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Singleton Coalition Graph Chains

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Abstract

Let G be graph with vertex set V and order n = |V|. A coalition in G is a combination of two distinct sets, $A \subseteq V$ and $B \subseteq V$, which are disjoint and are not dominating sets of G, but their union $A \cup B$ is a dominating set of G. A coalition partition of G is a partition $\mathcal{P} = \{S_1, \ldots, S_k\}$ of its vertex set V, where each set $S_i \in \mathcal{P}$ is either a dominating set of G with only one vertex, or it is not a dominating set but forms a coalition with some other set $S_i \in \mathcal{P}$. The coalition number C(G) is the maximum cardinality of a coalition partition of G. To represent a coalition partition \mathcal{P} of G, a coalition graph $CG(G, \mathcal{P})$ is created, where each vertex of the graph corresponds to a member of \mathcal{P} and two vertices are adjacent if and only if their corresponding sets form a coalition in G. A coalition partition \mathcal{P} of G is a singleton coalition partition if every set in \mathcal{P} consists of a single vertex. If a graph G has a singleton coalition partition, then G is referred to as a singleton-partition graph. A graph H is called a singleton coalition graph of a graph G if there exists a singleton coalition partition \mathcal{P} of G such that the coalition graph $CG(G, \mathcal{P})$ is isomorphic to H. A singleton coalition graph chain with an initial graph G_1 is defined as the sequence $G_1 \to G_2 \to G_3 \to \cdots$ where all graphs G_i are singleton-partition graphs, and $CG(G_i, \Gamma_1) = G_{i+1}$, where Γ_1 represents a singleton coalition partition of G_i . In this paper, we address two open problems posed by Haynes et al. We characterize all graphs G of order n and minimum degree $\delta(G) = 2$ such that C(G) = n. Additionally, we investigate the singleton coalition graph chain starting with graphs G where $\delta(G) < 2.$

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1 Introduction

Let G = (V(G), E(G)) be a graph with vertex set V(G) and edge set E(G), and of order n(G) = |V(G)| and size m(G) = |E(G)|. If the graph G is clear from context, we write V and E rather than V(G) and E(G). The open neighborhood $N_G(v)$ of a vertex v in G is the set of vertices adjacent to v, while the closed neighborhood of v is the set $N_G[v] = \{v\} \cup N_G(v)$. For a set $S \subseteq V(G)$, its open neighborhood is the set $N_G(S) = \bigcup_{v \in S} N_G(v)$, and its closed neighborhood is the set $N_G[S] = N_G(S) \cup S$. We denote the degree of v in G by $\deg_G(v) = |N_G(v)|$. The vertex v is called a full vertex if it is adjacent to all vertices $V \setminus \{v\}$, that is, if $N_G[v] = V$. An isolated vertex is a vertex of degree 0, and an isolate-free graph is a graph that contains no isolated vertex. The minimum and maximum degrees in G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. For a positive integer k, we let $[k] = \{1, \ldots, k\}$.

A subset D of V is called a *dominating set* in G if every vertex of G not in D is adjacent to at least one vertex of D. The minimum cardinality of a dominating set of G is the *domination* number of G, denoted by $\gamma(G)$. Domination in graphs is now very well studied in the literature. If $X, Y \subseteq V$, then set X dominates the set Y if $Y \subseteq N_G[X]$, that is, every vertex $y \in Y$ belongs to X or is adjacent to a vertex of X. A comprehensive treatment of domination in graphs can be found in the recent so-called "domination books" [6–9].

In 2020, Haynes, Hedetniemi, Hedetniemi, McRae, and Mohan [1] introduced and studied the concept of a *coalition* in a graph. They defined a pair of sets $A, B \subseteq V$ to be a *coalition* in G = (V, E) if neither A not B is a dominating sets in G, but $A \cup B$ is a dominating set of G. Such a pair A and B is said to form a coalition, and A and B are called *coalition partners*.

A vertex partition $\mathcal{P} = \{S_1, S_2, \ldots, S_k\}$ of V is a coalition partition of G, abbreviated a *c*partition in [1], if every set $S_i \in \mathcal{P}$ is either a dominating set of G with cardinality $|S_i| = 1$ or is not a dominating set of G but forms a coalition with some other set S_j in the partition \mathcal{P} . The maximum cardinality of a coalition partition of G is the coalition number of G, denoted by $\mathcal{C}(G)$. A motivation of this graph theoretic model of a coalition is given by Haynes et al. in their series of papers on coalitions in [1, 3-5]. In these papers, a special emphasis is on studying the coalition number in trees and cycles. Additionally, in [4], Haynes et al. introduced upper bounds on the coalition number of a graph in terms of the minimum and maximum degree.

In [1], Haynes et al. defined a coalition graph as follows. Given a coalition partition $\mathcal{P} = \{S_1, \ldots, S_k\}$ of a graph G, a graph called the *coalition graph* of G, denoted by $CG(G, \mathcal{P})$, is assigned to this partition, where the vertices correspond to the members of \mathcal{P} and two vertices are adjacent if and only if the corresponding members in \mathcal{P} form a coalition in the graph G. In [3], Haynes et al. examined coalition graphs in trees, paths, and cycles. Additionally, in [5] they showed that every graph is the coalition graph of some graph. In [1], Haynes et al. posed the following open problem.

Problem 1 ([1]) Characterize all graphs G of order n with C(G) = n.

In [2], Bakhshesh et al. characterized all graphs G of order n with $\delta(G) \leq 1$ whose coalition number is n, thereby partially solving Problem 1. They also identified all trees whose coalition number is n-1. In this paper, we make further progress in solving Problem 1. For this purpose, we characterize all graphs G of order n and $\delta(G) = 2$ satisfying C(G) = n.

In [5], Haynes et al. defined the following three concepts associated with a coalition graph: singleton-partition graph, singleton coalition graph and singleton coalition graph chain. Let G be a graph of order n. Let Γ_1 be the partition of V(G) such that every member of Γ_1 is a singleton set (a set with only one element). We call such a coalition partition of G a Γ_1 -partition of G. The graph G is a singleton-partition graph (SP-graph) if it has a Γ_1 -partition. If G is a SP-graph, then the coalition graph $CG(G, \Gamma_1)$ is called the singleton coalition graph (SC-graph) of G. A singleton coalition graph chain with an initial graph G_1 is defined as a sequence

$$\mathcal{S}_{G_1} := G_1 \to G_2 \to G_3 \to \cdots \to G_k$$

where all graphs G_i are singleton-partition graphs, and $\operatorname{CG}(G_i, \Gamma_1) = G_{i+1}$ for all $i \in [k-1]$. The length of the sequence \mathcal{S}_{G_1} is defined as k-1. If \mathcal{S}_{G_1} is an infinite sequence of graphs G_i , then its length is considered to be infinity (∞) . By convention, if all graphs G_i in \mathcal{S}_{G_1} are isomorphic, the length of \mathcal{S}_{G_1} is defined to be zero. We denote a sequence \mathcal{S}_{G_1} of maximum possible length (starting from the initial graph G_1) by G_1 -SC chain and we denote its length by $L_{\operatorname{SCC}}(G_1)$. For example, the C_4 -SC chain and C_5 -SC chain are given by

$$C_4 \to K_4 \to \overline{K}_4$$
 and $C_5 \to C_5 \to C_5 \to \cdots$,

respectively, while the P_3 -SC chain is given by the sequence

$$P_3 \to K_1 \cup K_2 \to P_3 \to K_1 \cup K_2 \to \cdots$$
.

Hence, $L_{SCC}(C_4) = 2$, $L_{SCC}(C_5) = 0$, and $L_{SCC}(P_3) = \infty$. In [5], Haynes et al. posed the following open problem.

Problem 2 ([5]) Investigate singleton coalition graph chains.

In addition to presenting Problem 2, Haynes et al. also posed the following question: "Do arbitrarily long singleton coalition graph chains exist?" In this paper, we show that there exist graphs G such that the length of the G-SC chain is infinity. Moreover, we characterize G-SC chains for all singleton coalition graph G with $\delta(G) \leq 2$.

2 Preliminaries

In this section, we present some known and preliminary results on the coalition number of graphs, as well as additional definitions. In [2], the authors characterize all SP-graphs G with $\delta(G) \in \{0, 1\}$. For the graphs with $\delta(G) = 0$, they proved the following result.

Theorem 1 ([2]) If G is a graph of order n with $\delta(G) = 0$, then C(G) = n if and only if $G \cong K_1 \cup K_{n-1}$.

For graphs with $\delta(G) = 1$ with exactly one full vertex, they proved the following result.

Theorem 2 ([2]) If G is a graph of order $n \ge 3$ with $\delta(G) = 1$ and with exactly one full vertex, then C(G) = n if and only if G is obtained from the graph $K_1 \cup K_{n-1}$ by adding an edge joining the isolated vertex to an arbitrary vertex of the complete graph K_{n-1} .

For the graphs with $\delta(G) = 1$ and with no full vertex, they defined a family \mathcal{F}_1 of graphs as follows.

Definition 1 ([2]) (The family \mathcal{F}_1) Let G be a graph constructed as follows. The vertex set of G consists of $\{x, y, w\}$ and two disjoint sets, P and Q, with $P \cap Q \cap \{x, y, w\} = \emptyset$ and $|P \cup Q| \ge 1$. If Q is non-empty, then $|Q| \ge 2$. To define the edge set E(G), we start by assigning y as the unique neighbor of x, and so $N_G(x) = \{y\}$. We then join w to all vertices in $P \cup Q$, and so $N_G(w) = P \cup Q$. For each vertex $p \in P$, we join it to all vertices in $(P \cup Q) \setminus \{p\}$. If Q is non-empty, we join y to all vertices in Q, and so $Q \cup \{x\}$ is a subset of N(y). Furthermore if $Q \neq \emptyset$, then we add edges between vertices in Q, including the possibility of adding no edge, in such a way that G[Q] does not contain a full vertex. Finally, we add any number of edges between y and vertices in P. Let G denote the resulting isolate-free graph. Let \mathcal{F}_1 be the family consisting of all such graphs G constructed in this way.

We note that if $G \in \mathcal{F}_1$ has order *n* and is disconnected, then $G \cong K_2 \cup K_{n-2}$. In [2], the authors presented the following result.

Theorem 3 ([2]) A graph G with $\delta(G) = 1$ and with no full vertex is a SP-graph if and only if $G \in \mathcal{F}_1$.

If G is a SP-graph and x and y are distinct vertices in G such that $\{x\}$ and $\{y\}$ form a coalition, then $\{x, y\}$ is a dominating set of G, and so $\{x, y\} \cap N[v] \neq \emptyset$ for every vertex $v \in V(G)$. We state this formally as follows.

Observation 1 If x and y are distinct vertices and are not full vertices in a SP-graph G, then $\{x\}$ and $\{y\}$ form a coalition if and only if $\{x, y\} \cap N[v] \neq \emptyset$ for every vertex $v \in V(G)$.

