + |Y_k|$.

Analogous to Lemma 7, we give a symmetric lemma for the set Y of G, whose proof follows similar lines, and is omitted.

Lemma 8. Let $S = (v_1, v_2, \dots, v_p)$ be a GD-sequence of G, which satisfies $Y \subseteq \hat{S}$. Then one of the following statements is true.

1. $\gamma_{qr}(G) = |Y|$

2. $\gamma_{gr}(G) = |X|$ 3. $\gamma_{gr}(G) = |Y| + |X_1|$

Lemma 9. If G = (X, Y, E) is a chain graph such that every GD-sequence of G is of type (c), then for any GD-sequence S of G, the following statements are true:

(1) $\cup_{r=1}^{i} X_r \subseteq \hat{S} \text{ and } \cup_{r=j}^{k} Y_r \subseteq \hat{S} \text{ for some } i, j \in [k].$ (2) Integers i and j satisfy exactly one of the following: (a) i < k, j > 1. (b) i = 1, j = 1. (c) i = k, j = k. (3) If i < k, j > 1, then $\gamma_{gr}(G) = \begin{cases} \sum_{r=1}^{i} |X_r| + \sum_{r=j}^{k} |Y_r| & :j = i \\ \sum_{r=1}^{i} |X_r| + \sum_{r=j}^{k} |Y_r| + 1 & :j = i+1 \end{cases}$ (4) If i = 1, j = 1, then $\gamma_{gr}(G) = |Y| + |X_1|$ (5) If i = k, j = k, then $\gamma_{gr}(G) = |Y| + |X_1|$

(5) If i = k, j = k, then $\gamma_{gr}(G) = |X| + |Y_k|$

Proof. Lemmas 1 and 2 ensure property (1). To prove property (2), we need to show that i = k, j = 1 cannot be true. So, assume that this is true. Then, Lemma 5 yields k = 1, a contradition proving that property (2) holds. Property (3) follows from Lemmas 3, 4, 5 and 6. Properties (4) and (5) can be proved using Lemmas 7 and 8.

We are ready to present an algorithm for computing a GD-sequence of a chain graph based on the above lemmas; see Algorithm 1.

By following the above discussion, note that $\gamma_{qr}(G) \in \mathcal{A}$, where

$$\mathcal{A} = \{n_1, n_2, n_1 + |Y_k|, n_2 + |X_1|, \sum_{r=1}^i |X_r| + \sum_{l=i}^k |Y_l|, \sum_{r=1}^i |X_r| + \sum_{l=i+1}^k |Y_l| + 1\},\$$

for some $i \in [k-1]$. Thus, the sequence returned by Algorithm 1 is a GD-sequence of G. It is easy to see that Algorithm 1 computes S in linear time, which is the time needed to compute the parts $X_1, \ldots, X_k, Y_1, \ldots, Y_k$. The following theorem readily follows.