Now, we assume that G is a SP-graph with $\delta(G) = 1$ and with no full vertex. By Theorem 3, $G \in \mathcal{F}_1$. We next characterize the SC-graph H of G. For this purpose, we define a family \mathcal{H}_1 of graphs.

Definition 2 (The family \mathcal{H}_1) This family comprises of all bipartite graphs H having two distinct parts $A_1 = \{x_1, y_1\}$ and $B_1 = P_1 \cup \{w_1\} \cup Q_1$, where w_1 is a vertex, and P_1 and Q_1 are sets of

vertices that satisfy the condition $|P_1 \cup Q_1| \ge 1$. Additionally, if $Q_1 \ne \emptyset$, then $|Q_1| \ge 2$. The edges of H are formed as follows: y_1 is adjacent to every vertex in B_1 , while x_1 is adjacent only to the vertices of P_1 and w_1 (as shown in Figure 1).



Figure 1: (a): A graph of \mathcal{H}_1 with $Q_1 \neq \emptyset$. (b): A graph of \mathcal{H}_1 with $Q_1 = \emptyset$.

The following notation is defined before proving the following theorem: Let G be an SP-graph and H the SC-graph of G. For a vertex x in G, we represent the corresponding vertex in H as \tilde{x} . We are now in a position to prove the following result.

Theorem 4 If $G \in \mathcal{F}_1$, then $CG(G, \Gamma_1) \in \mathcal{H}_1$.

Proof. Let $G \in \mathcal{F}_1$ with vertex set V = V(G) and the edge set E = E(G). Thus, $V = \{x, y\} \cup P \cup Q \cup \{w\}$, where the vertices x, y and w and the vertex subsets P and Q satisfy the conditions defined for the graphs in Definition 1 that belong to the family \mathcal{F}_1 . We note that x is a vertex of G with the minimum degree $\delta(G) = 1$, and $N_G(x) = \{y\}$. Let $H_1 = CG(G, \Gamma_1)$. By Observation 1, if $u \in P \cup \{w\}$, then $\{u\}$ and $\{x\}$ form a coalition, and so \tilde{u} is adjacent to \tilde{x} . If $u \in P \cup Q \cup \{w\}$, then $\{u\}$ and $\{y\}$ form a coalition, and so \tilde{u} is not adjacent to \tilde{u}' in H_1 . Since G has no full vertices, $\{x\}$ and $\{y\}$ do not form a coalition, and so \tilde{x} is not adjacent to \tilde{y} in H_1 . If $q \in Q$, then since G[Q] has no full vertex, $\{q\}$ and $\{x\}$ do not form a coalition, and so \tilde{q} is not adjacent to \tilde{x} in H_1 . By these properties of the graph H_1 , we infer that H_1 satisfies the conditions for the family \mathcal{H}_1 , and therefore $H_1 \in \mathcal{H}_1$. \Box

3 SP-graphs with minimum degree two

In this section, we characterize all graphs G of order n with $\delta(G) = 2$ and C(G) = n. We first present a family \mathcal{F}_2 of graphs.

Definition 3 (Family \mathcal{F}_2) Let G be a graph constructed as follows. The vertex set of G is defined as $V = \{x, y, z\} \cup L_1 \cup R_1 \cup R_2 \cup L_2 \cup W$, where L_1, R_1, R_2 , and L_2 are pairwise disjoint sets and

where the set W may intersect the set $L_1 \cup R_1 \cup R_2 \cup L_2$. The vertex x has degree 2 in G with y and z as its unique neighbors, and so $N_G(x) = \{y, z\}$. We now add additional edges to G as follows.

- (a) **The family** \mathcal{F}_2^1 . In this case, the sets W, L_1 , L_2 , and R_2 are defined to be empty and the set R_1 is non-empty. We join both vertices y and z to every vertex of R_1 , but do not add an edge between y and z. Further, we add edges between vertices in R_1 , including the possibility of adding no edge. The resulting graph G satisfies $\delta(G) = 2$ and has no full vertex. Let \mathcal{F}_2^1 be the family consisting of all such graphs G constructed in this way. An example of a graph in the family \mathcal{F}_2^1 is illustrated in Figure 2a.
- (b) The family \mathcal{F}_2^2 . In this case, the sets W, L_2 , and R_2 are defined to be empty and both sets L_1 and R_1 are non-empty. We join the vertex y to every vertex of $L_1 \cup R_1$, but do not add an edge between y and z. We join the vertex z to every vertex of R_1 but to no vertex of L_1 . Additionally, we add all edges between vertices in L_1 to form the clique $G[L_1]$. Further, we add edges between vertices in R_1 , including the possibility of adding no edge. Finally, we add any additional edges to G while maintaining a minimum degree of 2 (recall that the vertex x has degree 2 in G) and not creating any full vertex. The resulting graph G satisfies $\delta(G) = 2$ and has no full vertex. Let \mathcal{F}_2^2 be the family consisting of all such graphs G constructed in this way. An example of a graph in the family \mathcal{F}_2^2 is illustrated in Figure 2b.



Figure 2: (a): A graph in \mathcal{F}_2^1 (b): A graph in \mathcal{F}_2^2

(c) The family \mathcal{F}_2^3 . In this case, each of the sets L_1 , R_2 , and W is non-empty. Further the set $L_2 \subseteq W$, although possibly $L_2 = \emptyset$. We add all edges between vertices in W to form the clique G[W], and we join every vertex of W to all vertices of $L_1 \cup R_1 \cup R_2$. We join the vertex y to every vertex of $L_1 \cup R_1$, but do not add an edge between y and R_2 . We join the vertex z to every vertex of $R_1 \cup R_2$, but do not add an edge between z and L_1 . If R_1 is not empty, then we join each vertex in R_1 to all vertices in either L_1 or R_2 .

Now, either we do not add the edge yz, in which case we add all edges between vertices in L_1 to form the clique $G[L_1]$ and we add all edges between vertices in R_2 to form the clique $G[R_2]$, or we do add the edge yz, in which case we join each vertex in L_1 to all vertices in either L_1 or R_2 , and do the same for each vertex in R_2 .

Finally, we add any additional edges to G while maintaining a minimum degree of 2 (recall that the vertex x has degree 2 in G) and not creating any full vertex. The resulting graph G satisfies $\delta(G) = 2$ and has no full vertex. Let \mathcal{F}_2^3 be the family consisting of all such graphs G constructed in this way. Examples of graphs in the family \mathcal{F}_2^3 are illustrated in Figure 3.

The family \mathcal{F}_2 is defined as $\mathcal{F}_2 = \mathcal{F}_2^1 \cup \mathcal{F}_2^2 \cup \mathcal{F}_2^3$.



Figure 3: Two different graphs in \mathcal{F}_2^3 . (a): $(y, z) \notin E$. (b): $(y, z) \in E$.

We determine next the SP-cycles with no full vertex. The coalition number of a cycle is determined in [1]. In particular, Haynes et al. [1] showed that if G is a cycle C_n , then $C(C_n) = n$ if and only if $3 \le n \le 6$. Since C_3 has a full vertex, we consider here cycles C_n for $4 \le n \le 6$. If G is the 4-cycle xyazx, then taking $R_1 = \{a\}$, we infer that $G \in \mathcal{F}_2^1$. If G is the 5-cycle xyabzx, then taking $W = \{a, b\}, L_1 = \{a\}, R_2 = \{b\}$, and $R_1 = L_2 = \emptyset$, we infer that $G \in \mathcal{F}_2^3$. If G is the 6-cycle xyabczx, then taking $W = \{b\}, L_1 = \{a\}, R_2 = \{c\}$, and $R_1 = L_2 = \emptyset$, we infer that $G \in \mathcal{F}_2^3$. We state these observations formally as follows.

Observation 2 For $n \geq 3$, if C_n is a SP-graph, then $4 \leq n \leq 6$ and $G \in \mathcal{F}_2$.

We show next that if G is a SP-graph with $\delta(G) = 2$ that contains no full vertex, then G belongs to the family \mathcal{F}_2 .

Theorem 5 If G is a SP-graph with $\delta(G) = 2$ and with no full vertex, then $G \in \mathcal{F}_2$.

Proof. Let G = (V, E) be a SP-graph with $\delta(G) = 2$ and with no full vertex, and let G have order n, and so C(G) = n. Let x be a vertex of degree 2 in G with neighbors y and z, and so $N_G[x] = \{x, y, z\}$. Further, let \mathcal{P} be a singleton coalition partition of G. By Observation 1, all sets in $\mathcal{P} \setminus \{\{x\}, \{y\}, \{z\}\}$ form a coalition with at least one of $\{x\}, \{y\}$, or $\{z\}$, while no two sets in $\mathcal{P} \setminus \{\{x\}, \{y\}, \{z\}\}$ form a coalition. In what follows, we let $V_x = V \setminus \{x, y, z\}$, and so $V = N_G[x] \cup V_x$. We proceed further with a series of claims.

Claim 1 If $\{x\}$ forms a coalition with both $\{y\}$ and $\{z\}$, then $G \in \mathcal{F}_2$.

Proof. Suppose that $\{x\}$ forms a coalition with both $\{y\}$ and $\{z\}$. Thus, both $\{x, y\}$ and $\{x, z\}$ are dominating sets in G. Hence each of y and z is adjacent to all vertices in V_x . Since G has no full vertex, it follows that $yz \notin E$. For each vertex $u \in V_x$, the sets $\{u, y\}$ and $\{u, z\}$ are dominating sets in G, and so $\{u\}$ forms a coalition with both $\{y\}$ and $\{z\}$. Thus, letting $R_1 = V \setminus \{x, y, z\}$ and $W = L_1 = L_2 = R_2 = \emptyset$, we infer that $G \in \mathcal{F}_2^1$ (see Definition 3). Therefore, $G \in \mathcal{F}_2$.

Claim 2 If $\{x\}$ forms a coalition with exactly one of $\{y\}$ and $\{z\}$, then $G \in \mathcal{F}_2$.

Proof. Suppose that $\{x\}$ forms a coalition with exactly one of $\{y\}$ and $\{z\}$. Renaming the vertices y and z if necessary, we may assume that $\{x\}$ forms a coalition $\{y\}$ but not with $\{z\}$. Thus, $\{x, y\}$ is a dominating sets in G, implying that vertex y is adjacent to all vertices in V_x . Since G has no full vertex, it follows that $yz \notin E$. Since $\{x, z\}$ is not a dominating set, there exists a vertex in V_x that is not adjacent to z. Let $V_x = L_1 \cup R_1$, where L_1 and R_1 are the sets of vertices in V_x that are not adjacent to z and are adjacent to z, respectively. Necessarily, L_1 is non-empty since $\{x, z\}$ is not a dominating set. Let $u \in V_x$. If $u \in R_1$, then since $uz \in E$ and all vertices in V_x are adjacent to y, the sets $\{u\}$ and $\{y\}$ form a coalition. Now, suppose $u \in L_1$. Since z is not adjacent to u and y, the sets $\{u\}$ and $\{y\}$ do not form a coalition. Hence, $\{u\}$ must form a coalition with either $\{z\}$ or $\{x\}$. If $\{u\}$ forms a coalition with $\{x\}$, then all vertices in V_x are adjacent to u. If $\{u\}$ forms a coalition with $\{z\}$, then all vertices in U_x are adjacent to u. If $\{u\}$ forms a coalition with $\{z\}$, then all vertices in V_x are adjacent to u. If $\{u\}$ forms a coalition with $\{z\}$, then all vertices in V_x are adjacent to u. If $\{u\}$ forms a coalition with $\{z\}$, then all vertices in U_x are adjacent to u. If $\{u\}$ forms a coalition with $\{z\}$, then all vertices in U_x are adjacent to u. If $\{u\}$ forms a coalition with $\{z\}$, then all vertices in U_x are adjacent to u. If $\{u\}$ forms a coalition with $\{z\}$, then all vertices in U_x are adjacent to u. In both cases, $G[L_1]$ is a clique. By letting $W = L_2 = R_2 = \emptyset$, we infer that $G \in \mathcal{F}_2^2$ (see Definition 3). Therefore, $G \in \mathcal{F}_2$. (D)

Claim 3 If $\{x\}$ forms a coalition with neither $\{y\}$ nor $\{z\}$, then $G \in \mathcal{F}_2$.

Proof. Suppose that $\{x\}$ forms a coalition with neither $\{y\}$ nor $\{z\}$. Since \mathcal{P} is a singleton coalition partition of G, there exists a vertex $w \in V_x$ such that $\{x\}$ and $\{w\}$ form a coalition. Let W be the set of all vertices $w \in V_x$ such that $\{w\}$ and $\{x\}$ form a coalition. Thus, $\{x, w\}$ is a dominating set of G, implying that vertex w is adjacent to all vertices in $V_x \setminus \{w\}$. Hence, G[W] is a clique. Let L_1 be the set of all vertices in V_x which are adjacent to y and not are adjacent to z. Further let R_1 be the set of all vertices in V_x which are adjacent to both y and z, and let R_2 be the set of all vertices in V_x which are adjacent to z and not are adjacent to y. Let L_2 be the set of all vertices in V_x which are adjacent to z and not are adjacent to y. Let L_2 be the set of all vertices in V_x which are adjacent to z and not are adjacent to y. Let L_2 be the set of all vertices in V_x which are adjacent to z and not are adjacent to y. Let L_2 be the set of all vertices in V_x which are adjacent to z and not are adjacent to y. Let L_2 be the set of all vertices in V_x which are adjacent to z and not are adjacent to y. Let L_2 be the set of all vertices in V_x which are adjacent to z and not are adjacent to y. Let L_2 be the set of all vertices in V_x which are adjacent to neither y nor z (see Figure 4). We note that $L_1, R_1, R_2,$ and L_2 are pairwise disjoint. Since $W \subseteq V_x = L_1 \cup R_1 \cup R_2 \cup L_2$, we note that $W \cap (L_1 \cup R_1 \cup R_2 \cup L_2) \neq \emptyset$. Since $\{x, y\}$ is not a dominating set, $R_2 \cup L_2 \neq \emptyset$, and since $\{x, z\}$ is not a dominating set, $L_1 \cup L_2 \neq \emptyset$. We proceed further with two subclaims.

Claim 3.1 If $\{y\}$ and $\{z\}$ form a coalition, then $G \in \mathcal{F}_2$.

Proof. Suppose that $\{y\}$ and $\{z\}$ form a coalition. Thus, $\{y, z\}$ is a dominating set, and so in this case we note that $L_2 = \emptyset$.



Figure 4: Illustrating the proof of Claim 3.

Suppose that $yz \notin E$. We show firstly that $G[R_2]$ is a clique. Let $r_2 \in R_2$. Since y is adjacent to neither r_2 nor z, the sets $\{r_2\}$ and $\{z\}$ do not form a coalition. Thus, $\{r_2\}$ forms a coalition with $\{y\}$ or $\{x\}$. If $\{r_2\}$ forms a coalition with $\{x\}$, then $r_2 \in W$. By our earlier observations, G[W]is a clique, implying that r_2 is adjacent to all vertices in $R_2 \setminus \{r_2\}$. If $\{r_2\}$ forms a coalition with $\{y\}$, then $\{r_2, y\}$ is a dominating set of G, once again implying that r_2 is adjacent to all vertices in $R_2 \setminus \{r_2\}$. Hence, $G[R_2]$ is a clique. We show secondly that every vertex in R_1 is adjacent to all vertices of L_1 or R_2 (or both L_1 and R_2). Let $r_1 \in R_1$. If $\{r_1\}$ forms a coalition with $\{x\}$, then $r_1 \in W$. If $\{r_1\}$ forms a coalition with $\{y\}$, then every vertex of R_2 must be adjacent to r_1 . If $\{r_1\}$ forms a coalition with $\{z\}$, then every vertex of L_1 must be adjacent to r_1 . If $\{l_1\}$ forms a coalition with $\{x\}$, then $l_1 \in W$. Now, since $yz \notin E$ and z is not adjacent to l_1 , $\{l_1\}$ does not form a coalition with $\{y\}$. If $\{l_1\}$ forms a coalition with $\{z\}$, then l_1 must be adjacent to all other vertices of L_1 . Hence, $G[L_1]$ is a clique. From the above structural properties we infer that $G \in \mathcal{F}_2^3$ (see Definition 3). Therefore, in this case when $yz \notin E$, we have $G \in \mathcal{F}_2$.

Hence we may assume that $yz \in E$, for otherwise the desired result follows. We show that every vertex of R_2 is adjacent to all vertices of L_1 or $R_2 \setminus \{r_2\}$. Let $r_2 \in R_2$. If $\{r_2\}$ forms a coalition with $\{x\}$, then $r_2 \in W$. If $\{r_2\}$ forms a coalition with $\{z\}$, then r_2 is adjacent to all vertices in L_1 . If $\{r_2\}$ forms a coalition with $\{y\}$, then r_2 must be adjacent to all other vertices of R_2 . Hence, we conclude that every vertex of R_2 is adjacent to all vertices of L_1 or $R_2 \setminus \{r_2\}$. We show next that every vertex of R_1 is adjacent to all vertices of L_1 or R_2 . Let $r_1 \in R_1$. If $\{r_1\}$ forms a coalition with $\{x\}$, then $r_1 \in W$. If $\{r_1\}$ forms a coalition with $\{z\}$, then all vertices of L_1 must be adjacent to r_1 , and if $\{r_1\}$ forms a coalition with $\{y\}$, then all vertices of R_2 must be adjacent to r_1 . Hence, we conclude that every vertex of R_1 is adjacent to all vertices of R_2 must be adjacent to r_1 . Hence, we conclude that every vertex of R_1 is adjacent to all vertices of R_2 must be adjacent to r_1 . Hence, we conclude that every vertex of R_1 is adjacent to all vertices of R_2 must be adjacent to r_1 , and if $\{r_1\}$ forms a coalition with $\{y\}$, then all vertices of R_2 must be adjacent to l_1 , and if $\{l_1\}$ forms a coalition with $\{z\}$, then l_1 must be adjacent to all vertices $L_1 \setminus \{l_1\}$. Hence, we conclude that every vertex of L_1 is adjacent to all vertices of R_2 or $L_1 \setminus \{l_1\}$. From the above structural properties we infer that $G \in \mathcal{F}_2^3$ (see Definition 3). Therefore, once again we have $G \in \mathcal{F}_2$. (\Box)

Claim 3.2 If $\{y\}$ and $\{z\}$ do not form a coalition, then $G \in \mathcal{F}_2$.

Proof. Suppose that $\{y\}$ and $\{z\}$ do not form a coalition, implying that $L_2 \neq \emptyset$. We show that $L_2 \subseteq W$. Suppose, to the contrary, that $L_2 \setminus W \neq \emptyset$. Let $l_2 \in L_2 \setminus W$. Thus, $\{x\}$ and $\{l_2\}$ do not form a coalition, and so $\{l_2\}$ forms a coalition with $\{y\}$ or $\{z\}$. In both cases, since no vertex in L_2 is adjacent to y or z, the vertex l_2 is adjacent to every other vertex in L_2 . If $yz \notin E$, then since neither y nor z is adjacent to l_2 , the set $\{l_2\}$ does not form a coalition with $\{y\}$ or with $\{z\}$, a contradiction. Hence, $yz \in E$. Let $q \in L_1 \cup R_2 \cup R_2$. If $\{q\}$ forms a coalition with $\{x\}$, then $q \in W$, and so q is adjacent to l_2 . If $\{q\}$ forms a coalition with $\{y\}$ or $\{z\}$, then since l_2 is adjacent to l_2 . If $\{q\}$ forms a coalition with $\{y\}$ or $\{z\}$, then since l_2 is adjacent to every vertex of $L_1 \cup R_1 \cup R_2$. As observed earlier, l_2 adjacent to every other vertex of L_2 . Thus, l_2 adjacent to every other vertex of V_x , implying that $\{x\}$ and $\{l_2\}$ do form a coalition, and so $l_2 \in W$, a contradiction. Hence, $L_2 \subseteq W$. Proceeding analogously as in to the proof of Claim 3.1 when the set $L_2 = \emptyset$, we infer that $G \in \mathcal{F}_2^3$ (see Definition 3), and so $G \in \mathcal{F}_2$.

The proof of Claim 3 follows from Claims 3.2 and 3.2. (D)

The proof of Theorem 5 follows from Claims 1, 2 and 3. \Box

We next consider graphs G with $\delta(G) = 2$ that contain a full vertex.

Theorem 6 If G = (V, E) is a graph of order n with $\delta(G) = 2$, then the following properties hold.

- (a) If G contains exactly one full vertex, say f, then C(G) = n if and only if $G[V \setminus \{f\}] \in \mathcal{F}_1$.
- (b) If G contains exactly two full vertices, then C(G) = n if and only if $G \cong (K_1 \cup K_{n-3}) + K_2$.
- (c) If G contains at least three full vertices, then $G \cong C_3$.

Proof. (a) Suppose that G contains exactly one full vertex, say f. If C(G) = n, then since f is a full vertex, $C(G[V \setminus \{f\}]) = n-1$. Since G has exactly one full vertex, $G[V \setminus \{f\}]$ has no full vertices. Since the minimum degree of $G[V \setminus \{f\}]$ is 1, by Theorem 3 the graph $G[V \setminus \{f\}]$ belongs to the family \mathcal{F}_1 . Conversely, if $G[V \setminus \{f\}] \in \mathcal{F}_1$, then G has no full vertices and $C(G[V \setminus \{f\}]) = n-1$. Adding $\{f\}$ to the singleton coalition partition of $G[V \setminus \{f\}]$ yields a singleton coalition partition for G, whence C(G) = n.