Algorithm 1: GD-sequence of a chain graph

Input: A chain graph G = (X, Y, E) without isolated vertices along with a chain ordering $(x_1, ..., x_{n_1}, y_1, ..., y_{n_2})$ of V(G). **Output:** A GD-sequence S of G. **1** Find the parts $X_1, X_2, ..., X_k$ and $Y_1, Y_2, ..., Y_k$; **2** i = 0, sum[] = 0;**3** if $|Y_1| = 1$ then 4 $sum[0] = n_2 + |X_1|;$ 5 else **6** $sum[0] = n_2;$ 7 for i = 1 : k - 1 do **s** $\begin{bmatrix} sum[i] = \sum_{j=1}^{i} |X_j| + n_2 - \sum_{j=1}^{i} |Y_j| + 1; \end{bmatrix}$ 9 if $|X_k| = 1$ then **10** $sum[k] = n_1 + |Y_k|;$ 11 else **12** $sum[k] = n_1;$ 13 Find an index $i^* \in \{0, 1, 2, \dots, k\}$ for which sum[i] is maximum; 14 if $i^* == 0$ and $|Y_1| > 1$ then 15 $S \leftarrow (Y_k) \oplus (Y_{k-1}) \oplus \cdots \oplus (Y_1);$ 16 else if $i^* == 0$ and $|Y_1| = 1$ then if $k \ge 3$ then 17 $S \leftarrow (X_1) \oplus (Y_k) \oplus \cdots \oplus (Y_3) \oplus (Y_1) \oplus (Y_2);$ 18 else 19 $S \leftarrow (X_1) \oplus (Y_1) \oplus (Y_2);$ $\mathbf{20}$ 21 else if $i^* == k$ and $|X_k| > 1$ then $S \leftarrow (X_1) \oplus (X_2) \oplus \cdots \oplus (X_k);$ $\mathbf{22}$ 23 else if $i^* == k$ and $|X_k| = 1$ then if $k \geq 3$ then $\mathbf{24}$ $\overline{S} \leftarrow (X_1) \oplus (X_2) \oplus \cdots \oplus (X_{k-2}) \oplus (Y_k) \oplus (X_k) \oplus (X_{k-1});$ $\mathbf{25}$ else 26 $S \leftarrow (Y_2) \oplus (X_2) \oplus (X_1);$ $\mathbf{27}$ 28 else 29 choose a vertex $x \in X_{i^*+1}$ if $i^* > 1$ then 30 $S \leftarrow (X_1) \oplus (X_2) \oplus \cdots \oplus (X_{i^*-1}) \oplus (Y_k) \oplus (Y_{k-1}) \oplus \cdots \oplus (Y_{i^*+1}) \oplus x \oplus (X_{i^*});$ 31 else 32 33 **34** return S.

Theorem 4. Algorithm 1 outputs a GD-sequence S of G in linear time.

There is a connection between Grundy domination number of a graph and its independence number. Let A be an independent set of size $\alpha(G)$. By considering all vertices of A in

any order, we get a closed neighborhood sequence of G, which yields the well known bound $\gamma_{gr}(G) \geq \alpha(G)$. In addition, for a chain graph G, we prove that Grundy domination number is either $\alpha(G)$ or $\alpha(G) + 1$.

Theorem 5. If G is a chain graph, then $\gamma_{gr}(G) \in \{\alpha(G), \alpha(G) + 1\}$.

Proof. Let G be a chain graph. By using the notation established in this section, we claim that $\alpha(G) \in \{n_1, n_2, \sum_{j=1}^i |X_j| + \sum_{j=i+1}^k |Y_j|$ for some $i \in [k-1]\}$. To see this, let A be a maximum independent set of G. Three cases are possible. If $A \subseteq X$, then A = X implying that $\alpha(G) = n_1$; if $A \subseteq Y$, then A = Y implying that $\alpha(G) = n_2$. Now, if $A \cap X \neq \emptyset$ and $A \cap Y \neq \emptyset$, then one can easily infer that $A = (\bigcup_{j=1}^i X_j) \cup (\bigcup_{j=i+1}^k Y_j)$ for some $i \in [k-1]$ implying that $\alpha(G) = \sum_{j=1}^i |X_j| + \sum_{j=i+1}^k |Y_j|$. Letting $i_0 \in [k-1]$ be an index such that $\sum_{j=1}^{i_0} |X_j| + \sum_{j=i_0+1}^k |Y_j| \ge \sum_{j=1}^i |X_j| + \sum_{j=i_0+1}^k |Y_j|$ for each $i \in [k-1]$, we may write $\alpha(G) \in \{n_1, n_2, \sum_{j=1}^{i_0} |X_j| + \sum_{j=i_0+1}^k |Y_j|\}$.