(b) Suppose that G contains exactly two full vertices, say f_1 and f_2 . Let $G_{1,2} = G[V \setminus \{f_1, f_2\}]$. Since $\delta(G) = 2$, the graph $G_{1,2}$ contains an isolated vertex, and so $G_{1,2}$ has minimum degree zero. If C(G) = n, then $C(G_{1,2}) = n-2$, and so, by Theorem 1, we have $G_{1,2} \cong K_1 \cup K_{n-3}$. Reconstructing the graph G by adding back the full vertices f_1 and f_2 we have $G = (K_1 \cup K_{n-3}) + K_2$. Conversely, if $G = (K_1 \cup K_{n-3}) + K_2$, then C(G) = n.

(c) Suppose that G contains at least three full vertices, say f_1 , f_2 and f_3 . If $n \ge 4$, then $\delta(G) \ge 3$, contradicting our supposition that $\delta(G) = 2$. Hence, n = 3, whence $G = C_3$. \Box

Before we prove that if $G \in \mathcal{F}_2$, then C(G) = n, we will describe a family \mathcal{H}_2 of graphs. Thereafter, we establish a connection between $CG(G, \Gamma_1)$ and the family \mathcal{H}_2 , where $G \in \mathcal{F}_2$.

Definition 4 (Family \mathcal{H}_2) The family \mathcal{H}_2 consists of all graphs H which are constructed as follows.

- (a) The family \mathcal{H}_2^1 . Let H = (V, E) be the graph constructed as follows. Let $V = \{x', y', z'\} \cup R'_1$, with $R'_1 \neq \emptyset$. Add all edges between vertices in $\{x', y', z'\}$ to form a clique, and so $H[\{x', y', z'\}] = K_3$. Join every vertex of R'_1 to both y' and z'. Let R'_1 be an independent set in H. Further, we add edges joining the vertex x' to vertices in R'_1 , including the possibility of adding no additional edge incident to x' (except for the edges x'y' and x'z'). Let \mathcal{H}_2^1 be the family consisting of all such graphs H constructed in this way. An example of a graph in the family \mathcal{H}_2^1 is illustrated in Figure 5a.
- (b) The family \mathcal{H}_2^2 . Let H = (V, E) be the graph constructed as follows. Let $V = \{x', y', z'\} \cup R'_1 \cup L'_1$, with $R'_1 \neq \emptyset$ and $L'_1 \neq \emptyset$. Join the vertex y' to every vertex of R'_1 , and join the vertex z' to every vertex of L'_1 . We join the vertex y' to both vertices x' and z'. However, we do not add an edge between x' and z', and we add no edges between any vertex of L'_1 and y'. Let $L'_1 \cup R'_1$ be an independent set of H. Additional edges may be added to H, including the possibility of none. By construction, every vertex in L'_1 has degree 1 with z' as its unique neighbor or has degree 2 with x' and z' as its two neighbors. Let \mathcal{H}_2^2 be the family consisting of all such graphs H constructed in this way. An example of a graph in the family \mathcal{H}_2^2 is illustrated in Figure 5b.



Figure 5: (a): A graph in \mathcal{H}_2^1 (b): A graph in \mathcal{H}_2^2

(c) The family \mathcal{H}_2^3 . Let H = (V, E) be the graph constructed as follows. Let $V = \{x', y', z'\} \cup L'_1 \cup R'_1 \cup R'_2 \cup W'$, where $W' \neq \emptyset$ and where L'_1, R'_1, R'_2 and W' are pairwise disjoint sets. Further some, including the possibility of all, of the sets L'_1, R'_1 , and R'_2 may be empty. We add no edges between vertices in the set $L'_1 \cup R'_1 \cup R'_2 \cup W'$, and so this set is an independent set in H. We do not add the edges x'y' and x'z'. We join every vertex of R'_1 to at least one of the vertices y' and z'. We join x' to all vertices of W', but we do not add any edge joining x' to a vertex in the set $\{y', z'\} \cup L'_1 \cup R'_1 \cup R'_2$. We join each vertex of L'_1 and each vertex of R'_2 to at least one of the vertices y' and z'. Two examples of graphs in the family \mathcal{H}_2^3 are illustrated in Figure 6.

The family \mathcal{H}_2 is defined as $\mathcal{H}_2 = \mathcal{H}_2^1 \cup \mathcal{H}_2^2 \cup \mathcal{H}_2^3$.



Figure 6: Two graphs in \mathcal{H}_2^3 .

We are now in a position to prove the following result.

Theorem 7 If $G \in \mathcal{F}_2$, then G is a SP-graph and $CG(G, \Gamma_1) \in \mathcal{H}_2$.

Proof. Let $G \in \mathcal{F}_2$ have order n, and let G = (V, E). We adopt our notation in Definition 3 used to construct the graph G. Let $\{x, y, z\} \subseteq V$ and $N_G(x) = \{y, z\}$. In what follows, we let $V_x = V \setminus \{x, y, z\}$, and so $V = N_G[x] \cup V_x$.

If G is a SP-graph, that is, if C(G) = n, then we let $H_2 = CG(G, \Gamma_1)$, where recall that Γ_1 represents a singleton coalition partition of G. In this case (when C(G) = n), by Observation 1 for each vertex $u \in V \setminus \{x, y, z\}$, the set $\{u\}$ forms a coalition with at least one of the sets $\{x\}, \{y\}$, and $\{z\}$. Moreover, there is no coalition between any two sets $\{a\}$ and $\{b\}$ with $a, b \in V_x$. We also adopt our earlier notation that if G is a SP-graph, then for a vertex v in G, we represent the corresponding vertex in H_2 as \tilde{v} .

By Definition 3, the family \mathcal{F}_2 is defined as $\mathcal{F}_2 = \mathcal{F}_2^1 \cup \mathcal{F}_2^2 \cup \mathcal{F}_2^3$. Thus, G belongs to one of the families \mathcal{F}_2^1 or \mathcal{F}_2^2 or \mathcal{F}_2^3 . We proceed further with the following three claims.

Claim 4 If $G \in \mathcal{F}_2^1$, then C(G) = n and $H_2 \in \mathcal{H}_2^1$.

Proof. Suppose that $G \in \mathcal{F}_2^1$. In this case, both vertices y and z are adjacent to all vertices of R_1 . If $u \in R_1$, then $\{u\}$ forms a coalition with both $\{y\}$ and $\{z\}$. Moreover, any two sets of $\{x\}$, $\{y\}$ and $\{z\}$ form a coalition. Hence, C(G) = n. We now consider the graph $H_2 = CG(G, \Gamma_1)$. For each vertex $u \in R_1$, the vertex \tilde{u} in H_2 is adjacent to both \tilde{y} and \tilde{z} . Moreover, $H_2[\{\tilde{y}, \tilde{z}, \tilde{x}\}]$ is a clique. Thus, the graph H_2 belongs to the family \mathcal{H}_2^1 defined in Definition 4. Therefore, $H_2 \in \mathcal{H}_2$. (D)

Claim 5 If $G \in \mathcal{F}_2^2$, then C(G) = n and $H_2 \in \mathcal{H}_2^2$.

Proof. Suppose that $G \in \mathcal{F}_2^2$. Recall that in this case, $L_1 \neq \emptyset$ and $R_1 \neq \emptyset$. Let $l_1 \in L_1$. Since $G[L_1]$ is a clique, the vertex l_1 is adjacent to every other vertex of L_1 . Since the vertex z is adjacent

to all vertices in $R_1 \cup \{x\}$ and since the vertex l_1 dominates the set $L_1 \cup \{y\}$, the set $\{l_1, z\}$ is a dominating set of G, implying that the sets $\{l_1\}$ and $\{z\}$ form a coalition. Let $r_1 \in R_1$. Since the vertex y is adjacent to every vertex of $L_1 \cup R_1 \cup \{x\}$, and since the vertex r_1 is adjacent to the vertex z, the set $\{r_1, y\}$ is a dominating set of G, implying that the sets $\{r_1\}$ and $\{y\}$ form a coalition. Moreover since the vertex x is adjacent to the vertex z, the sets $\{x\}$ and $\{y\}$ form a coalition. Hence, C(G) = n.

We now consider the graph $H_2 = CG(G, \Gamma_1)$. Since $\{x\}$ and $\{y\}$ form a coalition, \tilde{x} is adjacent to \tilde{y} in H_2 , and since $\{y\}$ and $\{z\}$ form a coalition, \tilde{y} is adjacent to \tilde{z} . If $r_1 \in R_1$, then $\tilde{r_1}$ is adjacent to \tilde{y} . If $l_1 \in L_1$, then $\tilde{l_1}$ is adjacent to \tilde{z} . Since z is adjacent to neither y nor $l_1 \in L_1$, the set $\{y, l_1\}$ is not a dominating set, implying that $\tilde{l_1}$ is not adjacent to \tilde{y} . Since $L_1 \neq \emptyset$, the set $\{x, z\}$ is not a dominating set, and so $\{x\}$ and $\{z\}$ do not form a coalition, implying that \tilde{x} is not adjacent to \tilde{z} . If $r_1 \in R_1$, then since there may exists edges between L_1 and R_1 in G, we note that it is possible that \tilde{z} is adjacent to $\tilde{r_1}$. Thus, the graph H_2 belongs to the family \mathcal{H}_2^2 defined in Definition 4. Therefore, $H_2 \in \mathcal{H}_2$. (D)

Claim 6 If $G \in \mathcal{F}_2^3$, then C(G) = n and $H_2 \in \mathcal{H}_2^3$.

Proof. Suppose that $G \in \mathcal{F}_2^3$. Recall that by definition, $L_1 \neq \emptyset$, $R_2 \neq \emptyset$, and $W \neq \emptyset$, and that if $L_2 \neq \emptyset$, then $L_2 \subseteq W$. Further, W may intersects $L_1 \cup R_1 \cup R_2 \cup L_2$. Let $w \in W$. Since G[W] is a clique, and w is adjacent to all vertices $L_1 \cup R_1 \cup R_2 \cup L_2$, the set $\{w\}$ forms a coalition with $\{x\}$. We note that W may intersects $L_1 \cup R_1 \cup R_2 \cup L_2$. Let $R_1 \neq \emptyset$, and let $r_1 \in R_1$. By definition, the vertex r_1 is adjacent to all vertices of either L_1 or R_2 . If r_1 is adjacent to all vertices of L_1 , then $\{r_1\}$ and $\{z\}$ form a coalition, and if r_1 is adjacent to all vertices of R_2 , then $\{r_1\}$ and $\{y\}$ forms a coalition.

Suppose firstly that $yz \notin E(G)$. In this case, both $G[L_1]$ and $G[R_2]$ are cliques. Therefore, if $l_1 \in L_1$, then $\{z, l_1\}$ is a dominating set of G, and so $\{l_1\}$ and $\{z\}$ form a coalition. Moreover if $r_2 \in R_2$, then $\{y, r_2\}$ is a dominating set of G, and so $\{r_2\}$ and $\{y\}$ form a coalition.

Suppose secondly that $yz \in E(G)$. In this case, every vertex $l_1 \in L_1$ is adjacent to all vertices of either L_1 or R_2 , and every vertex $r_2 \in R_2$ is adjacent to all vertices of either L_1 or R_2 . If $l_1 \in L_1$ and l_1 is adjacent to all vertices of L_1 , then $\{l_1\}$ and $\{z\}$ form a coalition, and if l_1 is adjacent to all vertices of R_2 , then $\{l_1\}$ and $\{y\}$ form a coalition. Analogously, if $r_2 \in R_2$, then $\{r_2\}$ forms a coalition with either $\{z\}$ or $\{y\}$.

From the above properties we infer that C(G) = n. We now consider the graph $H_2 = CG(G, \Gamma_1)$. Since $L_1 \neq \emptyset$ and $R_2 \neq \emptyset$, the set $\{x\}$ does not form a coalition with $\{y\}$ or $\{z\}$. Thus, \tilde{x} is adjacent to neither \tilde{y} nor \tilde{z} in H_2 . If $w \in W$, then the vertex set $\{x\}$ forms a coalition with $\{w\}$, and so \tilde{w} is adjacent to \tilde{x} . If $R_1 \neq \emptyset$ and $r_1 \in R_1$, then $\{r_1\}$ forms a coalition with either $\{y\}$ or $\{z\}$, and so $\tilde{r_1}$ is adjacent to either \tilde{y} or \tilde{z} . If $yz \notin E(G)$, then for $l_1 \in L_1$ and $r_2 \in R_2$, we know that $\{l_1\}$ forms a coalition with $\{z\}$, and $\{r_2\}$ forms a coalition with $\{y\}$. Hence, $\tilde{l_1}$ is adjacent to \tilde{z} , and $\tilde{r_2}$ is adjacent to \tilde{y} . If $yz \in E(G)$, then for $l_1 \in L_1$ and $r_2 \in R_2$, we know that each of $\{l_1\}$ and $\{r_2\}$ form a coalition with either $\{y\}$ or $\{z\}$. Hence, both $\tilde{l_1}$ and $\tilde{r_2}$ are adjacent to either \tilde{z} or \tilde{y} . Let $W' = W, L'_1 = L_1 \setminus W, R'_1 = R_1 \setminus W, R'_2 = R_2 \setminus W$, and $x' = \tilde{x}, y' = \tilde{y}$, and $z' = \tilde{z}$. If $u \in W \setminus L_1 \cup R_1 \cup R_2$, then $\{u\}$ and $\{x\}$ do not form a coalition, and therefore x' is not adjacent to any any vertex of $L'_1 \cup R'_1 \cup R'_2$ in H_2 . Note that if $L_2 = \emptyset$, then \tilde{y} is adjacent to \tilde{z} in H_2 , and if $L_2 \neq \emptyset$, since there is a vertex in L_2 which has no neighbor in $\{y, z\}$, \tilde{y} is not adjacent to \tilde{z} in H_2 . Thus, the graph H_2 belongs to the family \mathcal{H}_2^3 defined in Definition 4. Therefore, $H_2 \in \mathcal{H}_2$. (D)

The proof of Theorem 7 follows from Claims 4, 5 and 6. \Box

4 G-SC chain

In this section, we characterize the G-SC chain for all SP-graphs G with $0 \le \delta(G) \le 2$.

4.1 G-SC chain with $\delta(G) = 0$

We first consider the case when the SP-graph G contains an isolated vertex. We shall prove the following theorem.

Theorem 8 If G is an SP-graph of order n with $\delta(G) = 0$, then the following properties hold.

- (a) If n = 1, then $G = K_1$, $L_{SCC}(G) = 0$, and the G-SC chain is $K_1 \to K_1 \to K_1 \to \cdots$
- (b) If n = 2, then $G = \overline{K}_2$, $L_{SCC}(G) = \infty$, and the G-SC chain is $\overline{K}_2 \to K_2 \to \overline{K}_2 \to K_2 \to \cdots$
- (c) If n > 3, then $G \cong K_1 \cup K_{n-1}$, $L_{SCC}(G) = 1$, and the G-SC chain is $G \to K_{1,n-1}$.
- (d) If n = 3, then $G = K_1 \cup K_2$, $L_{SCC}(G) = \infty$, and the G-SC chain is

$$K_1 \cup K_2 \to P_3 \to K_1 \cup K_2 \to P_3 \to \cdots$$

Proof. Let G be a SP-graph of order n with $\delta(G) = 0$. If n = 1, then $G = K_1$. In this case, the K_1 -SC chain is given by the sequence $K_1 \to K_1 \to K_1 \to \cdots$. This proves part (a).

(b) If n = 2, then $G = \overline{K}_2$. In this case, $CG(\overline{K}_2, \Gamma_1) = K_2$. Since $CG(K_2, \Gamma_1) = \overline{K}_2$, the \overline{K}_2 -SC chain is therefore given by the sequence

$$\overline{K}_2 \to K_2 \to \overline{K}_2 \to K_2 \to \cdots$$
.

(c) and (d): Suppose that $n \geq 3$. By Theorem 1, $G \cong K_1 \cup K_{n-1}$. We now consider the graph $\operatorname{CG}(G, \Gamma_1)$. Let q be the isolated vertex of G, and $\{x_1, \ldots, x_{n-1}\}$ be the vertex set of K_{n-1} . Thus, the set $\{x_i\}$ forms a coalition with $\{q\}$, and so \tilde{x}_i is adjacent to \tilde{q} in $\operatorname{CG}(G, \Gamma_1)$ for all $i \in [n-1]$. By Observation 1, there is no coalition between the sets of $\{x_i\}$ and $\{x_j\}$ for $1 \leq i < j \leq n-1$. Hence, $\operatorname{CG}(G, \Gamma_1)$ is the star $K_{1,n-1}$. We now consider the graph $K_{1,n-1}$, which has minimum degree 1 and contains exactly one full vertex. By Theorem 2, the graph $K_{1,n-1}$ is a SP-graph if and only if n = 3. Hence if n > 3, then the G-SC chain is given by the sequence $G \to K_{1,n-1}$, which proves part (c). Suppose, finally, that n = 3. In this case, $G = K_1 \cup K_2$ and $\operatorname{CG}(G, \Gamma_1) = K_{1,2} \cong P_3$. Since $\operatorname{CG}(P_3, \Gamma_1) = K_1 \cup K_2$, the G-SC chain for $G = K_1 \cup K_2$ is given by the sequence

$$K_1 \cup K_2 \to P_3 \to K_1 \cup K_2 \to P_3 \to \cdots$$

which completes the proof of part (d), and of Theorem 8. \Box

4.2 G-SC chain with $\delta(G) = 1$

In this section, we consider the case when the SP-graph G has minimum degree 1. We first consider the case when G contains a full vertex.

Theorem 9 If G is an SP-graph of order n with $\delta(G) = 1$ that contains a full vertex, then the following properties hold.

- (a) If n = 2, then $G = K_2$, $L_{SCC}(G) = \infty$, and the G-SC chain is $K_2 \to \overline{K}_2 \to \overline{K}_2 \to \overline{K}_2 \dots$
- (b) If n > 3 and $L_{SCC}(G) = 1$, then the G-SC chain is $G \to K_1 \cup K_{1,n-2}$.
- (c) If n = 3, then $G = P_3$, $L_{SCC}(G) = \infty$, and the G-SC chain is

$$P_3 \to K_1 \cup K_2 \to P_3 \to K_1 \cup K_2 \dots$$

Proof. Let G be a SP-graph of order n with $\delta(G) = 1$ that G contains at least one full vertex. If n = 2, then $G = K_2$. In this case since $CG(K_2, \Gamma_1) = \overline{K}_2$, the K_2 -SC chain is given by the sequence $K_2 \to \overline{K}_2 \to K_2 \to \overline{K}_2 \cdots$. This proves part (a).

(b) and (c): Suppose that $n \geq 3$. In this case, the graph G contains exactly one full vertex. By Theorem 2, G is obtained from $K_1 \cup K_{n-1}$ by adding an edge between the isolated vertex, say q, and a vertex, say x, in the complete graph K_{n-1} . We now consider $\operatorname{CG}(G, \Gamma_1)$. By Observation 1, if y is an arbitrary vertex of the complete graph K_{n-1} different from x, then the set $\{y\}$ forms a coalition with $\{q\}$, and so \tilde{y} is adjacent to \tilde{q} in $\operatorname{CG}(G, \Gamma_1)$. Moreover, no two vertices in the complete graph K_{n-1} , different from x, form a coalition. The full vertex x does not form a coalition with any other set. Therefore, $\operatorname{CG}(G, \Gamma_1) \cong K_1 \cup K_{1,n-2}$. However, $K_1 \cup K_{1,n-2}$ is a SP-graph if and only if n = 3. Hence if n > 3, then the G-SC chain is given by the sequence $G \to K_1 \cup K_{1,n-2}$, which proves part (b). Suppose, finally, that n = 3. In this case, $G \cong P_3$ and $\operatorname{CG}(P_3, \Gamma_1) = K_1 \cup K_2$. Since $\operatorname{CG}(K_1 \cup K_2, \Gamma_1) = P_3$, the G-SC chain for G is given by the sequence

$$P_3 \rightarrow K_1 \cup K_2 \rightarrow P_3 \rightarrow K_1 \cup K_2 \dots$$

which completes the proof of part (c), and of Theorem 9. \Box

We first next the case when the SP-graph G contains no full vertex.

Theorem 10 If G is a SP-graph of order n with $\delta(G) = 1$ that contains no full vertex, then the G-SC chain is one of the following chains, where H_1 is a graph in the family \mathcal{H}_1 .

(a) $G \to H_1$, (b) $G \to C_4 \to K_4 \to \overline{K}_4$, or (c) $G \to K_{2,n-2} \to K_1 + K_{1,n-2}$.

Proof. Let G be a SP-graph of order n with $\delta(G) = 1$ that contains no full vertex. By Theorem 3, $G \in \mathcal{F}_1$. By Theorem 4, $H_1 = CG(G, \Gamma_1) \in \mathcal{H}_1$. If H_1 is not a SP-graph, then the G-SC chain is the chain $G \to H_1$. Hence, we may assume that H_1 is a SP-graph. We now compute the SC-graph of H_1 . By construction, all graphs in the family \mathcal{H}_1 have order at least 4, implying that $n \geq 4$.