Algorithm 1 computes a GD-sequence of G and it turns out that $\gamma_{gr}(G) \in \{n_1, n_2, n_1 + |Y_k|, n_2 + |X_1|, \sum_{j=1}^i |X_j| + \sum_{j=i}^k |Y_j|, \sum_{j=1}^i |X_j| + \sum_{j=i+1}^k |Y_j| + 1\}$ for some $i \in [k-1]$. Note that $\gamma_{gr}(G) = n_1 + |Y_k|$ when $|X_k| = 1$ implying that $\gamma_{gr}(G) = \sum_{j=1}^{k-1} |X_j| + |Y_k| + 1$. Similarly, $\gamma_{gr}(G) = n_2 + |X_1|$ when $|Y_1| = 1$ implying that $\gamma_{gr}(G) = |X_1| + \sum_{j=2}^k |Y_k| + 1$. Now, if $\gamma_{gr}(G) = \sum_{j=1}^i |X_j| + \sum_{j=i}^k |Y_j|$ for some $i \in [k-1]$ then Lemma 5 ensures that $\gamma_{gr}(G)$ is either $\sum_{j=1}^i |X_j| + \sum_{j=i+1}^k |Y_j| + 1$ or $\sum_{j=1}^{i-1} |X_j| + \sum_{j=i}^k |Y_j| + 1$. Hence, $\gamma_{gr}(G) \in \{n_1, n_2, \sum_{j=1}^i |X_j| + \sum_{j=i+1}^k |Y_j| + 1$ for some $i \in [k-1]\}$. Since Algorithm 1 computes the GD-sequence by finding the maximum of the set $\{n_1, n_2\} \cup \{\sum_{j=1}^i |X_j| + \sum_{j=i+1}^k |Y_j| + 1: i \in [k-1]\}$, we have, $\gamma_{gr}(G) \in \{n_1, n_2, \sum_{j=1}^{i_0} |X_j| + \sum_{j=i_0+1}^k |Y_j|$, then we can write that $\alpha(G) \in \{n_1, n_2, t\}$ and $\gamma_{gr}(G) \in \{n_1, n_2, t+1\}$. Now, we consider three cases.

Case 1: $\gamma_{gr}(G) = n_1$:

In this case, $n_1 \ge n_2$ and $n_1 \ge t+1 > t$. This implies that $\alpha(G) = n_1$.

Case 2: $\gamma_{gr}(G) = n_2$:

In this case, $n_2 \ge n_1$ and $n_2 \ge t+1 > t$. This implies that $\alpha(G) = n_2$.

Case 3: $\gamma_{gr}(G) = t + 1$: In this case, if $\alpha(G) = n_1$ then $n_1 \ge t = \gamma_{gr}(G) - 1$. So, $\gamma_{gr}(G) \le \alpha(G) + 1$. Similarly, if $\alpha(G) = n_2$ then $n_2 \ge t = \gamma_{gr}(G) - 1$. So, $\gamma_{gr}(G) \le \alpha(G) + 1$. Otherwise, $\alpha(G) = t$ implying that $\gamma_{gr}(G) = \alpha(G) + 1$.

Therefore, $\gamma_{gr}(G) \in \{\alpha(G), \alpha(G) + 1\}.$

5 Conclusion

In this paper, we studied the GDD problem for bipartite graphs and co-bipartite graphs. We proved that the problem is NP-complete for bipartite graphs and efficiently solvable for chain graphs, which form a subclass of bipartite graphs. We also proved NP-completeness of the problem in co-bipartite graphs. To obtain a complete dichotomy, it would be interesting to find the status of the problem in some graph classes that lie between chain graphs and bipartite graphs.

Note that the GDD problem in the class of co-chain graphs (that is, the complements of chain graphs) is easily solvable. Indeed, in a co-chain graph G = (X, Y, E), one can find similar partitions of X and Y into k sets X_1, \ldots, X_k , and Y_1, \ldots, Y_k , respectively, that arises from the closed twin relation in G. Then, one can also immediately infer that $\gamma_{gr}(G) = k$. It would be interesting to see if there are some other known classes of graphs \mathcal{G} whose complement class $\overline{\mathcal{G}} = \{\overline{G} : G \in \mathcal{G}\}$ has similar status of the computational complexity of the GDD problem as \mathcal{G} . Two nice instances (with \mathcal{G} given by bipartite graphs and chain graphs) are presented in this paper.

Declarations

Conflict of interest: The authors have no conflicts of interest to declare that are relevant to the content of this article.

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