We show next that $Q_1 = \emptyset$. If $H_1 \cong P_4$, then it is obvious that $Q_1 = \emptyset$. Now, we assume that H_1 is not isomorphic to P_4 . Suppose, to the contrary, that $Q_1 \neq \emptyset$. In this case, $|Q_1| \ge 2$ and the graph H_1 is illustrated in Figure 1(a) and has vertex set $\{x_1, y_1\} \cup P_1 \cup \{w_1\} \cup Q_1$. Every vertex in Q_1 has degree 1 in H_1 , and so H_1 has at least $|Q_1| \ge 2$ vertices of degree 1. Since every connected graph in the family $\mathcal{F}_1 \setminus \{P_4\}$ has exactly one vertex of degree 1, namely the vertex x in Definition 1, the graph $H_1 \notin \mathcal{F}_1$. Thus by Theorem 3, the graph H_1 is not a SP-graph, a contradiction to our assumption.

Hence, $Q_1 = \emptyset$. In this case, the graph H_1 is illustrated in Figure 1(b). By the construction of graphs in the family \mathcal{H}_1 with $Q_1 = \emptyset$, we have $H_1 \cong K_{2,n-2}$. If n = 4, then $G \cong C_4$. As observed earlier, the C_4 -SC chain is given by $C_4 \to K_4 \to \overline{K}_4$. Hence, we may assume that $n \ge 5$.

Adopting our earlier notation when constructing graphs in the family \mathcal{H}_1 , we let $B_1 = P_1 \cup \{w_1\}$. Further, we let $A_1 = \{x_1, y_1\}$. Thus, H_1 is the complete bipartite graph $K_{2,n-2}$ with partite sets A_1 and B_1 , where $|A_1| = 2$ and $|B_1| = n - 2 \ge 3$. If $a \in A_1$ and $b \in B_1$, then the sets $\{a\}$ and $\{b\}$ form a coalition in H_1 . Moreover, the sets $\{x_1\}$ and $\{y_1\}$ form a coalition in H_1 . However, if b_1 and b_2 are two distinct vertices in B_1 , then since $n \ge 5$, the sets $\{b_1\}$ and $\{b_2\}$ do not form a coalition. Hence, the singleton coalition graph of H_1 is isomorphic to $K_1 + K_{1,n-2}$, where here '+' denotes the join operation. However, since the graph $K_1 + K_{1,n-2}$ has two full vertices, the coalition graph of $K_1 + K_{1,n-2}$ associated with any coalition partition has two isolated vertices, and is therefore not a SP-graph. Hence, in this case when $n \ge 5$, the G-SC chain is given by $G \to K_{2,n-2} \to K_1 + K_{1,n-2}$. \Box

4.3 G-SC chain with $\delta(G) = 2$

In this section, we consider the case when the SP-graph G has minimum degree 2. We first prove the following result.

Theorem 11 If G is a SP-graph of order n with $\delta(G) = 2$ that contains at least one full vertex, then $L_{SCC}(G) = 1$.

Proof. Suppose that G = (V, E) is a SP-graph of order n with $\delta(G) = 2$ that contains at least one full vertex. We note that $n \geq 3$. If n = 3, then $G = K_3$. In this case, $\operatorname{CG}(G, \Gamma_1) = \overline{K}_3$. Since \overline{K}_3 is not a SP-graph, the G-SC chain is the chain $K_3 \to \overline{K}_3$, and so $L_{\operatorname{SCC}}(G) = 1$. Hence we may assume that $n \geq 4$, for otherwise the desired result follows. Thus, G has at most two full vertices. Let $I = \operatorname{CG}(G, \Gamma_1)$. If G has two full vertices, then I has two isolated vertices, and so, by Theorem 1, the graph I is not a SP-graph, implying that in this case the G-SC chain is the chain $G \to I$, and so $L_{\operatorname{SCC}}(G) = 1$. Hence, we may assume that G has exactly one full vertex, say f. By Theorem 6, $G' = G[V \setminus \{f\}] \in \mathcal{F}_1$. Let $H' = \operatorname{CG}(G', \Gamma_1)$. By Theorem 4, $H' \in \mathcal{H}_1$. Since G has a full vertex, the singleton coalition graph I of G contains an isolated vertex, implying that $I \cong K_1 \cup H'$. If I is a SP-graph, then by Theorem 1, $H' \cong K_{n-1}$. However by the definition of family \mathcal{H}_1 (see Definition 2), the graph H' is not isomorphic to K_{n-1} , a contradiction. Hence, I is not a SP-graph, implying that the G-SC chain is once again the chain $G \to I$, and so $L_{\operatorname{SCC}}(G) = 1$. \Box

Suppose, next, that G is a SP-graph with $\delta(G) = 2$ that contains no full vertex. Let B =

 $CG(G, \Gamma_1)$. By Theorem 7, $B \in \mathcal{H}_2$. If B is not a SP-graph, then the G-SC chain is $G \to B$, and so $L_{SCC}(G) = 1$. Hence we may assume in what follows that B is an SP-graph. By Definition 4, the singleton coalition graph $B \in \mathcal{H}_2$ belongs to one of the three families \mathcal{H}_2^1 , \mathcal{H}_2^2 , or \mathcal{H}_2^3 . We consider the three possibilities in turn for the graph B. Throughout the following three lemmas, we adopt our notation in Definition 4.

Lemma 1 If $B \in \mathcal{H}_2^1$, then the G-SC chain is one of the following chains.

- (a) $G \to B \to \overline{K}_4$,
- (b) $G \to B \to \overline{K}_3 \cup K_2$,
- (c) $G \to B \to \overline{K}_2 \cup K_2$ or
- (d) $G \to B \to \overline{K}_2 \cup P_3$.

Proof. Suppose that the singleton coalition graph $B \in \mathcal{H}_2^1$ (as illustrated in Figure 5a). In this case, the graph *B* has either two full vertices (namely y' and z') or three full vertices (namely x', y', and z'). Thus, the singleton sets $\{z'\}$ and $\{y'\}$ (and also singleton set $\{x'\}$ when *B* has three full vertices) are included in any coalition partition of *B*. Let *B'* be the graph obtained from *B* by removing all its full vertices. Let *B'* have order n', and so n' = n - 2 or n' = n - 3. We note that *B* is a SP-graph if and only if *B'* is a SP-graph.

If n' = n - 3, then $B' \cong \overline{K}_{n'}$. It is clear that B' is a SP-graph if and only if $B' \cong \overline{K}_1$ or $B' \cong \overline{K}_2$. Then, $B \cong K_4$ or $B \cong \overline{K}_2 + K_3$. Hence, the G-SC chain is the sequence

$$G \to B \to \overline{K}_4$$
, if $B \cong K_4$,

or the sequence

$$G \to B \to \overline{K}_3 \cup K_2$$
, if $B \cong \overline{K}_2 + K_3$.

Now, we assume that n' = n - 2. By construction of graphs in the family \mathcal{H}_2^1 , either the vertex x' has degree 2 in B or the vertex x' has degree at least 3 in B. Further if x' has degree 2 in B, then every vertex in the independent set R'_1 has degree 2 and is adjacent to only y' and z' in B, while if x' has degree at least 3 in B, then we can partition the set R'_1 into two sets $R'_{1,2}$ and $R'_{1,3}$ where every vertex in $R'_{1,2}$ has degree 2 and is adjacent to only y' and z' in B, and where every vertex in $R'_{1,3}$ has degree 3 and is adjacent to only x', y' and z' in B. Hence if x' has degree 2 in B, then $B' = \overline{K}_{n'}$, while if x' has degree at least 3 in B, then $B' = \overline{K}_r \cup K_{1,n'-r-1}$ for some integer $r \ge 1$ where in this case $n' \ge r + 2$. Note that since n' = n - 2, we must have $r \ge 1$. In both cases, the graph B' contains at least one isolated vertex.

By our earlier observations, B is a SP-graph if and only if B' is a SP-graph. Furthermore, the graph B' contains at least one isolated vertex. By Theorem 1, B' is a SP-graph if and only if $B' \cong K_1 \cup K_{n'-1}$. As observed earlier, either $B' = \overline{K}_{n'}$ or $B' = \overline{K}_r \cup K_{1,n'-r-1}$ for some integer $r \ge 1$ where in this case $n' \ge r+2$. Therefore, B' is a SP-graph if and only if $B' \cong \overline{K}_2$ or $B' \cong K_1 \cup K_2$.

Reconstructing the graph B from the graph B' by adding back the deleted vertices y' and z', we infer that B is a SP-graph if and only if it is isomorphic to the graph $K_4 - e$ obtained from a

 K_4 by removing an edge (illustrated in Figure 7(a)) or the graph of order 5 obtained from K_4 by adding a new vertex and joining it to two vertices of the complete graph (illustrated in Figure 7(b)). Moreover, if $B = K_4 - e$, then $\operatorname{CG}(B, \Gamma_1) \cong \overline{K}_2 \cup K_2$, while if B is obtained from K_4 by adding a new vertex of degree 2, then $\operatorname{CG}(B, \Gamma_1) \cong \overline{K}_2 \cup P_3$. In both cases, $\operatorname{CG}(B, \Gamma_1)$ is not a SP-graph. Therefore, the G-SC chain is given by $G \to B \to \overline{K}_2 \cup K_2$ or by $G \to B \to \overline{K}_2 \cup P_3$. \Box



Figure 7: (a): The graph B with $B' \cong \overline{K}_2$. (b). The graph B with $B' \cong K_1 \cup K_2$.

Let M_1, M_2 , and M_3 be three graphs depicted in Figure 8. Moreover, let M_4 be obtained from M_3 by adding an edge from one of the leaf neighbors of the vertex of degree n - 2 in M_3 to the vertex of degree 2 in M_3 .



Figure 8: (a): The graph M_1 . (b): The graph M_2 . (c): The graph M_3 .

We are now in a position to present the following lemma.

Lemma 2 If $B \in \mathcal{H}_2^2$, then the G-SC chain is one of the following chains, where H_1 is a graph in the family \mathcal{H}_1 .

$$\begin{array}{ll} (\mathrm{a}) & G \to B \to H_1, \\ (\mathrm{b}) & G \to B \to C_4 \to K_4 \to \overline{K}_4. \\ (\mathrm{c}) & G \to B \to K_{2,n-2} \to K_1 + K_{1,n-2}, \\ (\mathrm{d}) & G \to M_1 \to \left(\overline{K}_2 + (K_1 \cup K_2)\right) \to \overline{K}_2 + K_3 \to \overline{K}_3 \cup K_2 \end{array}$$

(e) $G \to M_2 \to \overline{K}_2 + K_3 \to \overline{K}_3 \cup K_2$, (f) $G \to B \to K_3 \circ K_1$, (g) $G \to B \to M_3$, (h) $G \to B \to M_4$, (i) $G \to B \to K_2 + \overline{K}_{n-2}$, or (j) $G \to B \to (K_2 + \overline{K}_{n-2}) + e$.

Proof. Suppose that the singleton coalition graph $B \in \mathcal{H}_2^2$ (as illustrated in Figure 5b). In this case, $L'_1 \neq \emptyset$ and $R'_1 \neq \emptyset$. Further the set $L'_1 \cup R'_1$ is an independent set and every vertex of L'_1 is adjacent to z' and possibly adjacent to x'. By construction, we have $n(B) \geq 5$ and $1 \leq \delta(B) \leq 2$. By assumption, B is an SP-graph.

Suppose firstly that $\delta(B) = 1$. Based on the construction of B, there is no full vertex in B. Thus by Theorem 10, the *B*-SC chain is given by the chain in part (a), (b) or (c) in the statement of Theorem 10, where H_1 is a graph in the family \mathcal{H}_1 . Thus, the *G*-SC chain is given by part (a), (b) or (c) in the statement of Lemma 2.

Hence, we may assume that $\delta(B) = 2$, for otherwise the desired result of the lemma follows. Recall that by construction, the vertex y' is not adjacent to any vertex of L'_1 but is adjacent to every vertex of R'_1 . Hence in this case when $\delta(B) = 2$, the vertex x' must be adjacent to all vertices of L'_1 (see Figure 5b), and so every vertex in L'_1 has degree 2 in B (with x' and z' as its two neighbors). Further, every vertex in R'_1 is adjacent to y' and to at least one of x' and z'.

Claim 7 If $|L'_1| = |R'_1| = 1$, then the G-SC chain is given by part (d) or (e) in the statement of Lemma 2.

Proof. Suppose that $|L'_1| = |R'_1| = 1$. Let $R'_1 = \{r_1\}$ and $L'_1 = \{z_1\}$. By our earlier observations, the vertex z_1 has degree 2 with x' and z' as its two neighbors. Further, the vertex r_1 is adjacent to y' and to at least one of x' and z'. If r_1 is adjacent to exactly one of x' and z', then $B \cong M_1$. If r_1 is adjacent to both x' and z', then $B = M_2$. In both cases, B is a SP-graph. If $B \cong M_1$, then the G-SC chain is the sequence

 $G \to M_1 \to (\overline{K}_2 + (K_1 \cup K_2)) \to \overline{K}_2 + K_3 \to \overline{K}_3 \cup K_2,$

while if $B \cong M_2$, then the G-SC chain is the sequence

$$G \to M_2 \to \overline{K}_2 + K_3 \to \overline{K}_3 \cup K_2.$$

Thus in this case when $|L'_1| = |R'_1| = 1$, the G-SC chain is given by part (d) or (e) in the statement of Lemma 2. (D)

By Claim 7, we may assume that $|L'_1| \ge 2$ or $|R'_1| \ge 2$, for otherwise the desired result of the lemma follows. Thus, $L'_1 \cup R'_1$ is an independent set of cardinality at least 3 in *B*. Hence if *u* and *v* are two arbitrary vertices in $L'_1 \cup R'_1$, the sets $\{u\}$ and $\{v\}$ do not form a coalition.

Let $r_1 \in R'_1$. Since y' is not adjacent to any vertex of L'_1 , the set $\{r_1\}$ forms a coalition with only $\{x'\}$ or $\{z'\}$. If $\{r_1\}$ forms a coalition with $\{x'\}$, then z' must be adjacent to r_1 , and x' must be adjacent to all vertices of $R'_1 \setminus \{r_1\}$. If $\{r_1\}$ forms a coalition with $\{z'\}$, then x' must be adjacent to r_1 , and z' must be adjacent to all vertices of $R_1 \setminus \{r_1\}$. This property holds for all vertices in R'_1 . Thus if $|R'_1| \ge 3$, then we infer that all edges between R'_1 and $\{x', z'\}$ are present, except for possibly one edge. If $|R'_1| = 2$, then we infer that both x' and z' are adjacent to at least one vertex in R'_1 and, as observed earlier, every vertex in R'_1 is adjacent to at least one of x' and z'. In particular, when $|R'_1| = 2$, we infer that all edges between R'_1 and $\{x', z'\}$ are present, except for possibly at most two edges.

Claim 8 If all edges between R'_1 and $\{x', z'\}$ are present, except for exactly two edges, then the G-SC chain is given by part (f) in the statement of Lemma 2.

Proof. Suppose that all edges between R'_1 and $\{x', z'\}$ are present, except for exactly two edges. In this case, $R'_1 = \{r_1, r_2\}$ and, renaming vertices if necessary, we may assume that r_1 is adjacent to x' but not to z', and r_2 is adjacent to z' but not to x'. Thus, $\{r_1\}$ forms a coalition with $\{z'\}$, while $\{r_2\}$ forms a coalition with $\{x'\}$. If $|L'_1| \ge 2$ and $z_1 \in L'_1$, then the set $\{z_1\}$ cannot form a coalition with any other singleton set in B, contradicting our assumption that B is an SP-graph. Hence, $|L'_1| = 1$. Let $L'_1 = \{z_1\}$. We now infer that $\{z_1\}$ forms a coalition with only the set $\{y'\}$, the set $\{x'\}$ forms a coalition with each of the sets $\{y'\}$ and $\{z'\}$, and the set $\{y'\}$ and $\{z'\}$ form a coalition. However, there are no additional coalitions in B. Hence in this case, the singleton coalition graph of B is given by $\operatorname{CG}(B, \Gamma_1) \cong K_3 \circ K_1$, where $K_3 \circ K_1$ denotes the corona of K_3 as illustrated in Figure 9. However, $K_3 \circ K_1$ is not a SP-graph, and so in this case the G-SC chain is the sequence $G \to B \to K_3 \circ K_1$ given by part (f) in the statement of Lemma 2. (D)



Figure 9: The corona $K_3 \circ K_1$ of K_3

By Claim 8, we may assume that we may assume that all edges between R'_1 and $\{x', z'\}$ are present, except for at most one edge.

Claim 9 If all edges between R'_1 and $\{x', z'\}$ are present, except for exactly one edge, then the G-SC chain is given by part (g) or (h) in the statement of Lemma 2.

Proof. Suppose that all edges between R'_1 and $\{x', z'\}$ are present, except for exactly one edge. By symmetry, and renaming x' and z' if necessary, we may assume that x' is not adjacent to exactly one vertex of R'.

Suppose that $|L'_1| \ge 2$. In this case, the graph B is a SP-graph, and the singleton coalition graph of B is the graph M_3 . As an illustration, if B is the graph illustrated on the left hand side of

Figure 10, then the singleton coalition graph of B is the graph $CG(B, \Gamma_1) = M_3$ illustrated on the right hand side of Figure 10. However in this case, there is no vertex \tilde{u} in the graph $CG(B, \Gamma_1)$ such that $\{\tilde{z}_1\}$ forms a coalition with $\{\tilde{u}\}$, implying that $CG(B, \Gamma_1)$ is not a SP-graph, and therefore the G-SC chain is the sequence $G \to B \to M_3$ given by part (g) in the statement of Lemma 2.



Figure 10: Left: the graph B. Right: the graph $CG(B, \Gamma_1)$.

Hence, we may assume that $|L'_1| = 1$. Let $L'_1 = \{z_1\}$. Since $|L'_1| + |R'_1| \ge 3$, we note that $|R'_1| \ge 2$. In this case, the set $\{z_1\}$ forms a coalition with each of the sets $\{y'\}$ and $\{z'\}$, while all other coalitions from the previous case when $|L'_1| \ge 2$ remain unchanged. The graph B is once again a SP-graph, and the singleton coalition graph of B is the graph M_4 . In the example in Figure 10 in the case when $L'_1 = \{z_1\}$, the graph M_4 is obtained from the graph M_3 on the right hand side of Figure 10 by adding the edge between the vertices $\tilde{y'}$ and $\tilde{z_1}$. However noting that $|R'_1| \ge 2$, we once again infer that there is no vertex \tilde{u} in the graph $CG(B, \Gamma_1)$ such that $\{\tilde{z_1}\}$ forms a coalition with $\{\tilde{u}\}$, implying that $CG(B, \Gamma_1)$ is not a SP-graph, and therefore the G-SC chain is the sequence $G \to B \to M_4$ given by part (h) in the statement of Lemma 2. (D)

We now return to the proof of Lemma 2 one final time. By Claim 9, we may assume that we may assume that all edges between R'_1 and $\{x', z'\}$ are present.

Suppose that $|L'_1| \geq 2$. In this case, the set $\{y'\}$ forms a coalition with only the sets $\{x'\}$ and $\{z'\}$. Each of the sets $\{x'\}$ and $\{z'\}$ form a coalition with every other singleton set in the partition Γ_1 . Moreover, each vertex in $R'_1 \cup L'_1$ forms a coalition with only the sets $\{x'\}$ and $\{z'\}$. Thus in this case, the graph B is a SP-graph, and the singleton coalition graph of B is the graph $\operatorname{CG}(B,\Gamma_1) = K_2 + \overline{K}_{n-2}$. However, the graph $K_2 + \overline{K}_{n-2}$ contains two full vertices and is not a singleton coalition graph noting that $n \geq 6$. As an illustration, if B is the graph illustrated on the left hand side of Figure 11, then the singleton coalition graph of B is the graph $\operatorname{CG}(B,\Gamma_1) = K_2 + \overline{K}_{n-2}$ illustrated on the right hand side of Figure 11. Therefore, $\operatorname{CG}(B,\Gamma_1)$ is not a SP-graph, and the G-SC chain is the sequence $G \to B \to K_2 + \overline{K}_{n-2}$ given by part (i) in the statement of Lemma 2.

Hence, we may assume that $|R'_1| \ge 2$ and $|L'_1| = 1$. Let $L'_1 = \{z_1\}$. In this case, the set $\{z_1\}$ forms a coalition with each of the sets $\{x'\}$, $\{y'\}$ and $\{z'\}$, while all other coalitions from the previous case when $|L'_1| \ge 2$ remain unchanged. Thus in this case, the graph B is a SP-graph, and the singleton coalition graph of B is the graph $\operatorname{CG}(B, \Gamma_1) = (K_2 + \overline{K}_{n-2}) + e$ obtained from $K_2 + \overline{K}_{n-2}$ by adding an edge between two vertices in the independent set. (In the example of Figure 11 in the case when $L'_1 = \{z_1\}$, the edge added to the graph on the right hand side of Figure 11 is the edge joining $\{\tilde{z}_1\}$ and $\{\tilde{y'}\}$.) However, the graph $(K_2 + \overline{K}_{n-2}) + e$ contains two full vertices and



Figure 11: Left: the graph B. Right: the graph $CG(B, \Gamma_1)$.

is not a singleton coalition graph noting that $n \ge 6$. Therefore, $CG(B, \Gamma_1)$ is not a SP-graph, and the *G*-SC chain is the sequence $G \to B \to (K_2 + \overline{K}_{n-2}) + e$ given by part (j) in the statement of Lemma 2. This completes the proof of Lemma 2. \Box

We consider next the case when the graph B belongs to the family \mathcal{H}_2^3 .

Lemma 3 If $B \in \mathcal{H}_2^3$, then the G-SC chain is one of the following chains, where H_1 is a graph in the family \mathcal{H}_1 .

$$\begin{array}{ll} (a) \ G \rightarrow B \rightarrow H_1, \\ (b) \ G \rightarrow B \rightarrow C_4 \rightarrow K_4 \rightarrow \overline{K}_4, \\ (c) \ G \rightarrow B \rightarrow K_{2,n-2} \rightarrow K_1 + K_{1,n-2}, \\ (d) \ G \rightarrow C_5 \rightarrow C_5 \rightarrow \cdots, \\ (e) \ G \rightarrow M_1 \rightarrow \left(\overline{K}_2 + (K_1 \cup K_2)\right) \rightarrow \overline{K}_2 + K_3 \rightarrow \overline{K}_3 \cup K_2, \\ (f) \ G \rightarrow \overline{K}_3 + \overline{K}_2 \rightarrow \overline{K}_2 + K_3 \rightarrow \overline{K}_3 \cup K_2, \\ (g) \ G \rightarrow \left(\overline{K}_2 + (K_1 \cup K_2)\right) \rightarrow \overline{K}_2 + K_3 \rightarrow \overline{K}_3 \cup K_2, \\ (h) \ G \rightarrow B \rightarrow \mathrm{CG}(B, \Gamma_1) \rightarrow \overline{K}_4, \\ (i) \ G \rightarrow B \rightarrow \mathrm{CG}(B, \Gamma_1) \rightarrow \overline{K}_2 \cup K_2, \\ (k) \ G \rightarrow B \rightarrow \mathrm{CG}(B, \Gamma_1) \rightarrow \overline{K}_2 \cup P_3, \\ (l) \ G \rightarrow B \rightarrow \mathrm{CG}(B, \Gamma_1) \rightarrow K_{2,n-2} \rightarrow K_1 + K_{1,n-2}, \\ (o) \ G \rightarrow B \rightarrow \mathrm{CG}(B, \Gamma_1) \rightarrow K_{2,n-2} \rightarrow K_1 + K_{1,n-2}, \\ (o) \ G \rightarrow B \rightarrow M_1 \rightarrow \left(\overline{K}_2 + (K_1 \cup K_2)\right) \rightarrow \overline{K}_2 + K_3 \rightarrow \overline{K}_3 \cup K_2, \\ (p) \ G \rightarrow B \rightarrow M_2 \rightarrow \overline{K}_2 + K_3 \rightarrow \overline{K}_3 \cup K_2, \\ (q) \ G \rightarrow B \rightarrow \mathrm{CG}(B, \Gamma_1) \rightarrow M_3, \\ (s) \ G \rightarrow B \rightarrow \mathrm{CG}(B, \Gamma_1) \rightarrow M_4, \\ (t) \ G \rightarrow B \rightarrow \mathrm{CG}(B, \Gamma_1) \rightarrow K_2 + \overline{K}_{n-2}, \\ (u) \ G \rightarrow B \rightarrow \mathrm{CG}(B, \Gamma_1) \rightarrow K_2 + \overline{K}_{n-2}, \\ (v) \ G \rightarrow K_{3,n-3} \rightarrow K_{3,n-3} \rightarrow \cdots, or \\ (w) \ G \rightarrow (K_1 \cup K_2) + \overline{K}_{n-3} \rightarrow P_3 + \overline{K}_{n-3} \rightarrow K_1 \cup K_{2,n-3}. \end{array}$$

Proof. Suppose that the singleton coalition graph $B = CG(G, \Gamma_1) \in \mathcal{H}_2^3$ (as illustrated in Figure 6a). Recall that by assumption, the graph G is a SP-graph with $\delta(G) = 2$ and with no full vertex, and so by Theorem 5 we have $G \in \mathcal{F}_2$. Since $CG(G, \Gamma_1) = B \in \mathcal{H}_2^3$, we know from the proof of Theorem 7 that $G \in \mathcal{F}_2^3$ (as illustrated in Figure 3). Adopting our notation to define the families \mathcal{F}_2^3 and \mathcal{H}_2^3 , let $X = R_1 \cup R_2 \cup L_1 \subset V(G)$ and let $X' = R'_1 \cup L'_1 \cup R'_2 \subset V(B)$.

We show firstly that $X' = \emptyset$. Suppose, to the contrary, that $|X'| \ge 1$. Suppose that |X'| = 1. By construction of the graph $G \in \mathcal{F}_2^3$, each vertex of W is adjacent to all vertices of $X \cup L_2$. Further, G[W] is a clique. Since |X'| = 1, we have $|(X \cup L_2) \setminus W| = 1$. Let $(X \cup L_2) \setminus W = \{g\}$, and so $\tilde{g} \notin W'$. By our earlier properties of vertices in the set W, the set $\{g\} \cup W$ forms a clique. The sets $\{g\}$ and $\{x\}$ form a coalition in G. Therefore, by the construction of B, the vertex $\tilde{g} \in W'$, a contradiction. Hence, $|R'_1 \cup L'_1 \cup R'_2| \ge 2$. We may assume, without loss of generality, that $R'_1 \neq \emptyset$. Let $q_1, q_2 \in X'$. By construction of $B \in \mathcal{H}_2^3$ in Definition 4, we have $W' \neq \emptyset$ and x' is adjacent only to all vertices of W'. Thus since x' is not adjacent to any of the vertices q_1, q_2, y' and z', the set $\{q_1\}$ is not a coalition partner of $\{u\}$, where u is a vertex of $B \setminus \{q_1\}$. Hence, B is not a SP-graph, which is a contradiction.

Therefore, $X' = \emptyset$, implying that the vertex set of the graph $B \in \mathcal{H}_2^3$ is $V(B) = \{x', y', z'\} \cup W'$. Recall that W' is an independent set, and that each vertex in W' is adjacent to x'. Therefore if $w' \in W'$, then in the graph B we have $N(w') \subset \{x', y'z'\}$, and so $\delta(B) \leq 3$.

Suppose that $\delta(B) = 0$. Since x' is adjacent to all vertices of W', in this case we must have $\deg_B(z') = 0$ or $\deg_B(y') = 0$. By assumption, B is a SP-graph, and so by Theorem 1, we have $B \cong K_1 \cup K_{n-1}$. However since x' is adjacent to neither y' nor z', such a construction of B in this case is not possible. Hence, $1 \leq \delta(B) \leq 3$.

Suppose that $\delta(B) = 1$. If |W'| = 1, then by the construction of $G \in \mathcal{F}_2^3$, |W| = 1, and since x' is not adjacent to both y' and z' in B, $\delta(G) = 0$, which is a contradiction. Hence, $|W'| \ge 2$, implying by construction that there is no full vertex in B. Hence applying Theorem 10 to the SP-graph B, the B-SC chain is one of the chains given in part (a), (b) or (c) in the statement of Theorem 10. Thus, the G-SC chain is given by Part (a), (b) or (c) in the statement of Lemma 3. Hence, we may assume that $\delta(B) \ge 2$.

Suppose that $\delta(B) = 2$. Since x' is only adjacent to all vertices of W' and $\delta(B) = 2$, we have $|W'| \ge 2$. Suppose that |W'| = 2, and so B has order 5. All possible SP-graphs of order 5 are C_5 , $M_1, \overline{K}_3 + \overline{K}_2$ and $(\overline{K}_2 + (K_1 \cup K_2))$. Thus, in this case the G-SC chain is given by Part (d), (e), (f) or (g) in the statement of Lemma 3.

Hence, we may assume that in this case $\delta(B) = 2$, we have $|W'| \ge 3$. Since B is an isolate-free SP-graph, by Theorem 5 we have $B \in \mathcal{F}_2$. Thus, $B \in \mathcal{F}_2^1 \cup \mathcal{F}_2^2 \cup \mathcal{F}_2^3$. We show that $B \notin \mathcal{F}_2^3$. Suppose, to the contrary, that $B \in \mathcal{F}_2^3$. Consider the vertices x, y and z, and the sets W, R_1, L_1, R_2 and L_2 used in the definition of \mathcal{F}_2^3 (see Definition 3).

Since $|W'| \ge 3$, the vertex x' has degree at least 3 in B. Since the vertex x has degree 2 in \mathcal{F}_2^3 , the vertex $x' \ne x$. If x = z' and y = y', then $z \in W'$. Hence, $W = \{x\}$ and $W' = X \cup L_2$. Since W'is an independent set of B, there are no edges between the vertices of $X \cup L_2$, which contradicts the definition of the family \mathcal{F}_2^3 . If x = z', $y \in W'$ and $z \in W'$, then $y', z' \notin W$. Hence, $W \subseteq W'$. If $|W \cap W'| = 1$, then it is not hard to see that $R_2 = \emptyset$ or $L_1 = \emptyset$ (see Definition 3), which is a contradiction. If $|W \cap W'| > 1$, then since W' is an independent set, G[W] is not a clique, which is a contradiction. The other cases for the selections of x, y, x, and z analogously leads to a contradiction. Hence, $B \notin \mathcal{F}_2^3$. Thus, $B \in \mathcal{F}_2^1 \cup \mathcal{F}_2^2$. Hence by Theorem 7, $CG(B, \Gamma_1) \in \mathcal{H}_2$ and by Definition 4, we infer that $CG(B, \Gamma_1) \in \mathcal{H}_2^1$ or $CG(B, \Gamma_1) \in \mathcal{H}_2^2$. Hence, using Lemmas 1 and 2, the G-SC chain is given by Part (l), (m), (n), (o), (p), (q), (r), (s), (t), or (u) in the statement of Lemma 3.

Hence, we may assume that $\delta(B) = 3$. If y' is not adjacent to z', then $B \cong K_{3,n-3}$. In this case, the *G*-SC chain is given by Part (v) in the statement of Lemma 3. If y' is adjacent to z', then $B \cong (K_1 \cup K_2) + \overline{K}_{n-3}$. In this case, the *G*-SC chain is given by Part (w) in the statement of Lemma 3. This completes the proof of Lemma 3. \Box

As an immediate consequence of Lemmas 1, 2, and 3, we have the following result.

Theorem 12 If G is a SP-graph of order n with $\delta(G) = 2$ that contains no full vertex, then $L_{SCC}(G) = \infty$ or $L_{SCC}(G) \leq 5$.

5 Conclusion

In this paper, we have addressed two open problems posed by Haynes et al (see [1,5]). The first problem was to characterize all graphs G of order n and minimum degree $\delta(G) = 2$, such that C(G) = n. The second problem was to investigate singleton coalition graph chains. We showed that there exist singleton coalition graph chains with the infinite length. Moreover, we characterized the singleton coalition graph chains starting with the graphs G satisfying $\delta(G) \leq 2$. It would be interesting to characterize the singleton coalition graph chains starting with the graphs G satisfying $\delta(G) \geq 3$.

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